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Estimation of Travel Time Based on Vehicle-Tracking Models

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ESTIMATION OF TRAVEL TIME BASED ON VEHICLE-TRACKING MODELS

by

Anuj Nayyar

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ABSTRACT

STUDY OF HYPERBOLIC SYSTEM OF EQUATIONS TO STUDY TRAVEL TIME FOR TRAFFIC MODELLING

by

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In this thesis we study the travel time problem based on the known traffic density model. Using the conservation law, we model the travel time function by a boundary value problem of a non homogeneous linear hyperbolic equation. The equation is transformed into an initial value hyperbolic equation, and the well-posedness of the problem is discussed. The mathematical analysis for both density and travel problems are given. We also derive the analytic solutions for several special cases of traffic density. Numerical schemes are proposed for solving for travel time problem. Several numerical examples are presented and error analysis on the solutions obtained is performed to illustrate the rates of convergence of the numerical schemes.

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INTRODUCTION

1.1 Motivation and Research Goal

Since $1920's$, time required to travel a given route between two points have been used by transportation engineers as a fundamental measure for both evaluation of the performance of existing transportation facilities and planning the new ones. However, with the advent of Intelligent Transportation Systems and the introduction of advanced traffic surveillance technologies, there has been an increasing interest for developing more accurate real-time travel time estimation techniques that can be used in the context of Advanced Traveler Information Systems (ATIS). In all the major studies done till now, the focus has been on using the continuously available density and volume data from sensors and use them for travel time evaluation. In addition to providing the travelers with expected travel times, real-time travel time information can also be used to detect the occurrence of accidents and to devise efficient traffic routing strategies. The current study focuses on presenting an improved theoretical approach for the accurate estimation of travel times on a given route based on the real-time traffic data, using a vehicle tracking model.

Although real-time traffic data is now widely available in most of the urban areas in the US, accurate estimation of route travel times still requires a robust and theoretically sound technique that can make efficient use of this raw traffic data obtained from sensors. This thesis attempts to develop a robust methodology that is based on the fundamentals of traffic flow theory by taking advantage of the bivariate relationship between speed and density, speed and flow, and flow and density. Moreover, instead of treating traffic as a point process, it models traffic as continuous process by employing the well-known fluid flow analogy of traffic. This spatial treatment of the traffic process using fluid flow analogy for estimating route travel times departs from most of the travel time estimation techniques proposed in the past that model traffic as a point process by using time-series or other statistical models.

The current thesis work presents a detailed discussion of the proposed travel time estimation methodology and the theory employed to develop this methodology. The other major concern of this research is the study of the resulting system of partial differential equations. We discuss the type of the system, well posedness of the solution and also discuss a few numerical schemes that can be used to solve the system. The corresponding results obtained are also shown.

1.2 Previous Work

In this section a brief discussion of various studies for travel time evaluation have been presented. Most of the studies done are based on direct evaluation of travel time or travel time correction. The results are evaluated with direct utilization of traffic sensor data that is collected on road segments fragmented over space and time. The work done in most studies is on developing algorithms for error correction and travel time prediction.

- In a study done by Xiaoyan Zhang and John A. Rice [4], linear models are developed for travel time evaluation based on the available freeway sensor data. The short term models use the available data on a stretch of freeway till a certain time and use it to predict the travel time between times for future times. The algorithm uses a linear model to calcluate and compare results for various future times.

- In another study done by T. Oda [5] at Matsushita Communication Industrial Co., Ltd., Japan, an approach is proposed to predict the travel time by using the changes in traffic conditions from origin to destination. The traffic parameters that are being considered for use in travel time estimation are traffic volume and occupancy times. An Auto-Regression Model is used for travel time prediction. A total of four travel times are proposed - predicted travel time, measured travel time, calculated travel time and total subsection travel time. These four travel times are evaluated for a road segment, and the results are compared.

- Another study done on Travel-Time prediction on freeway corridors [6] discusses nonlinear models for estimation of traffic parameters on freeways. The travel times are evaluated after observing speed data on freeways in Orlando, Florida. Multivariable prediction schemes are developed based on speed, occupancy and volume data that is collected using the loop detectors.

