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Ascent and decompression of viscous vesicular magma in a volcanic conduit

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Abstract. During eruption, lava domes and flows may become unstable and generate dangerous explosions. Fossil lava-filled eruption conduits and ancient lava flows are often characterized by complex internal variations of gas content. These observations indicate a need for accurate predictions of the distribution of gas content and bubble pressure in an eruption conduit. Bubbly magma behaves as a compressible viscous liquid involving three different pressures: those of the gas and magma phases, and that of the exterior. To solve for these three different pressures, one must account for expansion in all directions and hence for both horizontal and vertical velocity components. We present a new two-dimensional finite element numerical code to solve for the flow of bubbly magma. Even with small dissolved water concentrations, gas overpressures may reach values larger than 1 MPa at a volcanic vent. For constant viscosity the magnitude of gas overpressure does not depend on magma viscosity and increases with the conduit radius and magma chamber pressure. In the conduit and at the vent, there are large horizontal variations of gas pressure and hence of exsolved water content. Such variations depend on decompression rate and are sensitive to the “exit” boundary conditions for the flow. For zero horizontal shear stress at the vent, relevant to lava flows spreading horizontally at the surface, the largest gas overpressures, and hence the smallest exsolved gas contents, are achieved at the conduit walls. For zero horizontal velocity at the vent, corresponding to a plug-like eruption through a preexisting lava dome or to spine growth, gas overpressures are largest at the center of the vent. The magnitude of gas overpressure is sensitive to changes of magma viscosity induced by degassing and to shallow expansion conditions in conduits with depth-dependent radii.

1. Introduction

Key volcanological phenomena such as degassing and lava dome explosions depend on the dynamics of magma ascent toward Earth’s surface [Newhall and Melson, 1983; Newman et al., 1988; Sparks, 1997]. Many different effects are involved, such as large viscosity varia-

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Gas exsolution and bubble growth in magmas have been studied by a number of authors [Sparks, 1978; Proussevitch et al., 1993; Navon et al., 1998; Barclay et al., 1995; Lyakhovsky et al., 1996]. It has been shown that gas overpressure may develop inside bubbles due to the large viscosity of natural magmas [Sparks, 1978]. Most of these studies have relied on “shell” models, such that one bubble surrounded by its small volume of melt expands under prescribed decompression conditions [Proussevitch et al., 1993; Lyakhovsky et al., 1996]. However, decompression depends on both the ascent rate and on density changes due to pressure release. Thus the rates of decompression and flow are coupled together and must be solved for simultaneously for given boundary conditions. Such a self-consistent solution for the evolution of gas pressures in volcanic eruptions has not been attempted.

Viscous compressible flows seem to be a neglected topic in fluid dynamics, presumably due to limited applicability in fields other than physical volcanology, and hence there is a need for some systematic research in this area. In a previous study [Massol and Jaupart, 1999] we have presented a simplified model to investigate the conditions which lead to an overpressured gas phase at a volcanic vent. We showed that gas overpressure varies horizontally in the eruption conduit and is an increasing function of eruption rate. This study captures the basic physical principles involved but relies on several simplifying assumptions which must be assessed with a general solution. For example, it was assumed that motion is purely vertical, whereas lateral pressure variations are likely to drive flow in the horizontal direction. Furthermore, the solutions could only be obtained for constant and small compressibility, whereas volatile exsolution and gas expansion lead to a significant increase of compressibility as pressure decreases. Another interesting problem arises when specifying boundary conditions at the conduit exit, where expansion conditions are sensitive to flow in all directions. For these reasons, we have developed a finite element numerical code to solve for compressible volcanic flows in two dimensions. The code was written to handle complex and large variations of rheological properties as well as vertical variations of conduit size.

The purpose of the present study is twofold. First, we derive the full set of governing equations for compressible viscous flows, discuss the boundary conditions required for a solution and describe the numerical method implemented. Second, we investigate the novel dynamical aspects of such flows. For the sake of clarity we consider first cases with constant viscosity and a straight conduit. We show how the flow characteristics depend on boundary conditions and input parameters, such as reservoir pressure and conduit radius. We study in detail how exit conditions affect eruption behavior and investigate the effects of variations of conduit radius and changes of magma viscosity due to degassing. Volcanological implications are discussed briefly at the end of the paper.

## 2. Flow of Bubbly Magma

### 2.1. Three Different Pressures

In a bubbly flow, gas bubbles expand and hence are at a different pressure than surrounding liquid. Furthermore, because of the flow the liquid pressure is not equal to the country rock pressure. As we shall see, upon exit, the finite rate of expansion implies that the liquid pressure is not equal to the atmospheric pressure. Thus, in general, one must solve for two different pressures, corresponding to the liquid and gas phases. To illustrate the novel physics of such flows, we take the simplest rheological equation which allows a self-consistent solution. The most general rheological law for a Newtonian compressible fluid is

\[
\tau = p\delta - 2\mu e + \frac{2}{3}\mu K (\nabla \cdot \vec{v}) \delta \quad (1)
\]

where \(\tau\) is the stress tensor, \(e\) is the deformation rate tensor, \(\delta\) is the identity tensor, and \(\vec{v}\) is the velocity field. Three viscosity coefficients are involved, the shear viscosity \(\mu\), the bulk viscosity \(K\), and coefficient \(\lambda\) which combines the two. The bulk flow pressure \(p_b\) is the sum of the thermodynamic pressure \(p\) and viscous stresses due to expansion:

\[
p_b = p - K (\nabla \cdot \vec{v}) . \quad (3)
\]

We neglect the effects of volatile species diffusion and assume bulk equilibrium between gas and liquid. This assumption is valid for the relatively slow flows of relevance to lava domes [Lyakhovsky et al., 1996; Navon et al., 1998]. Assuming thermodynamic equilibrium, the gas pressure is the relevant pressure for solubility relationships and for the equation of state. Magmas are weakly compressible compared to gas and can be taken as incompressible liquids. In this case, the above rheological equation can be derived from a “shell” model of bubbly magma [Prud’homme and Bird, 1978] (details can be found in Appendix A). For typical volcanic bubble sizes, one may neglect surface tension, and thermodynamic pressure \(p\) is equal to the gas pressure in a bubble \(p_g\):

\[
p = p_g . \quad (4)
\]

The shell model leads to an explicit equation for bulk viscosity \(K\):

\[
K = \frac{2}{3}\mu \frac{1 - \alpha}{\alpha} , \quad (5)
\]
where $\alpha$ is the volume fraction of gas and $\mu$ is the viscosity of magma. Given the approximations of the shell model, this relationship is valid only at small values of the gas volume fraction. There are unfortunately no reliable theory and experimental determinations for $K$ at all values of the gas volume fraction [Bagdassarov and Dingwell, 1993], and we shall carry out many calculations for $K = 0$. By definition, $K$ is positive, and hence calculations made for $K = 0$ lead to underestimate the values of gas pressure.

