Modeling, Analysis and Control of Fuzzy Systems

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Abstract

For the development of the field of fuzzy control systems, techniques for modeling fuzzy systems need to be developed, which makes analysis of the system and the design of control laws systematic. In this paper, a new model of fuzzy systems is proposed which is a variation of "Tagaki and Sugeno's fuzzy model". Analysis of this model in terms of stability, controllability, observability etc. is much simpler. It also makes model-based control design easier, while retaining the derivations of connections of fuzzy blocks for piecewise continuous polynomial membership functions. Although the model is easier to analyze, it can represent highly nonlinear dynamics.

1. Introduction

Fuzzy logic can be used effectively to deal with uncertainty in decision making processes. Fuzzy control is based on the fuzzy set theory proposed by Zadeh [1], [2] and [3]. There are three major ways to design fuzzy controllers. In the first method, the controller tries to emulate a human-like control action by tranforming linguistic terms into fuzzy variables [4-7]. The second method is to develop heuristic based fuzzy controllers. In the third method, the plant is represented as a fuzzy system and a control is designed by analyzing the fuzzy model.

There are many ways of modeling a fuzzy system [8-13]. In this paper, a variation of the model described in [9] is proposed which makes stability analysis and design of fuzzy control systems simpler. The analysis and design using the model of [9] is detailed in [13].

This paper describes the new model, explains how the fuzzy block structure [13], needed as an important building block for fuzzy systems control theory, is preserved and shows the simplicity of system analysis and control design.

2. Fuzzy Modeling

A fuzzy model of a dynamic process is written in terms of fuzzy implications. Keeping the same notation as [13], the model is described by:

$$\begin{aligned}L^i:\ &\text{If } x(k) \text{ is } A^i_1,\dots,x(k-n+1) \text{ is } A^i_n \text{ and}\\ &u(k) \text{ is } B^i_1,\dots,u(k-m+1) \text{ is } B^i_m\\ \text{then}\ &x^i(k+1)=a^i_0+a^i_1x(k)+.\\ &..+b^i_1u(k-n+1)+\dots+b^i_mu(k-m+1)\end{aligned} \tag{1}$$

Here, L^i refers to the ith implication ($i = 1,\dots,p$); p is the number of implications; $x(k)\dots x(k-n+1)$ are state variables; $u(k),\dots,u(k-m+1)$ are input variables; $A^i_j(j=1,\dots,n)$ and $B^i_q(q=1,\dots m)$ are fuzzy sets with piecewise continuous polynomial membership functions (see [13] for details); $a^i_j(j=0,\dots,n)$ and $b^i_q(q=i,\dots,m)$ are constant real parameters.

The output of the fuzzy model is given by

$$x(k+1)=x^r(k+1) \tag{2}$$

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where $r = \max_i w^i$ (3)

The weight w^i is given by

$$w^i = \prod_{j=1}^{n} A_j^i(x(k-j+1)) \prod_{q=1}^{m} B_q^i(u(k-q+1)) \quad (4)$$

In matrix form Eq.(1) can be represented as

$$L^i : \text{If } \mathbf{x}(k) \text{ is } \mathbf{J}^i \text{ and } \mathbf{u}(k) \text{ is } \mathbf{Q}^i$$

$$\text{then } x^i(k+1) = a_0^i + \mathbf{A}_i\mathbf{x}(k) + \mathbf{B}_i\mathbf{u}(k) \quad (5)$$

where, $\mathbf{x}(k) = [x(k)...x(k-n+1)]^T$, $\mathbf{u}(k) = [u(k)...u(k-m+1)]^T$, $\mathbf{J}^i = [A_1^i...A_n^i]^T$, $\mathbf{Q}^i = [B_1^i...B_m^i]^T$, $\mathbf{A}_i = [a_1^i...a_n^i]$ and $\mathbf{B}_i = [b_1^i...b_m^i]$.

Although the implication equations are linear, the overall system, composed of all the implications can get highly nonlinear.

Example Let us define membership functions A_1^1 and B_1^1 as shown in Fig.1. A fuzzy implication L^1 could be as follows:

If $x(k)$ is A_1^1 and $u(k)$ is B_1^1 then

$x(k+1) = 0.2 + 0.5x(k) + 0.75u(k)$

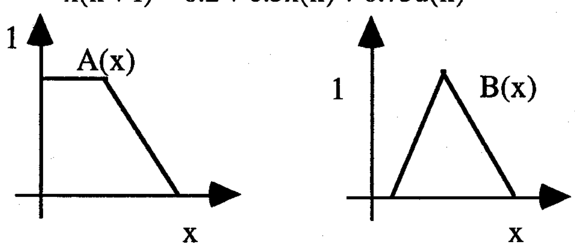

Fig. 1. Membership functions

3. Fuzzy Block Manipulations

A fuzzy block represents a fuzzy input-output relationship of Eq.(1) and can be shown as a block diagram like in Fig.2 [13]. The two important properties of fuzzy blocks are:

(1) The implication equations are linear and

(2) the membership functions of implications are piecewise continuous polynomial functions.

