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Zero-sum magic graphs and their null sets

Samuel M. Hansen
University of Nevada, Las Vegas

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ZERO-SUM MAGIC GRAPHS AND THEIR NULL SETS

by

Samuel M. Hansen

Bachelor of Science in Mathematics
University of Wisconsin, Green Bay
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A thesis submitted in partial fulfillment of
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**Samuel M. Hansen**

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**Master of Science in Mathematical Sciences**

Ebrahim Salehi, Committee Chair

Peter Shiue, Committee Member

Hossein Tehrani, Committee Member

Fatma Nasoz, Graduate Faculty Representative

Ronald Smith, Ph. D., Vice President for Research and Graduate Studies and Dean of the Graduate College

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ABSTRACT

Zero-Sum Magic Graphs and Their Null Sets

by

Samuel M. Hansen

Dr. Ebrahim Salehi, Examination Committee Chair
Professor of Mathematics
University of Nevada, Las Vegas

For any $h \in \mathbb{N}$, a graph $G = (V, E)$, with vertex set $V$ and edge set $E$, is said to be $h$-magic if there exists a labeling $l : E(G) \to \mathbb{Z}_h - \{0\}$ such that the induced vertex labeling $l^+ : V(G) \to \mathbb{Z}_h$ defined by

$$l^+(v) = \sum_{uv \in E(G)} l(uv)$$

is a constant map. When this constant is 0 we call $G$ a zero-sum $h$-magic graph. The null set of $G$ is the set of all natural numbers $h \in \mathbb{N}$ for which $G$ admits a zero-sum $h$-magic labeling. A graph $G$ is said to be uniformly null if every magic labeling of $G$ induces zero sum. In this thesis we will identify the null sets of certain classes of Planar Graphs.
I wish to thank my advisor, Ebrahim Salehi. Without his guidance, support, and expertise, this thesis would not have been possible. I would like to thank my father who has supported me throughout my education, often to his own detriment. Many thanks also, to my mother and sisters who have always been there with words of encouragement when I needed them. Special thanks also go to my thesis committee members Peter Shiue, Hossein Tehrani, and Fatma Nasoz for their time and thoughts. Finally I would like to thank my fellow graduate students for being around when I needed it and talking to me even when they did not want to, with particular thanks going to Cody Palmer who was kind enough to help with copy editing.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABSTRACT</td>
<td></td>
<td>iii</td>
</tr>
<tr>
<td>ACKNOWLEDGEMENTS</td>
<td></td>
<td>iv</td>
</tr>
<tr>
<td>LIST OF FIGURES</td>
<td></td>
<td>vi</td>
</tr>
<tr>
<td>CHAPTER 1</td>
<td>Introduction</td>
<td>1</td>
</tr>
<tr>
<td>1.1</td>
<td>Magic-Labelings</td>
<td>1</td>
</tr>
<tr>
<td>1.2</td>
<td>Integer-Magic Labelings</td>
<td>3</td>
</tr>
<tr>
<td>CHAPTER 2</td>
<td>Zero-Sum Magic Graphs</td>
<td>4</td>
</tr>
<tr>
<td>2.1</td>
<td>Zero-Sum Magic</td>
<td>4</td>
</tr>
<tr>
<td>CHAPTER 3</td>
<td>Null Sets of Certain Planar Graphs</td>
<td>8</td>
</tr>
<tr>
<td>3.1</td>
<td>Null sets of Wheels</td>
<td>8</td>
</tr>
<tr>
<td>3.2</td>
<td>Null sets of Fans</td>
<td>11</td>
</tr>
<tr>
<td>3.3</td>
<td>Null sets of Double Wheels</td>
<td>15</td>
</tr>
<tr>
<td>3.4</td>
<td>Null sets of Double Fans</td>
<td>16</td>
</tr>
<tr>
<td>3.5</td>
<td>Prisms and n-Prisms</td>
<td>18</td>
</tr>
<tr>
<td>3.6</td>
<td>Anti-Prisms and n-Anti-Prisms</td>
<td>20</td>
</tr>
<tr>
<td>3.7</td>
<td>Null sets of Grids</td>
<td>22</td>
</tr>
<tr>
<td>3.8</td>
<td>Null sets of Bowties</td>
<td>23</td>
</tr>
<tr>
<td>3.9</td>
<td>Null sets of Axles</td>
<td>26</td>
</tr>
<tr>
<td>BIBLIOGRAPHY</td>
<td></td>
<td>30</td>
</tr>
<tr>
<td>VITA</td>
<td></td>
<td>32</td>
</tr>
</tbody>
</table>
LIST OF FIGURES

Figure 1.1: A Z-magic graph .................................................. 1
Figure 1.2: An example of a non-magic graph .......................... 2

Figure 2.1: Finding the null set of this graph seems difficult. .......... 5
Figure 2.2: Graph $G$ is constructed by six copies of $K_4$. ............. 5
Figure 2.3: The graph $\theta_{3,4,7}$ ............................................. 6
Figure 2.4: Alternating label in a magic labeling ......................... 7

Figure 3.1: A zero-sum labeling of $W_3, W_4$ and $W_6$. ............... 10
Figure 3.2: Two zero-sum-magic labeling of $W_5$. ..................... 11
Figure 3.3: The four-spoke extension of a wheel ......................... 11
Figure 3.4: A typical magic labeling of $F_3$. ............................... 12
Figure 3.5: The fan $F_n$ ($n = 7$). ........................................... 13
Figure 3.6: A zero-sum magic labelings of $F_4, F_5$ and $F_6$. .......... 14
Figure 3.7: The three-blade extension of a fan ............................ 14
Figure 3.8: A zero-sum magic labeling of $DW_3$ and $DW_4$ ............ 15
Figure 3.9: The four blade extension for a double wheel ................. 16
Figure 3.10: A zero-sum magic labeling of $DF_4$ and $DF_5$ ............ 17
Figure 3.11: The four blade extension for a double fan .................. 18
Figure 3.12: A zero-sum magic labeling $P_2C_4$ .......................... 19
Figure 3.13: A zero-sum magic labeling $P_3C_3$ .......................... 20
Figure 3.14: $AP_4$ ............................................................... 20
Figure 3.15: A zero-sum magic labeling of $P_{4,5}$ ........................ 22
Figure 3.16: A zero-sum magic labeling $BT_2, BT_3,$ and $BT_4$ .......... 25
Figure 3.17: The three blade extension of a bowtie ........................ 25
Figure 3.18: A zero-sum magic labeling $AX_3, AX_4,$ and $AX_5$ .......... 27
Figure 3.19: The three blade extension of an axle ........................ 28
CHAPTER 1
INTRODUCTION

1.1 Magic-Labelings

In this thesis all graphs are connected, finite, simple, and undirected. For graph theory notations and terminology not directly defined in this thesis, we refer readers to [2]. For an abelian group $A$, written additively, any mapping $l : E(G) \to A - \{0\}$ is called a labeling. Given a labeling on the edge set of $G$ one can induce a vertex set labeling $l^+ : V(G) \to A$ by

$$l^+(v) = \sum_{uv \in E(G)} l(uv).$$

A graph $G$ is said to be $A$-magic if there is a labeling $l : E(G) \to A - \{0\}$ such that for each vertex $v$, the induced vertex label is a constant map.