2.2. Governing Equations

We assume equilibrium degassing conditions and take the following solubility law:

$$x = s \sqrt{p},$$

where $x$ is the mass fraction of volatiles which may be dissolved in the melt at pressure $p$ and $s$ is a coefficient determined from experiment. For water in silicic melts, we take $s = 4.11 \times 10^{-6} \text{ Pa}^{-1/2}$. Note that solubility is written in terms of gas pressure and not of bulk flow pressure. The density of the magma/gas mixture is given by [Jaupart and Tait, 1990]

$$\rho = \left[ \frac{1}{\rho_1} \frac{1 - x_0}{1 - x} + \frac{1}{\rho_g} \left( \frac{x_0 - x}{1 - x} \right) \right]^{-1},$$

where $x_0$ is the initial water concentration in the melt, $\rho_1$ is magma density, and $\rho_g$ is gas density. In this paper, we consider that water is the only volatile species and use the ideal gas law for $\rho_g$.

Neglecting cooling due to surrounding rocks and assuming steady state conditions, the conservation equations are

$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r \rho u) + \frac{\partial}{\partial z} (\rho w) = 0,$$

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial r} + \rho w \frac{\partial u}{\partial z} = \frac{\partial p}{\partial r} + 2\mu \frac{1}{r} \left( \frac{\partial u}{\partial r} - \frac{u}{r} \right) + \frac{\partial}{\partial r} \left( 2\mu \frac{\partial u}{\partial z} + \lambda \nabla^2 u \right) + \frac{\partial}{\partial z} \left[ \mu \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right) \right],$$

$$\rho \frac{\partial w}{\partial t} + \rho u \frac{\partial w}{\partial r} + \rho w \frac{\partial w}{\partial z} = \frac{\partial p}{\partial z} + \frac{1}{r} \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right) + \frac{\partial}{\partial z} \left( 2\mu \frac{\partial w}{\partial z} + \lambda \nabla^2 w \right) - \rho g,$$

where $u$ and $w$ are the radial and vertical velocity components. The advection terms have been retained for the sake of generality, even though they are likely to be small at the small Reynolds numbers of lava flow eruptions. Note that several new terms are introduced which are neglected in classical volcanic flow models.

In particular, there are horizontal velocity components and vertical gradients of the normal stress component $\tau_{zz}$. We shall see that compressibility induces horizontal flow even if the conduit has straight vertical walls.

2.3. Boundary Conditions

For simplicity, we assume that the densities of country rock and bubble-free magma are equal. We consider fixed pressure conditions at the top of a magma reservoir located at depth $H$ beneath Earth's surface:

$$p_0 = p_a + \rho g H + \Delta P,$$

where $\Delta P$ is an overpressure due to the previous history of replenishment and crystallization. For a straight conduit the incompressible flow solution is very well understood and there is no need to investigate it further. Thus we start the calculations at the saturation pressure $p_s$ such that

$$x_0 = s \sqrt{p_s},$$

where $x_0$ is the initial volatile concentration of magma. The saturation pressure is reached at some depth $h$ which depends on $Q$, the eruption mass flux. $Q$ depends on $\Delta P$ and must be solved for.

The computational domain extends from $z = 0$ (at the bottom) to $z = h$ at the top. In practice, $h$ is not known a priori and is fixed at some arbitrary value. $Q$ is then solved for, and the corresponding value of $\Delta P$ is calculated using the incompressible flow solution between depths $H$ and $h$:

$$Q = \rho_0 \alpha^4 \left( \frac{p_0 - p_s - \rho g}{H - h} \right).$$

Four boundary conditions are required in each direction $(r, z)$. No slip is allowed at the rigid nondeformable conduit walls:

At $r = a$,
$$w = u = 0.$$

At the conduit axis, by symmetry,
$$\frac{\partial w}{\partial r} = 0,$$
$$u = \frac{\partial w}{\partial r} = 0.$$

Below the saturation level, conditions are those of an incompressible liquid flowing in a conduit with straight walls, such that pressure is uniform in a horizontal cross section and that there are no horizontal velocity components. Thus, at $z = 0$, there is no vertical gradient of vertical velocity and the vertical normal stress $\tau_{zz}$ is equal to the saturation pressure:

At $z = 0$,
$$\tau_{zz} = p_s,$$
$$u = 0.$$

At the top of the conduit ($z = h$), flow conditions depend on how lava spreads away from the vent. As usual, boundary conditions are in fact simplifications
Figure 1. Integration domain and boundary conditions. The computational domain is limited to the compressible part of the flow, such that pressures are below the saturation value. Note that at the top there are two possibilities. Variable \( u = 0 \) corresponds to plug flow or spine growth. Variable \( \tau_{rz} \) corresponds to a lava flow spreading horizontally away from the vent.

Figure 1 illustrates the different boundary conditions used. In one-dimensional models with no horizontal velocity, exit boundary conditions are considerably simpler because pressures in both the liquid and gas phases are equal and equal to the external pressure, which is known. In the present model, these two pressures must be calculated as part of the solution. We shall see that, at the two ends of the integration domain, flow conditions vary rapidly over vertical distances called the "entry" and "exit" lengths. Note that, at the vent, the flow pressure is larger than the atmospheric value because the flow is expanding at that level.

2.4. Scaling Considerations

In order to obtain dimensionless governing equations, we first subtract the hydrostatic equilibrium equation for a static volatile-free magma column, with density equal to \( \rho_0 \) everywhere. Height and radial distance are scaled to the total conduit length \( H \) and to the conduit radius \( a \), respectively. Velocities may be scaled with the value for Poiseuille flow, \( a^2 \Delta P/\mu_0 H \), where \( \mu_0 \) is the viscosity of saturated magma. The governing equations introduce two dimensionless numbers, the conduit aspect ratio \( a/H \) and a Reynolds number

\[
Re = \frac{\rho_0 a^2 \Delta P}{\mu_0^2 H}. \tag{20}
\]

A volcanic conduit is such that \( a/H \ll 1 \). Standard dimensional arguments then imply that horizontal velocity components are smaller than vertical ones and that horizontal derivatives dominate over vertical ones. Furthermore, for nonexplosive eruptions, the Reynolds number is small and hence inertial terms are neglected [Wilson et al., 1980; Jaupart and Tait, 1990; Woods, 1995]. This leads to what can be called the "standard" effusive eruption model. In fact, velocity and pressure vary by large amounts over a small vertical extent near the top of the conduit [Massol and Jaupart, 1999]. In this case, the relevant vertical scale is not \( H \) and corresponds to the height over which density varies significantly. Gas pressure varies in both directions, and its value at the vent cannot be fixed a priori, implying that no simple height scale can be extracted from the equation of state. The gas overpressure at the vent probably takes values that are small compared to the overall pressure difference between top and bottom; however, this is the variable we seek to estimate, and it is difficult to simplify the equations without affecting the reliability of the result. For these reasons, we have left the equations in their dimensional form and have kept all terms.