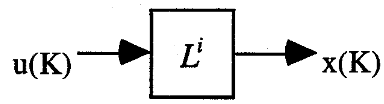

Fig. 2. A fuzzy block

There are two types of fuzzy block connections defined: feedforward as shown in Fig.3 or feedback as shown in Fig. 4. The implications resulting from these connections are described next.

3.1 Feedforward Connection

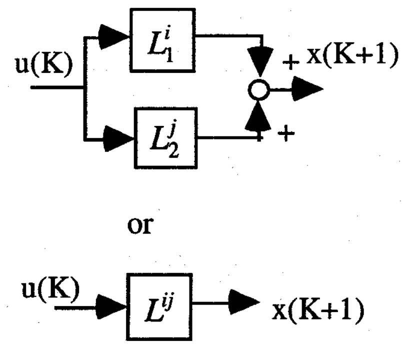

Fig. 3 Feedforward Connection

If L_1^i and L_2^j blocks for a feedforward connection are defined as follows:

$$L_1^i : \text{If } \mathbf{x}(k) \text{ is } \mathbf{P}^i \text{ and } \mathbf{u}(k) \text{ is } \mathbf{Q}^i$$

$$\text{then } x^i(k+1) = a_0^i + \mathbf{A}_i\mathbf{x}(k) + \mathbf{B}_i\mathbf{u}(k) \quad (6)$$

$$L_2^j : \text{If } \mathbf{x}(k) \text{ is } \mathbf{M}^j \text{ and } \mathbf{u}(k) \text{ is } \mathbf{N}^j$$

$$\text{then } x^j(k+1) = c_0^j + \mathbf{C}_j\mathbf{x}(k) + \mathbf{D}_j\mathbf{u}(k) \quad (7)$$

where $i = 1,...,l_1$, $j = 1,...,l_2$, $\mathbf{x}(k) = [x(k)...x(k-n+1)]^T$, $\mathbf{u}(k) = [u(k)...u(k-m+1)]^T$, $\mathbf{P}^i = [A_1^i...A_n^i]^T$, $\mathbf{Q}^i = [B_1^i...B_m^i]^T$, $\mathbf{A}_i = [a_1^i...a_n^i]$, $\mathbf{B}_i = [b_1^i...b_m^i]$, $\mathbf{M}^j = [C_1^j...C_n^j]^T$, $\mathbf{N}^j = [D_1^j...D_m^j]^T$, $\mathbf{C}_j = [c_1^j...c_n^j]$ and $\mathbf{D}_j = [d_1^j...d_m^j]$ (8)

then the implications of the equivalent block are given by:

$$L^{ij} : \text{If } \mathbf{x}(k) \text{ is } (\mathbf{P}^i \text{ and } \mathbf{M}^j) \text{ and } \mathbf{u}(k) \text{ is } (\mathbf{Q}^i \text{ and } \mathbf{N}^j)$$

$$\text{then } x^{ij}(k+1) = (a_0^i + c_0^j) + (\mathbf{A}_i + \mathbf{C}_j)\mathbf{x}(k) + (\mathbf{B}_i + \mathbf{D}_j)\mathbf{u}(k) \quad (9)$$

where the operation "and" implies point by point multiplication of corresponding membership functions. Note that the multiplication of two or more piecewise continuous polynomial functions results in a piecewise continuous polynomial function.

Proof: Let the weights of L^i and L^j systems be w^i and v^j, respectively. The output from the L^i system is $x_1(k+1)$ and the output from the L^j system is $x_2(k+1)$.

$$x_1(k+1) = a_0^q + \mathbf{A}_q\mathbf{x}(k) + \mathbf{B}_q\mathbf{u}(k)$$

where $q = \max_i w^i$

$$x_2(k+1) = a_0^r + \mathbf{A}_r\mathbf{x}(k) + \mathbf{B}_r\mathbf{u}(k)$$

where $r = \max_j v^j$

$$x(k+1) = x_1(k+1) + x_2(k+1)$$

$$x(k+1) = (a_0^q + c_0^r) + (\mathbf{A}_q + \mathbf{C}_r)\mathbf{x}(k) + (\mathbf{B}_q + \mathbf{D}_r)\mathbf{u}(k)$$

This is equivalent to Eq.(9), because the maximum weight of the combined system is $w^q v^r$.