- A research by W. Chen, S. C. Wong, C. W. Shu, p. Zhang [7] discusses the front tracking algorithm for solving the Lighthill-Whitham-Richard Traffic Model with piecewise fundamental diagrams.

- A research done by P. Kachroo, K. Ozbay, A. G. Hobeika [10] discusses the estimation of travel time on freeways using the macroscopic modeling. The models in both time and space are developed in lumped and dumped settings and travel time algorithms are developed.

Most of the work highlighted here focussed on having the data for density continuously available for study. This involved limitations in prediction of travel time. The current work focusses on overcoming this shortcoming by using just the initial data available and use it for travel time prediction and estimation.

1.3 Current Work

The current work is mainly focused on developing a theory for travel time estimation based on the physical traffic model that is available and is widely used. The work done can be broadly separated into chapters which follow. The various aspects of the study can be briefly outlined as follows.

In Chapter 2, the entire traffic system is discussed on which the entire study is based. The underlying traffic density equation is discussed in detail. From the traffic density model, the travel time equation is formulated and the corresponding system is discussed. Chapter 3 presents a detailed analysis of equivalence of the traffic density equation to the inviscid Burgers' equation under suitable substitutions. The properties and conditions under which the system holds a unique solution are outlined. The initial value and the boundary value conditions are discussed, which allow the system to have a solution in the domain of its definition.

The exact solutions for the travel time problems are mathematically evaluated in Chapter 4 for some special cases of initial density value profile. The shock waves that originate due to various density profiles, and the travel time solutions in those respective cases are evaluated and studied. Moving on to Chapter 5, techniques are discussed to devise numerical solutions to evaluate travel time solutions for other cases than discussed in Chapter 4. The solutions are evaluated for various initial density profiles, and the corresponding errors are evaluated. The stability and convergence of the scheme are discussed by studying the errors obtained.

CHAPTER 2

SYSTEM MODELLING

The first step in the design of estimators for travel time is to model the system dynamics appropriately. In this chapter we discuss the system for traffic modeling and formulate the equations defining the system. There are various traffic flow models that can be used for this, the three main models commonly used for the study are as follows.

1. *Microscopic Model:* This model treats every vehicle as an individual, each of which can be modeled using ordinary differential equations. The road can be discretized into cells and the cells can be then assumed to contain a vehicle, moving with its corresponding velocity.

2. *Macroscopic Model:* This model uses the idea of fluid dynamics and treats the traffic flow as one entity. Rather than treating each vehicle as a separate entity, properties like density of traffic, the mean velocity, etc. are studied and modeled using partial differential equations.

3. *Mesoscopic Model:* This model uses the idea of defining a probabilistic function which expresses the chances of having a vehicle at a certain position at a certain time, running with a certain velocity.

For our study, the macroscopic model is used, where we consider one dimensional traffic flow, i.e. traffic flowing in one direction (and overtaking not allowed).

2.1 Conservation Laws

Consider the traffic flow of cars on a highway with one lane, where the density $\rho(t, x)$ is used to study the nature of traffic flow, where x is the position of cars and the time $t > 0$. For a particular spatial interval (x_1, x_2), at a particular time t, the number of cars is given by $\int_{x_1}^{x_2} \rho(t, x)dx$.

Let $v(t, x)$ be the velocity. At a particular time and instant, the flow of traffic $q(t, x)$ can be given by the product of the velocity and the density at that instant, i.e.,

$$q(t, x) = \rho(t, x)v(t, x) \tag{2.1}$$

Now since the rate of change of number of cars in (x_1, x_2) is given by the difference in flows at x_1 and x_2. Hence the relation between traffic density and traffic flow can be stated in a differential equation as

$$\int_{x_1}^{x_2} \rho(x, t_2)dx - \int_{x_1}^{x_2} \rho(x, t_1)dx = \int_{t_1}^{t_2} q(x_1, t)dt - \int_{t_1}^{t_2} q(x_2, t)dt \tag{2.2}$$