2.5. Numerical Method

The governing equations have been solved with a finite element numerical method. The basic structure of
Table 1. Table of Input Parameters a

<table>
<thead>
<tr>
<th>Boundary condition,</th>
<th>u = 0</th>
<th>( \tau_x = 0 )</th>
<th>( \tau_y = 0 )</th>
<th>( \tau_z = 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass flux, kg s(^{-1})</td>
<td>(3.38 \times 10^6)</td>
<td>(3.6 \times 10^6)</td>
<td>(3.8 \times 10^6)</td>
<td>(10.6 \times 10^6)</td>
</tr>
<tr>
<td>(p(0, H)), MPa</td>
<td>0.56</td>
<td>0.67</td>
<td>0.42</td>
<td>0.34</td>
</tr>
<tr>
<td>(p(a, H)), MPa</td>
<td>0.1</td>
<td>1.5</td>
<td>1.5</td>
<td>1.1</td>
</tr>
<tr>
<td>(\Delta P), MPa</td>
<td>8.85</td>
<td>9.50</td>
<td>9.80</td>
<td>2.71 \times 10^1</td>
</tr>
<tr>
<td>Re</td>
<td>0.13</td>
<td>0.14</td>
<td>0.15</td>
<td>0.42</td>
</tr>
</tbody>
</table>

aIn all calculations, \(H = 1000\) m, \(a = 25\) m, \(\mu = 10^6\) Pa s and \(x_0 = 0.5\) wt %.

the code has been borrowed from Pepper and Heinrich [1992], and we have added new terms in the momentum equation corresponding to the new rheology. The equations are solved in their weak formulation, and details can be found in Appendix B.

3. Compressible Flow Dynamics

In this section, we illustrate the novel dynamical aspects of viscous compressible flows using one particular exit boundary condition \(u = 0\). This boundary condition is adopted implicitly in the standard one-dimensional eruption model. We use the following values for the various variables: \(H = 1000\) m, \(a = 25\) m, and \(x_0 = 0.5\) wt % (Table 1).

3.1. Basic Principles

A simplified model for small and constant compressibility allows an analytical solution, which is useful to understand the behavior of compressible viscous flows [Massol and Jaupart, 1999]. Horizontal velocity components are neglected, and the equation of state for the mixture is

\[
\rho = \rho_0 \left[1 + \beta (p - p_0)\right],
\]

where \(\beta\) stands for compressibility. With these simplifications, gas overpressure varies in the horizontal direction with a maximum value at the conduit axis and reaches its largest value \(p_H\) at the vent:

\[
\frac{p_H - p_0}{p_0 - p_a} = \beta \Delta P = \frac{K + \frac{1}{2} \mu \frac{a^2}{H^2}}{8 \mu} = D.
\]

Equation (22) illustrates two effects which are intuitively obvious. The gas overpressure at the vent is an increasing function of driving pressure \(\Delta P\). It also increases with compressibility \(\beta\). This equation further shows a key point about magma viscosity \(\mu\). The second viscosity coefficient \(K\) is proportional to magma viscosity (equation (5)), and hence \(\mu\) can be cancelled from ratio \((K + \frac{1}{2} \mu)/8 \mu\) in (22). Here two effects counterbalance one another. Increasing magma viscosity decreases the eruption mass flux and hence the decompression rate, which acts to decrease gas overpressure. However, increasing magma viscosity simultaneously acts to impede gas expansion and hence to increase gas overpressure. In this simple model, therefore, gas overpressure does not depend on magma viscosity.

The simple model sheds light on some basic principles of compressible flow dynamics at the cost of important simplifications and must be assessed with a full 2-D model and more complicated equations of state. According to (7), compressibility is not small even at small initial volatile concentrations. For example, for \(x_0 = 0.5\) wt %, the mixture density varies by more than 1 order of magnitude between the saturation and atmospheric pressures. Furthermore, compressibility varies with pressure.

3.2. Horizontal Velocity

Starting from \(z = 0\) at depth \(h\) beneath Earth’s surface, horizontal velocity develops due to compressibility (Figure 2). The vertical velocity profile differs slightly from the Poiseuille profile of the incompressible solution due to horizontal flow and horizontal variations of density (Figure 3). The horizontal velocity at midheight of the compressible part of the flow (at \(z = h/2\)) is a small fraction of the maximum vertical velocity (0.5%). This is due to the fact that, in this particular calculation, horizontal velocity values are set to zero at the top of the conduit. We show below that for the other exit boundary condition (\(\tau_z = 0\)), values of horizontal velocity become comparable to those of vertical velocity.

3.3. Horizontal Variations of Pressure

Pressure is not constant in a horizontal cross section (Figure 4). For the \(u = 0\) exit boundary condition, pressure is greater at the center than at the edges along the entire height of the conduit. With little horizontal motion, gas pressure variations are due to horizontal variations of decompression rate in the conduit. For a material point the decompression rate in steady flow is
Figure 2. Radial profiles of horizontal velocity at different heights above the saturation level. Parameters used in this calculation are listed in Table 1. Note that the velocity profile changes over a small vertical distance.

\[
\frac{\text{Dp}}{\text{Dt}} = \frac{\partial p}{\partial t} + \frac{\partial p}{\partial z}.
\]  
(23)

For small horizontal velocities,

\[
\frac{\text{Dp}}{\text{Dt}} \approx \frac{\partial p}{\partial z}.
\]  
(24)

Vertical velocity \( w \) varies from a maximum at the conduit axis to zero at the walls and hence so does the decompression rate. Because of this, gas pressure is greater at the axis than at the walls. This simple behavior breaks down at shallow levels beneath the vent for the zero shear stress exit boundary condition. In that case, as discussed below, a large horizontal flow component generates a different horizontal gas pressure profile.

3.4. New Terms in the Momentum Balance

In the vertical momentum equation (10) we separate the viscous terms in two groups:

\[
V_1 = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial w}{\partial r} \right),
\]  
(25)

\[
V_2 = \frac{\mu}{r} \frac{\partial u}{\partial z} + \mu \frac{\partial^2 u}{\partial z \partial r} + (2\mu + \lambda) \frac{\partial^2 w}{\partial z^2} + \lambda \left( \frac{\partial^2 u}{\partial r \partial z} + \frac{1}{r} \frac{\partial u}{\partial z} \right).
\]  
(26)

The first group corresponds to what is retained in the standard 1-D model, and the second group has all the additional viscous terms. We find that the \( V_1/V_2 \) ratio typically reaches values of 0.25 near the top of the conduit. The standard model fails because vertical gradients of velocity and horizontal velocities are not negligible.

