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Abstract

In this paper, the two dimensional (2-D) frequency sampling filter system function described in reference [1] is further developed for frequency sampling filters that have linear phase and frequency sampling filters that have linear phase and fourfold symmetry. The resulting system functions are computationally more efficient for implementing frequency sampling filters with linear phase and frequency sampling filters with linear phase and fourfold symmetry than the system function described in reference [1].

1 Introduction

A 2-D linear phase filter implemented by direct convolution uses the filter's impulse response as coefficients. If a 2-D linear phase filter has a region of support, R_N, where

$$R_N = \{(n_1,n_2): 0 \le n_1 \le N_1\text{-}1,\ 0 \le n_2 \le N_2\text{-}1\},$$

and n_1 and n_2 are members of the set of integers ($n_1,n_2 \in I$), then the filter's impulse response has the form

$$h(n_1,n_2) = h(N_1\text{-}1\text{-}n_1,N_2\text{-}1\text{-}n_2).$$

If the filter is implemented using direct convolution, approximately $N_1N_2/2$ multiplies are required to compute each output sample. To reduce the number of multiplies required by a direct convolution implementation of a linear phase filter, fourfold symmetry conditions are often imposed on linear phase filters. If a 2-D fourfold symmetric linear phase filter has support over the region, R_N, then the filter's impulse response has the form

$$h(n_1,n_2) = h(N_1\text{-}1\text{-}n_1,N_2\text{-}1\text{-}n_2) = h(N_1\text{-}1\text{-}n_1,n_2) = h(n_1,N_2\text{-}1\text{-}n_2)$$

and a direct convolution implementation of the filter requires $L_1L_2/4$ multiplies per output sample where $L_i = N_i+1$ when N_i is odd and $L_i = N_i$ when N_i is even for $i = 1, 2$.

As a filter's passband narrows and its stopband requirements become more stringent, the values of N_1 and N_2 increase and consequently the number of multiplies per output sample also increase. Thus, increased filter requirements can substantially increase the computational requirements needed by a direct convolution implementation of the filter. Unlike direct convolution implementations, frequency sampling filters use frequency samples, which are specific frequency response values from the filter's frequency response, as coefficients in the filter's implementation. The frequency sampling filter design technique discussed in this paper interpolates a frequency response from a set of N_1N_2 samples from the filter's desired frequency response. Therefore as the filter's passband narrows and the filter's stopband increases, the computational requirements of a frequency sampling filter can decrease.

2 Two Dimensional Frequency Sampling Filters

Consider a filter which has an impulse response, $h(n_1,n_2)$, a region of support R_N where

$$R_N = \{(n_1,n_2): 0 \le n_1 \le N_1\text{-}1,\ 0 \le n_2 \le N_2\text{-}1\} \quad n_1, n_2 \in I$$

and a frequency response, $H(e^{j\omega_1},e^{j\omega_2})$ for $(\omega_1,\omega_2) \in R_C$, where

$$R_C = \{(\omega_1,\omega_2): 0 \le \omega_1 < 2\pi,\ 0 \le \omega_2 < 2\pi\} \quad \omega_1, \omega_2 \in \text{the set of real numbers}$$

Suppose we approximate $H(e^{j\omega_1},e^{j\omega_2})$ for $(\omega_1,\omega_2) \in R_C$ by a discrete set of values taken from the frequency response. Let $H(k_1,k_2)$[1] for $(k_1,k_2) \in R_K$ where

$$R_K = \{(k_1,k_2): 0 \le k_1 \le N_1\text{-}1,\ 0 \le k_2 \le N_2\text{-}1\} \quad k_1, k_2 \in I$$

represent this discrete set of values such that

$$H(k_1,k_2) = H(e^{j\omega_1},e^{j\omega_2})\Big|_{\omega_1=\frac{2\pi}{N_1}k_1,\,\omega_2=\frac{2\pi}{N_2}k_2}.$$

