Adaptive Control of Decouplable Systems and Nonlinear Flight Control Systems

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Adaptive Control of Decouplable Systems and Nonlinear Flight Control Systems

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Abstract

This paper treats the question of output trajectory tracking in nonlinear systems in the presence of uncertainty. It is assumed that the nominal system is decouplable using state variable feedback. An adaptive control law is derived such that in the closed-loop system, output vector asymptotically converges to the given reference trajectory. The controller includes a dynamic system in the feedback path. This result is applied to design a flight control system to control roll angle, angle of attack and sideslip in rapid, nonlinear maneuvers of aircraft.

Introduction

Considerable effort has been made in decoupling control of nonlinear systems [1-2]. The derivation of results in these papers require complete knowledge of the system. In a realistic situation, the mathematical model of the system is not completely known. Thus there is a need to design controllers for systems in the presence of uncertainty. Nonlinear flight control systems have recently been designed in literature [3-8].

We consider the trajectory tracking control of a class of nonlinear systems which can be decoupled by state variable feedback. It is assumed that there are uncertain variations in the system parameters and unknown disturbance inputs acting on the system. Based on the Lyapunov stability theory, a nonlinear adaptive control law is derived such that in the closed-loop system the error between the reference trajectory and the output of the system asymptotically tends to zero. The control input is the sum of a decoupling control signal $u_d$ and a control signal $u_a$ which nullifies the effect of uncertainty in
the system. The control law $u_d$ decouples the system when there is no uncertainty in the system. The derivation of the control law is based on a result of [9].

This design approach is applied to synthesize a flight control system for the trajectory control of roll angle, angle of attack, and sideslip in large, rapid maneuvers of aircraft using aileron, rudder and elevator deflections. In this study it is assumed that roll angle and angle of attack represent the basic lateral and longitudinal variables the pilot would like to control.

Problem Formulation

Consider a nonlinear system described by the equation of the form

$$\dot{x}(t) = A(x, t) + \Delta A(x, t) + (B(x, t) + \Delta B(x, t))u(t)$$

$$y(t) = (c_1(x, t), \ldots, c_m(x, t))^T$$

(1)

where the vectors $x(t), u(t),$ and $y(t)$ are real functions of time and of dimension $n \times 1, m \times 1$ and $m \times 1$, respectively. It is assumed that $A, B,$ and $C$ are analytic functions of the variables $x$ and $t$, and the functions $\Delta A$ and $\Delta B$ which represent uncertainty in the system are continuously differentiable with respect to $x$ and $t$. (Often the arguments of functions are suppressed for simplicity). Then nominal system is obtained by setting $\Delta A = 0$ and $\Delta B = 0$ in (1). It is assumed that the solution of (1) exists for any initial condition $(x(t_0, t_0) \in X \times [0, \infty) = M$ where the closed, bounded set $X \subset \mathbb{R}^n$ denotes the state space of the system.

The systems to be considered here are those which can be decoupled by state variable feedback when the uncertain functions $\Delta A$ and $\Delta B$ are zero. A closed-loop system is said to be decoupled if each output is independently controlled by a single input.

The following operators are useful in the sequel.

$$A\xi(x, t) = \frac{\partial c_i}{\partial t}(x, t) + [\frac{\partial c_i}{\partial x}(x, t)] A(x, t)$$

(2)

$$\frac{\partial c_i}{\partial x} = (\frac{\partial c_i}{\partial x_1}, \ldots, \frac{\partial c_i}{\partial x_n})$$

$$A^i c_i(x, t) = A(A^{i-1} c_i)(x, t)$$

$$A^0 c_i(x, t) = c_i(x, t).$$

Let $\alpha_i$ be the least nonnegative integer $j$ such that $[\partial A^j c_i(x, t)/\partial x] B(x, t) \neq 0$ for each $(x, t) \in M$.

We are interested in the class of nonlinear systems for which the following assumptions hold.

