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Generalizations of Pascal's Triangle: A Construction Based Approach

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GENERALIZATIONS OF PASCAL'S TRIANGLE:
A CONSTRUCTION BASED APPROACH

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ABSTRACT

Generalizations of Pascal's Triangle:
A Construction Based Approach

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The study of this paper is based on current generalizations of Pascal's Triangle, both the expansion of the polynomial of one variable and the multivariate case. Our goal is to establish relationships between these generalizations, and to use the properties of the generalizations to create a new type of generalization for the multivariate case that can be represented in the third dimension.

In the first part of this paper we look at Pascal's original Triangle with properties and classical applications. We then look at contemporary extensions of the triangle to coefficient arrays for polynomials of the forms:

$$(x^k + x^{k-1} + \cdots + x + 1)^n$$

and

$$(x_1 + x_2 + \cdots + x_{k-1} + x_k)^n$$

We look at construction of the resulting objects, properties and applications. We then relate the two objects together through substitution and observe a general process in which to do so.

In the second part of the paper I observe an application of the current generalizations to the classical problem "The Gambler's Game of Points" to games of alternative point structures. The paper culminates with a generalization I have made for a particular case of the second equation, moving the current four dimensional generalization into the third dimension for observation and study. We see the relationships of this generalization to those from our overview in part one, and develop the main theorem of study from the construction of its arrangement. From this theorem we are able to derive several interesting combinatorial identities from our construction.

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Yang Hui (1261) listed the coefficients of $(a + b)^n$ up to the sixth power, and credited this expansion from the *Shih-so suan-shu* of Chia Hsien (1100), where it is called "the tabulation system for unlocking binomial coefficients"[4]*. This early Binomial Triangle was also given in 1303 in Chu Shih-chieh's *Precious Mirror of the Four Elements*. The tabulation system used is the same as that used in Europe, and thus the Binomial Triangle can properly be attributed to China sometime around 1100A.D.

It was from Persia, however, that we can trace lines back from the European figuration of the binomial coefficients. The *Al-bahir* of Al-Samawal, who died around 1180 B.C., is reported to contain a calculation of the coefficients, the method of which is attributed to Al-Karaji sometime soon after 1007. An early form of the Binomial Theorem was given by Al-Kashi (1427) along with the Binomial Triangle up to the ninth power in his *Key of Arithmetic*. However, it is Pascal who gave us the proof of these results.

Blaise Pascal was born on June 19, 1623 in Clermont, and in 1631 was moved by his father Etienne Pascal to Paris to receive a better education. Etienne, a mathematician himself, was one of the founders of Marin Mersennes's "Academy", where Pascal began study at fourteen. At age sixteen he produced his famous *Essay pour les coniques* based on the work of Desargues.

In 1654, Pascal wrote his famous *Traité du triangle arithmétique.* His interest was combinatorial, having solved the gambler's Problem of Points. The problem was regarding the division of stakes between players when a game has to be left unfinished. We address the solution of this problem at a later part of this paper.

The *Traité du triangle arithmétique* was published in 1665, from papers found amongst Pascal's papers after his death. Pascal's work defined the triangle with corollaries, applied it in the theory of both figurate numbers and combinations, and gave applications including finding the powers of binomial expressions.

*- All historical information in the preceding section is attributed to [4].

In this paper we take a close look at Pascal's Triangle, particularly interested in its construction. We will then look at the construction of two major generalizations, which we will refer to as the T-Triangles and T-Simplices. Relating these constructions together and observing their basic attributes will allow us to then observe a new construction, which I have named Pascal's Square and Pyramid. This new construction has many interesting properties of its own, and in particular the construction of the Square will provide us immediate results of combinatorial identities, several of them of famous nature but not before shown through this construction. I believe that many such results can come from similar constructions, a discussion which we leave later as concluding remarks.

1.1 Overview and Properties

The Triangle

Although Pascal himself used a rectangular version of this triangle based on the figurate numbers (visited later), the contemporary version of arrangement is that of an equilateral triangle of the binomial coefficients $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, where each row represents the exhaustion of coefficients of a particular n, beginning with $n = 0$ and increasing by one with each subsequent row, where in any given row n, $k = 0,1,2, \dots, n$. The first few lines of the triangle are as follows:

n=0						1					
n=1					1		1				
n=2				1		2		1			
n=3			1		3		3		1		
n=4		1		4		6		4		1	
n=5	1		5		10		10		5		1

Figure 1: Pascal's Triangle

Each row can also be seen as the coefficients of the expansion given by the Binomial Theorem, $(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$, something worth noting in exploring the properties of the triangle.

Properties

Pascal's Triangle has many interesting and convenient properties, most of which deal with its symmetry, but all of which hold an innate beauty.

Theorem: The Pascal Triangle has the following properties:

1. Every row starts and terminates with 1.

2. Every row is symmetric about its center, and thus the triangle as a whole is symmetric about the vertical line running through its center.

3. The sum of the numbers on any row n is 2^n.

4. The sum of the rows 0 through m is $2^{m+1} - 1$.

5. Every entry in each row is the sum of the numbers to its left and right on the previous row.

Proofs:

1. This comes from the identities $\binom{n}{0} = \binom{n}{n} = 1$.

2. The symmetry comes directly from the observation that $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k}$.

3. This is easily seen from setting both variables to 1 in the Binomial Theorem, as $2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} (1)^{n-k} (1)^k = \sum_{k=0}^{n} \binom{n}{k}$.

4. Follows directly from (3) with the additional note that $\sum_{k=0}^{m} 2^k = 2^{m+1} - 1$.

5. This is Pascal's Identity given as Lemma 4 in Part II of his Treatise:

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$$

While this result was known before Pascal's writing of it, he was the first to prove it. He did so using combinations, with what was arguably one of the first proofs using induction:

"For consider any particular one of the $(n+1)$ things: $[\binom{n}{r}]$ gives the number of combinations that contain it, whilst $[\binom{n}{r+1}]$ gives the number that exclude it, the two numbers together giving the total. Pascal does not give a formal proof by induction but uses the above argument with $n = 3$ and $r = 1$, indicating its generality".[4]

This last property is often used to construct the triangle quickly, and is familiar to many in the use of quickly calculating the coefficients of binomial expansion to some n. This can also be seen to be true by the action of multiplication in a certain way. Consider:

$$(x+y)^3 = (x+y)(x+y)^2 = (x+y)(x^2+2xy+y^2)$$

We can think of constructing the third row from the second by finishing out this multiplication in a way that each step gives only, and all of, one of the resultant terms. We do this by flipping $x+y$ and following the multiplication as follows:

$$\begin{array}{llll}
y+x & y+x & y+x & y+x \\
\quad x^2+2xy+y^2 & x^2+2xy+y^2 & x^2+2xy+y^2 & x^2+2xy+y^2 \\
\hline
\quad x^3 & x^2y+2x^2y & \quad 2xy^2+xy^2 & \qquad\qquad y^3 \\
\hline
x^3 & 3x^2y & 3xy^2 & \qquad\qquad y^3
\end{array}$$

Figure 2: Multiplication Construction of Pascal's Triangle

This process works because of our coefficients of 1 in our linear term $x+y$. We generalize this idea later in the construction of the T-triangle.

There are many more properties to this triangle, but since the intended focus of this paper is generalization, I shall stop here and show a few of them as applications.

1.2 Applications

Row n of Pascal's Triangle can be used to compute nth powers of 11. This comes directly from the Binomial Theorem, as $11^n = (10+1)^n = \sum_{k=0}^{n}\binom{n}{k}(10)^{n-k}(1)^k$. For example, $11^4 = (10+1)^4 = 1\cdot 10^4 + 4\cdot 10^3 + 6\cdot 10^2 + 4\cdot 10^1 + 1\cdot 10^0 = 14641$. Once we move beyond a power of four the process changes slightly, since once a coefficient exceeds 10 we cannot simply list the coefficients in order, but rather need to "carry over" any increments of 10 in the kth coefficient as increments of 1 in the $(k-1)st$ coefficient. For example, $11^5 = (10+1)^5 = 1\cdot 10^5 + 5\cdot 10^4 + 10\cdot 10^3 + 10\cdot 10^2 + 5\cdot 10^1 + 1\cdot 10^0$. Since $10\cdot 10^3 = 1\cdot 10^4$ and $10\cdot 10^2 = 1\cdot 10^3$, so we have $11^5 = 1\cdot 10^5 + 6\cdot 10^4 + 1\cdot 10^3 + 0\cdot 10^2 + 5\cdot 10^1 + 1\cdot 10^0 = 161051$. Writing it out this way is tedious and simply illustrates the method. Typically one can simply look at the line of the triangle and do this simple calculation to very quickly come to a result.

This can be generalized to the power of any number beginning and ending with 1s, where all intermediate digits are 0s. The process is the same, one just needs to keep in mind what our "base" is, and write each term appropriately. For example, in calculating $(101)^4 = (100+1)^4$, we represent all values from the $4th$ row as numbers with two terms, and carry over numbers when the coefficients exceed 100 in higher powers. So $(101)^4 = 104060401$.

I also include here two famous applications of the Triangle given by Pascal. One to an unsolved problem at the time, and the second to a problem presented by him with a solution, and my suspicions to his reasoning.