![Figure 1.1](image)

Figure 1.1. A $\mathbb{Z}$-magic graph.

In general, a graph $G$ may have multiple labelings that show the graph is $A$-magic. For example, if $|A| > 2$ and $l : E(G) \to A - \{0\}$ is a magic labeling of $G$ with sum $c$, then $l : E(G) \to A - \{0\}$, the inverse labeling of $l$, defined by $l(uv) = -l(uv)$ will be another magic labeling of $G$ with sum $-c$. A graph $G = (V, E)$ is called fully magic if it is $A$-magic for every abelian group $A$. For example, every regular graph is fully
A graph \( G = (V, E) \) is called \textit{non-magic} if for every abelian group \( A \), the graph is not \( A \)-magic. The most obvious example of a non-magic graph is \( P_n \) \((n \geq 3)\), the path of order \( n \). As a result, any graph with a path pendant of length at least two would be non-magic. Here is another example of a non-magic graph: Consider the graph \( H \) in Figure 1.2. Given any abelian group \( A \), a potential magic labeling of \( H \) is illustrated in that figure. The condition \( l^+(u) = l^+(v) \) implies that \( 6x + y = 7x + y \) or \( x = 0 \), which is not an acceptable magic labeling. Thus \( H \) is not \( A \)-magic.

![Figure 1.2](image_url)  

\textit{Figure 1.2.} An example of a non-magic graph.

Certain classes of non-magic graphs are presented in [1].

The original concept of \( A \)-magic graph originated with J. Sedlacek [16] [17], who defined \( A \)-magic-graphs to be a graph with a real-valued edge labeling such that:

1. distinct edges have distinct nonnegative labels

2. the sum of the labels of the edges incident to a particular vertex is the same for all vertices.

Jenzy and Trenkler [4] proved that a graph \( G \) is magic if and only if every edge of \( G \) is contained in a (1-2)-factor. \( Z \)-magic graphs were considered by Stanley [18] [19], who pointed out that the theory of magic labeling can be put into the more general context
of linear homogeneous diophantine equations. Recently, there has been considerable research articles in graph labeling, interested readers are directed to [3, 20]. For convenience, the notation 1-magic will be used to indicate $\mathbb{Z}$-magic and $\mathbb{Z}_h$-magic graphs will be referred to as $h$-magic graphs. Clearly, if a graph is $h$-magic, it is not necessarily $k$-magic ($h \neq k$).

1.2 Integer-Magic Labelings

**Definition 1.1.** For a given graph $G$ the set of all positive integers $h$ for which $G$ is $h$-magic is called the **integer-magic spectrum** of $G$ and is denoted by $IM(G)$.

Since any regular graph is fully magic, then it is $h$-magic for all positive integers $h \geq 2$; therefore, $IM(G) = \mathbb{N}$. On the other hand, the graph $H$, Figure 1.2 is non-magic, therefore $IM(H) = \emptyset$. The integer-magic spectra of certain classes of graphs created through the amalgamation of cycles and stars have been identified in [6] and [7] the integer-magic spectra of the trees of diameter at most four have been completely characterized. Also, the integer-magic spectra of certain other graphs have been studied in [3, 8, 9, 10, 13, 14, 15].
2.1 Zero-Sum Magic

**Definition 2.1.** An $h$-magic graph $G$ is said to be $h$-zero-sum (or just zero-sum) if there is a magic labeling of $G$ in $\mathbb{Z}_h$ that induces a vertex labeling with sum 0. The graph $G$ is said to be uniformly zero-sum if any magic labeling of $G$ induces 0 sum.

A direct result of this definition is that any graph that has an edge pendant is not zero-sum.

**Definition 2.2.** The null set of a graph $G$, denoted by $N(G)$, is the set of all natural numbers $h \in \mathbb{N}$ such that $G$ is $h$-magic and admits a zero-sum labeling in $\mathbb{Z}_h$.

Here are some well known results concerning null sets of graphs by E. Salehi in [13, 12]

**Theorem 2.1.** If $n \geq 4$, then $N(K_n) = \begin{cases} \mathbb{N} & \text{if } n \text{ is odd} \\ \mathbb{N} - \{2\} & \text{if } n \text{ is even} \end{cases}

**Theorem 2.2.** Let $m, n \geq 2$. Then

$$N(K(m, n)) = \begin{cases} \mathbb{N} & \text{if } m + n \text{ is even;} \\ \mathbb{N} - \{2\} & \text{if } m + n \text{ is odd.} \end{cases}$$

**Definition 2.3.** An $h$-magic graph $G$ is said to be uniformly null if every $h$-magic labeling of $G$ induces 0 sum.

**Theorem 2.3.** The bipartite graph $K(m, n)$ is uniformly null if and only if $|m - n| = 1$.

One can introduce a number of operations among zero-sum graphs which produce magic graphs. Here is an example of one such operation.
Definition 2.4. Given \( n \) graphs \( G_i, i = 1, 2, \ldots, n \), the chain \( G_1 \circ G_2 \circ \cdots \circ G_n \) is the graph in which one of the vertices of \( G_i \) is identified with one of the vertices of \( G_{i+1} \). If \( G_i = G \), we use the notation \( \circ G^n \) for the \( n \)-link chain all of whose links are \( G \).

Observation 2.1. If graphs \( G_i \) have zero sum, so does the chain \( G_1 \circ G_2 \circ \cdots \circ G_n \), hence it is a magic graph. Moreover, if \( G_i = G \), then the null set of the chain \( \circ G^n \) is the same as \( N(G) \).

![Figure 2.1](image1.png)

**Figure 2.1.** Finding the null set of this graph seems difficult.

![Figure 2.2](image2.png)

**Figure 2.2.** Graph \( G \) is constructed by six copies of \( K_4 \).
Theorem 2.4. \( N(C_n) = \begin{cases} \mathbb{N} & \text{if } n \text{ is even} \\ 2\mathbb{N} & \text{if } n \text{ is odd} \end{cases} \)

For any three positive integers \( \alpha < \beta \leq \gamma \), the theta graph \( \theta_{\alpha,\beta,\gamma} \) consists of three edge disjoint paths of length \( \alpha, \beta \) and \( \gamma \) having the same endpoints, as illustrated in Figure 2.3. Theta graphs are also known as cycles with a \( P_k \) chord.