3.2 Feedback Connection

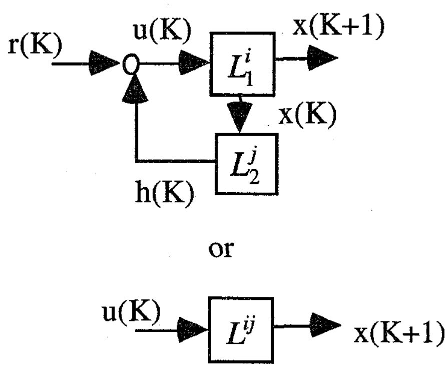

Fig. 4 Feedback Connection

If L_1^i and L_2^j blocks for a feedback connection are defined as follows:

$$L_1^i : \text{If } \mathbf{x}(k) \text{ is } \mathbf{P}^i \text{ and } \mathbf{u}(k) \text{ is } \mathbf{Q}^i$$

$$\text{then } x^i(k+1) = a_0^i + \mathbf{A}_i\mathbf{x}(k) + \mathbf{B}_i\mathbf{u}(k) \qquad (10)$$

$$L_2^j : \text{If } \mathbf{x}(k) \text{ is } \mathbf{M}^j \text{ and } \mathbf{u}(k) \text{ is } \mathbf{N}^j$$

$$\text{then } h^j(k) = c_0^j + \mathbf{C}_j\mathbf{x}(k) \qquad (11)$$

where $i = 1,\ldots,l_1$, $j = 1,\ldots,l_2$, $r(k)$ is the reference input, $u(k) = r(k)\text{-}h(k)$, $\mathbf{x}(k) = [x(k)\ldots x(k-n+1)]^T$, $\mathbf{u}(k) = [u(k)\ldots u(k-m+1)]^T$, $\mathbf{P}^i = [A_1^i\ldots A_n^i]^T$, $\mathbf{Q}^i = [B_1^i\ldots B_m^i]^T$, $\mathbf{A}_i = [a_1^i\ldots a_n^i]$, $\mathbf{B}_i = [b_1^i\ldots b_m^i]$, $\mathbf{M}^j = [C_1^j\ldots C_n^j]^T$, $\mathbf{N}^j = [D_1^j\ldots D_m^j]^T$, $\mathbf{C}_j = [c_1^j\ldots c_n^j]$ and $\mathbf{D}_j = [d_1^j\ldots d_m^j]$ (12)

then the implications of the equivalent block are given by:

$$L^{ij} : \text{If } \mathbf{x}(k) \text{ is } (\mathbf{P}^i \text{and } \mathbf{M}^j) \text{ and } \mathbf{v}(k) \text{ is } (\mathbf{Q}^i \text{and } \mathbf{N}^j)$$

$$L^{ij} : \text{If } \mathbf{x}(k) \text{ is } (\mathbf{P}^i \text{and } \mathbf{M}^j) \text{ and } \mathbf{v}(k) \text{ is } (\mathbf{Q}^i \text{and } \mathbf{N}^j)$$

$$\text{then } x^{ij}(k+1) = a_0^i + \mathbf{A}_i\mathbf{x}(k) + \mathbf{B}_i\mathbf{v}_j(k) \qquad (13)$$

where $\mathbf{v}(k) = [r(k) - e(\mathbf{x}(k)),\ldots,r(k-m+1) - e(\mathbf{x}(k-m+1))]^T$

and function $e(\mathbf{x}(k))=h(k)$ is the input-output relation for the L^j block.

Proof: Let the weights of L^i and L^j systems be w^i and v^j, respectively. The output from the L^i system is $x(k+1)$ and the output from the L^j system is $h(k)$.

$$x(k+1) = a_0^q + \mathbf{A}_q\mathbf{x}(k) + \mathbf{B}_q\mathbf{u}(k)$$

where $q = \max_i w^i$

$$h(k) = c_0^r + \mathbf{C}_r\mathbf{x}(k)$$

where $r = \max_j v^j$

$$\mathbf{u}(k) = \mathbf{r}(k) - \mathbf{h}(k)$$

which leads us to Eq.(13), because the maximum weight of the combined system is $w^q v^r$.