Alternatively, the above equation can be stated in double integral form as

$$\frac{d}{dt} \int_{x_1}^{x_2} \rho(t, x)dx = q(x_1, t) - q(x_2, t) \tag{2.3}$$

Consequently, the equation can be restated as

$$\int_{x_1}^{x_2} [\rho(x,t_2) - \rho(x,t_1)]dx = \int_{t_1}^{t_2} [q(x,t_1) - q(x,t_2)]dx$$

If $\rho(t,x)$ and $q(t,x)$ are differentiable functions, then we have

$$\rho(x,t_2) - \rho(x,t_1) = \int_{t_1}^{t_2} \frac{\partial}{\partial t}\rho(t,x)dt,$$
$$q(t,x_2) - q(t,x_1) = \int_{x_1}^{x_2} \frac{\partial}{\partial x}q(t,x)dx.$$

Thus we get the equation

$$\int_{x_1}^{x_2} \int_{t_1}^{t_2} \left(\frac{\partial}{\partial t}\rho(t,x) + \frac{\partial}{\partial x}q(t,x)\right) dtdx = 0.$$

Since this must hold for all intervals of x and t, the integrals can be dropped to get the differential form of conservation law as

$$\frac{\partial}{\partial t}\rho(t,x) + \frac{\partial}{\partial x}q(t,x) = 0$$

This traffic model is known as the Lighthill-Whitham-Richards Model[8] [9]. With the specification (2.1), the conservation law for the density $\rho(t,x)$ can be restated as

$$\frac{\partial}{\partial t}\rho(t,x) + \frac{\partial}{\partial x}\big(\rho(t,x)v(t,x)\big) = 0, \quad \forall\ t > 0,\ -\infty < x < \infty. \tag{2.4}$$

Concerning the model about relation between the traffic speed $v(t,x)$ and density $\rho(t,x)$, we shall adopt the Greenshield's model, which is the most commonly used macroscopic stream model. The Greenshield's model simply assumes a linear speed-density relationship:

$$v(t,x) = v_f \left(1 - \frac{\rho(t,x)}{\rho_m}\right), \tag{2.5}$$

where v_f is the free flow speed and ρ_m is the jam density. Both the ρ_m and v_f are known parameters. As evident from the equation (2.5), when density becomes zero, speed approaches the free flow speed(i.e. $v \to v_f$ as $\rho \to 0$); and when speed becomes 0(vehicles are stationary), the density approaches the jam density(i.e. $\rho \to \rho_m$ as $v \to 0$) (see Fig.2.1).

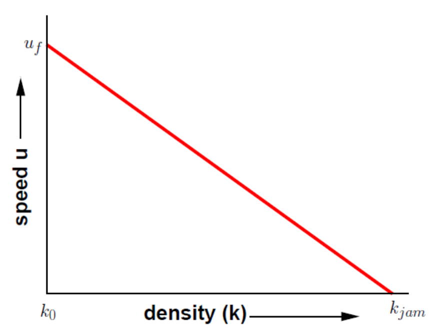

Figure 2.1: Relation between speed and density

Now the relationship between flow and density is given by

$$q(t,x) = v_f \rho(t,x) \left(1 - \frac{\rho(t,x)}{\rho_m}\right)$$

The above equation depicts a parabolic relationship between density and flow of traffic (see Fig. 2.2).

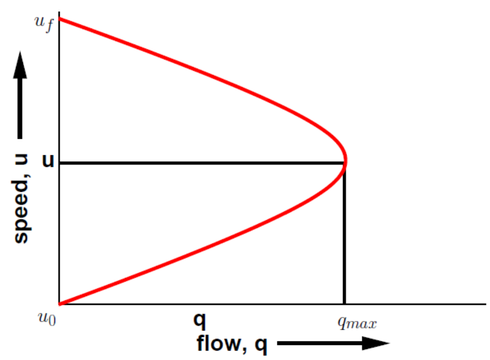

Figure 2.2: Relation between speed and flow

After Substituting (2.5) to (2.4), the conservation law for the density $\rho(t,x)$ becomes

$$\frac{\partial}{\partial t}\rho(t,x) + \frac{\partial}{\partial x}\left(v_f \rho(t,x)\left(1 - \frac{\rho(t,x)}{\rho_m}\right)\right) = 0, \quad \forall\, t \geq 0, \quad -\infty < x < \infty \tag{2.6}$$

With the specified initial value $\rho(0,x)$, we have formulated an initial value problem for the density $\rho(t,x)$.