![Figure 2.3. The graph \( \theta_{3,4,7} \).](image)

Theorem 2.5.

\[
N(\theta_{\alpha,\beta,\gamma}) = \begin{cases} \mathbb{N} - \{2\} & \text{if } \alpha, \beta, \gamma \text{ have the same parity} \\ 2\mathbb{N} - \{2\} & \text{otherwise} \end{cases}
\]

When \( k \) copies of \( C_n \) share a common edge, it will form an \( n \)-gon book of \( k \) pages and is denoted by \( B(n,k) \).

Theorem 2.6.

\[
N(B(n,k)) = \begin{cases} \mathbb{N} & \text{n is even, } k \text{ is odd} \\ \mathbb{N} - \{2\} & \text{n and } k \text{ are both even} \\ 2\mathbb{N} - \{2\} & \text{n is odd, } k \text{ is even} \\ 2\mathbb{N} & \text{n and } k \text{ both are odd} \end{cases}
\]

Lemma 2.1. (Alternating label) Let \( u_1, u_2, u_3 \) and \( u_4 \) be four vertices of a graph \( G \) that are adjacent \( (u_1 \sim u_2 \sim u_3 \sim u_4) \) and \( \deg u_2 = \deg u_3 = 2 \). Then in any magic labeling of \( G \) the edges \( u_1u_2 \) and \( u_3u_4 \) have the same label.
Given $k \geq 2$ the positive integers $a_1 < a_2 \leq a_3 \leq \cdots \leq a_k$, the generalized theta graph $\theta(a_1, a_2, \cdots, a_k)$ consists of $k$ edge disjoint paths of lengths $a_1, a_2, \cdots, a_k$ having the same initial and terminal points.

When discussing magic labeling of a generalized theta graph $G = \theta(a_1, a_2, \cdots, a_k)$, the alternating label lemma (2.1), allows us to assume that $a_i = 2$ or $3$. For convenience, we will use $\theta(2^m, 3^n)$ to denote the generalized theta graph which consists of $m$ paths of even lengths and $n$ paths of odd lengths.

**Theorem 2.7.** Following the above notations, for any two non-negative integers $m, n$

\[
N(\theta(2^m, 3^n)) = \begin{cases} 
2\mathbb{N} - \{1 - (-1)^{m+n}\} & \text{if } m = 1 \text{ or } n = 1; \\
\mathbb{N} - \{1 - (-1)^{m+n}\} & \text{otherwise.}
\end{cases}
\]
CHAPTER 3
NULL SETS OF CERTAIN PLANAR GRAPHS

3.1 Null sets of Wheels

For \( n \geq 3 \), wheels, denoted \( W_n \), are defined to be \( C_n + K_1 \), where \( C_n \) is the cycle of order \( n \). The integer-magic spectra of wheels are determined in [10].

**Theorem 3.1.** If \( n \geq 3 \), then \( IM(W_n) = \mathbb{N} - \{1 + (-1)^n\} \).

In this section we determine the null sets of wheels. Since the degree set of the \( W_n \) is \( \{3, n\} \), \( W_n \) cannot have a zero-sum magic labeling in \( \mathbb{Z}_2 \). Therefore, for any \( n \geq 3 \), \( 2 \notin N(W_n) \). Let \( u_1 \sim u_2 \sim \cdots \sim u_n \sim u_1 \) be the vertices of the cycle \( C_n \) and \( u \) the center vertex of the wheel. In some cases, for convenience, we may use \( u_{n+1} \) for \( u_1 \) and \( u_{-1}, u_0 \) for \( u_{n-1}, u_n \), respectively. The following observation will be useful in finding the null sets of wheels.

**Observation 3.1.** If \( l : E(W_n) \to \mathbb{Z}_h \) \((h \neq 2)\) is a zero-sum magic labeling, then

\[
2\left(l(u_1u_2) + l(u_2u_3) + \cdots + l(u_{n-1}u_n) + l(u_nu_1)\right) \equiv 0 \pmod{h}.
\]

**Proof.** Let \( l : E(W_n) \to \mathbb{Z}_h \) be the edge labeling that provides zero-sum. Clearly, \( l^*(u) = 0 \) implies that sum of the labels of all spokes is 0. Also, \( l^*(u_k) = 0 \) \((1 \leq k \leq n)\). Therefore,

\[
\sum_{k=1}^{n} l^*(u_k) = 2 \sum l(u_iu_{i+1}) + l^*(u) \\
= 2\left(l(u_1u_2) + l(u_2u_3) + \cdots + l(u_{n-1}u_n) + l(u_nu_1)\right) \equiv 0.
\]

**Observation 3.2.** For every \( n \geq 3 \), \( 3 \in N(W_n) \) if and only if \( n \equiv 0 \pmod{3} \).
Proof. If \( n \equiv 0 \pmod{3} \), then we label all the edges of \( W_n \) by 1 and this provides a zero-sum in \( \mathbb{Z}_3 \). Now suppose \( n \not\equiv 0 \pmod{3} \) and let \( l : E(W_n) \to \mathbb{Z}_3 \) be any magic labeling of \( W_n \) with zero-sum. Then by Observation 3.6, the sum of the labels of the outer edges is 0. Since the outer edges cannot all be labeled 1 (or 2), two adjacent outer edges would have labels 1 and 2. This implies that the spoke adjacent to these two outer edges must have label 0, which is not an acceptable label.

Observation 3.3. If \( W_n \) is zero-sum \( h \)-magic, so is \( W_{kn} \) for every \( k \in \mathbb{N} \).

Proof. Following the notations used above, let \( u_1 \sim u_2 \sim \cdots \sim u_n \sim u_1 \) be the vertices of the cycle \( C_n \) and \( u \) the center vertex of \( W_n \). For \( W_{kn} \), let \( v_1 \sim v_2 \sim \cdots \sim v_{kn} \sim v_1 \) be the vertices of \( C_{kn} \) and \( v \) be its center vertex. Also, assume that \( f : E(W_n) \to \mathbb{Z}_h \) is a magic labeling of \( W_n \) with 0 sum. Now define \( g : E(W_{kn}) \to \mathbb{Z}_h \) by \( g(vv_m) = f(uu_i) \) whenever \( m \equiv i \pmod{n} \) and \( g(v_mv_{m+1}) = f(u_iu_{i+1}) \) whenever \( m \equiv i \pmod{n} \).