Assumption 1. Each $\alpha_i < \infty, i = 1, \ldots, m$ and the $m \times m$ matrix $B^*(x, t)$
is nonsingular at each \((x,t) \in \mathcal{M}\), where

\[
B^*(x,t) = \begin{bmatrix}
\left(\frac{\partial}{\partial x} A^{\alpha_1} c_1(x,t)\right) B(x,t) \\
\vdots \\
\left(\frac{\partial}{\partial x} A^{\alpha_m} c_m(x,t)\right) B(x,t)
\end{bmatrix}
\]

(3)

\[
= [B_1^*(x,t))^T, \ldots, (B_m^*(x,t))^T]^T.
\]

Assumption 2. For \(i = 1, \ldots, m; j = 1, \ldots, \alpha_i - 1\) and \((x,t) \in \mathcal{M}\).

\[
\begin{bmatrix}
\frac{\partial}{\partial x}(A^j c_i(x,t)) \\
\frac{\partial}{\partial x}(A^j c_i(x,t))
\end{bmatrix}
\Delta A(x,t) = 0
\]

(4)

Since \(B^*\) is nonsingular on \(\mathcal{M}\), according to Assumption 1, the nominal system can be decoupled by state variable feedback [1,2]. Assumption 2 implies that uncertain functions do not appear in the \(j\)th derivative of \(y_i\) (denoted as \(y_i^{(j)}\)) along the solutions of (1), \(j = 1, \ldots, \alpha_i; i = 1, \ldots, m\), where \(y = (y_1, \ldots, y_m)^T\).

Using the definition of \(\alpha_i\) and in view of Assumption 2 it follows that the derivatives of \(y_i(t)\) along the trajectories of (1) are given by \((i = 1, \ldots, m)\)

\[
y_i^{(j)}(t) = A^j c_i(x,t), j = 0, 1, \ldots, \alpha_i
\]

\[
y_i^{(\alpha_i+1)}(t) = A^*_i(x,t) + \Delta A^*_i(x,t)
\]

(5)

\[
+ [B_i^*(x,t) + \Delta B_i^*(x,t)] u(t)
\]

where

\[
\Delta A^*_i = \left[\frac{\partial}{\partial x} A^{\alpha_i} c_i(x,t)\right] \Delta A(x,t)
\]

(6)

\[
\Delta B_i^* = \left[\frac{\partial}{\partial x} A^{\alpha_i} c_i(x,t)\right] \Delta B(x,t).
\]

Let an analytic function \(y_r(t) = (y_{r1}(t), \ldots, y_{rm}(t))^T\) be a given reference output trajectory which is to be tracked by system (1). Let \(\tilde{y} = (y_1 - y_{r1}, \ldots, y_m - y_{rm})^T\) be the tracking error, \(\tilde{y}_i^{(j)} = d^j \tilde{y}_i/dt^j, Y(t) = (y_{r1}^{(\alpha_1+1)}(t), \ldots, y_{rm}^{(\alpha_m+1)}(t))^T\), \(A^* = (A^*_1, \ldots, A^*_m)^T\), and \(\Delta A^* = (\Delta A^*_1, \ldots, \Delta A^*_m)^T\).

A decoupling control law \(u = u_d\) is given by

\[
u_d(x,t) = (B^*(x,t))^{-1}(-A^*(x,t) + Y(t))
\]

(7)

\[
- (B^*(x,t))^{-1} \left[ \sum_{i=1}^{\alpha_1+1} k_{i1} \tilde{y}_1^{(i-1)}(t) \right]
\]

where \(k_{ij}\) are constants. The control law \(u_d\) is obtained as a function of \(x\) by substituting the derivatives of \(y_i\) from (5) in (7). In the closed-loop system (1) and (7) when \(\Delta A^* = 0\) and \(\Delta B^* = 0\), one has

\[
\tilde{y}_i^{(\alpha_i+1)} + k_{i,\alpha_i+1} \tilde{y}_i^{(\alpha_i)} + \ldots + k_{ii} \tilde{y}_i = 0
\]

(8)
It follows from (8) that responses \( y_i; i = 1, \ldots, m \) are decoupled. In the presence of uncertainty, additional unknown coupling terms appear in (8) and exact decoupling is not possible.

We are interested in deriving a control law \( u = u_d + (B^*)^{-1}u_a \) such that in the closed-loop system the tracking error \( \tilde{y} \) tends to zero as \( t \to \infty \) in spite of the uncertainty in the system.