The impulse response, $h(n_1,n_2)$ for $(n_1,n_2) \in R_N$, which interpolates a frequency response through the set of frequency samples, $H(k_1,k_2)$ for $(k_1,k_2) \in R_K$, can be determined from the inverse discrete Fourier transform (IDFT),

$$h(n_1,n_2) = \frac{1}{N_1N_2}\sum_{k_1=0}^{N_1-1}\sum_{k_2=0}^{N_2-1} H(k_1,k_2)e^{j\frac{2\pi}{N_1}n_1k_1}e^{j\frac{2\pi}{N_2}n_2k_2}. \tag{1}$$

If we let $H(z_1,z_2)$ represent the system function of the filter with the impulse response, $h(n_1,n_2)$ for $(n_1,n_2) \in R_N$, then $H(z_1,z_2)$ can be represented by

$$H(z_1,z_2) = \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1} h(n_1,n_2)z_1^{-n_1}z_2^{-n_2} \tag{2}$$

By substituting the Equation (1) into Equation (2), interchanging the order of summation and performing the summation over the n_1 and n_2 indices [1], $H(z_1,z_2)$ becomes

$$H(z_1,z_2) = \frac{1-z_1^{-N_1}}{N_1}\frac{1-z_2^{-N_2}}{N_2}\sum_{k_1=0}^{N_1-1}\sum_{k_2=0}^{N_2-1}\frac{H(k_1,k_2)}{\left(1-e^{j\frac{2\pi}{N_1}k_1}z_1^{-1}\right)\left(1-e^{j\frac{2\pi}{N_2}k_2}z_2^{-1}\right)} \tag{3}$$

Equation (3) has the form of an interpolation formula. The complex polynomial, $H(z_1,z_2)$, interpolates a polynomial through the points $H(k_1,k_2)$ for $(k_1,k_2) \in R_K$ so that

$$H(z_1,z_2)\Big|_{z_1=e^{j\frac{2\pi}{N_1}k_1},\,z_2=e^{j\frac{2\pi}{N_2}k_2}} = H(k_1,k_2)$$

1. $H(k_1,k_2)$ represents the system function, $H(z_1,z_2)$, evaluated at $z_1 = e^{j2\pi k_1/N_1}$ and $z_2 = e^{j2\pi k_2/N_2}$. Although this notation is a mathematical faux pas, it is commonly used throughout the literature and this paper.

Thus a frequency sampling filter's frequency response is designed to pass through a set of N_1N_2 frequency samples.

Equation (3) can be expressed in a computationally more efficient form if we constrain the filter to have a real impulse response and linear phase. Because $H(k_1,k_2)$ is complex, it can be written as

$$H(k_1,k_2) = |H(k_1,k_2)|e^{j\theta(k_1,k_2)} \quad \text{for} \quad (k_1,k_2) \in R_K.$$

where $\theta(k_1, k_2)$ is the phase of $H(k_1, k_2)$. A filter with a real impulse response will have a DFT and frequency samples of the form

$$|H(k_1, k_2)| = |H(N_1-k_1, N_2-k_2)|$$

$$\theta(k_1, k_2) = -\theta(N_1-k_1, N_2-k_2)$$

It can be shown[2] that the phase of a linear phase FIR filter with a region of support R_N is

$$\arg\left[H(e^{j\omega_1}, e^{j\omega_2})\right] = -\omega_1\left(\frac{N_1-1}{2}\right) - \omega_2\left(\frac{N_2-1}{2}\right)$$

Therefore, the phase of the frequency samples is

$$\theta(k_1,k_2) = -\frac{2\pi}{N_1}k_1\left(\frac{N_1-1}{2}\right) - \frac{2\pi}{N_2}k_2\left(\frac{N_2-1}{2}\right)$$