Then for the induced vertex labeling \( g^* : V(W_{kn}) \to \mathbb{Z}_h \) we have \( g^*(v) = kf^*(u) = 0 \). Moreover, given any \( v_m \) let \( m = qn + r \) (\( 0 \leq r \leq n - 1 \)). Then \( g^*(v_m) = g(v_{m-1}v_m) + g(v_mv_{m+1}) + g(vv_m) = f(u_{r-1}u_r) + f(u_ru_{r+1}) + f(uu_r) = f^*(u_r) = 0 \).

Therefore, \( g \) is a magic labeling of \( W_{kn} \) with 0 sum.

Corollary 3.1. For any \( n \geq 1 \), \( N(W_{3n}) = \mathbb{N} - \{2\} \).

Proof. Note that \( W_3 \cong K_4 \), for which we have \( N(W_3) = \mathbb{N} - \{2\} \). Therefore, by 3.3, \( N(W_{3n}) = \mathbb{N} - \{2\} \).

Lemma 3.1. For any \( n \geq 3 \), \( \mathbb{N} - \{2, 3\} \subset N(W_n) \).

Proof. To prove the lemma we consider the following four cases:
Case 1. Suppose \( n \equiv 0 \pmod{4} \) or \( n = 4p \) for some \( p \in \mathbb{N} \).

A zero-sum magic labeling of \( W_4 \) is provided in Figure 3.1, which indicates that for every \( h > 3 \), the graph \( W_4 \) admits a zero-sum magic labeling in \( \mathbb{Z}_h \), where \(-2 \equiv 2^{-1}\). Therefore, by Observation 3.3, \( W_{4p} \) has a zero-sum magic labeling in \( \mathbb{Z}_h \). That is \( \mathbb{N} - \{2, 3\} \subset N(W_{4p}) \).

![Figure 3.1](image)

**Figure 3.1.** A zero-sum labeling of \( W_3, W_4 \) and \( W_6 \).

Case 2. Suppose \( n \equiv 1 \pmod{4} \) or \( n = 4p + 1 \) for some \( p \in \mathbb{N} \). We proceed by induction on \( p \) and show that

"for any \( p \), there is a zero-sum magic labeling for \( W_{4p+1} \). Moreover, in this labeling at least one of the outer edges have label 1."

Let \( p = 1 \). In Figure 3.2(A), a zero-sum magic labeling of \( W_5 \) in \( \mathbb{Z}_4 \) is provided. Also, Figure 3.2(B) indicates that \( W_5 \) admits a zero-sum magic labeling in \( \mathbb{Z}_h \) for all \( h \geq 5 \), where \(-1 \equiv 1^{-1}, -2 \equiv 2^{-1} \) and \(-3 \equiv 3^{-1}\).

Now, assume that the statement is true for \( W_{4p+1} \) and let \( u_1u_2 \) be the outer edge of \( W_{4p+1} \) that has label 1. Then we eliminate this edge and insert the four-spoke extension, which is given in Figure 3.3 in such a way that the vertices \( z, v \) and \( w \) of this extension be identified with the central vertex \( u \) and vertices \( u_1, u_2 \) of \( W_{4p+1} \),
respectively. This provides the desired zero-sum magic labeling for $W_{4p+5}$.

![Figure 3.2. Two zero-sum-magic labeling of $W_5$.](image)

**Figure 3.2.** Two zero-sum-magic labeling of $W_5$.

![Figure 3.3. The four-spoke extension of a wheel.](image)

**Figure 3.3.** The four-spoke extension of a wheel.

An argument similar to the one presented in case 2, will also work for the remaining two cases:

**Case 3.** Suppose $n \equiv 2 \pmod{4}$ or $n = 4p + 2$ for some $p \in \mathbb{N}$.

**Case 4.** Suppose $n \equiv 3 \pmod{4}$ or $n = 4p + 3$ for some $p \in \mathbb{N}$.

We summarize the main result of this section in the following theorem:

**Theorem 3.2.** For any $n \geq 3$, $\mathcal{N}(W_n) = \begin{cases} \mathbb{N} - \{2\} & \text{if } n \equiv 0 \pmod{3} \\ \mathbb{N} - \{2, 3\} & \text{if otherwise} \end{cases}$

### 3.2 Null sets of Fans

For $n \geq 2$, Fans, denoted $F_n$, are defined to be $P_n + K_1$, where $P_n$ is the path of order $n$. In this section we determine the null sets of Fans. Since the degree set of the $F_n$ is
\{2, 3, n\}, it cannot have a magic labeling in \(\mathbb{Z}_2\). Therefore, for any \(n \geq 3\), \(2 \notin N(F_n)\).

Note that \(F_2 \equiv C_3\), and we know that \(N(F_2) = 2\mathbb{N}\). Also, a typical magic labeling of \(F_3 \cong K_4 - e\) is illustrated in Figure 3.4(A), for which we require that \(a + b - z = a + b + z\) or \(2z \equiv 0 \pmod{h}\); that is, \(h\) has to be even. On the other hand, if \(h = 2r\), then \(F_3\) admits a zero-sum magic labeling in \(\mathbb{Z}_h\), as indicated in Figure 3.4(B). Therefore, \(N(F_3) = 2\mathbb{N} - \{2\}\). For the general case, let \(u_1 \sim u_2 \sim \cdots \sim u_n\) be the vertices of the path \(P_n\) and \(u\) the central vertex of the fan. We call the edges \(uu_i\) (\(1 \leq i \leq n\)) blades of the fan \(F_n\). The following observation will be useful in finding the null sets of fans.

![Figure 3.4. A typical magic labeling of \(F_3\).](image)

**Observation 3.4.** If \(l : E(F_n) \to \mathbb{Z}_h\) (\(h \neq 2\)) is a zero-sum magic labeling, then

\[
2\left(l(u_1u_2) + l(u_2u_3) + \cdots + l(u_{n-1}u_n)\right) \equiv 0 \pmod{h}.
\]

Proof. Let \(l : E(W_n) \to \mathbb{Z}_h\) be the edge labeling that provides zero-sum. Clearly, \(l^*(u) = 0\) implies that sum of the labels of all blades is 0. Also, \(l^*(u_k) = 0\) (\(1 \leq k \leq n\)). Therefore,

\[
\sum_{k=1}^{n} l^*(u_k) = 2 \sum l(u_iu_{i+1}) + l^*(v)
\]

\[
= 2\left(l(u_1u_2) + l(u_2u_3) + \cdots + l(u_{n-1}u_n)\right) \equiv 0.
\]

**Theorem 3.3.** \(N(F_2) = 2\mathbb{N}\), \(N(F_3) = 2\mathbb{N} - \{2\}\) and for any \(n \geq 4\),
\[ N(F_n) = \begin{cases} \mathbb{N} - \{2\} & \text{if } n \equiv 1 \pmod{3}; \\ \mathbb{N} - \{2, 3\} & \text{otherwise.} \end{cases} \]

\textbf{Proof.} First let us first consider if \( n \geq 2, \ 3 \in N(F_n) \). We know from above that \( 3 \not\in N(F_n) \) for \( n = 2, 3 \). Suppose \( n \geq 4 \) and \( n \equiv 1 \pmod{3} \). Then we label all the edges of \( P_n \) by 2, the two outer blades by 1 and all other blades by 2, as illustrated in Figure 3.5. This is a zero-sum magic labeling of \( F_n \).