### Adaptive Trajectory Control

Let

\[
z = (\tilde{y}_1, \ldots, \tilde{y}_i^{(a_i)}, \ldots; \tilde{y}_m, \ldots, \tilde{y}_m^{(a_m)})^T
\]

where \( z \in \mathbb{R}^p \), and \( p = m + \sum_{i=1}^{m} a_i \). In view of (5), (9) defines a mapping from \( M \times \mathbb{R}^m \) to \( \mathbb{R}^p \) where \( (x, t) \in M \). We assume that \( z \in N \subset \mathbb{R}^p \), where \( N \) is a sufficiently large closed and bounded set.

In view of (5), and (7), the differential equation for \( z \) can be written as

\[
\dot{z} = Ez + Fw
\]

where \( E = \text{diag}(E_i); f = \text{diag}(F_i); i = 1, \ldots, m; E_i \) is a \((a_i + 1) \times (a_i + 1)\) matrix; \( F_i = [0, \ldots, 0, 1]^T \) is a \((a_i + 1)\) vector; \( w = \Delta A^*(x, t) + \Delta B^*[f(x, z, t) + (B^*(x, t))^{-1}u_a] + u_a; f(x, z, t) = u_d(x, t) \) and

\[
E_i = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 1 \\
-k_i1 & -k_i2 & -k_i3 & \cdots & -k_{i,a_i+1}
\end{bmatrix}
\]

The parameters \( k_{ij} \) are selected such that the eigenvalues of \( E \) have negative real parts. Thus given any positive definite symmetric matrix \( Q \) (denoted as \( Q > 0 \)) there exists a unique solution \( P > 0 \) of the Lyapunov equation

\[
E^T P + PE = -Q.
\]

In view of the special structure of (10), we choose \( Q = \text{diag}(Q_{ii}) \), and \( Q_{ii} > 0, 1 = 1, \ldots, m \). Let \( P = \text{diag}(P_{ii}), i = 1, \ldots, m \). Then (11) gives

\[
E_i^T P_{ii} + P_{ii}E_i = -Q_{ii}, \quad i = 1, \ldots, m.
\]

Define \( v = (v_1, \ldots, v_m)^T = F^TPz \), and \( z_i = (\tilde{y}_i, \ldots, \tilde{y}_i^{(a_i)})^T \). Then it follows that \( v_i = F_i^TP_{ii}z_i \).

To this end, it is essential to obtain certain bounds on uncertain functions.

**Assumption 3.** There exist functions \( \gamma_1(x, t) \) and \( \gamma_2(x, z, t) \), and constants \( \gamma_0 \) and \( \gamma_{20} \) such that for each \( (x, t) \in M \), and \( z \in N \)

\[
\begin{align*}
\|\Delta B^*(x, t)(B^*(x, t))^{-1}\| & \leq \gamma_1(x, t) < \gamma_0 < 1 \\
\|\Delta A^*(x, t) + \Delta B^*(x, t)f(x, z, t)\| & \leq \gamma_2(x, z, t) \leq \gamma_{20}
\end{align*}
\]
In order to obtain the structure of the controller, we compute

\[ u_a^T w = u_a^T [\Delta A^*(x, t) + \Delta B^* f(x, z, t) + \{\Delta B^*(B^*(x, t))^{-1} + I\} u_a] \]

\[ \geq ||u_a||^2 (1 - \gamma_1(x, t)) - ||u_a|| \gamma_2(x, z, t) \]

\[ \geq (1 - \gamma_0) ||u_a||[||u_a|| - \Pi(x, z, t, \beta)] \]

where \( \Pi(x, z, t, \beta) = \gamma_2(x, z, t)/(1 - \gamma_0) \) and \( \beta \in \mathbb{R}^k \).

**Assumption 4:** There exists functions \( h_0(x, z, t), h(x, z, t) \) and \( \beta > 0 \) such that for \((x, t) \in M\) and \( z \in N\),

\[ \Pi(x, z, t, \beta) = h_0(x, z, t) + \beta^T h(x, z, t) \] \hspace{1cm} (14)

For the derivation of the adaptive control law it is assumed that the functions \( h_0 \) and \( h \) are known but the constant vector \( \beta \) is unknown which depends on the uncertainty in the system.