2.2 Travel Time

Travel time estimation algorithms and functions can be designed using the distributed or lumped parameter models of the traffic. We will develop appropriate models using the different modeling paradigms. This section will present models without forecasting that will approximate the travel time based on only the current state values. The next section will present more accurate estimators that will use forecasting.

It is a performance measure used in transportation used to define the time taken to reach a certain fixed point(destination) from an initial position. The quantity depends upon the other parameters that define the system parameters such as density ρ and flow q which are both functions of position x and time t. The current methodologies used to compute the travel time broadly fall in the following two categories.

- Link Measurement Approach - using active test vehicles between points of interest.

- Point measurement approach - inferred indirectly from other sensors , such as loop detectors or video cameras.

These methods can be classified as being either direct or indirect methods. With the link measurement method, the travel time is the observed parameter, and hence is a direct method. On the other hand the point measurement approach is an indirect observation method as the observed parameters are used to deduce the travel time rather than directly measuring it.

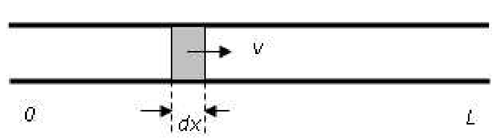

Figure 2.3: Infinitesimal Section

Let $u(t, x)$ be the travel time function for a vehicle at point x and time t to reach point $x = L$. For an infinitesimal road segment of length dx at x (see Fig. 2.3), the time taken to traverse it would be du, given by

$$du = -\frac{dx}{v(\rho(t, x))},$$

i.e.

$$\frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dx = -\frac{dx}{v(\rho(t, x))}.$$

Recall that $v = \frac{dx}{dt}$. We have

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = -1,$$

which is the conservation law for the travel time $u(t, x)$. Replacing v by its specification in equation (2.5), we have

$$\frac{\partial u}{\partial t} + \left(v_f(1 - \frac{\rho(t, x))}{\rho_m}\right)\frac{\partial u}{\partial x} = -1, \ \forall\ 0 < x < L, \quad t \geq 0. \tag{2.7}$$

Once the vehicle reaches L, the value of $u(t, x)$ is 0, which provides the following boundary condition

$$u(t, L) = 0 \ \forall\ t \geq 0. \tag{2.8}$$

Another way of deriving the system is by intuitively observing that the change Δx in position is directly proportional to the change Δu in travel time. In fact, for a vehicle moving with velocity $v(t, x)$, the change in position $\Delta x = x(t + \Delta t) - x(t) \approx$

$v(t, x)\Delta t$. It is apparent that the travel time decreases by Δt, i.e., $\Delta u \approx -\Delta t$. Hence we have

$$\Delta x \approx -v(t, x)\Delta u, \tag{2.9}$$

i.e,.

$$u(t + \Delta t, x + \Delta x) - u(t, x) \approx -\frac{1}{v(t, x)}\Delta x,$$

which can be rewritten as

$$\frac{u(t + \Delta t, x + \Delta x) - u(t, x + \Delta x)}{\Delta t} + \frac{u(t, x + \Delta x) - u(t, x)}{\Delta x} \cdot \frac{\Delta x}{\Delta t} \approx -\frac{1}{v(t, x)}\frac{\Delta x}{\Delta t}.$$

Letting $\Delta t \to 0$, we obtain the equation (2.7).

CHAPTER 3

SYSTEM ANALYSIS

In this chapter we analyse the equations for both $\rho(t,x)$ and $u(t,x)$ derived in the previous chapter and discuss some transformations that allow the system to be further studied in terms of other classical equations.