![Figure 3.5. The fan \( F_n \) (\( n = 7 \)).](image)

Next, suppose \( n \not\equiv 1 \pmod{3} \) and let \( l : E(F_n) \rightarrow \mathbb{Z}_3 \) be a zero-sum magic labeling. By Observation 3.4 we require that \( l(u_1u_2) + l(u_2u_3) + \cdots + l(u_{n-1}u_n) \equiv 0 \pmod{3} \), which implies that at least two adjacent edges of \( P_n \) are labeled 1 and 2. But this will force the label of the blade adjacent to these edges be 0, which is not an acceptable label. Therefore, such a zero-sum magic labeling does not exist and \( n \geq 2, \ 3 \in N(F_n) \) if and only if \( n \equiv 1 \pmod{3} \).

To finish the proof of the theorem we consider the following three cases:

\textbf{Case 1.} Suppose \( n \equiv 1 \pmod{3} \) or \( n = 3p + 1 \) for some \( p \in \mathbb{N} \). We proceed by induction on \( p \) and show that

"for any \( p \), there is a zero-sum magic labeling for \( F_{3p+1} \). Moreover, in this
labeling at least one of the edges of $P_n$ has label 2.”

Let $p = 1$. In Figure 3.6 a zero-sum magic labeling of $F_4$ is provided in $\mathbb{Z}_h$ for all $h \geq 4$, where $-1 \equiv 1^{-1}$, $-2 \equiv 2^{-1}$ and $-3 \equiv 3^{-1}$.

![Figure 3.6. A zero-sum magic labelings of $F_4$, $F_5$ and $F_6$.](image)

Now, assume that the statement is true for $F_{3p+1}$ and let $u_i u_{i+1}$ be the edge of $P_{3p+1}$ that has label 2. Then we eliminate this edge and insert the three-blade extension, which is given in Figure 3.7 in such a way that the vertices $z, v$ and $w$ of this extension be identified with vertices $u, u_i, u_{i+1}$ of $F_{3p+1}$, respectively. This provides the desired zero-sum magic labeling for $F_{3p+4}$.

![Figure 3.7. The three-blade extension of a fan.](image)

An argument similar to the one presented in case 1, will also work for the remaining two cases:
**Case 2.** Suppose \( n \equiv 2 \pmod{3} \) or \( n = 3p + 2 \) for some \( p \in \mathbb{N} \).

**Case 3.** Suppose \( n \equiv 0 \pmod{3} \) or \( n = 3p \) for some \( p \in \mathbb{N} - \{0\} \).

### 3.3 Null sets of Double Wheels

For \( n \geq 3 \), double wheels, denoted by \( DW_n \), are the cycle \( C_n(u_1 \sim u_2 \sim \cdots \sim u_n \sim u_1) \) together with two additional vertices \( v \) and \( w \) that are connected to all vertices of the cycle. In this section we determine the null sets of double wheels.

**Theorem 3.4.** For all \( n \geq 3 \)

\[
N(DW_n) = \begin{cases} 
\mathbb{N} & \text{if } n \equiv 0 \pmod{2}; \\
\mathbb{N} - \{2\} & \text{if } n \equiv 1 \pmod{2}
\end{cases}
\]

**Proof.** To prove the Theorem we consider the following two cases:

**Case 1.** Suppose \( n \equiv 1 \pmod{2} \) or \( n = 2p + 1 \) for some \( p \in \mathbb{N} \). First observe that if \( n \) is odd then the degree set of \( DW_n \) is \( \{4, n\} \) and \( DW_n \) is not \( \mathbb{Z}_2 \)-magic, therefore can not have a zero-sum labeling in \( \mathbb{Z}_2 \). Now we proceed by induction on \( p \) and show that

"for any \( p \), there is a zero-sum magic labeling for \( DW_{2p+1} \). Moreover, in this labeling at least one of the edges of \( C_n \) has label 1."

![Figure 3.8. A zero-sum magic labeling of \( DW_3 \) and \( DW_4 \)](image)
In Figure 3.8, a zero-sum magic labeling of $DW_3$, $p = 1$, is provided in $\mathbb{Z}_h$ for all $h \geq 3$, where $-1 \equiv 1^{-1}$ and $-2 \equiv 2^{-1}$. Now, assume that the statement is true for $DW_{2p+1}$ and let $u_iu_{i+1}$ be the edge of $C_{2p} + 1$ that has label 1. Then we eliminate this edge and insert the four-blade extension, given in Figure 3.9, with vertices $a$, $b$, $c$, and $d$ such that $a$ and $d$ are identified with $u_i$ and $u_{i+1}$ respectively. This provides the desired zero-sum magic labeling for $DW_{2p+3}$.

**Figure 3.9.** The four blade extension for a double wheel

**Case 2.** Suppose $n \equiv 0 \pmod{2}$ or $n = 2p$ for some $p \in \mathbb{N} - \{0\}$. Then to prove $2 \in N(DF_n)$ one must observe that the degree set of $DW_{2p} = \{4, n\}$. Since $n$ is even a labeling of 1 on all edges yields a zero-sum-magic labeling and $2 \in N(DW_n)$. Finally, an argument similar to one presented in case 1 will suffice to show that for $k \geq 3$, $k \in N(DW_n)$.

3.4 Null sets of Double Fans

For $n \geq 2$, double fans, denoted by $DF_n$, are the path $C_n(u_1 \sim u_2 \sim \cdots \sim u_n)$ together with two additional vertices $v$ and $w$ that are connected to all vertices of
the path. In this section we determine the null sets of Double Fans. Since the degree
set of the $DF_n$ is $\{3, 4, n\}$, it cannot have a magic labeling in $\mathbb{Z}_2$. Therefore, for any
$n \geq 2$, $2 \not\in N(DF_n)$. Note that $DF_2 \equiv F_3$, and we know that $N(DF_2) = 2\mathbb{N} - \{2\}.$
Also note, $DF_3 \equiv W_4$ and we know that $N(DF_3) = \mathbb{N} - \{2, 3\}$.