The difference with the standard model is more dramatic for the radial momentum equation which is usually simplified to \( \partial p/\partial r \approx 0 \). For this equation the extra terms are all significant and introduce horizontal pressure variations.

We found that the "bulk" Reynolds number given by

\[ Re = Q/\alpha u, \]

where \( Q \) is the mass flux of the eruption,

Figure 3. Radial profile of gas pressure at the vent. For this exit boundary condition \( u = 0 \) the gas pressure has a maximum (0.56 MPa) at the axis. Parameters for this calculation are listed in Table 1.
allows an appropriate estimate of the ratio between the advection and viscous terms in the momentum equation (see Table 1). Thus at low Reynolds number the advection terms can be neglected everywhere in the conduit.

3.5. Mass Discharge Rate

The mass eruption rate is slightly smaller than the value for an incompressible liquid with the same viscosity, as in the simplified analytical model of Massol and Jaupart [1999]. The difference is typically a few percent, because compressibility affects the flow over a small vertical distance and hence has little influence on the bulk momentum balance over the whole conduit length. For this reason the mass flux increases almost linearly as a function of chamber overpressure (Figure 5), as in the incompressible case. The mass flux has the same dependence on conduit radius and magma viscosity as the incompressible solution, as shown by the results of Tables 2 and 3.

3.6. Gas Overpressure at the Vent

We find that magma viscosity does not affect the value of the exit pressure (Table 2), as predicted by the simple analytical model. The effect of conduit radius on gas overpressure is smaller than implied by (22) (Table 3). This is due to the fact that, for bubbly magma, compressibility increases with decreasing pressure. As the conduit radius is decreased, the mass flux decreases, which acts to decrease gas overpressure. However, the "local" value of compressibility simulta-

<table>
<thead>
<tr>
<th>Viscosity, $\mu$ (Pa s)</th>
<th>$\Delta P$ (MPa)</th>
<th>Mass flux, $p(0,H)$ (kg s$^{-1}$)</th>
<th>$p(0,H)$ (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^6$</td>
<td>9.8</td>
<td>$3.8 \times 10^6$</td>
<td>0.42</td>
</tr>
<tr>
<td>$10^7$</td>
<td>9.8</td>
<td>$3.8 \times 10^5$</td>
<td>0.42</td>
</tr>
<tr>
<td>$10^8$</td>
<td>9.8</td>
<td>$3.8 \times 10^4$</td>
<td>0.42</td>
</tr>
</tbody>
</table>
Table 3. Effect of Conduit Radius on Ascent Conditions for $H = 1000$ m, $\mu = 10^6$ Pa s, and $x_0 = 0.5$ wt %

<table>
<thead>
<tr>
<th>Boundary Condition</th>
<th>Radius, $r_0$ (m)</th>
<th>$\Delta P$, MPa</th>
<th>Mass Flux, $q_0$, kg s$^{-1}$</th>
<th>$p(0, H)$, MPa</th>
<th>$p(a, H)$, MPa</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u = 0$</td>
<td>5</td>
<td>7.1</td>
<td>$4.3 \times 10^3$</td>
<td>0.27</td>
<td>0.10</td>
</tr>
<tr>
<td>$\tau_{rz} = 0$</td>
<td>5</td>
<td>7.2</td>
<td>$4.4 \times 10^3$</td>
<td>0.20</td>
<td>0.80</td>
</tr>
<tr>
<td>$u = 0$</td>
<td>25</td>
<td>8.9</td>
<td>$3.38 \times 10^6$</td>
<td>0.56</td>
<td>0.10</td>
</tr>
<tr>
<td>$\tau_{rz} = 0$</td>
<td>25</td>
<td>9.8</td>
<td>$3.8 \times 10^6$</td>
<td>0.42</td>
<td>1.5</td>
</tr>
</tbody>
</table>

neously increases, which has the opposite effect. The end result is that, even for a low mass flux due to a small conduit radius, the values of gas overpressure remain significant. As expected, gas overpressure increases with the reservoir overpressure because of the induced increase of decompression rate (Figure 6). Setting bulk viscosity $K$ to nonzero acts to impede bubble expansion further and hence to increase gas overpressure (Figure 7).

Mass discharge rate depends on magma viscosity, conduit radius, and chamber pressure, but only the latter two parameters are relevant for the magnitude of gas pressure at the vent.

4. Volcanic Eruption Conditions

We now discuss several effects which come into play in volcanic systems and which act on the magnitude of gas overpressure.

4.1. Exit Boundary Conditions

We have so far discussed one particular exit boundary condition ($u = 0$) which is not valid for an eruption which feeds a lava flow horizontal spreading away from the vent. For zero shear stress at the vent the solution changes dramatically and exhibits large horizontal velocities. In the example of Figure 8 the horizontal velocity amounts to 30% of the maximum vertical component. The distributions of velocity and gas pressure in this particular solution are illustrated in Plate 1. Particularly noteworthy is the fact that in the vicinity of the vent, pressure is maximum at the conduit walls. This is exactly the opposite of what is achieved with the $u = 0$ boundary condition. At depth, however, the two different flows have the same features: gas pressure is largest at the conduit axis and horizontal velocity components are very small.

The peculiar pressure distribution at shallow levels for the $\tau_{rz} = 0$ boundary condition may be understood as follows. For zero bulk viscosity $K$, the continuity of normal stress reads

$$p(r, h) = p_a + \frac{4}{3} \mu \frac{\partial w}{\partial z} - \frac{2}{3} \mu \left( \frac{\partial u}{\partial r} + \frac{u}{r} \right).$$  \hspace{1cm} (27)

Below the vent the flow accelerates ($\partial w / \partial z > 0$) and hence $p > p_a$. At the wall, $u = w = 0$, and hence
Figure 8. Radial profiles of velocity at the vent for boundary condition $\tau_r = 0$. The horizontal velocity represents almost 30% of the vertical velocity. Parameters used in this calculation are listed in Table 1.

\[ p(a, h) = p_a - \frac{2}{3} \frac{\partial u}{\partial r} \bigg|_{r=a} . \]  

(28)

The horizontal velocity must drop to zero at the conduit walls, and hence $\frac{\partial u}{\partial r}|_{r=a} < 0$, which implies that $p(a, h) > p_a$. In the example of Figure 9, the conduit margins are almost brought back to the saturation pressure, which implies bubble resorption and compression, and hence very small values of vesicularity. For a smaller conduit radius (5 m) the eruption mass flux is smaller and gas overpressures are smaller (Table 3). In this case, pressures at the conduit margins are not brought to saturation values but remain significantly larger than the atmospheric value (Table 3). The value of gas overpressure at the center of the vent increases as a function of chamber overpressure, as for the $u = 0$ boundary condition (Figure 6 and Table 3).