The Gambler's Game of Points

Consider a game of chance with two players, interrupted before completion. The game is based on points gained by either player with a successful event of equal chance, the first player to reach a total number of points being the winner. Say, for example, the game is based on flipping a coin, where a flip of "heads" awards a point to player A, and a flip of "tails" to player B. The problem is to determine how to split the stakes between the two players based on current standings should the game need to end before completion.

Suppose that in order to win, A needs a points and B needs b points. Pascal and Fermat solved the problem initially by the method of "combinations", noting that at most ($a + b$ - 1) more tosses will settle the game, and that this resulted in 2^{a+b-1} possible games of equal probability.[4] These could then be classified as games which A or B would be the winner, and the resultant proportion should be the same in which the stakes should be split. With this construction, the order of heads or tails is not important, only the total number after the ($a + b$ - 1) tosses. Furthermore, once a tosses are in player A's favor, player B can have at most (b - 1) in theirs, which makes calculation almost trivial with the properties of the Triangle in mind. At the time of the solution, however, Pascal had not yet finished his treatise, and used enumeration to exemplify the general case with an example where a = 2 and b = 3. ($a + b$ -1) = 4 tosses at most are needed to complete the game, with 16 possible games. They are enumerated, and Pascal writes in a letter to Fermat dated 24 August 1654:

"because the first player needs two wins, he must win the game whenever there are two [heads]: thus there are 11 [games] for him; and because the second player needs three, he must win the game whenever there are three [tails]: therefore there are 5 [games] for him".[4] In this case it should be decided that the first player receive $\frac{11}{16}$ ths of the stakes, and the second player $\frac{5}{16}$ ths.

Now, with the Triangle in our understanding, we can simply look at the row correspondent to $(x + y)^{a+b-1}$, and consider x to signify a point for A and y a point for B. Then the coefficients of terms with a power of x equal to or higher than *a* indicate a game in favor of player A, where those with a power of y equal to or higher than *b* indicate a game in favor of player B. That is, we have:

$\sum_{n=0}^{b-1}\binom{a+b-1}{n} = \alpha$ games in favor of player A, and:

$\sum_{n=b}^{a+b-1}\binom{a+b-1}{n} = \beta$ games in favor of player B.

The stakes should then be split with $\frac{\alpha}{2^{a+b-1}}$ to player A and $\frac{\beta}{2^{a+b-1}}$ to player B.

In the example given by Pascal in his letter, we see that we do indeed have:

$\sum_{n=0}^{2}\binom{4}{n} = 1 + 4 + 6 = 11$ games in favor of the first player, and:

$\sum_{n=3}^{4}\binom{4}{n} = 1 + 4 = 5$ games in favor of the second player.

Although Pascal did not blatantly identify what we know as the Binomial Distribution at this time, he implicitly did so in his application of his triangle to this Problem of Points, a problem clearly using the distribution of total outcomes with an event of probability $p = \frac{1}{2}$. He wrote "Proper calculation masters fickle fortune", and concerning the Problem of Points "each player always has assigned to him precisely what justice demands".[4]

The Gambler's Ruin

The "Gambler's Ruin" was a problem posed by Pascal to his friend Fermat some time after their correspondence and solution of the Problem of Points, likely as he thought it to be too unfriendly to enumeration that he would be able to demonstrate the power of his Triangle, and his notion of expectations. Fermat was quickly able to solve the problem when posed, giving a range in which the odds must lie as the numbers were quite large, and Pascal then gave his exact solution in response. Although the method used by either of them is unknown, there has been much speculation on what they may have been with the tendencies of the two in mind. The speculation I have seen regarding Pascal's method have been algebraic, using his notion of expectation. I would like to pose here an alternative method that I believe he may have employed, simply by direct observation of the properties and symmetry of his Triangle.

I give the problem here as described by Carcavi:

"Let two men play with three dice, the first player scoring a point whenever 11 is thrown, and the second whenever 14 is thrown. But instead of the points accumulating in the ordinary way, let a point be added to a player's score only if his opponent's score is nil, but otherwise let it instead be subtracted from his opponent's score. It is as if opposing points form "pairs", and annihilate each other, so that the trailing player always has zero points. The winner is the first to reach twelve points; what are the relative chances of each player winning?"[4]

As foreshadowed above, this problem is quickly concluded upon observation of the symmetry of the triangle. It is a simple matter of calculation to see that the relative odds of tossing an 11 verses 14 are 9:5, which here I denote as $p: q$. Using the combinatorial properties of the triangle formed by $(p+q)^n$, we see that a possible conclusion of the game occurs first on the row with $n = 12$. In fact, there are two endings of the game on this row, represented by p^{12} and q^{12}, being wins in favor of the first player and second player, respectively. These are the "blowout" games, where only one player ever obtains a point, and here we see the relative odds of winning for the players are simply $p^{12}: q^{12}$.

The next possible resolution to the game happens at 14 tosses, where one player has won 13 of them. Here we look at the line correspondent to $n = 14$, specifically at the terms $14p^{13}q$ and $14q^{13}p$. Since two of the wins represented by each term actually stem from wins that already occurred after twelve tosses, looking at wins **at** 14 tosses gives the relative odds $12p^{13}q: 12pq^{13}$, which of course reduces down to $p^{12}: q^{12}$. We note, however, that since the included previous wins are symmetrically included on both sides, it would be equivalent to simply consider the relative odds provided by the terms themselves, $14p^{13}q: 14pq^{13}$.

Continuing in this fashion, we quickly see that the relative odds for either player at our possible number of tosses that can end a game, $n = 12 + 2k, k = 0,1,2, \ldots$, are simply $\binom{n}{k}p^{n-k}q^k: \binom{n}{n-k}p^k q^{n-k}$. With our properties of the binomial coefficient and symmetry of the triangle, this always resolves to $p^{12}: q^{12}$, which of course we now see are the relative odds of each player winning regardless of the amount of tosses made to conclude the game.

It is with this observation of symmetry in the Triangle that I believe Pascal made his solution. More accurately, I believe that Pascal posed the problem itself with this symmetry in mind, giving him an opportunity to exhibit the power of his prized object.

1.3 Generating Fibonacci Numbers with Skewed Diagonals

The Fibonacci Numbers are found to be present in Pascal's Triangle using the result known as Lucas' Formula:[6]

$$F_n = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-i-1}{i}$$

Which can be seen as taking the sums of certain northeastern diagonals in the following way:

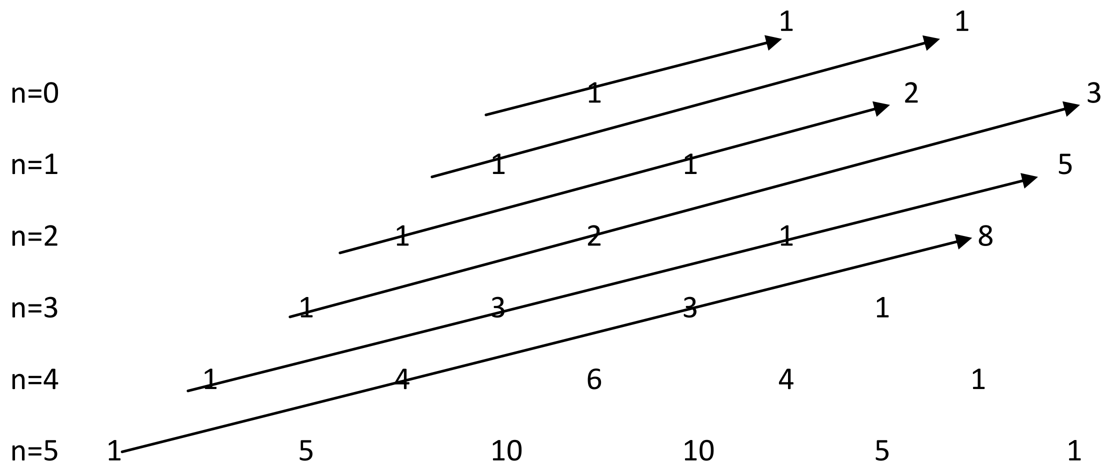

Figure 3: Fibonacci Numbers on Pascal's Triangle

A result which we will see an analogue to in the following generalization of Pascal's Triangle.

Chapter 2: Current Generalizations with Applications

2.1 The T-Triangle

Just as Pascal's Triangle can be seen as the expansion of $(x + y)^n$, we can construct two dimensional arrays for the coefficients of the expansion $(x^2 + xy + y^2)^n$ or $(x^3 + x^2y + xy^2 + y^3)^n$. For simplicity, we can think of these triangles as simply the expansions of $(x^2 + x + 1)^n$, $(x^3 + x^2 + x + 1)^n$, etc.. Since the homogeneous nature of the equations given in two unknowns is easily related to the case where $y = 1$, as the coefficients of $x^k y^{n-k}$ in the prior are the same as x^k in the latter. We can illustrate the construction of these triangles with that same method of multiplication that we exhibited in the binomial case.