Derivation of e(.): Consider the feedback connection given by Eq.10-13. The final output of L^j block is given by

$$h(k) = c_0^r + \mathbf{C}_r\mathbf{x}(k) \qquad (14)$$

$$\text{where } r = \max_j v^j \qquad (15)$$

Now, v^j is $h(k)$ dependent and is given by

$$v^j = \prod_{p=1}^{n} C_p^j(x(k-p+1)) D_1^j(r(k)-h(k)) \prod_{q=2}^{m} D_q^i(r(k-q+1)-h(k-q+1)) \quad (16)$$

Since $D_1^j(r(k)-h(k))$ is a continuous piecewise polynomial function, Eq. 14-16 can be locally solved for $h(k)$ so that $h(k)=e(\mathbf{x}(k))$. Notice that both the connections, feedforward and feedback, retain the two important fuzzy block properties.

The number of fuzzy imlications for block connections can get very large, hence some model reduction schemes should be utilized. Some schemes detailed in [13] can be used for this model also.

4. Analysis

Since the output of the system is given by only one of the implication equations at a time, depending on the weights, the overall system stability is simply dependent on the stability of its subsystems. In other words, if all the subsystems (i.e the implication equations) are stable, the overall system is too.

The same argument goes for the controllability and observability of the overall system. If all the subsystems are controllable and observable, then the whole system is controllable and observable too.

If any of the subsystems is unstable, uncontrollable or unobservable, then the overall system is also unstable, uncontrollable or unobservable.

This property of this new model makes the analysis of the system and design of controllers for this system very convenient.

5. Controller Design

Since the stability of the subsystems guarantees the stability of the overall system, a feedforward or a feedback system can be designed using the derivations of section-3 so that each of the subsystems is stable with satisfactory dynamics.

A simple example of control design is given next.

Example Let the membership functions of fuzzy sets A and B be as shown in Fig.5

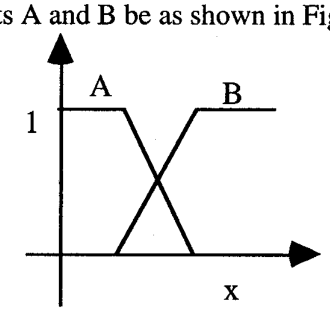

Fig. 5 Membership Functions

The fuzzy system to be controlled is given by

L^1 : If $x(k)$ is A then
$x(k+1) = 0.1x(k) - 0.05x(k-1) + 0.25u(k)$

L^2 : If $x(k)$ is B then
$x(k+1) = 0.15x(k) - 0.05x(k-1) + 0.5u(k)$

If $\mathbf{x}(k) = [x(k-2)\ x(k-1)\ x(k)]^T$, we can write

$$L^1 : \mathbf{x}(k+1) = \mathbf{A}_1\mathbf{x}(k) + \mathbf{B}_1 u(k)$$
$$L^2 : \mathbf{x}(k+1) = \mathbf{A}_2\mathbf{x}(k) + \mathbf{B}_2 u(k)$$

where

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -0.05 & 0.25 \end{bmatrix}, \mathbf{B}_1 = \begin{bmatrix} 0 \\ 0 \\ 0.1 \end{bmatrix},$$

$$\mathbf{A}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -0.05 & 0.15 \end{bmatrix}, \mathbf{B}_2 = \begin{bmatrix} 0 \\ 0 \\ 0.5 \end{bmatrix}.$$

Notice that matrices $[\mathbf{B}_1\ \mathbf{A}_1\mathbf{B}_1\ \mathbf{A}_1^2\mathbf{B}_1]$ and $[\mathbf{B}_2\ \mathbf{A}_2\mathbf{B}_2\ \mathbf{A}_2^2\mathbf{B}_2]$ are full rank, hence [14] both the subsystems are controllable, which in turn makes the overall system controllable.

We could design our fuzzy control as

$$R^1: \text{If } x(k) \text{ is } A \text{ then } u(k) = 0.4x(k)$$

$$R^2: \text{If } x(k) \text{ is } B \text{ then } u(k) = 0.1x(k)$$

Using the feedback fuzzy connection technique developed in section-3, we can see that the controlled system is stable with same dynamics all the time given by

$$x(k+1) = 0.2x(k) - 0.05x(k-1).$$

If instead of L_2 we had

$$L^2: \text{If } x(k) \text{ is } B \text{ then}$$
$$x(k+1) = 0.15x(k) - 0.05x(k-1)$$

then the subsystem-2 would be uncontrollable but stable, so we would control only subsystem-1.

6. Conclusions

A new model for fuzzy systems based on "Takagi and Sugeno's fuzzy model" is developed, which retains all the maneuverabilty of the old model and renders the analysis of the system very easy and control design convenient. Using this model, concepts of stability, controllability, observability etc. can be analyzed for fuzzy systems, which makes the science of fuzzy control much more than just emulating human-like control.

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