Figure 9 compares the horizontal profiles of gas pressure at the vent for the two boundary conditions for almost the same mass flux. Plates 2 and 3 compare the full distributions of gas pressure in the volcanic conduit. A transition zone is shown on Figure 10 which occurs at the two thirds of the conduit where the velocity changes from pointing toward the axis to pointing toward the wall. For the $\tau_r = 0$ exit boundary condition, the flow field diverges toward the conduit edges. For the $u = 0$ exit boundary condition, horizontal velocities are small everywhere and the particle paths are almost vertical (Plate 2).

Changing the exit boundary condition does not affect the qualitative features of the flow but affects the magnitude and distribution of gas pressure in the erupting lava. For spine growth or plug flow out of the vent, completely degassed lava erupts only from the edges of the vent at the atmospheric pressure. For a lava flow, gas bubbles are overpressured everywhere at the vent.

4.2. Variable Conduit Radius

Comparing results for the two exit boundary conditions emphasizes the role played by expansion conditions at shallow levels. Thus one expects that they also depend on conduit shape. Consider for example a conduit which flares upward. For a comparison with a straight conduit we have used the same values of magma viscosity and initial dissolved water content. The distribution of gas pressure in the conduit has the same characteristics, with a zone of high values near the walls at shallow levels (Plate 4). The very shape of the conduit facilitates expansion and the end result is that gas overpressures are smaller. This effect is obviously a function of the aperture angle for the conduit. Conversely, a conduit which narrows upward leads to larger values of gas overpressure than a straight conduit.

4.3. Variable Magma Viscosity

Magma viscosity depends on the amount of dissolved volatiles. The above solutions emphasize large horizontal variations of gas pressure in the flow, which imply horizontal variations of dissolved volatile contents. In turn, such variations imply variations of magma viscosity, which are likely to influence flow behavior and expansion. We use an empirical equation from Hess and Dingwell [1996]

\[ \log \mu = \left[ -3.545 + 0.833 \ln (x) \right] + \frac{9601 - 2368 \ln (x)}{T - [195.7 + 32.25 \ln (x)]} \]

(29)

where $x$ is the amount of water dissolved in weight percent and $T$ is the temperature in kelvins (Figure 11).

Figure 9. Radial profiles of gas pressure at the vent for the two different exit boundary conditions. The two pressure profiles are totally different. The maximum overpressure is found at the edges for zero shear stress and at the center for zero horizontal velocity. Parameters used in this calculation are listed in Table 1.
Figure 10. Radial profiles of gas pressure and horizontal velocity at midheight in the computational domain for the two different exit boundary conditions. Note that the radial pressure gradient is positive in the region where the horizontal velocity is positive. Parameters used in this calculation are listed in Table 1.

The dependence is strongest at small concentrations, and hence at small pressures, which is particularly relevant to the present study because it focuses on shallow processes below an eruptive vent.

For comparison with the previous calculations, we have rescaled the above equation (29) so that it yields the same viscosity of $10^6$ Pa s for the initial water content of 0.5 wt %. The increase of viscosity which occurs as degassing proceeds impedes expansion and generates higher values of gas pressure than in the constant viscosity case (Plate 5 and Figure 12).

5. Implications for Volcanic Eruptions

5.1. Magnitude of Gas Overpressure at a Vent

This study was not intended to duplicate a true eruption but to show how compressibility affects flow conditions and to identify key variables. The magnitude of gas overpressure depends on a host of effects and variables such as the variation of magma viscosity due to degassing and the shape of the eruptive conduit at shallow levels. Changes of pressure in the magma chamber induce variations of mass discharge rate and gas pressure. Calculations were presented for a single volatile concentration of 0.5 wt %. For such small values of volatile content the compressibility of magma does not affect the mass flux because it only affects flow conditions at shallow levels over a very small portion of the total ascent path. However, it has important effects on eruption conditions because it may generate large values of gas pressure at the vent. For higher volatile concentrations, gas overpressures would be larger, as

Figure 11. Liquid viscosity as a function of dissolved water content for a leucogranitic melt [from Hess and Dingwell, 1996]. Note the very large viscosity variation when water contents are less than about 1%.

Figure 12. Pressure profiles at the conduit exit in the case of variable and constant melt viscosity. Parameters of the calculations are listed in Table 1. Note that pressure is higher for the variable viscosity case which leads to a large increase of viscosity at shallow depths (low dissolved volatile content).
Plate 1. Distribution of gas pressure in the volcanic conduit for zero horizontal shear stress at the vent. The portion of conduit shown corresponds to the compressible part of the flow above the exsolution level, which is 40 m high in this example. Note the zone of high gas pressure near the conduit walls below the vent. Parameters used in this calculation are listed in Table 1.

Plate 2. Pressure contours and particle paths in the volcanic conduit for \( u = 0 \) at the vent. The portion of conduit shown corresponds to the compressible part of the flow above the exsolution level, which is 40 m high in this example. Dots represent new positions after 1 s. Note that particle paths are almost vertical everywhere. Parameters used in this calculation are listed in Table 1.

Plate 3. Pressure contours and particle paths in the volcanic conduit for \( \tau_{xx} \) at the vent. The portion of conduit shown corresponds to the compressible part of the flow above the exsolution level. Dots represent new positions after 1 s. Note that after a height of approximately 25 m, particle paths are oriented toward the conduit walls. Parameters used in this calculation are listed in Table 1.
shown by the simple analytical model (equation (22)). The volume fraction of gas would also increase and the flow would eventually undergo fragmentation. In this paper, we have emphasized the difference between gas pressure, flow pressure, and the exterior pressure, and all questions regarding the behavior of the gas phase must be addressed with a compressible flow model such as ours. Neglect of gas overpressure implies an overestimation of gas volume fraction and eruption velocity.

For the same conduit radius and chamber pressure, two magmas with different viscosities are erupted with different mass discharge rates but the same values of gas overpressure. Thus when evaluating the explosive potential of an effusive eruption, knowledge of the mass flux is not sufficient. One may wonder whether significant overpressures may be reached for the small mass discharge rates of effusive eruptions. For example, typical values for the mass discharge at the recent Unzen and Soufriere Hills eruptions were $4 \times 10^3$ kg s$^{-1}$ [Nakada et al., 1999; Robertison et al., 2000]. For the latter the conduit radius was about 30 m and the initial volatile concentration was as large as 5 wt % [Melnik and Sparks, 2001]. A study of this specific eruption would require an investigation of many parameters, which is outside the scope of this paper. In particular, the large volatile concentration implies large values of gas content at the vent which were not observed, and one must probably invoke separated gas flow [Jaupart and Allègre, 1991; Melnik and Sparks, 2001]. For the sake of example, it is nevertheless useful to evaluate the value of gas overpressure for the parameters of this paper ($H = 1000$ m, $a = 25$ m, $x_0 = 0.5$ wt %). For those the mass discharge of $4 \times 10^3$ kg s$^{-1}$ may be obtained with a reservoir overpressure of 10 MPa and an average magma viscosity of $10^9$ Pa s. Extrapolating the results of Table 2 to higher values of magma viscosity, we obtain an estimate of 0.4 MPa for gas overpressure at the vent. This estimate corresponds to a constant viscosity calculation for $K = 0$ and for the $u = 0$ exit boundary condition and hence must be considered as a lower bound. Such an overpressure is sufficient to drive a dome explosion [Fink and Kieffer, 1993].