2.1.1 Construction

Consider $(x^2 + x + 1)^2 = (x^2 + x + 1)(x^2 + x + 1)$. Observing the pattern in coefficients using our "flipping" method gives:

$$
\begin{array}{ccccc}
1 + x + x^2 & 1 + x + x^2 & 1 + x + x^2 & 1 + x + x^2 & 1 + x + x^2 \\
x^2 + x + 1 & x^2 + x + 1 & x^2 + x + 1 & x^2 + x + 1 & x^2 + x + 1 \\
\hline
x^4 & x^3 + x^3 & x^2 + x^2 + x^2 & x + x & 1 \\
\hline
x^4 & 2x^3 & 3x^2 & 2x & 1
\end{array}
$$

Figure 4: Multiplication Construction of the T-Triangle

Repeating this process with this new result gives:

$(x^2 + x + 1)^3 = x^6 + 3x^5 + 6x^4 + 7x^3 + 6x^2 + 3x + 1$

We see through the illustration of this process that the coefficients of the new line of our desired triangle are the sums of three consecutive coefficients of the preceding line, similar to our binomial case. The first few lines of the triangle of coefficients of $(x^2 + x + 1)^n$ are given in figure 5, and similarly the first few lines of the coefficients for $(x^3 + x^2 + x + 1)^n$ are given in figure 6.

In the same way that we construct the coefficients for the triangle of the polynomial with three terms, we can quickly construct this second triangle. We obtain a coefficient in the nth row by summing the four coefficients centered above it in the $(n - 1)st$ row, considering any coefficient to the left or right of the triangle to be 0.

n=0						1					
n=1					1	1	1				
n=2				1	2	3	2	1			
n=3			1	3	6	7	6	3	1		
n=4		1	4	10	16	19	16	10	4	1	
n=5	1	5	15	30	45	51	45	30	15	5	1

Figure 5: The 3-Triangle

n=0	1												
n=1	1	1	1	1									
n=2	1	2	3	4	3	2	1						
n=3	1	3	6	10	12	12	10	6	3	1			
n=4	1	4	10	20	31	40	44	40	31	20	10	4	1

Figure 6: The 4-Triangle

This construction is consistent for the powers of any length of polynomial of this type. We will call the first triangle the 3-Triangle, the second the 4-Triangle, and so on, the number indicating the number of terms in the polynomial being considered.

2.1.2 Properties and Applications

We first list some of the basic properties of the T-Triangle.

Theorem: Let $\binom{n,T}{k}$ denote the kth coefficient of the nth row (n and k beginning at 0) of the T-Triangle generated by a polynomial of T terms. For $m < 0$ or $m > (T-1)n$ we define $\binom{n,T}{m} = 0$.

1. $\binom{n,T}{0} = \binom{n,T}{(T-1)n} = 1$

2. There are $(T-1)n + 1$ entries in the nth row of the corresponding T-Triangle.

3. $\binom{n,T}{k} = \sum_{i=k-T+1}^{k} \binom{n-1,T}{i}$

(For example, $\binom{3,4}{5} = \sum_{i=2}^{5} \binom{2,4}{i} = 2 + 3 + 4 + 3 = 12$)

4. $\sum_{i=0}^{(T-1)n} \binom{n,T}{i} = T^n$, i.e. the sum of coefficients of the nth row of a given T-Triangle is T^n.

(This is easily seen by the substitution of $x = 1$ into the respective polynomial.)

5. A given T-Triangle is symmetric about the vertical line passing through its center. That is:

$$\binom{n,T}{k} = \binom{n,T}{(T-1)n-k}$$

6. Observing that the polynomial $(x^{T-1} + \cdots + x + 1)^n = \frac{(x^T-1)^n}{(x-1)^n}$, we can use this generating function to find a formula for $\binom{n,T}{k}$ in terms of binomial coefficients. Expanding the numerator and denominator with the binomial theorem and collecting the coefficients of x^k gives:

$$\binom{n,T}{k} = \sum_j (-1)^j \binom{n}{j} \binom{n-1+k-Tj}{n-1}$$

Proofs: Properties 1-5 can be found in [6], property 6 in [3].

We also note an observation given in [5], that the first T coefficients of row n in the T-Triangle match the first T coefficients in row n of the $(T+1)$-Triangle. That is: $\binom{n,T}{0} = \binom{n,T+1}{0}, \binom{n,T}{1} = \binom{n,T+1}{1}, \dots, \binom{n,T}{T} = \binom{n,T+1}{T}$, and these coefficients are also equal to the first T coefficients on the $(n-1)st$ diagonal of the 2-Triangle (Pascal's Triangle).

Applications

A power of a number with every digit equal to 1 can be found using a T-Triangle in a similar way as we found the powers of 11 with Pascal's Triangle. To find $(111 \dots 11)^n$ where our number has T digits, we refer to the nth row of the T-Triangle considering each coefficient to be a single digit in our result. We "carry over" coefficients of more than one digit, letting every 10 in our kth coefficient be a 1 in our $(k-1)$st coefficient. This follows in a similar fashion to our previous examples, as $(111 \dots 11)^n = ((10)^{T-1} + \cdots + 10 + 1)$.

For example, to find $(111)^3$ we look at the row for $n = 3$ in our 3-Triangle:

n=3	1	3	6	7	6	3	1

And see that $(111)^3 = 1367631$.

To find $(1111)^4$ we look at the row for $n = 4$ in our 4-Triangle:

n=4	1	4	10	20	31	40	44	40	31	20	10	4	1

And after "carrying over" multiples of 10 find $(1111)^4 = 152354833141$.

This method can also be applied to numbers of alternating 1s and 0s, or numbers of 1s with the same number of 0s between each by changing our base from 10 to 100, 1000, etc.. Considering each coefficient from the table to be one digit less than our base, as is clear from our previous explanation of the method.

For example,

$(10101)^5 = ((100)^2 + (100) + 1)^5 = 1051530455145301505 01$ from the row $n = 5$ in our 3-Triangle.

$(1001001001)^3 = 1003006010012012010006003001$ from the row $n = 3$ in our 4-Triangle.

This process can be used in numbers of other bases as well, although for smaller bases the "carrying over" procedure can be more tedious.

$$(111)_2^3 = (1367631)_2 = (101010111)_2$$

The T-Triangle is also a natural setting for many enumeration problems. Consider the following problems given by Richard Bollinger in his 1993 paper[3] on the subject, which are presented here using our notation. Enumerate the number of ways $W(s, n, m)$:

a) that a given sum, *s*, can be thrown with *n* fair *m*-sided dice;

b) of solving the equation $x_1 + x_2 + \cdots + x_n = s$ in positive integers not exceeding a given integer *m*;

c) of compositions (ordered partitions) of s into exactly *n* positive parts with no part greater than *m*;

d) that *s* identical objects can be placed in *n* cells with each cell containing at least one object and at most *m* objects.

These are equivalent questions taken in different contexts, but if we consider the generating function $\phi(s,n,m,x) = (x^m + x^{m-1} + \ldots + x^2 + x)^n = x^n(x^{m-1} + \cdots + x + 1)^n$, then we can relate the solution to our T-Triangles by observing:

$$\phi(s,n,m,x) = x^n \sum_{k=0}^{(m-1)n} \binom{n,m}{k} x^k = \sum_{k=0}^{(m-1)n} \binom{n,m}{k} x^{k+n}$$

Which gives an immediate result for our coefficient of x^s, and the formula:

$$W(s,n,m) = \binom{n,m}{n-s}$$

For example, the number of ways of obtaining the sum $s = 26$ with $n = 6$ ordinary $(m = 6)$ dice is [3]:

$$W(26,6,6) = \binom{6,6}{20} = \binom{6,6}{10} = 2247$$

J.D. Bankier wrote a paper [2] based on a similar problem to Bollinger's (d) above, where we look at the number of ways k objects can be placed in n cells, allowing at most m objects to fall into a given cell, a problem which is the same as Bollinger's with the exception of the requirement of at least one item in each cell. Bankier also achieved this result using T-Triangles. Using generating functions we see that our solution can be found with the generating function $(x^m + \cdots + 1)^n$, each factor representing a cell, and the exponent of x the number of objects placed in that cell, yielding the solution $\binom{n,m+1}{k}$.

2.1.3 Generating N-bonacci Numbers with Skewed Diagonals

We can define a series of numbers correspondent to any natural number T, the case where $T = 2$ the classical Fibonacci Numbers. We define the cases for T=3 and T=4, the general concept then being clear.

Let F_n^3 be defined recursively by $F_1^3 = F_2^3 = 1$, $F_3^3 = 2$, $F_k^3 = F_{k-1}^3 + F_{k-2}^3 + F_{k-3}^3$ for $k > 3$. The first few "tribonacci"[6] numbers then, are: 1, 1, 2, 4, 7, 13, 24.

Let F_n^4 be defined recursively by $F_1^4 = F_2^4 = 1$, $F_3^4 = 2$, $F_4^4 = 4$, and $F_k^4 = F_{k-1}^4 + F_{k-2}^4 + F_{k-3}^4 + F_{k-4}^4$ for $k > 4$. The first few "quadronacci"[6] numbers then, are: 1, 1, 2, 4, 8, 15, 29.

In general, $F_n^T = \sum_{i=n-T}^{n-1} F_i^T$ for $n > 1$, where $F_1^T = 1$, and $F_k^T = 0$ for $k \leq 0$.