If a linear phase FIR filter has a region of support R_N where N_1 is even, then $H(e^{j\pi}, e^{j\omega_2}) = 0$ [2]. This implies that

$$H\left(\frac{N_1}{2}, k_2\right) = 0 \text{ for } N_1 \text{ even.}$$

Similarly,

$$H\left(k_2, \frac{N_2}{2}\right) = 0 \text{ for } N_2 \text{ even.}$$

Substituting these constraints into Equation (3) yields,

$$\begin{aligned} H(z_1,z_2) = \frac{1-z_1^{-N_1}}{N_1}\frac{1-z_2^{-N_2}}{N_2}\Bigg\{ & \frac{H(0,0)}{\left(1-z_1^{-1}\right)\left(1-z_2^{-2}\right)} \\ & + \sum_{k_1=1}^{M_1} \frac{(-1)^{k_1} 2|H(k_1,0)|\cos\left(\frac{\pi k_1}{N_1}\right)\left(1-z_1^{-1}\right)}{\left(1-z_2^{-1}\right)\left[1-2\cos\left(\frac{2\pi}{N_1}k_1\right)z_1^{-1}+z_1^{-2}\right]} + \sum_{k_2=1}^{M_2} \frac{(-1)^{k_2} 2|H(0,k_2)|\cos\left(\frac{\pi k_2}{N_2}\right)\left(1-z_2^{-1}\right)}{\left(1-z_1^{-1}\right)\left[1-2\cos\left(\frac{2\pi}{N_2}k_2\right)z_2^{-1}+z_2^{-2}\right]} \\ & + \sum_{k_1=1}^{N_1-1}\sum_{k_2=1}^{M_2} \frac{(-1)^{k_1+k_2} 2|H(k_1,k_2)|\left[\cos\left(\frac{\pi k_1}{N_1}+\frac{\pi k_2}{N_2}\right)\left(1+z_1^{-1}z_2^{-1}\right) - \cos\left(\frac{\pi k_1}{N_1}-\frac{\pi k_2}{N_2}\right)\left(z_1^{-1}-z_2^{-1}\right)\right]}{\left[1-2\cos\left(\frac{2\pi}{N_1}k_1\right)z_1^{-1}+z_1^{-2}\right]\left[1-2\cos\left(\frac{2\pi}{N_2}k_2\right)z_2^{-1}+z_2^{-2}\right]}\Bigg\} \end{aligned} \quad (4)$$

where $M_2 = (N_2-1)/2$ when N_2 is odd and $M_2 = N_2/2 - 1$ when N_2 is even.

Equation (4) can be expressed in a computationally more efficient form if we also constrain the filter to have fourfold symmetry. A filter with fourfold symmetry will have a DFT of the form

$$|H(k_1, k_2)| = |H(k_1, N_2-k_2)| = |H(N_1-k_1, k_2)| = |H(N_1-k_1, N_2-k_2)|.$$

Substituting these constraints into Equation (4) yields,

$$H(z_1,z_2) = \frac{1-z_1^{-N_1}}{N_1}\frac{1-z_2^{-N_2}}{N_2}\Bigg\{\frac{H(0,0)}{\left(1-z_1^{-1}\right)\left(1-z_2^{-2}\right)}$$
$$+\sum_{k_1=1}^{M_1}\frac{(-1)^{k_1}2|H(k_1,0)|\cos\left(\frac{\pi k_1}{N_1}\right)\left(1-z_1^{-1}\right)}{\left(1-z_2^{-1}\right)\left[1-2\cos\left(\frac{2\pi}{N_1}k_1\right)z_1^{-1}+z_1^{-2}\right]}$$
$$+\sum_{k_2=1}^{M_2}\frac{(-1)^{k_2}2|H(0,k_2)|\cos\left(\frac{\pi k_2}{N_2}\right)\left(1-z_2^{-1}\right)}{\left(1-z_1^{-1}\right)\left[1-2\cos\left(\frac{2\pi}{N_2}k_2\right)z_2^{-1}+z_2^{-2}\right]}$$
$$+\sum_{k_1=1}^{M_1}\frac{(-1)^{k_1}2\cos\left(\frac{\pi k_1}{N_1}\right)\left(1-z_1^{-1}\right)}{\left[1-2\cos\left(\frac{2\pi}{N_1}k_1\right)z_1^{-1}+z_1^{-2}\right]}\sum_{k_2=1}^{M_2}\frac{(-1)^{k_2}2|H(k_1,k_2)|\cos\left(\frac{\pi k_2}{N_2}\right)\left(1-z_2^{-1}\right)}{\left[1-2\cos\left(\frac{2\pi}{N_2}k_2\right)z_2^{-1}+z_2^{-2}\right]}\Bigg\} \quad (5)$$