5.2. Variations of Gas Overpressure During Eruption

The dramatic effect of exit boundary conditions suggests that eruption conditions may be unsteady depending on the behavior of lava above the vent. In early stages of dome growth, the free boundary condition ($\tau_{r z} = 0$) probably provides the most realistic approximation of exit conditions. In this case, relatively degassed magma issuing from the conduit center gets emplaced on top of overpressured magma coming from the conduit walls. With time, the presence of a thick dome over the vent is likely to change the exit conditions. Dome growth may proceed internally, by emplacement of new lava inside the dome (endogenous growth), or externally by lava extruding at the top of the dome (exogenous growth). Colder and partially crystallized lava offers strong resistance to horizontal spreading. The weakest region in a dome is at the top of the vent because it is there that the lava pile is thickest and most efficiently insulated from the surroundings. This may act to channel the flow vertically, leading to exogenous growth. The $u = 0$ boundary condition is the most appropriate for this case and implies a strong horizontal pressure gradient across the plug, with a maximum at the center. At the surface, such a pressure gradient implies lateral expansion, with the undegassed, and hence less viscous, center flowing outward. This may be responsible for the peculiar crease structures which characterize such phases of dome growth [Anderson and Fink, 1990]. In a dome, depending on its thickness, overpressured gas bubbles may expand during flow on the ground, which may account for explosions documented at many volcanoes, most recently at Soufriere Hills, Montserrat, and Lascar, Chile [Matthews et al., 1997]. One key effect is that for given conduit and magma composition, gas overpressures increase with increasing eruption rate. One may predict that in a time sequence of increasing eruption rate, dome explosions become more frequent. This has been documented during the recent eruption of Soufriere Hills, Montserrat [Cole et al., 1998].

The significant differences of gas overpressure between the $u = 0$ and $\tau_{r z} = 0$ exit boundary conditions, as well as the various regimes of dome growth, suggest that the coupling between the conduit and surface flows plays a key role. Such complex behavior probably cannot be adequately understood using a single exit boundary condition. One important effect to bear in mind is that different values of gas overpressure imply different values of volumetric discharge rate when the mass flux does not change.

5.3. Mule Creek Vent, New Mexico

At Mule Creek, New Mexico, it is possible to observe a fossil eruption conduit over a total vertical extent of 300 m [Stasiuk et al., 1996]. Lava preserved inside this conduit fed a small lava flow or dome, part of which can still be seen today. Remarkable features of this unit are large horizontal variations of vesicularity and almost vesicle-free margins. The data cannot be explained by simple equilibrium thermodynamic models for the known volatile content of the magma [Stasiuk et al., 1996]. Stasiuk et al. [1996] proposed that vesicular magma was permeable and able to leak gas to the surrounding country rock, but there are several difficulties with this explanation, as discussed by Jaupart [1998]. The present study shows that gas may be significantly overpressured and hence that equilibrium thermodynamics are not appropriate. Furthermore, it predicts that, in the case of horizontal spreading away from the vent, gas pressure increases, and hence vesicularity decreases, toward the conduit walls. In the examples shown in Plates 3 and 5, the model actually predicts
that at shallow levels all gas bubbles are resorbed near the conduit walls, exactly as observed at Mule Creek. Thus compressible effects offer an alternative explanation for the observations.

6. Conclusion

At pressures smaller than the saturation threshold, gas bubbles nucleate and grow in magma and induce large amounts of expansion. To solve for the flow of such a compressible viscous mixture, one must account for both horizontal and vertical velocity components, as well as for complex boundary conditions at the conduit exit and large variations of magma viscosity with dissolved water content. To achieve these aims, a flexible finite element numerical code has been implemented. Numerical solutions demonstrate that significant values of gas overpressure (i.e., larger than 1 MPa) may develop in effusive eruptions. Upon exit, there are large variations of gas overpressure depending on distance from the conduit axis, implying the simultaneous eruption of magma batches degassed to varying degrees. For given conduit dimensions and magma composition, the magnitude of gas overpressure is an increasing function of chamber pressure, and eruption rate. For given conduit dimensions, chamber pressure, and initial magma volatile content, gas overpressure is independent of magma viscosity. The solutions are sensitive to expansion conditions at shallow levels and depend on the flow regime away from the vent. Alternating phases of exogenous and endogenous growth should be characterized by different values of gas volume fraction in lava.

Appendix A: Shell Model for a Mixture of Gas and Liquid

We briefly repeat the main steps from Prud'homme and Bird [1978] to determine the behavior of a compressible mixture undergoing expansion. The mixture properties must be equivalent to those of a gas bubble surrounded by a shell of incompressible viscous liquid (see Figure A1). This analysis establishes the relationship between the pressure of gas inside a bubble and the variables of the equivalent compressible fluid.

A1. Spherical Cell of Compressible Liquid

A compressible material whose rheological equation is given by (1) fills a spherical cell of radius $R$. During expansion, velocity is purely radial and density is constant in the cell and hence

$$\rho = \rho_0 \left( \frac{R_0}{R} \right)^3. \quad (A1)$$

Mass conservation and radial momentum conservation equations are

$$\frac{\partial p}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \tau_{rr} \right) = 0, \quad (A2)$$

$$-\frac{\partial p_m}{\partial r} - \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \tau_{rr} \right) + \tau_{\theta\theta} + \tau_{\phi\phi} = 0, \quad (A3)$$

where $v_r$ is the radial velocity, $\tau$ is the stress tensor, and $p_m$ is the pressure. From (A2) and (A1),

$$v_r = \frac{r}{R} \dot{R}. \quad (A4)$$

Using (A4), we obtain

$$\tau_{rr} = \tau_{\theta\theta} = \tau_{\phi\phi} = -3K \frac{\dot{R}}{R}, \quad (A5)$$

and (A3) leads to

$$\frac{\partial p_m}{\partial r} = 0. \quad (A6)$$

The normal stress at the outer edge of the cell is thus

$$[\Pi_{rr}]_{r=R} = p_m - 3K \frac{\dot{R}}{R}. \quad (A7)$$

A2. Gas Bubble in a Liquid Shell

A spherical bubble of radius $b$ is surrounded by a shell of incompressible liquid with viscosity $\mu_l$. In the incompressible liquid the governing equations are

$$\frac{\partial}{\partial r} \left( r^2 v_r \right) = 0, \quad (A8)$$

$$-\frac{\partial p_l}{\partial r} + \mu_l \frac{\partial}{\partial r} \left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 v_r \right) \right] = 0. \quad (A9)$$

Integration of (A8) leads to

$$v_r = \dot{R} \left( \frac{R^2}{r^2} \right) = \dot{b} \left( \frac{b^2}{r^2} \right). \quad (A10)$$

For this velocity field one finds again that $\partial p_l/\partial r = 0$. At the liquid-gas interface,

$$p_b = \frac{2\sigma}{b} + p_f + \tau_{rr}, \quad (A11)$$

where $p_b$ is the pressure of gas in the bubble. This leads to
Plate 4. Pressure contours and particle paths in a conduit which flares upward. Dots represent new positions after 1 s. Parameters used in this calculation are listed in Table 1.