Then we can generalize all such numbers using our T-Triangle with the following formula:*

$$F_n^T = \sum_{r \geq 0} \binom{n-r, T}{r}$$

*-generalized from results found in [6]

This formula can be exhibited with the same northeast diagonal pattern on our T-Triangles. The two examples we have given below:

1 1 2 4

n=0 1 7

n=1 1 1 1

n=2 1 2 3 2 1

n=3 1 3 6 7 6 3 1

n=4 1 4 10 16 19 16 10 4 1

1 1 2 4

n=0 1 8

n=1 1 1 1 1

n=2 1 2 3 4 3 2 1

n=3 1 3 6 10 12 12 10 6 3 1

n=4 1 4 10 20 31 40 44 40 31 20 10 4 1

Figure 7: N-bonacci Numbers on T-Triangles

2.1.4 The T-Triangle and Generalizing the *Gambler's Game of Points*

In an application which I believe to be original, we can also generalize the "Gambler's Game of Points" problems for games with a more complicated point structure. Suppose for example, that the game being played has three outcomes. One in which player A receives two points, one in which player B receives two points, and one in which they each receive a point (note that outcomes resulting in no points for either player can be omitted, and we can consider the problem to consist only of throws in which points are awarded). Suppose that player A needs 2 points to win, and player B 4 points. Then the highest number of possible throws remaining to end the game is 3, all resulting in a clear victory or tie. With this in mind, we can look at our third row of our 3-Triangle (with corresponding monomials added for emphasis):

n=3 $\quad 1x^6 \quad 3x^5y \quad 6x^4y^2 \quad 7x^3y^3 \quad 6x^2y^4 \quad 3xy^5 \quad 1y^6$

And we see that the first four terms correspond to wins in favor of player A, the last two wins for player B. The term $6x^2y^4$ corresponds to a tie, and thus we need to consider the order in which tosses were made. This term is generated from the previous line of the triangle by the three terms $\{3x^2y^2 \;,\; 2xy^3 \;,\; y^4\}$, the first term a win for player A, the second term the true tie term, and the third term a win for player B. The tie term can be split 1:1 between the players, as omitting it gives favor to player A. Thus the stakes for A:B can be split in the ratio 21:6.

2.2 Pascal's Simplices

Consider the Multinomial Theorem:[6]

Let $x_1, x_2, \dots, x_k$ be any k real variables, and n any nonnegative integer. Then

$$(x_1 + x_2 + \ \dots + x_k)^n = \sum \frac{n!}{i_1!\, i_2! \dots i_k!} x_1^{i_1} x_2^{i_2} \dots x_k^{i_k}$$

where the sum is taken over every $i_1, i_2, \dots, i_k \geq 0$ with $i_1 + i_2 + \cdots + i_k = n$.

When $k = 2$, this reduces down to the Binomial Theorem. The arrangements of the coefficients of the expansion of a multinomial are known generally as Pascal's Simplices. The 2-Simplex is Pascal's original Triangle, the 1-Simplex is a single line (all of 1s, the coefficients of $(x_1)^n$), and the 0-Simplex is a single point.

2.2.1 Construction

I shall focus on construction of the 3-Simplex and 4-Simplex, the general idea being clear. We will use the terms "slice", "row" or "layer" to indicate the sub-arrays of coefficients for a particular value of n in a respective object. We will sometimes use two of these terms to differentiate in context, for example "the second row of the third layer of the nth slice of the 4-Simplex". Before proceeding to the objects themselves, we give a simple theorem which helps makes some later relations more apparent.

Theorem: The sum of coefficients in the nth slice of the T-Simplex is T^n.

Proof: Plugging in $x_1 = x_2 = \cdots = x_T = 1$ to the multinomial formula above gives us this result immediately.

Pascal's Tetrahedron

The 3- Simplex is sometimes referred to as Pascal's Pyramid, but as I reserve this name for a later topic, I shall refer to it (in a sense, more properly, due to its form) as Pascal's Tetrahedron.

Pascal's Tetrahedron is a three dimensional array of the coefficients of the multinomial expression $(x_1 + x_2 + x_3)^n$, where the nth horizontal slice corresponds to the power n, just as in the 2-Simplex. The top is a coefficient of one, given by $n = 0$, and the base is a triangular array of the coefficients given by the largest n considered. The slices follow the procession of the triangular numbers in size: 1, 3, 6, 10, 15, 21, ..., with the pure powers of a single term lying along the edges. The faces of the Tetrahedron are themselves Pascal's Triangles of two variables, with the mixed terms on the interior of the slices.

Making a slice using Pascal's Triangle

We can use Pascal's Triangle to create a slice of the 3-Simplex by observing the following identity:

$$(x_1 + x_2 + x_3)^n = \left((x_1 + x_2) + x_3\right)^n = \sum_{k=0}^{n} \binom{n}{k} (x_1 + x_2)^{n-k} x_3^k$$

The idea is, we can create a Pascal's Triangle from $(x_1 + x_2)^{n-k}, k = 0,1, \dots, n$, and then multiply the rows by the corresponding coefficient of $\binom{n}{k} x_3^k$. I illustrate this idea below with the slice where $n = 3$.

$$
\begin{array}{lclccccccccl}
x_3^3 & \times & (& & & & 1 & & & & &) \\
3x_3^2 & \times & (& & & x_1 & + & x_2 & & & &) \\
3x_3^1 & \times & (& & x_1^2 & + & 2x_1x_2 & + & x_2^2 & & &) \\
1 & \times & (& x_1^3 & + & 3x_1^2x_2 & + & 3x_1x_2^2 & + & x_2^3 & &)
\end{array}
$$

Figure 8: 3-Simplex Slice Construction

Multiplying through and removing operators, we see the $n = 3$ slice to be:

$$
\begin{array}{ccccccc}
 & & & x_3^3 & & & \\
 & & 3x_3^2x_1 & & 3x_3^2x_2 & & \\
 & 3x_3x_1^2 & & 6x_1x_2x_3 & & 3x_3x_2^2 & \\
x_1^3 & & 3x_1^2x_2 & & 3x_1x_2^2 & & x_2^3
\end{array}
$$

Figure 9: 3-Simplex Slice of Terms

With the idea in hand, we can quickly construct the slice for n of the 3-Simplex by creating the Pascal's Triangle up to the row for n, and then multiplying each row through by the corresponding value of the line on its base. For example, the slice for $n = 4$ is as follows:

1

4 4

6 12 6

4 12 12 4

1 4 6 4 1

Figure 10: 3-Simplex Slice of Coefficients

The slices have a beautiful three way symmetry with respect to the various terms, and when determining which coefficient is assigned to each term it is enough to note that the pure powers of a single variable occur in the corners, and the power of that variable reduces by one for each (respectively oriented) line it travels from "its" corner. This is easily seen in our example above with the powers of the variables present. The illustration in figure 11 is the first several layers of the Tetrahedron, taken from a paper by John F. Putz.[7]

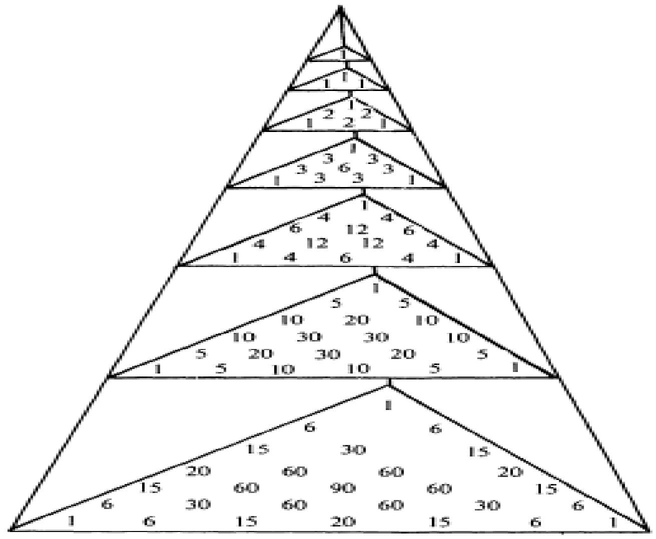

Figure 11: The 3-Simplex

The 4-Simplex

Just as the 3-Simplex can be viewed as a series of Pascal-like triangles, the 4-Simplex is a series of 3-Simplices. Similar to our assessment before, this is seen in the identity:

$$(x_1 + x_2 + x_3 + x_4)^n = \sum_{k=0}^{n} \binom{n}{k} (x_1 + x_2 + x_3)^{n-k} x_4^k$$

A "slice" then of the 4-Simplex corresponding to a particular value of n would be a tetrahedron of order n, each subsequent layer k (with $k = 0$ the base of the tetrahedron) is multiplied through entirely by the corresponding $\binom{n}{k} x_4^k$. A tetrahedron with four way symmetry, each of the four corners being a pure power of a single variable, with the power reducing by one with each (respectively oriented) triangular layer traveled away from a variable's respective corner. We see this construction of the "slice" of the 4-Simplex corresponding to a particular value of n is analogous to our previous construction to the same of the 3-Simplex.

The 5-Simplex then is a series of 4-Simplices, a slice corresponding to a particular value of n being a series of Tetrahedrons of four variables, each multiplied through by a respective binomial coefficient attached to the natural power of the fifth variable. This generalization carries us through the Simplices of any order.