**Theorem 3.5.** $N(DF_2) = 2\mathbb{N} - \{2\}$, $N(DF_3) = \mathbb{N} - \{2, 3\}$, and for any $n \geq 4,$

$$N(DF_n) = \mathbb{N} - \{2\}$$

**Proof.** To prove the theorem we consider the following two cases:

**Case 1.** Suppose $n \equiv 0 \pmod{2}$ or $n = 2p$ for some $p \in \mathbb{N} + 1 - \{0\}.$ We proceed by
induction on $p$ and show that

“for any $p$, there is a zero-sum magic labeling for $DF_{2p}$. Moreover, in this
labeling at least one of the edges of $P_n$ has label 2.”

![Figure 3.10. A zero-sum magic labeling of $DF_4$ and $DF_5$](image)

In figure 3.10, a zero-sum magic labeling of $DF_4$, $p = 2$, is provided in $\mathbb{Z}_h$ for all
$h \geq 3$, where $-1 \equiv 1^{-1}$ and $-2 \equiv 2^{-1}$. Now, assume that the statement is true for
$DF_{2p}$ and let $u_iu_{i+1}$ be the edge of $P_{2p}$ that has label 2. Then we eliminate this edge
and insert the four-blade extension, given in Figure 3.11 with vertices $a$, $b$, $c$, and
d such that $a$ and $d$ are identified with $u_i$ and $u_{i+1}$ respectively. This provides the
desired zero-sum magic labeling for $DF_{2p+2}$.

![Figure 3.11. The four blade extension for a double fan](image)

An argument similar to the one presented in case 1, will also work for the remaining case:

**Case 2.** Suppose $n \equiv 1 \pmod{2}$ or $n = 2p + 1$ for some $p \in \mathbb{N} + 1$.

3.5 Prisms and n-Prisms

For $k \geq 3$ a Prism of order $k$, denoted $P_2C_k$, is $P_2 \times C_k$ In other words, two identical
copies of $C_k$, $u_1 \sim u_2 \sim \cdots \sim u_k \sim u_1$ and $v_1 \sim v_2 \sim \cdots \sim v_k \sim v_1$, with additional
edges connecting $u_i$ and $v_i$ for all $i$. For $k \geq 3$ and $n \geq 3$ an n-Prism of order $k$, is
defined to be $P_nC_k = C_k \times P_n$. The degree set of $P_2C_k$ is $\{3\}$ and the degree set of
$P_nC_k$ is $\{3, 4\}$. Therefore $P_2C_k$ and $P_nC_k$ cannot have a zero-sum magic labeling in
$\mathbb{Z}_2$, which implies $2 \not\in N(P_2C_k)$ and $2 \not\in N(P_nC_k)$.
Theorem 3.6. For any $k \geq 3$, $N(P_2C_k) = \mathbb{N} - \{2\}$

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{figure3_12.png}
\caption{A zero-sum magic labeling $P_2C_4$}
\end{figure}

Proof. $P_2C_k$ can be drawn as two cycles of order $k$, one within the other and oriented in the same way, with edges connected the corresponding vertices of the cycles. Label all of the edges of the cycles with 1, and all of the connecting edges with -2. Then all of the vertices have two edges labeled 1 and one edge labeled -2 incident. This is a zero-sum magic labeling of $P_2C_k$ in $\mathbb{Z}_h$ for all $h \geq 3$, where $-2$ stands for the inverse of 2. $\square$

Theorem 3.7. For any $k \geq 3$ and $n \geq 3$, $N(P_nC_k) = \mathbb{N} - \{2\}$

Proof. $P_nC_k$ can be drawn as $n$ cycles of order $k$, within each other and oriented in the same way, with edges connecting the corresponding vertices. Label all of the edges of the outermost cycles with 1, all the edges of the interior cycles with 2, and all of the connecting edges with -2. Then all of the vertices on the outermost cycles have two edges labeled 1 and one edge labeled -2 incident and all the vertices on the interior cycles have two edges labeled 2 and two edges labeled -2 incident. This is a zero-sum
magic labeling of $P_2C_k$ in $\mathbb{Z}_h$ for all $h \geq 3$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure313}
\caption{A zero-sum magic labeling $P_3C_3$}
\end{figure}

3.6 Anti-Prisms and n-Anti-Prisms

For $k \geq 3$ an anti-Prism of order $k$, denoted by $AP_k$, is two identical copies of $C_k$, $C_k$: $u_1 \sim u_2 \sim \cdots \sim u_k \sim u_1$ and $C'_k$: $v_1 \sim v_2 \sim \cdots \sim v_k \sim v_1$ with additional edges $u_iv_i$ and $u_iv_{i-1}$ (mod $n$) for $i = 1 \cdots n$, as illustrated in Figure 3.14.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure314}
\caption{$AP_4$}
\end{figure}
For $k \geq 3$ and $n \geq 3$, an $n$-Anti-Prism of order $k$, $n$-$AP_k$, is $n$ identical copies of $C_k$, $C_k^{(1)}, C_k^{(2)}, \cdots, C_k^{(n)}$, where any two consecutive cycles form an Anti-Prism.

**Theorem 3.8.** For any $k \geq 3$, $N(\text{AP}_k) = N$

*Proof.\* $AP_k$ can be drawn in a planar fashion as two cycles of order $k$, one within the other with edges connected the corresponding vertices. If all the cycle edges are labeled with a 1 and all the edges connecting the two cycles are labeled with a -1, then all vertices with have two edges labeled with a 1 and two edges labeled with a -1. This is a zero-sum magic labeling of $AP_k$ in $\mathbb{Z}_h$ for all $h \geq 2$, where $-1$ stands for the inverse of 1. 

**Theorem 3.9.** For any $k \geq 3$ and $n \geq 3$, $N(n$-$AP_k) = N$

*Proof.\* $n$-$AP_k$ can be drawn in a planar fashion as $n$ cycles of order $k$, drawn one within another with edges connected the corresponding vertices. Label the edges in the following fashion: since all of the vertices are either degree 4 or degree 6 if we label ever edges with a 1 that is clearly a zero-sum labeling in $\mathbb{Z}_2$. Also if we label the outermost cycle edges with 1’s, all of the inner cycle edges with 2’s and all edges connecting cycles with -1’s, then every vertex on the outermost cycles will have two edges labeled with 1 and two edges labeled with -1 and all inner cycle vertices will have two edges labeled with 2 and four edges labeled with -1. This is a zero-sum magic labeling of $n$-$AP_k$ in $\mathbb{Z}_h$ for all $h \geq 3$. 

\[ \square \]
3.7 Null sets of Grids

For \( n \geq 2 \) and \( k \geq 2 \), an \( n \) by \( k \) Grid, \( P_{n,k} \), is \( P_n \times P_k \). In this section we determine the null sets of Grids. Since the degree set of \( G_{n,k} \), for \( n \geq 3 \) and \( k \geq 3 \), is \( \{2, 3, 4\} \) and the degree set of \( G_{n,2} \), \( n \geq 3 \), and \( G_{2,k} \), \( k \geq 3 \), is \( \{2, 3\} \), \( P_{n,k} \), where \( n \geq 3 \) or \( k \geq 3 \), cannot have a magic labeling in \( \mathbb{Z}_2 \) if \( n \geq 3 \) or \( k \geq 3 \). Therefore \( 2 \not\in N(P_{n,k}) \) if \( n \geq 3 \) or \( k \geq 3 \). Note that \( P_{2,2} \equiv C_4 \), and we therefore know that \( N(P_{2,2}) = \mathbb{N} \).