3.1 Analysis of the Density Equation

The initial value problem for the density $\rho(t,x)$ derived in §2.1 of Chapter 2 can be summarized as follows:

$$\frac{\partial}{\partial t}\rho(t,x) + \frac{\partial}{\partial x}\left(v_f\rho(t,x)\left(1-\frac{\rho(t,x)}{\rho_m}\right)\right) = 0, \quad 0 < x < L,\ t > 0, \tag{3.1}$$

$$\rho(0,x) = \rho_0(x), \quad -\infty < x < \infty, \tag{3.2}$$

where $\rho_0(x)$ is a known function of x. For simplicity, we shall use ρ instead of $\rho(t,x)$ in the following. Notice that

$$\begin{aligned} v_f\rho\left(1-\frac{\rho}{\rho_m}\right) &= \frac{v_f}{\rho_m}\left(\left(\frac{\rho_m}{2}\right)^2 - \left(\frac{\rho_m}{2}\right)^2 + \rho_m\rho - \rho^2\right) \\ &= \frac{v_f}{\rho_m}\left(\frac{\rho_m}{2}\right)^2 - \frac{v_f}{\rho_m}\left(\rho - \frac{\rho_m}{2}\right)^2. \end{aligned}$$

We can rewrite equation (3.1) to

$$\frac{\partial}{\partial t}\rho - \frac{\partial}{\partial x}\left(\frac{v_f}{\rho_m}\left(\rho - \frac{\rho_m}{2}\right)^2\right) = 0,$$

i.e.,

$$-\frac{\partial}{\partial t}\left(\frac{\rho_m}{2} - \rho\right) - \frac{\partial}{\partial x}\left(\frac{v_f}{\rho_m}\left(\rho - \frac{\rho_m}{2}\right)^2\right) = 0.$$

Thus we have

$$\frac{\partial}{\partial t}\left(\frac{\rho_m}{2} - \rho\right) + \frac{\partial}{\partial x}\left(\frac{v_f}{\rho_m}\left(\frac{\rho_m}{2} - \rho\right)^2\right) = 0.$$

Multiplying the above equation by $\frac{2v_f}{\rho_m}$, we get

$$\frac{\partial}{\partial t}\left(v_f\left(1 - \frac{2\rho}{\rho_m}\right)\right) + \frac{1}{2}\frac{\partial}{\partial x}\left(v_f\left(1 - \frac{2\rho}{\rho_m}\right)\right)^2 = 0.$$

Let

$$\phi = v_f\left(1 - \frac{2\rho}{\rho_m}\right). \tag{3.3}$$

Then the problem (3.1)–(3.2) for ρ become the following initial value problem of the Burgers' equation for ϕ:

$$\phi_t + \frac{1}{2}(\phi^2)_x = 0,\ 0 < x < L,\ t > 0, \tag{3.4}$$

$$\phi(0, x) = v_f\left(1 - \frac{2\rho_0(x)}{\rho_m}\right), \quad -\infty < x < \infty. \tag{3.5}$$

$$\tag{3.6}$$

Consider the characteristic curves for the system:

$$\frac{dx}{dt} = \phi(t, x(t)), \quad x(0) = x_0.$$

Since $\phi(t, x)$ is constant along the characteristic curves of the equation. Then we have

$$\frac{dx}{dt} = \phi(0, x_0) = \phi_0(x_0).$$

Hence

$$x(t) = x_0 + \phi_0(x_0)t, \quad t \geq 0.$$

Since the first equation states that the characteristic curves are simply straight lines, the value of the function $\phi(t, x)$ does not change along these lines in the t-x plane.

Consider the characteristic curves originating from two points x_1 and x_2. Since the values $\phi_0(x_1)$ and $\phi_0(x_2)$ can be different, the curves can intersect for some time t. The characteristics cross at any time $\phi_x(0, x)$ is negative. In the case the characteristic curves intersect, the function $\phi(t, x)$ can become multi-valued at that time. In that case, the continuity of the solution ceases, and shocks may form. The shocks, when they arise are associated with a shock speed s, which is given at a certain time by Rankine-Hugonoit jump condition[12] and is governed by:

$$s = \frac{\frac{1}{2}{\phi_1}^2 - \frac{1}{2}{\phi_2}^2}{\phi_1 - \phi_2} = \frac{1}{2}\left(\phi_1 + \phi_2\right).$$

where ϕ_1 and ϕ_2 are the values of ϕ just to the right and left at that point. Since a discontinuity arises at that point, $\phi(t,x)$ is not differentiable at that point. To overcome these issues, the concept of weak solutions has to be combined with some physical phenomenon to obtain a unique solution. We consider the viscous Burgers' equation by inducing the viscosity term - ϕ_{xx} to get

$$\frac{\partial}{\partial t}\phi(t,x) + \frac{1}{2}\frac{\partial}{\partial x}\left(\phi^2\right) = \epsilon\phi_{xx}$$

When $\epsilon \to 0$, the smooth solution obtained from the above equation converges to the solution of the inviscid Burgers' equation. This solution $\phi(t,x)$ is also the entropy solution for the problem. Hence we shall use the Lax-Friedrich's scheme to solve the Burgers' equation numerically which gives the vanishing viscosity solution to the problem.

3.2 Analysis of the Travel Time Equation

The boundary value problem for the travel time equation $u(t,x)$ derived in §2.1 of Chapter 2 is as follows:

$$u_t + vu_x = -1, \quad 0 < x < L, \quad t > 0, \tag{3.7}$$

$$u(t,L) = 0, \quad t \geq 0, \tag{3.8}$$

where

$$v(t,x) = v_f \left(1 - \frac{\rho(t,x)}{\rho_m}\right).$$

Let

$$w(t,x) = u(t,x) + t.$$

Then problem (3.7)–(3.8) becomes

$$w_t + v w_x = 0, \quad 0 < x < L, \quad t > 0, \tag{3.9}$$

$$w(t,L) = t, \quad t \geq 0. \tag{3.10}$$

Consider the variable transformations:

$$\tau = L - x, \quad y = t.$$

Let $p(\tau, y) = w(y, L - \tau)$. Since

$$w_t = p_y, \quad w_x = -p_\tau,$$

we have the following initial value problem for p:

$$p_\tau - \theta(\tau, y) p_y = 0, \quad y > 0, \quad 0 < \tau < L, \tag{3.11}$$

$$p(0,y) = y, \quad y > 0, \tag{3.12}$$

where

$$\theta(\tau, y) = \frac{1}{v(y, L-\tau)}.$$

Consider the characteristic curves for equation (3.11):

$$\frac{dy}{d\tau} = -\theta(\tau, y(\tau)), \quad y(0) = y_0, \tag{3.13}$$

where $y_0 > 0$. If $v(t, x)$ satisfies the Lipschitz condition with respect to t, i.e.

$$|v(t_1, x) - v(t_2, x)| \leq c\,|t_1 - t_2|\,, \; 0 < x < L, \quad t_1, t_2 > 0.$$

then since $\theta(\tau, y)$ is Lipschitz with respect to y. Then the Initial Value Problem described by (3.13) has a unique solution for all $y > 0,\; 0 < \tau < L$. Let $y(\tau; y_0)$ be its solution. Since $p(\tau, y)$ is constant along this curve, we have

$$p(\tau, y(\tau; y_0)) = p(0, y_0) = y_0. \tag{3.14}$$

Assume that velocity function $v(t, x)$ is bounded below by a positive number v^*, i.e.,

$$v(t, x) \geq v^*, \quad \forall x \in [0, L], \quad t \geq 0.$$

Recall that $v(t, x)$ is bounded by v_f. Then

$$\frac{1}{v_f} \leq \theta(\tau, y) \leq \frac{1}{v^*}, \quad y \geq 0, \quad 0 \leq \tau \leq L.$$