2.2.2 Relation to the Figurate Triangle

In a letter dated 8 June 1712, "Montmort remarked to Nicholas Bernoulli that he had discovered what no-one he knew had yet found, that the number of terms in a q-nomial raised to the *p*th power is given by the figurate number f_{q-1}^{p+1}"(p. 118). [4]

The figurate numbers are formed from the integers, themselves being formed in the same fashion by a string of 1s, with the formula $f_n^l = \sum_{k=1}^{l-1} f_{n-1}^k$. The first few rows are listed below.

Consider the triangular numbers ($n = 2$ below): 1, 3, 6, 10, 15, 21, 28... . These numbers are clearly the procession of the number of coefficients present in the nth slice of the 3-Simplex.

Similarly, we can derive from these numbers the tetrahedral numbers, in progression 1, 4, 10, 20, 35, 56, 84, ... where the kth tetrahedral number is the sum of the first k triangular numbers. These form the intuitive total number of nodes (or lattice points, here we use the term nodes to indicate the lattice points on which the coefficients lie in the array) in the tetrahedron as a whole up to and including the nth slice.

We continue in this fashion to notate the number of total nodes in the T-Simplex, and realize a construction known as the "Figurate Triangle".[4]

T\ l	1	2	3	4	5	6	7	.	.	.
0	1	1	1	1	1	1	1	.	.	.
1	1	2	3	4	5	6	7	.	.	.
2	1	3	6	10	15	21	28	.	.	.
3	1	4	10	20	35	56	84	.	.	.
4	1	5	15	35	70	126	210	.	.	.

Figure 12: The Figurate Triangle

Here we see each row represents the number of vertices in the T-Simplex, of order l. That is, the truncated T-Simplex terminating at the slice determined by l. Each number is found in the same fashion as the triangular and tetrahedral numbers, the lth number of the Tth row being the sum of the first l numbers in the preceding row. This comes intuitively, as the 3-Simplex is simply a series of triangles, the 4-Simplex a series of tetrahedrons, the 5-Simplex a series of 4-Simplices, and so on. So although Montmort did not give much detail, he had indeed correctly associated the number of terms in a q-nomial raised to the pth power with the figurate numbers. From this observation we also note that the number of terms in the nth slice of the 3-Simplex (or any tetrahedron in a higher Simplex) is the nth triangular number. These occur in both the third row and column of the Figurate Triangle, giving the coefficient $\binom{n+2}{2}$.

While the figurate triangle has been well known for millennia, and formulas for the sums of the triangular numbers have been found on a papyrus from Ancient Egypt dated 300 B.C[4], we of course realize this to again be Pascal's Triangle without much difficulty. In fact, this is included in the material of *Traité du triangle arithmétique,* and many such triangles have been used historically for results involving the extraction of roots from expressions of the form $(a + b)^n$. Stifel even derived a formula from his own figurate triangle for the binomial coefficient in 1545[4], the first known person to do so in the West. In a sense this table is misplaced in this document as the figurate numbers are really the basis for the study of the binomial, but I insert it here more for reference on the size of the Simplices.

Chapter 3: T-Triangles from Simplex Slices

The slices of the Simplices have a very direct relationship to our T-Triangles. We know from our previous theorems that the sums of coefficients in the kth row in our T-Triangle matches that of the kth slice in our T-Simplex. It is easily seen that the T-Triangles we have discussed earlier are a particular case of the Simplices with the substitution $x_i = x^{T-i}$, e.g. $(x_1 + x_2 + x_3) = (x^2 + x + 1)$. Thus we can think of a line corresponding to a particular n in any T-Triangle as this substitution in the corresponding slice for that n in our T-Simplex. For example, we see that our slice created before (changing the roles of x_2 and x_3 to match our substitution) of our 3-Simplex for $n = 3$:

$$\begin{array}{ccccccc}
 & & & x_2^3 & & & \\
 & & 3x_2^2x_1 & & 3x_2^2x_3 & & \\
 & 3x_2x_1^2 & & 6x_1x_2x_3 & & 3x_2x_3^2 & \\
x_1^3 & & 3x_1^2x_3 & & 3x_1x_3^2 & & x_3^3
\end{array}$$

Figure 13: 3-Simplex Slice of Terms

becomes:

$$\begin{array}{ccccccc}
 & & & x^3 & & & \\
 & & 3x^4 & & 3x^2 & & \\
 & 3x^5 & & 6x^3 & & 3x & \\
x^6 & & 3x^4 & & 3x^2 & & 1 \\
\hline
x^6 & 3x^5 & 6x^4 & 7x^3 & 6x^2 & 3x & 1
\end{array}$$

Figure 14: Substituted 3-Simplex Slice

Our row corresponding to $n = 3$ in our 3-Triangle.

This relationship between our 3-Triangle and 3-Simplex is clear for any corresponding n. Dropping our variables, we can see this process with only the coefficients for $n = 4$:

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 4 & & 4 & & & \\
 & & 6 & & 12 & & 6 & & \\
 & 4 & & 12 & & 12 & & 4 & \\
1 & & 4 & & 6 & & 4 & & 1 \\
\hline
1 & 4 & 10 & 16 & 19 & 16 & 10 & 4 & 1
\end{array}
$$

Figure 15: Summing the 3-Simplex Slice

As our T increases this process becomes more tedious, as our slices of the T-Simplex are not as easily generated. For example, we show the slice of the 4-Simplex with $n = 3$ and our general substitution method in figure 16. These triangles can be imbedded upon each other according to the powers of x as in figure 17. Summing after substitution gives us a line of coefficients, which of course is the corresponding row to $n = 3$ in our 4-Triangle.

x^9	x^8	x^7	x^6	x^5	x^4	x^3	x^6	x^5	x^4	x^3	x^2
			1								
		3		3					3		
	3		6		3			6		6	
1		3		3		1	3		6		3

x^3	x^2	x			1
	3				
3		3			1

Figure 16: Substituted 4-Simplex Slice

x^9	x^8	x^7	x^6	x^5	x^4	x^3	x^2	x	1
			1				3		
		3		3	3	3		3	
	3		6	6	3	6			
1		3	3	3	6	1	3		1
1	3	6	10	12	12	10	6	3	1

Figure 17: Imbedded 4-Simplex Slice and Sum

Recall that for $T = 5$ a slice of our T-Simplex is a series of Tetrahedrons (i.e. a 4-Simplex multiplied through by respective binomial coefficients), so a similar process would entail doing the above with all slices of the altered 4-Simplex generated. This process is simplified in our case, since we already know that the slices of the general 4-Simplex correspond to our rows in the 4-Triangle. So for $n = 3$ we multiply rows by the respective binomial coefficient $\binom{n}{k}$ in a similar way to our construction of our slice of the Tetrahedron, and realize a new correspondence between the nth row of a T-Triangle and the first n rows of the $(T-1)$-Triangle. Using our previous substitution for x_1, x_2, x_3 and setting $x_4 = x^3$:

n=0	$\binom{3}{3}x^{12}$					1					
n=1	$\binom{3}{2}x^{8}$				1	1	1	1			
n=2	$\binom{3}{1}x^{4}$			1	2	3	4	3	2	1	
n=3	$\binom{3}{0}x^{0}$	1	3	6	10	12	12	10	6	3	1

Figure 18: Modified 4-Triangle

Multiplying through and aligning with corresponding powers of x gives:

$\underline{x^{12}}$	$\underline{x^{11}}$	$\underline{x^{10}}$	$\underline{x^{9}}$	$\underline{x^{8}}$	$\underline{x^{7}}$	$\underline{x^{6}}$	$\underline{x^{5}}$	$\underline{x^{4}}$	$\underline{x^{3}}$	$\underline{x^{2}}$	$\underline{x^{1}}$	1
1												
	3	3	3	3								
		3	6	9	12	9	6	3				
			1	3	6	10	12	12	10	6	3	1
1	3	6	10	15	18	19	18	15	10	6	3	1

Figure 19: Modified 4-Triangle Overlay by Term

The line corresponding to $n = 3$ in our 5-Triangle.

This leads to a very nice identity in the case where $x = 1$, following from our fact that the sum of the coefficients of the nth row of any T-Triangle is T^n:

$$k^n = (k-1)^n + \binom{n}{1}(k-1)^{n-1} + \cdots + \binom{n}{n-1}(k-1) + \binom{n}{n} = \sum_{i=0}^{n} \binom{n}{i}(k-1)^i$$

An identity which admittedly is much more easily proved with the Binomial Theorem using $k^n = \big((k-1)+1\big)^n$.

When looking at the connection between the Simplices and the T-Triangles, it is also interesting to note that the N-bonacci numbers we have seen in our T-Triangle have also been observed in the Tetrahedron by John F. Putz in his paper[7]. He gives the following illustration, the diagonal plane in the below related to our T-Triangle diagonals by the above relationships.