**Theorem 3.10.** For any \( n \geq 3 \) or \( k \geq 3 \), \( \mathbb{N} - \{2\} \subset N(P_{n,k}) \)

**Proof.** You can think of a Grid as being \( k \) \( P_n \)'s oriented horizontally and connected together by \( n \) \( P_k \)'s. Label all of the edges in the following way: the two outer \( P_n \)'s with 1's, the \( k - 2 \) inner \( P_n \)'s with 2's, the two outer \( P_{k+1} \)'s with \(-1\)'s, and the the \( n - 2 \) outer \( P_k \)'s with \(-2\)'s.

![Figure 3.15. A zero-sum magic labeling of \( P_{4,5} \)](image)

Then the corner vertices will have edges incident to them having values 1 and \(-1\), the non-corner vertices on the outer \( P_n \)'s will have edges incident to them having values 1, 1, and \(-2\), the non-corner vertices on the outer \( P_k \)'s will have edges incident
to them having values $-1$, $-1$, and $2$, and all interior vertices will have two edges incident to them having value $2$ and two others having value $-2$. This is a zero-sum magic labeling of $G_{n,k}$ in $\mathbb{Z}_h$ for all $h \geq 3$, where $-1$ and $-2$ stand for the inverses of $1$ and $2$ respectively.

\[3.8 \text{ Null sets of Bowties}\]

For $n \geq 2$, Bowties, $BT_n$, are two identical copies of $F_n$ that are connected together at the $F_n$’s $K_1$ nodes. Since the degree set of the $BT_n$ is $\{2, 3, n + 1\}$, $BT_n$ cannot have a zero-sum magic labeling in $\mathbb{Z}_2$. Therefore, for any $n \geq 2$, $2 \not\in N(BT_n)$. Let $u_1 \sim u_2 \sim \cdots \sim u_n$ be the vertices of the path of one of the $F_n$s, $u$ the $K_1$ vertex of the corresponding fan, and $v$ the $K_1$ of the other fan. The following observation will be useful in finding the null sets of bowties.

**Observation 3.5.** If $l : E(BT_n) \to \mathbb{Z}_h$ is a zero-sum magic labeling, then

\[2 \left( l(u_1u_2) + l(u_2u_3) + \cdots + l(u_{n-1}u_n) \right) \equiv l(uv) \pmod{h}.\]

**Proof.** Let $l : E(BT_n) \to \mathbb{Z}_h$ be the edge labeling that provides zero-sum. Clearly, $l^*(u) = 0$ implies that sum of the labels of all spokes is $-l(uv)$. Also, $l^*(u_k) = 0$ ($1 \leq k \leq n$) and $l^*(u) = 0$. Therefore,

\[
\sum_{k=1}^{n} l^*(u_k) = 2 \sum_{k=1}^{n} l(u_{i}u_{i+1}) + \left( l^*(u) - l(uv) \right)
\]

\[= 2 \sum_{k=1}^{n} l(u_{i}u_{i+1}) - l^*(uv)
\]

\[\Rightarrow 2 \left( l(u_1u_2) + l(u_2u_3) + \cdots + l(u_{n-1}u_n) \right) \equiv l(uv).\]

**Theorem 3.11.** For any $n \geq 2$,
\[
N(BT_n) = \begin{cases} 
\mathbb{N} - \{2, 3\} & \text{if } n \equiv 1 \pmod{3}; \\
\mathbb{N} - \{2\} & \text{otherwise}.
\end{cases}
\]

Proof. First let us consider if \( n \geq 2, \, 3 \not\in N(BT_n) \). For \( n \equiv 1 \pmod{3} \) label all the path edges with 1’s, all the interior spokes with 1’s, all the exterior spokes with 2’s, and the \( uv \) edge with a 1 if \( n \equiv 0 \pmod{3} \) or 2 if \( n \equiv 2 \pmod{3} \). Then the end vertices of the path have an edge labeled 1 and an edge labeled 2 incident, the interior path vertices have three edges labeled 1 incident, and if \( n \equiv 0 \pmod{3} \) \( u \) and \( v \) have two edges labeled 2, \( n - 1 \) edges labeled 1 incident or if \( n \equiv 2 \pmod{3} \) \( u \) and \( v \) have three edges labeled 2 and \( n - 2 \) edges labeled 1 incident. These are both clearly zero-sum labeling. Suppose \( n \equiv 1 \pmod{3} \) and let \( l : E(BT_n) \to \mathbb{Z}_3 \) be any magic labeling of \( BT_n \) with zero-sum. Then by Observation 3.5, twice the sum of the labels of the path edges is \( l(uv) \). Since there are \( n - 1 \) path edges, which is equivalent to 0 (mod 3), they cannot all be labeled 1 (or 2) since that would imply that \( l(uv) \) is zero, two adjacent path edges would have labels 1 and 2. This implies that the spoke adjacent to these two outer edges must have label 0, which is not an acceptable label. Therefore, \( n \geq 2, \, 3 \not\in N(BT_n) \) if and only if \( n \equiv 1 \pmod{3} \).

To finish the proof of the theorem we consider the following three cases:

**Case 1.** Suppose \( n \equiv 1 \pmod{3} \) or \( n = 3p + 1 \) for some \( p \in \mathbb{N} \). We proceed by induction on \( p \) and show that

“for any \( p \), there is a zero-sum magic labeling for \( BT_{3p+1} \). Moreover, in this labeling at least one of the edges on both \( P_n \)'s has label 1.”

In figure 3.16, a zero-sum magic labeling of \( BT_4, \, p = 1 \), is provided in \( \mathbb{Z}_h \) for all \( h \geq 4 \), where \(-2 \equiv 2^{-1} \) and \(-3 \equiv 3^{-1} \).
Figure 3.16. A zero-sum magic labeling $BT_2$, $BT_3$, and $BT_4$

Now, assume that the statement is true for $BT_{3p+1}$ and let $u_i u_{i+1}$ be the edge of $P_{3p+1}$ that has label 1. Then we eliminate this edge and insert the four-blade extension, given in Figure 3.17, with vertices $a$, $b$, $c$, $d$, and $e$, such that $a$ and $e$ are identified with $u_i$ and $u_{i+1}$ respectively. This provides the desired zero-sum magic labeling for $BT_{3p+4}$.