We select a control law of the form

\[ u_a(t) = -\Pi(t, x, z, \hat{\beta}) s(t, x, z, \hat{\beta}, \epsilon) \]

\[ \dot{\hat{\beta}}(t) = L h(x, z, t) ||v(z)||, \dot{\beta}(t_0) \epsilon(0, \infty)^k \]

\[ \dot{\epsilon}(t) = -l \epsilon(t), \epsilon > 0, \epsilon(t_0) > 0 \] \hspace{1cm} (15)

where \( L \in \mathbb{R}^{k \times k} \) is diagonal with positive elements, and the function \( s \) is given by

\[ s(t, x, z, \hat{\beta}) = sat[\Pi(t, x, z, \hat{\beta}) v(z) / \epsilon] \] \hspace{1cm} (16)

where

\[ sat \eta = \begin{cases} \eta, ||\eta|| \leq 1 \\ \eta/||\eta||, ||\eta|| > 1 \end{cases} \]

Now we state the following result.

**Theorem 1.** Consider the closed-loop system (1), (7) and (15). Suppose that in the closed-loop system the trajectory \( x(t) \) beginning at \( (x_0, t_0) \in M \) remains in \( X \) for all \( t \geq t_0 \). Then in the closed-loop system, \( z(t) \to 0 \) as \( t \to \infty \).

**Proof:** Consider a Lyapunov function

\[ V(\epsilon, z, \hat{\beta}) = z^T p z + \frac{1}{2} (1 - \gamma_0) \hat{\beta}^T L^{-1} \hat{\beta} + (1 - \gamma_0) l^{-1} \epsilon \] \hspace{1cm} (17)

where \( \hat{\beta} = \hat{\beta} - \beta \). Then one can show that along the trajectory of the closed-loop system

\[ \dot{V} \leq -z^T Q z \] \hspace{1cm} (18)

Since proof can be completed by following the steps of [9] the details are not given here.
Flight Control System Design

For this study the mathematical model of the aircraft has been taken from [10]. The equations of motion are given by

\[
\begin{bmatrix}
\dot{p} \\
\dot{q} \\
\dot{r} \\
\dot{\alpha} \\
\dot{\beta} \\
\dot{\phi}
\end{bmatrix} =
\begin{bmatrix}
l_\beta \beta + l_\gamma q + l_\rho r + (l_\phi \alpha \beta + l_\tau r) \Delta \alpha + l_\rho \rho - i_1 qr \\
\bar{m}_\alpha \Delta \alpha + \bar{m}_q q + i_2 pr - \bar{m}_p \beta + \bar{m}_g (g/V)(\cos \theta \cos \phi - \cos \theta_0) \\
n_\beta \beta + n_r r + n_\rho \rho + n_\rho \rho \rho \Delta \alpha - i_3 pq + n_q q \\
q - p \beta + z_{\alpha} \Delta \alpha + (g/V)(\cos \theta \cos \phi - \cos \theta_0) \\
y_{\beta} \beta + p (\sin \alpha_0 + \Delta \alpha) - r \cos \alpha_0 + (g/V) \cos \theta \sin \phi \\
p + q \tan \theta \sin \phi + r \tan \theta \cos \phi \end{bmatrix}
\begin{bmatrix}
\delta a \\
\delta r \\
\delta e
\end{bmatrix}
\]

\[
\begin{bmatrix}
\dot{\alpha} \\
\dot{\beta} \\
\dot{\phi}
\end{bmatrix}
= A(x) \Delta A(x) + (B(x) + \Delta B(x))u
\]

where state vector and control vector are

\[
x = [p, q, r, \alpha, \beta, \phi, \theta]^T, \quad u = [\delta a, \delta r, \delta e]^T
\]

\[
\bar{m}_{\delta a} = l_\delta a + l_\alpha \delta \Delta \alpha \quad \text{and} \quad \bar{n}_{\delta a} = n_{\delta a} + n_{\alpha \delta} \Delta \alpha
\]

The output vector to be controlled is the linear function of \( x \) given by

\[
y = (\phi, \beta, \alpha)^T = C(x).
\]

The mathematical model of the airplane response in (19) ignores speed changes and contains only a rudimentary representation of aerodynamic nonlinearities. Although the assumption of constant speed in large maneuvers is unrealistic, these simplifications are in no way essential and are used only to make the example more tractable. Speed could be considered as variable and could be decoupled from the other responses by including a throttle control, while introducing more complete nonlinear aerodynamics would simply increase the computational difficulties.