Plate 5. Pressure field in the volcanic conduit for magma viscosity which varies as a function of dissolved water content as specified in (29). Height zero corresponds to the exsolution level. Parameters of the calculations are listed in Table 1.
\[ p_f = p_b - \frac{2\sigma}{b} - 4\mu \frac{R^2 \ddot{R}}{b^3}, \quad (A12) \]

\[ \Pi_{rr} \bigg|_{r=R} = p_b - \frac{2\sigma}{b} - 4\mu \frac{R R^2}{b^3} \left(1 - \frac{b^3}{R^3}\right) \quad (A13) \]

**A3. Gas Pressure and the Second Viscosity Coefficient**

Identifying terms in the two normal stress expressions (A7) and (A13), we obtain

\[ p_m = p_b - \frac{2\sigma}{b}, \quad (A14) \]

\[ K = 4\mu \frac{1 - \alpha}{3}, \quad (A15) \]

where \( \alpha = \frac{b^3}{R^3} \) is the volume fraction of gas in the cell. For typical bubble sizes, surface tension can be neglected. Bulk viscosity \( K \) introduces a difference between gas pressure and the bulk flow pressure (see (3)).

A key point is that the above reasoning does not depend on the mass of gas inside the bubble and can therefore be used for bubble growth driven by diffusion. Depending on the diffusion coefficient and the decompression rate, volatile concentration may not be uniform in the liquid phase. Local thermodynamic equilibrium at the bubble wall implies that gas pressure depends on the local volatile concentration in the adjacent liquid as specified by the solubility law. The average volatile concentration in the liquid phase is larger than the local value at the interface. Thus if we define an effective gas pressure to be such that it corresponds to an equilibrium with the average volatile concentration, it is larger than the gas pressure in the bubble. In a rough approximation this effect can be accounted for by increasing the bulk viscosity coefficient \( K \). In this sense therefore the estimates of gas overpressures given in the paper must be treated as lower bounds. For the relatively small decompression rates associated with dome eruptions, however, the assumption of bulk equilibrium between gas and liquid is valid [Lyakhovsky et al., 1996; Navon et al., 1998].

**Appendix B: Finite Element Numerical Method**

**B1. Weak Formulation**

Given \( \mathcal{L} \) a partial differential operator, we seek solutions \( u(x) \) for which

\[ \mathcal{L} u = f \quad (B1) \]

at every \( x \) in \( \Omega \). Variable \( u(x) \) is as an element of a Hilbert space \( \mathcal{H} \). We obtain an approximate solution:

\[ \hat{u}(x) = \sum_{i=1}^{G} u_i N_i(x), \quad (B2) \]

where \( N \) stands for a weighting function. The solution is not known everywhere in the integration domain but only at the nodes of the computational grid. Recalling the definition of the inner product,

\[ <u, v> = \int_{\Omega} uv \, d\Omega, \quad (B3) \]

the weak solution \( \hat{u} \) is obtained with the Petrov-Galerkin weighted residual method:

\[ <\mathcal{L}\hat{u} - f, N> = 0. \quad (B4) \]

Here the weighting functions are identical to the interpolation functions, except for the advection terms. For those the weighting functions are modified as follows in order to avoid artificial numerical diffusion [Brueckner and Heinrich, 1991]:

\[ W_i = N_i + \frac{\gamma h_e}{2|V|} \left(\frac{\partial N_i}{\partial r} + \frac{\partial N_i}{\partial z}\right), \quad (B5) \]

where \( h_e \) is the element size, \(|V|\) is the average velocity over an element, and coefficient \( \gamma \) is defined as

\[ \gamma = \coth \frac{\Lambda}{2} - \frac{2}{\Lambda}, \quad (B6) \]

\[ \Lambda = \frac{\rho Re|V|h_e}{\mu}. \quad (B7) \]

The interpolation functions \( N_i \) are bilinear (Appendix C). For each rectangular element there are four weighting functions, and each conservation equation is written 4 times in the form of (B4).

From the governing equations, we obtain

\[ \int_{\Omega} \rho \frac{\partial u}{\partial t} N d\Omega = \int_{\Omega} (\nabla \cdot \mathbf{v}) N d\Omega - \int_{\Omega} \rho \frac{\partial N}{\partial t} d\Omega - \int_{\Omega} \rho \frac{\partial u}{\partial r} W d\Omega - \int_{\Omega} \rho \frac{\partial u}{\partial z} W d\Omega, \quad (B8) \]

\[ \int_{\Omega} \rho \frac{\partial w}{\partial t} N d\Omega = \int_{\Omega} (\nabla \cdot \mathbf{v}) \cdot \mathbf{e}_z \cdot N d\Omega - \int_{\Omega} \rho \frac{\partial N}{\partial z} d\Omega - \int_{\Omega} \rho \frac{\partial w}{\partial r} W d\Omega - \int_{\Omega} \rho \frac{\partial w}{\partial z} W d\Omega, \quad (B9) \]

\[ \int_{\Omega} \frac{\partial \rho}{\partial t} N d\Omega = -\int_{\Omega} u \frac{\partial \rho}{\partial r} W d\Omega - \int_{\Omega} \frac{\partial \rho}{\partial z} W d\Omega - \int_{\Omega} \frac{\partial \rho}{\partial r} W N d\Omega - \int_{\Omega} \frac{\partial \rho}{\partial z} W N d\Omega, \quad (B10) \]

where \( \Omega \) is the integration domain. Using Green's theorem, the two momentum equations can be rewritten as
\[
\int_{\Gamma} \rho \frac{\partial u}{\partial t} N d\Omega = \int_{\Gamma} (N \mathbf{\bar{\alpha}} \cdot \mathbf{e}_r) \cdot n d\Gamma
- \int_{\Omega} (\nabla N) \cdot (\mathbf{\bar{\alpha}} \cdot \mathbf{e}_r) d\Omega - \int_{\Omega} \frac{\partial \mathbf{\bar{\alpha}}}{\partial t} N d\Omega
- \int_{\Omega} \rho u \frac{\partial w}{\partial t} W d\Omega - \int_{\Omega} \rho w \frac{\partial u}{\partial t} W d\Omega - \int_{\Omega} \rho g N d\Omega,
\]