After integrating the ODE in (3.13), we have

$$y(\tau; y_0) = y_0 - \int_0^\tau \theta(s, y(s; y_0))ds.$$

For fixed $\tau \in (0, L]$ and $y > 0$, define

$$f(y_0) = y_0 - \int_0^\tau \theta(s, y(s; y_0))ds - y.$$

Notice that $f(0) < 0$ and $f(y_0) > 0$ when y_0 is sufficiently large. By the Intermediate Value Theorem, there is a $y_0 = y_0(\tau, y)$ such that $y(\tau; y_0) = y$. Hence, the solution for the initial value problem (3.11)–(3.12) can be defined by

$$p(\tau, y) = y_0(\tau, y).$$

In particular, we can solve the problem when v is a function of x only. Since θ is a function of τ only now, we can solve the initial problem (3.13) to get

$$y(\tau) = y_0 - \int_0^\tau \theta(s)ds.$$

Thus, by (3.14), we have

$$p\left(\tau, y_0 - \int_0^\tau \theta(s)ds\right) = y_0,$$

which means that

$$p(\tau, y) = y + \int_0^\tau \theta(s)ds.$$

Then we have

$$w(t, x) = t + \int_0^{L-x} \frac{1}{v(s)}ds.$$

Hence, the travel time is given by

$$u(t, x) = \int_0^{L-x} \frac{1}{v(s)}ds. \tag{3.15}$$

Thus we have shown that the solution for the PDE in $p(\tau, y)$ exists and is unique. Also in case the velocity profile $v(t, x)$ is a function of x only, the solution can be obtained explicitly using (3.15). This is in direct agreement with the model we used to model the PDE in Chapter 2.

CHAPTER 4

EXACT SOLUTIONS FOR SPECIAL CASES

In this chapter, we discuss the solutions $u(t,x)$ for the travel time PDE in some special cases. We consider the cases that the initial density profile $\rho(0,x)$ is constant or a piecewise constant function with two values ρ_1 and ρ_2.

4.1 Constant Density

Let the density $\rho(t,x)$ be equal to a constant ρ^*. Then the problem (2.7)–(2.8) becomes

$$\frac{\partial u}{\partial t} + \left(v_f \left(1 - \frac{\rho^*}{\rho_m} \right) \right) \frac{\partial u}{\partial x} = -1, \quad 0 < x < L, \ t > 0, \tag{4.1}$$

$$u(t,L) = 0, \quad t > 0. \tag{4.2}$$

It is easy to find its solution as follows:

$$u(t,x) = c_1 \left(x - v_f \left(1 - \frac{\rho^*}{\rho_m} \right) t \right) - t + c_2, \tag{4.3}$$

where c_1 and c_2 are constants to be determined by the system parameters and the boundary conditions. Since $\rho(t,x)$ is constant, the velocity $v(t,x)$ is constant as well and thus the total time taken for a vehicle to go from 0 to L would be

$$T = \frac{L}{v_f\left(1 - \frac{\rho^*}{\rho_m}\right)}. \tag{4.4}$$

Combining with the boundary condition (2.8), we get the constants as

$$c_1 = -\frac{T}{L}, \quad c_2 = T.$$

Thus $u(t,x)$ for this special case can be explicitly given by

$$u(t,x) = T\left(1 - \frac{x}{L}\right). \tag{4.5}$$

The solution obtained is linear with x and independent of t, which agree with the model that was used to derive the PDE for $u(t,x)$ in Chapter 2.

4.2 Piecewise Constant Initial Density

In this section, we analyze the exact solutions for $u(t,x)$ when the initial density profile $\rho(0,x)$ is a piecewise constant function. Write

$$\rho(0,x) = \begin{cases} \rho_1, & x \leq X^* \\ \rho_2, & X^* < x \leq L, \end{cases} \tag{4.6}$$

whre X^* is a given point.

4.2.1 $\rho_1 > \rho_2$

In the case where the density ρ_1 is greater than $\rho_2$the entropy solution for the density $\rho(t,x)$ can be found out to be

$$\rho(t,x) = \begin{cases} \rho_1, & x - X^* \leq v^* t \\ \rho_2, & x - X^* > v^* t. \end{cases} \tag{4.7}$$

where v^* is the shock speed velocity given by

$$v^* = v_f \left(1 - \frac{\rho_1 + \rho_2}{\rho_m}\right). \tag{4.8}$$

Without loss of generality we can assume that $\rho_m < \rho_1 + \rho_2$, which implies that the shock speed $v^* \geq 0$. Also the velocities for the two sections can be found to be v_1 and v_2 defined as follows:

$$v_i = v_f \left(1 - \frac{\rho_i}{\rho_m}\right), \quad i = 1, 2,$$

Then we have

$$v(t,x) = \begin{cases} v_1, & x - X^* \leq v^* t, \\ v_2, & x - X^* > v^* t. \end{cases} \tag{4.9}$$