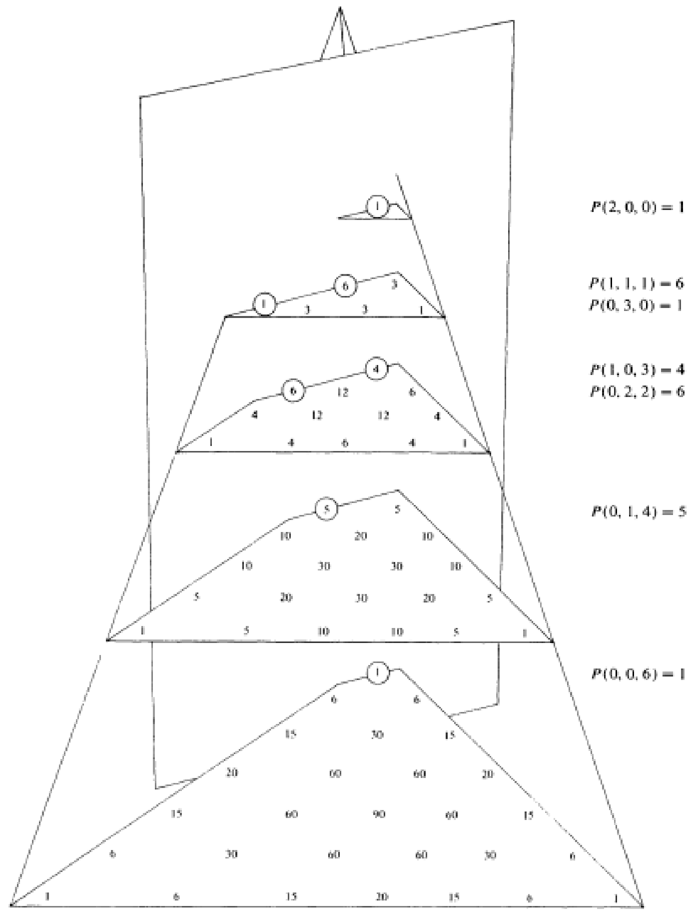

Figure 20: N-bonacci Numbers on the 3-Simplex

Chapter 4: A New Generalization of the 4-Simplex - Pascal's Square and Pyramid

From the 0-Simplex, we have seen that each consecutive Simplex is generated by an additional point in a new dimension. But what if we considered all Simplices to be in the third dimension, where a new point was added in the third dimension to construct the next Simplex, with interaction filling the space bounded by all vertex points? It is in this mindset that I proceed with the following generalization, which I believe to be new.

While the 4-Simplex and higher order Simplices have a very distinct beauty, I have been making attempts to create three dimensional objects in which to arrange their coefficients that carry the same properties as the 3-Simplex (Pascal's Tetrahedron). In the 3-Simplex we saw that the top of the entire object corresponded clearly to $n = 0$, and the sides of the object were all Pascal Triangles of two variables. In fact, all possible Pascal Triangles of two variables are present. This is also true for the Simplices of higher order, but to see it (with our dimensionally limited visualization) requires one to consider the "top" to be the peak of the smallest Tetrahedron, 4-Simplex, .. etc as the top of the entire object, and so see the Pascal Triangles of two variables one must do rearrangement depending on the expansion used in the original construction. Where as in the Tetrahedron, assigning one of the variables to be 0 immediately reduces the form to the original Pascal's Triangle, in higher orders this action requires different forms of "stacking" after reduction depending on which variables are chosen to be 0. Also, as we have seen, relating a slice of a T-Simplex to its respective T-triangle becomes increasingly difficult as we travel to higher and higher dimensions.

The Simplices as we have constructed them in this paper are done so based completely on the triangular numbers 1, 3, 6, 10, 15, 21, ... We also have what are known as the square numbers, representing latticed squares of steadily increasing size. These numbers are, in progression, 1, 4, 9, 16, 25, ... In a certain sense, we are simply creating a square with two triangles, one having length 1 less on each side as the larger. With this relationship we can correspond a slice of the 4-Simplex (the tetrahedron generated by n) to a set of overlaying squares, each constructed by two consecutive layers of the slice beginning from the bottom.

The construction gives an $(n+1) \times (n+1)$ square of $(n+1)^2$ coefficients, an $(n-1) \times (n-1)$ square with $(n-1)^2$ coefficients, and so on until reaching 1 or 0. Overlaying these we see the arrangement of coefficients in a single square, where each node of the square contains the number of coefficients equal to the shortest distance to the nearest side. This construction gives us confirmation that the number of coefficients present in the square is the same as that in the tetrahedral slice of the 4-Simplex.

As to which coefficients lie in which node, we can link the generation of this square to the idea of the generation of different layers of the 3-Simplex. In the 3-Simplex, each consecutive layer is a consecutive triangle with respect to the triangular numbers, and each coefficient is generated by the addition of the three coefficients in the triangle above it. We can think of this process similar to that in the original Pascal's Triangle, and I will make another comparison for visualization. For those familiar with the long-standing game show "The Price is Right", there was a game played by contestants known as "Plinko". In this game, there was a wall of pegs, with a disc dropped above the top row. Each time the disc fell to a subsequent row, there were two possible directions for the disc to fall. For a given starting point, the distribution of ways that the disc would end at a particular space in the final row is directly related to the proportion of coefficients in the Pascal Triangle.

In her Master's Thesis[1], Iowa State University student Katie Asplund used the Pascal Triangle to give an analysis of the game "Plinko" in depth. She was able to calculate the expectation of winnings given any starting point for the disc, as well as the probabilities of each particular prize amount.

In the 3-Simplex, we could think of the coefficients being generated in a similar way. If a ball is placed on the top of the tetrahedron, and at each point the ball had three directions in which to travel, the coefficients in each slice of the tetrahedron would correspond directly to the number of ways the ball could reach any particular node on that level. The three directions here correspond directly to our three variables, the power of a variable increasing if movement is in the direction of the corner assigned to that variable.

It is in this way that I construct the Square. If we consider a pyramid of such squares, all centered about a central axis and with diagonals directly above one another, we can generate the coefficients by observing the ways a ball could arrive at a particular node, given the opportunity to move in four different directions when traveling to the next layer with each direction being toward a particular corner. The first several layers generated in this way would look as follows, corresponding to a layer n, where the peak is $n = 0$.

$n = 0$	1					
$n = 1$	1	1				
	1	1				
$n = 2$	1	2	1			
	2	4	2			
	1	2	1			
$n = 3$	1	3	3	1		
	3	9	9	3		
	3	9	9	3		
	1	3	3	1		
$n = 4$	1	4	6	4	1	
	4	16	24	16	4	
	6	24	36	24	6	
	4	16	24	16	4	
	1	4	6	4	1	
$n = 5$	1	5	10	10	5	1
	5	25	50	50	25	5
	10	50	100	100	50	10
	10	50	100	100	50	10
	5	25	50	50	25	5
	1	5	10	10	5	1

Figure 21: Pascal Square Layers of the Pyramid

It is clear that any of the four sides give the coefficients for possible horizontal or vertical movement to a particular row or column. For example, the first row gives all possible horizontal movement with vertical movement remaining constant, the first column all possible vertical movement with horizontal movement constant. So, for the central coefficients, we can easily calculate this in a multiplication table fashion, giving the number of ways to travel to a particular node horizontally times the number of ways to travel to that node vertically.

To relate this to our tetrahedral slice, we can relate motion in any diagonal direction as being an increase in the power of the variable assigned to that direction. Recall that a tetrahedral slice of our 4-Simplex is a four way symmetric object with a pure power of a single variable at each vertex, and the binomial correspondence between any two variables lying along the edge connecting the respective vertices. If we position the tetrahedron so that the variable x_1 has its pure power on the top, then each consecutive layer would consist of terms with the power of x_1 reduced by 1. With our relation to motion, each layer would give the coefficients of motion in the Pyramid a fixed number of times towards the corner of x_1, with all possible combinations of movement otherwise. With this correspondence, we see that we can overlay the layers of the tetrahedron in the following manner, using $n = 4$ as an example:

$n = 4$

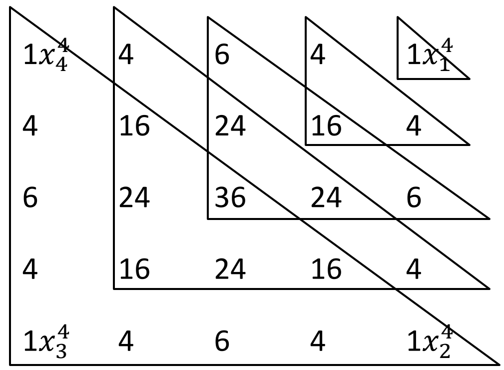

Figure 22: Triangular Overlay of the Pascal Square

Here each larger triangle contains a single layer of coefficients with reduced powers of x_1. This overlay is exactly in line with our expectation from our original idea, where each node contains a number of terms equal to the shortest distance to the nearest side. The 3×3 square has two terms in each node along its sides, and the 1×1 square as a single point in the center has three terms.

This overlaying also beautifully displays the number of ways the ball could travel given a particular distance traveled in a particular direction. The overlapping of terms comes from the fact that traveling towards two opposite corners "cancels" the movement. e.g. the movement of $x_1^3 x_3$ leads to the same node as $x_1^2 x_2 x_4$, as $x_1 x_3$ and $x_2 x_4$ both correspond to no diagonal movement at all.

The square so far is not as related as we would like it to be to the multinomial distribution, as we have not yet found a way to "extract" particular coefficients from the sum present at each node. But with our overlaying observation, we can use the properties of the layers in our tetrahedron to reveal a formula for any given node.