Let \( \hat{A}(x) = A(x) + \Delta A(x) \) and \( \hat{B} = B(x) + \Delta B(x) \). For simplicity in notation, let \( \hat{A}(x) = (f_p, f_q, f_r, f_a, f_{\beta}, f_{\phi}, f_\Theta)^T \). Now we compute the \( \alpha \) parameters and matrices \( A^* + \Delta A^* \) and \( B^* + \Delta B^* \). Using the definition of operators in (2), gives

\[
\hat{A}c(x) = (\hat{A}c_1(x), \hat{A}c_2(x), \hat{A}c_3(x))^T
\]

\[
= (f_\phi, f_\beta, f_\alpha)^T
\]

\[
(\partial C(x)/\partial x)\hat{B}(x) = \begin{bmatrix}
y_{\delta a} & y_{\delta r} & 0 \\
0 & 0 & z_{\delta e}
\end{bmatrix}
\]

\[
(\partial C(x)/\partial x)\hat{B}(x) = \begin{bmatrix}
y_{\delta a} & y_{\delta r} & 0 \\
0 & 0 & z_{\delta e}
\end{bmatrix}
\]

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For simplicity, the effect of control forces is neglected here, since \( y_{\delta a}, y_{\delta r}, \) and \( z_{se} \) are small. Thus we set \( [\partial C/\partial x] \dot{B} = 0 \) and we proceed to compute \( \alpha_i \). It easily follows that \( \alpha_1 = \alpha_2 = \alpha_3 = 1 \),

\[
\begin{align*}
A^*(x) + \Delta A^*(x) &= \hat{A}^2 C_i(x) \\
B^*(x) + \Delta B^*(x) &= [\partial(AC(x)/\partial x)]B(x). \\
\end{align*}
\]

(22)

The expressions for these matrices are computed easily. The matrices \( A^* \) and \( B^* \) are obtained at a given nominal flight condition. The perturbation matrices \( \Delta A^* \) and \( \Delta B^* \) represent the contribution of uncertainty when the flight condition changes. The region of interest in the state space for the control system design is the one in which \( B^*(x) \) is nonsingular.

We are interested in designing a control system to follow reference trajectories generated by a command generator of the form

\[
y_t^{(3)}(t) + g_{c22}y_t^{(2)} + g_{c11}y_t^{(1)} + g_{c00}(y_r - y_t^*) = 0
\]

(23)

where \( y_r = (\phi_r, \beta_r, \alpha_r)^T \). The parameters \( g_{cii} \) are selected by equating the characteristic polynomial of (23) to a standard third-order polynomial

\[
s^3 + g_{c22}s^2 + g_{c11}s + g_{c00} = (s + \lambda_c)(s^2 + 2\zeta_c\omega_{nc}s + \omega_{nc}^2).
\]

(24)

The parameters \( \lambda_c, \zeta_c, \) and \( \omega_{nc} \) are chosen to obtain desirable command trajectories \( y_r \). Let \( \phi = \phi - \phi_r, \beta = \beta - \beta_r, \alpha = \alpha - \alpha_r \).

The control law (7), and (15) is easily determined by using the expressions for \( B^* \) and \( A^* \) evaluated at the nominal flight condition and it takes the form

\[
\begin{align*}
u(t) &= (B^*(x))^{-1}(-A^*(x) + u_a) + (B^*(x))^{-1} \\
& \times \begin{bmatrix} \ddot{\phi}_r - k_{11}\ddot{\phi} - k_{12}\ddot{\beta} \\ \ddot{\beta}_r - k_{21}\ddot{\beta} - k_{22}\ddot{\alpha} \\ \ddot{\alpha}_r - k_{31}\ddot{\alpha} - k_{32}\ddot{\phi} \end{bmatrix}. \\
\end{align*}
\]

(25)

For simplicity, we take \( k_{i,j} = k_{l,j}; \ j = 1, 2; \ i, l = 1, 2, 3. \) Now we determine \( u_a \) using (15).

Conclusions

For a class of decouplable systems, an adaptive control law was derived. The adaptive controller includes a dynamic compensator in the feedback path. In the closed-loop system, the output vector asymptotically converges to the reference trajectories in spite of the uncertainty in the system. Based on this result, an adaptive control law was derived to control roll angle, angle of attack, and sideslip angle of aircraft in rapid, nonlinear maneuvers.

References