where $M_i = (N_i-1)/2$ when N_i is odd and $M_i = N_i/2 - 1$ when N_i is even for $i = 1, 2$.

3 Computational Advantage of Frequency Sampling Filters

When most of the frequency sampling filter's frequency samples, the $H(k_1,k_2)$'s, are exactly zero, most of the frequency sampling filter's resonators[2] do not need to be realized. Therefore, in the case of a narrow band filter where only a small number of the filter's frequency samples are non-zero, the resulting structure may require fewer arithmetic operations than the direct convolution structure.

The frequency sampling filters described by Equations (4) and (5), require exact pole zero cancellation on the 4-D unit sphere in the 4-D complex space. Exact pole zero cancellation is generally not possible when the filter is implemented with finite word lengths. An uncanceled pole on the unit circle will cause the filter to be unstable. To prevent this instability, rz_i^{-1}, $r < 1$, can be substituted for z_i^{-1} in the frequency sampling filter system functions

2. The term *resonator* is used in this paper to denote a system which has either a single pole or a complex conjugate pair of poles on or near the unit circle.

described by Equations (4) and (5) for i = 1, 2. To keep the frequency response of the new filter using rz_i^{-1} as close as possible to the frequency response of the original filter, r is chosen near to the value 1.

Substituting rz_i^{-1} for z_i^{-1} guarantees the stability of a frequency sampling filter at the cost of increasing its computational requirements. If implemented with z_i^{-1} replaced by rz_i^{-1}, a linear phase frequency sampling filter requires at most 8 multiplies per resonator, and a linear phase frequency sampling filter with fourfold symmetry requires at most 6 multiplies per resonator If only K of the frequency samples are non-zero, then a linear phase frequency sampling structure requires approximately 8K multiplies per output sample, and a linear phase frequency sampling filter with fourfold symmetry requires approximately 6K multiplies per output sample. If a linear phase FIR filter with region of support R_N is implemented using direct convolution, it requires approximately $N_1N_2/2$ multiplies per output sample; and if a linear phase FIR filter with fourfold symmetry and region of support R_N is implemented using direct convolution, it requires approximately $N_1N_2/4$ multiplies per output sample. Therefore, if we wish to implement a linear phase filter with region of support R_N, a frequency sampling filter can implement the filter more efficiently (in the sense of fewer multiplies) than a direct convolution implementation when $8K < N_1N_2/2$ or $K < N_1N_2/16$; and if we wish to implement a linear phase filter with fourfold symmetry and region of support R_N, a frequency sampling filter can implement the filter more efficiently than a direct convolution implementation when $6K < N_1N_2/4$ or $K < N_1N_2/24$.

4 Summary

In this paper, a system function for linear phase frequency sampling filters and a system function for linear phase frequency sampling filters with fourfold symmetry were developed. Their computational requirements were compared to the computational requirements of a direct convolution implementation of a filter with an identical region of support.

References

[1] Hu, James V. and Lawrence R. Rabiner, "Design Techniques for Two-Dimensional Filters," IEEE Transactions on Audio and Electroacoustics, Vol. AU-20, No. 4, Oct 1972, pp. 249-257.

[2] Rabiner, Lawrence R and Bernard Gold, Theory and Application of Digital Signal Processing, New Jersey, Prentice Hall, Inc., 1975.