Observe that along the first row, all coefficients fall along a peak of a triangle, giving them the proportion 1 to the coefficient along the edge. In the second row, each pair of consecutive coefficients falls into the second line of a triangle, giving the coefficients in the same triangle the proportion 1:1. Because of the symmetry of the construction, the first four coefficients also follow the proportion 1:3:3:1 in a triangle generated by the corner x_4, and the last four coefficients follow the proportion 1:3:3:1 in a triangle generated by the corner x_2.

With the four way symmetry of the square, and different arrangements of triangles, we can calculate the coefficients of the different terms in each node with the following result.

Main Theorem: Let $N(n,k,l)$ denote the node in the Pascal's Square corresponding to n, where $N(n,0,0)$ is any corner of the square, and k,l correspond to either (but different) directions parallel to the sides of the square. Then the set $S(n,k,l)$ of coefficients corresponding to unique terms of the multinomial expansion in $N(n,k,l)$ is:

$$S(n,k,l) = \{\binom{n}{k}\binom{k}{x}\binom{n-k}{l-x}\} \; for \; x = 0,1,\dots,l$$

Due to the symmetry, the roles of l and k can be interchanged, and the formula for coefficients nicely gives zero for extraneous values of x, generating only $\min\{l+1, k+1, n-k+1, n-l+1\}$ coefficients as desired.

Proof: Without loss of generality, let us consider the case where $l \leq k$, and both $k,l \leq \frac{n+1}{2}$. In any other case we can reorient our initial corner to make this the case.

Consider the triangular layering originating from the upper right corner as in our previous diagram, where k is the horizontal component and l the vertical. Then each coefficient in $S(n,k,l)$ is present in the lth row of a slice of the Tetrahedron, the slice being multiplied through by a constant. There are l such slices intersecting at the node $N(n,k,l)$, wherein the smallest slice is the kth slice of the Tetrahedron multiplied through by $\binom{n}{k}$, and the desired coefficient is the first in the lth row. The second smallest is the $(k+1)$st slice multiplied through by $\binom{n}{k+1}$ with the desired coefficient the second in the lth row and so on, the lth and largest slice being the $(k+l)$st slice of the Tetrahedron multiplied through by $\binom{n}{k+l}$ with the desired coefficient the lth in the lth row.

Then the first coefficient is $\binom{n}{k}\binom{k}{l}\binom{l}{0}$, the second is $\binom{n}{k+1}\binom{k+1}{l}\binom{l}{1}$, and continuing in this fashion the final coefficient is $\binom{n}{k+l}\binom{k+l}{l}\binom{l}{l}$. In all cases we can rewrite these as $\binom{n}{k+x}\binom{k+x}{l}\binom{l}{x}, x = 0,1, \dots l$ and see that;

$$\binom{n}{k+x}\binom{k+x}{l}\binom{l}{x} = \frac{n!}{(k+x)!\,(n-k-x)!}\frac{(k+x)!}{l!\,(k+x-l)!}\frac{l!}{x!\,(l-x)!}$$

$$= \frac{n!}{k!\,(n-k)!}\frac{k!}{(l-x)!\,(k+x-l)!}\frac{(n-k)!}{x!\,(n-k-x)!}$$

$$= \binom{n}{k}\binom{k}{l-x}\binom{n-k}{x}, x = 0,1, \dots, l$$

Now letting $y = l - x$;

$$\binom{n}{k}\binom{k}{l-x}\binom{n-k}{x} = \binom{n}{k}\binom{k}{y}\binom{n-k}{l-y}, y = 0,1, \dots, l$$

Completing the proof.

Corollary: With our above theorem, the final equality in its proof and symmetries of the square we see that our square also produces the immediate identities:

$$\sum_{x=0}^{l}\binom{n}{k+x}\binom{k+x}{l}\binom{l}{x} = \sum_{x=0}^{l}\binom{n}{k}\binom{k}{l-x}\binom{n-k}{x} = \sum_{x=0}^{l}\binom{n}{k}\binom{k}{x}\binom{n-k}{l-x} = \binom{n}{k}\binom{n}{l}$$

$$for\ any\ non-negative\ integer\ n, with\ 0 \leq k, l \leq n$$

In particular, the last two resolve to the identities:

$$\sum_{x=0}^{l}\binom{k}{l-x}\binom{n-k}{x} = \sum_{x=0}^{l}\binom{k}{x}\binom{n-k}{l-x} = \binom{n}{l}$$

$$for\ any\ non-negative\ integer\ n, with\ 0 \leq k, l \leq n$$

Which is known as **Vandermonde's Identity**[9] or Vandermonde's Convolution, an identity which can be proved is several different ways but to my knowledge has not before been proven by this construction. Also, we do not have the restriction of $l \leq k$ given by [9]. And we also see that the first sum gives us:

$$\sum_{x=0}^{l}\binom{n}{k+x}\binom{k+x}{l}\binom{l}{x} = \binom{n}{k}\binom{n}{l}$$

$$for\ any\ non-negative\ integer\ n, with\ 0 \leq k, l \leq n$$

A rank 3 combinatorial identity, which is an expansion formula for the product of binomial coefficients given by John Riordan in [8](pp. 15). There it is derived by repeatedly using the Vandermonde convolution. This method of derivation comes as no surprise to us, as we have seen from our construction they are very closely related.

As an example, let's consider the node $N(6,4,3)$ which is the same as $N(6,2,3)$. We see that we have:

$$\sum_{x=0}^{3}\binom{6}{2+x}\binom{2+x}{3}\binom{3}{x}=\binom{6}{2}\binom{2}{3}\binom{3}{0}+\binom{6}{3}\binom{3}{3}\binom{3}{1}+\binom{6}{4}\binom{4}{3}\binom{3}{2}+\binom{6}{5}\binom{5}{3}\binom{3}{3}$$
$$=0+(20)(1)(3)+(15)(4)(3)+\ (6)(10)(1)=60+180+60$$
$$=300=(15)(20)=\binom{6}{2}\binom{6}{3}$$

$$\sum_{x=0}^{3}\binom{6}{4+x}\binom{4+x}{3}\binom{3}{x}=\binom{6}{4}\binom{4}{3}\binom{3}{0}+\binom{6}{5}\binom{5}{3}\binom{3}{1}+\binom{6}{6}\binom{6}{3}\binom{3}{2}+\binom{6}{7}\binom{7}{3}\binom{3}{3}$$
$$=(15)(4)(1)+(6)(10)(3)+\ (1)(20)(3)+0=60+180+60$$
$$=300=(15)(20)=\binom{6}{4}\binom{6}{3}$$

Our rank 3 identity working beautifully regardless of the representative node chosen.

And we have all versions of our Vandermonde's Identity:

$$\sum_{x=0}^{l}\binom{6}{4}\binom{4}{3-x}\binom{6-4}{x}=\binom{6}{4}\binom{4}{3}\binom{2}{0}+\binom{6}{4}\binom{4}{2}\binom{2}{1}+\binom{6}{4}\binom{4}{1}\binom{2}{2}+\binom{6}{4}\binom{4}{0}\binom{2}{3}$$
$$=(15)(4)(1)+(15)(6)(2)+(15)(4)(1)+0=60+180+60=300$$
$$=(15)(20)=\binom{6}{4}\binom{6}{3}$$

$$\sum_{x=0}^{l}\binom{6}{2}\binom{2}{3-x}\binom{6-2}{x}=\binom{6}{2}\binom{2}{3}\binom{4}{0}+\binom{6}{2}\binom{2}{2}\binom{4}{1}+\binom{6}{2}\binom{2}{1}\binom{4}{2}+\binom{6}{2}\binom{2}{0}\binom{4}{3}$$
$$=0+(15)(1)(4)+(15)(2)(6)+(15)(1)(4)=60+180+60=300$$
$$=(15)(20)=\binom{6}{2}\binom{6}{3}$$

$$\sum_{x=0}^{l}\binom{6}{4}\binom{4}{x}\binom{6-4}{3-x}=\binom{6}{4}\binom{4}{0}\binom{2}{3}+\binom{6}{4}\binom{4}{1}\binom{2}{2}+\binom{6}{4}\binom{4}{2}\binom{2}{1}+\binom{6}{4}\binom{4}{3}\binom{2}{0}$$
$$=0+(15)(4)(1)+(15)(6)(2)+(15)(4)(1)=60+180+60=300$$
$$=(15)(20)=\binom{6}{4}\binom{6}{3}$$

$$\sum_{x=0}^{l}\binom{6}{2}\binom{2}{x}\binom{6-2}{3-x}=\binom{6}{2}\binom{2}{0}\binom{4}{3}+\binom{6}{2}\binom{2}{1}\binom{4}{2}+\binom{6}{2}\binom{2}{2}\binom{4}{1}+\binom{6}{2}\binom{2}{3}\binom{4}{0}$$
$$=(15)(1)(4)+(15)(2)(6)+(15)(1)(4)+0=60+180+60=300$$
$$=(15)(20)=\binom{6}{2}\binom{6}{3}$$

In all cases, the set of coefficients in our node is:

$$S(6,4,3) = S(6,2,3) = \{60,60,180\}$$

Using $n = 4$ as our example again, we see that the square with coefficients of individual terms listed would look like the following:

$n = 4$

1	4	6	4	1
4	4,12	12,12	12,4	4
6	12,12	6,6,24	12,12	6
4	4,12	12,12	12,4	4
1	4	6	4	1

Figure 23: Square with Separated Coefficients

To see a more intuitive format for generating coefficients, consider the dual triangular overlay displayed as follows:

$n = 4$

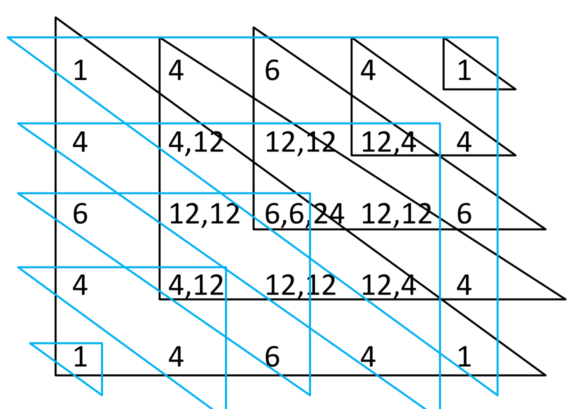

Figure 24: Square with Dual Triangle Overlay

We can see that the overlay of "Pascalian" rows gives a very nice method to find the distinct coefficients of a row quickly. For example, to calculate all coefficients of the second row, we use the first value as an "anchor". We can calculate out coefficients in proportion to either the second or fourth row of Pascal's Triangle, and then use these results as anchors to calculate out the other row of proportions. For example, the second row:

4	12	12	4		or	4	4			
	4	12	12	4			12	12		
								12	12	
									4	4

Figure 25: Coefficient Separation

Where each "column" represents a particular node in that row. Similarly, we can calculate the distinct coefficients of the third row, both anchors and rows in the proportion 1:2:1 respective to the relevant triangles:

6	12	6		
	12	24	12	
		6	12	6

Figure 26: Alternative Coefficient Separation

This quick and easy to see method is the origination of the above theorem.

Now taking our construction of a single square, the pyramid formed by consecutive squares forms the Pascal Pyramid, a three dimensional representation of our 4-Simplex. Along each side of the pyramid, and across each diagonal, are Pascal Triangles of two variables. Each diagonal half of the pyramid contains a Tetrahedron of three variables, four in total. Setting any combination of variables to 0 immediately reduces the construction to the relevant Simplex of the remaining variables, without any rearrangement. It is in the resolution of this object to its finer forms where I find the true beauty of the construction.

One final correspondence with the Square that I will make, is that the substitution for our x_i of the decreasing powers of x that form the monomial which generates our 4-Triangle resolves each node to contain a single term, in tandem with other nodes along a skewed diagonal (of "slope" -2). Adding along the skewed diagonal produces our respective line of the 4-Triangle. I display this with our $n = 4$ square in figure 26. With the correctly oriented triangle overlay, for the above case the smallest triangle in the bottom left corner, we see that this is the same process that we saw in Section 3. Viewing it in this way, however, gives us a nice formula for the coefficients of our 4-Triangle:

$$\binom{n,4}{k} = \sum_{x} \binom{n}{k-2x}\binom{n}{x}$$

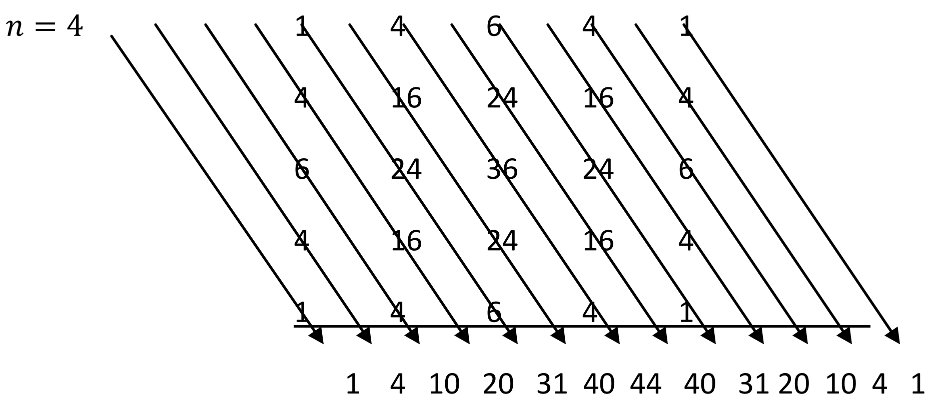

Figure 27: Skewed Diagonals on the Square

Noting that the binomial product will be zero once x exceeds $\frac{k}{2}$. In particular, taking all such diagonals on the square, or by simply using Property (5) in 2.1.2, we have:

$$\sum_{k=0}^{k=3n} \sum_{x=0}^{n} \binom{n}{k-2x} \binom{n}{x} = 4^n$$

Where the indices act more as a restriction in evaluation, as in many of the possible values of x the product is zero.

This same concept, along with our work relating the constructions in part (3) gives us a similar formula for the coefficients of the 3-Triangle. If instead of the centered orientation we arrange the triangle with one side vertical, we can use the same diagonals ("slope" of -2) as above in our square to resolve to the row in our 3-Triangle. I give a visualization of this below:

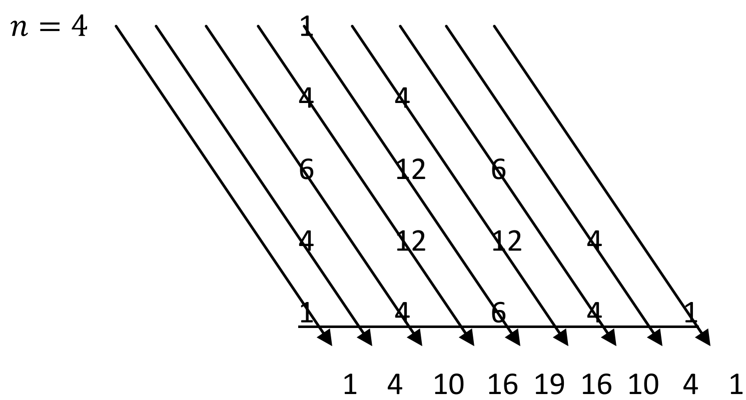

Figure 28: Skewed Diagonals on 3-Simplex Slices

The coefficient sums along the diagonals being the same as summing in the "columns" in part (3). This gives us the coefficient formula:

$$\binom{n,3}{k} = \sum_{x=0}^{n} \binom{n-x}{k-2x} \binom{n}{x}$$

Requiring $n - x$ in the first binomial coefficient instead of just the n we had for the formula of $\binom{n,4}{k}$ due to the columns now being consecutive rows of the Pascal Triangle instead of the same row repeated as in the square. These formulas are the constructively derived cousins of the coefficient formula given as Property (7) in 2.1.2, which was done with the use of generating functions. Finally, analogous to the above from the square, we have:

$$\sum_{k=0}^{k=2n} \sum_{x=0}^{n} \binom{n-x}{k-2x} \binom{n}{x} = 3^n$$

It is interesting to note that the diagonals used in the above triangle are the same that would normally generate the Fibonacci Numbers in a standard (un-truncated) Pascal Triangle that had not had its rows multiplied through as we have done.

Chapter 5: Discussion on Similar Generalizations for Higher Simplices

The pentagonal numbers, 1, 5, 12, 22, 35,... the kth pentagonal number being the kth square number added to the $(k-1)$th triangular number. Formulated geometrically, one can quickly convince themselves that the kth pentagon could be formed from the kth square with a triangle as a hat, then stretching the upper two vertices of the square to the sides with some internal shifting to give symmetry to the diagonals.

While I have not had success in generating a general Pascal's Pentagon in the same way as the square, or using the traditional pentagonal numbers, I have had some success using nested pentagons, maintaining our conventions of sides and diagonals of the major pentagon containing the Pascal Triangles of two variables. Although this generalization is possible, and the 5-way symmetry quite entertaining, the construction is currently unrefined, so I omit it at this time. The 5-directional "movement" does not correspond geometrically with the construction, and the observations that we made with the square and pyramid are not so distinct. Should one find the proper application and method, I believe that such a construction can be made for any T-gon corresponding to the T-Simplex. Below is the pentagon with overlaying squares as we used our triangles before. The coefficients of the pentagon align in this way, but the coefficients in the squares sometimes "split", which has been the major cause for the end of the generalization here. However, the general idea of overlaying "squares" has potential, with the overlay looking somewhat as the below diagram.

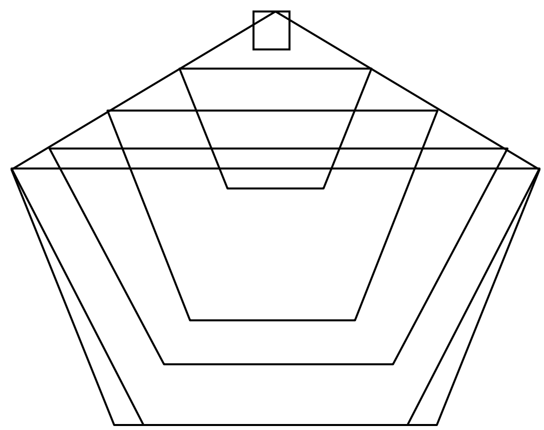

Figure 29: The Pascal Pentagon

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