Figure 3.17. The three blade extension of a bowtie.

An argument similar to the one presented in case 1, will also work for the remaining two cases:

**Case 2.** Suppose $n \equiv 2 \pmod 3$ or $n = 3p + 2$ for some $p \in \mathbb{N}$.

**Case 3.** Suppose $n \equiv 0 \pmod 3$ or $n = 3p$ for some $p \in \mathbb{N} - \{0\}$.
3.9 Null sets of Axles

For $n \geq 3$, Axles, $AX_n$, are two identical copies of $W_n$ that are connected together at the $W_n$'s’ $K_1$ nodes. Since the degree set of the $AX_n$ is $\{3, n + 1\}$, $AX_n$ cannot have a zero-sum magic labeling in $\mathbb{Z}_2$. Therefore, for any $n \geq 2$, $2 \not\in N(AX_n)$. Let $u_1 \sim u_2 \sim \cdots \sim u_n \sim u_1$ be the vertices of the cycle of one of the $W_n$'s (in some cases, for convenience, we may use $u_{n+1}$ for $u_1$ and $u_{-1}, u_0$ for $u_{n-1}, n_n$, respectively.), $u$ the $K_1$ vertex of the corresponding wheel, and $v$ the $K_1$ of the other wheel.

**Observation 3.6.** If $l : E(AX_n) \to \mathbb{Z}_h$ is a zero-sum magic labeling, then

$$2\left(l(u_1u_2) + l(u_2u_3) + \cdots + l(u_{n-1}u_n + l(u_nu_1))\right) \equiv l(uv) \pmod{h}.$$ 

**Proof.** Let $l : E(AX_n) \to \mathbb{Z}_h$ be the edge labeling that provides zero-sum. Clearly, $l^*(u) = 0$ implies that sum of the labels of all spokes is $-l(uv)$. Also, $l^*(u_k) = 0$ ($1 \leq k \leq n$). Therefore,

$$\sum_{k=1}^{n} l^*(u_k) = 2 \sum_{k=1}^{n} l(u_iu_{i+1}) + \left(l^*(u) - l(uv)\right) = 2 \sum_{k=1}^{n} l(u_iu_{i+1}) - l^*(uv) \Rightarrow 2\left(l(u_1u_2) + l(u_2u_3) + \cdots + l(u_{n-1}u_n)\right) \equiv l(uv).$$

\[\square\]

**Theorem 3.12.** For any $n \geq 3$,

$$N(AX_n) = \begin{cases} \mathbb{N} - \{2, 3\} & \text{if } n \equiv 0 \pmod{3} \\ \mathbb{N} - \{2\} & \text{otherwise} \end{cases}$$

**Proof.** First let us consider if $n \geq 3$, $3 \not\in N(AX_n)$. For $n \not\equiv 0 \pmod{3}$ label all the cycle and spoke edges with 1’s and the edge $uv$ with a 1 if $n \equiv 2 \pmod{3}$ or 2 if $n \equiv 1 \pmod{3}$. Then the cycle vertices of the wheel have three edges labeled 1 incident and
$u$ and $v$ have $n$ edges labeled 1 and one edge labeled 2 incident if $n \equiv 1 \pmod{3}$ or $n + 1$ edges labeled 1 if $n \equiv 2 \pmod{3}$. These are both clearly zero-sum labelings.

Suppose $n \equiv 1 \pmod{3}$ and let $l : E(AX_n) \to \mathbb{Z}_3$ be any magic labeling of $BT_n$ with zero-sum. Then by Observation 3.6, twice the sum of the labels of the cycle edges is $l(uv)$. Since there are $n$ cycle edges, which is equivalent to 0 (mod 3), they cannot all be labeled 1 (or 2) since that would imply that $l(uv)$ is zero, two adjacent path edges would have labels 1 and 2. This implies that the spoke adjacent to these two outer edges must have label 0, which is not an acceptable label. Therefore, $n \geq 3$, $3 \not\in N(AX_n)$ if and only if $n \equiv 0 \pmod{3}$.

![Figure 3.18. A zero-sum magic labeling $AX_3$, $AX_4$, and $AX_5$.](image)
To finish the proof of the theorem we consider the following three cases:

**Case 1.** Suppose \( n \equiv 0 \pmod{3} \) or \( n = 3p \) for some \( p \in \mathbb{N} \). We proceed by induction on \( p \) and show that

“for any \( p \), there is a zero-sum magic labeling for \( AX_{3p} \). Moreover, in this labeling at least one of the edges on both \( W_n \)’s has label 1.”

In figure 3.18, a zero-sum magic labeling of \( AX_3 \), \( p = 1 \), is provided in \( \mathbb{Z}_h \) for all \( h \geq 4 \), where \(-2 \equiv 2^{-1} \) and \(-3 \equiv 3^{-1} \).

Now, assume that the statement is true for \( AX_{3p} \) and let \( u_i u_{i+1} \) be the edge of \( AX_{3p} \) that has label 1. Then we eliminate this edge and insert the four-blade extension, given in Figure 3.19 with vertices \( a, b, c, d, \) and \( e \), such that \( a \) and \( e \) are identified with \( u_i \) and \( u_{i+1} \) respectively. This provides the desired zero-sum magic labeling for \( AX_{3p+4} \).

![Figure 3.19. The three blade extension of an axle.](image-url)

An argument similar to the one presented in case 1, will also work for the remaining two cases:

**Case 2.** Suppose \( n \equiv 1 \pmod{3} \) or \( n = 3p + 1 \) for some \( p \in \mathbb{N} \).
Case 3. Suppose $n \equiv 2 \pmod{3}$ or $n = 3p + 2$ for some $p \in \mathbb{N} - \{0\}$. □
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VITA

Graduate College
University of Nevada, Las Vegas

Samuel M. Hansen

Degrees:
  Bachelor of Science in Mathematics, 2008
  University of Wisconsin, Green Bay

Special Honors and Awards:
  Shortlist Best Educational Podcast Edublog Awards 2010 for Math/Maths: 5136 Miles of Mathematics Podcast
  Student Commencement Speaker Graduating Class May, 2008 University of Wisconsin, Green Bay
  Rising Phoenix Award for Fiction: ”Today on Book of the Day: Your Story on Shuffle” Spring 2008, Sheepshead Review

Publications:

Thesis Title:
  Zero-Sum Magic Graphs and Their Null Sets

Thesis Examination Committee:
  Chairperson, Ebrahim Salehi, Ph.D.
  Committee Member, Peter Shiue, Ph.D.
  Committee Member, Hossein Tehrani, Ph.D.
  Graduate Faculty Representative, Fatma Nasoz, Ph.D.