where \(\Gamma\) is the boundary of the integration domain. These equations are solved for each element. The boundary integrals need only be solved for the elements located at the boundaries of the computational domain, because they represent edge forces which cancel one another when two neighboring elements are considered. Equations (B10)-(B12) are written in matrix form as follows:

\[
(M) \{\hat{u}\} = (S), \quad (B13)
\]
\[
(M) \{\hat{w}\} = (S)_W, \quad (B14)
\]
\[
[D] \{\hat{\rho}\} = (S)_\rho, \quad (B15)
\]

where \((S)_i\) stand the right-hand sides of (B11), (B12), and (B10), respectively. Matrices \((M)\) and \((D)\) are defined as follows:

\[
M_{ij} = \int_{\Omega_i} N_i N_j d\Omega, \quad \text{(B16)}
\]
\[
D_{ij} = \int_{\Omega_i} N_i N_j \rho d\Omega, \quad \text{(B17)}
\]

**B2. "Lumped Mass" Approximation**

In order to solve the above equations, one must invert two matrices. One time-saving procedure is to "lump" the mass matrices into diagonal matrices, such that the condition of mass conservation is satisfied [Pepper and Heinrich, 1992; Zienkiewicz and Taylor, 1991]:

\[
\sum_i M^l_{ij} = \int_{\Omega_i} N_i N_j d\Omega. \quad \text{(B18)}
\]

One technique is the row sum method, such that

\[
M^l_{ij} = \sum_j M_{ij}, \quad \text{(B19)}
\]

where \(M^l\) is the lumped matrix. The physical meaning of this procedure is that the total mass of an element is distributed amongst the four nodes. With the lumped mass approximation,

\[
(M)^{-1} = (M^l)^{-1}, \quad \text{(B20)}
\]

and (B13), (B14), and (B15) are simplified according to the following scheme:

\[
\{u_{n+1}\} = \{u_n\} + \Delta t [(M^l)^{-1}] S_U. \quad \text{(B21)}
\]

This approximation may not be accurate when solving for a true transient. Here, however, time is an artificial variable used to iterate toward steady state.

**B3. Accuracy and Convergence**

We have used two different convergence criteria. For each variable the absolute difference between two successive iterations was kept below a small threshold value,

\[
|X_{n+1} - X_n| < \varepsilon_1. \quad \text{(B22)}
\]

We have also verified that values of the mass flux at the top and bottom of the conduit, noted \(Q_t\) and \(Q_b\), respectively, are very close to one another:

\[
\frac{|Q_b - Q_t|}{Q_b} < \varepsilon_2. \quad \text{(B23)}
\]

We have taken \(\varepsilon_1 = 10^{-8}\) and \(\varepsilon_2 = 10^{-2}\). We further verified that results obtained with two different grids differ by less than 1% (Table B1).

**Appendix C: Interpolation Functions**

For a rectangular element with sides of lengths \(l_1\) and \(l_2\) in the \((r, z)\) directions we use a bilinear interpolation function \(\Phi\):

\[
\Phi(r, z) = \alpha_1 + \alpha_2 r + \alpha_3 z + \alpha_4 rz. \quad \text{(C1)}
\]

at node \(i\). \(\Phi\) may be rewritten as a function of the four nodal values and four other bilinear functions called \(N_1\), \(N_2\), \(N_3\) and \(N_4\):

\[
\Phi(r, z) = N_1(r, z) \Phi_1 + N_2(r, z) \Phi_2 + N_3(r, z) \Phi_3 + N_4(r, z) \Phi_4, \quad \text{(C3)}
\]

---

**Table B1. Accuracy Tests**

<table>
<thead>
<tr>
<th>Grid, (N_r \times N_z)</th>
<th>Mass Flux, (\text{kg s}^{-1})</th>
<th>(p(0, H)), MPa</th>
</tr>
</thead>
<tbody>
<tr>
<td>40 \times 64</td>
<td>(3.38 \times 10^6)</td>
<td>0.565</td>
</tr>
<tr>
<td>80 \times 128</td>
<td>(3.37 \times 10^6)</td>
<td>0.564</td>
</tr>
</tbody>
</table>
such that
\[
N_1(r, z) = \frac{1}{4l_1 l_2} (l_1 - r)(l_2 - z), \quad (C4)
\]
\[
N_2(r, z) = \frac{1}{4l_1 l_2} (l_1 + r)(l_2 - z), \quad (C5)
\]
\[
N_3(r, z) = \frac{1}{4l_1 l_2} (l_1 + r)(l_2 + z), \quad (C6)
\]
\[
N_4(r, z) = \frac{1}{4l_1 l_2} (l_1 - r)(l_2 + z). \quad (C7)
\]

**Notation**

- \( \tau \): stress tensor, Pa.
- \( \varepsilon \): deformation rate tensor, s\(^{-1} \).
- \( \delta \): identity tensor.
- \( \vec{v} \): velocity field, m s\(^{-1} \).
- \( w \): vertical velocity component, m s\(^{-1} \).
- \( u \): horizontal velocity component, m s\(^{-1} \).
- \( \mu \): shear viscosity of the magma + gas mixture, Pa s\(^{-1} \).
- \( \mu_l \): shear viscosity of the liquid, Pa s\(^{-1} \).
- \( \mu_s \): shear viscosity of the saturated magma, Pa s\(^{-1} \).
- \( K \): bulk viscosity, Pa s\(^{-1} \).
- \( \lambda \): second coefficient of viscosity, Pa s\(^{-1} \).
- \( p_b \): bulk flow pressure, Pa.
- \( p \): thermodynamic pressure, Pa.
- \( p_g \): gas pressure, Pa.
- \( p_s \): pressure in the magma chamber, Pa.
- \( p_a \): atmospheric pressure, Pa.
- \( \Delta p \): exsolution pressure, Pa.
- \( \sigma \): surface tension, N m\(^{-1} \).
- \( \rho \): magma + gas mixture density, kg m\(^{-3} \).
- \( \rho_{\text{sat}} \): saturated magma density, kg m\(^{-3} \).
- \( \rho \): magma density, kg m\(^{-3} \).
- \( \rho_g \): gas density, kg m\(^{-3} \).
- \( x \): mass fraction of volatiles dissolved in the magma.
- \( x_0 \): initial mass fraction of volatiles dissolved in the magma.
- \( \alpha \): volume fraction of gas.
- \( \beta \): compressibility, Pa\(^{-1} \).
- \( H \): conduit height, m.
- \( h \): height of the compressible part, m.
- \( a \): conduit radius, m.

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