With the initial assumption of $\rho_m > \rho_1 > \rho_2$, we have that $v_2 > v_1 > v^*$. Using the straight line relation of the the line followed by the shock speed wave, the time at

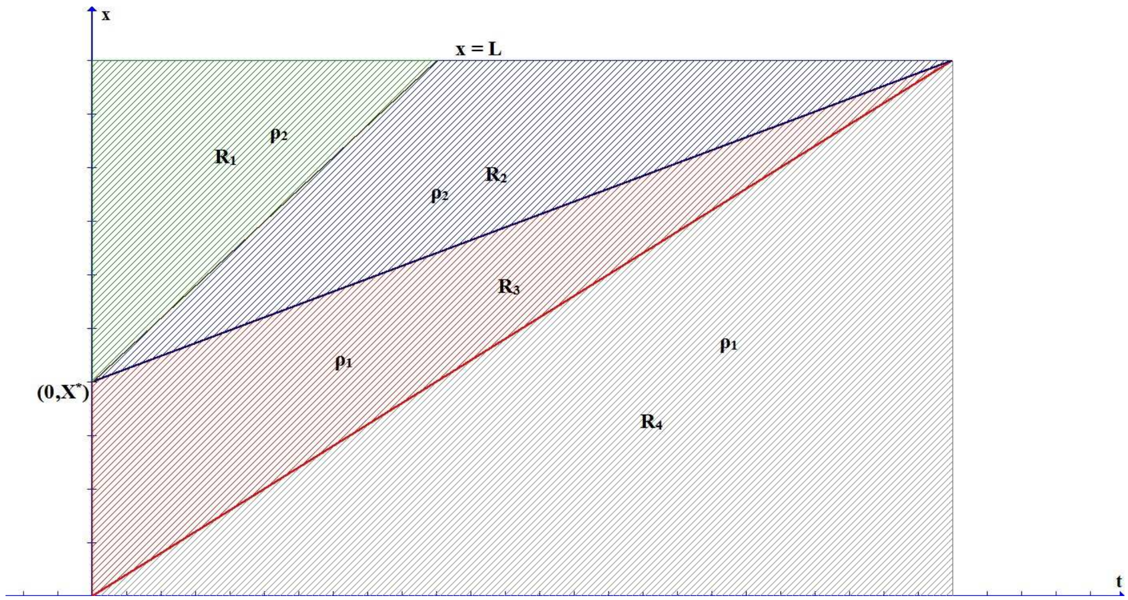

Figure 4.1: The characteristics for a two value initial density case over the domain in (t, x)

which the wave hits $X = L$ is found to be $t^* = \frac{L-X^*}{v^*}$. The domain $\mathbb{D}$ for $u(t, x)$ can be given by

$$\mathbb{D} = [0, \infty) \times [0, L]. \tag{4.10}$$

The domain of solution can be divided into regions given by $R_i, 1 \leq i \leq 4$ as shown in the figure below. In the region R_1, the velocity of the vehicle will be given by v_2 and hence the trajectory can be given by a straight line. Thus $u(t, x)$ in that region can be given by

$$u(t, x) = \frac{L - x}{v_2}.$$

In the region R_2, the velocity of any vehicle could be given by v_1, and hence the trajectory would be a straight line giving the solution to $u(t,x)$ as

$$u(t,x) = \frac{L-x}{v_1}.$$

For the regions denoted by R_2 and R_3, the solution needs to be calculated in two pieces. For R_3, any vehicle that starts there has to pass through R_2 to reach L. From a starting point (t,x) in R_3, the vehicle trajectory follows a straight line with a slope v_1 until it hits the shock wave at the point (t',x'). From the point (t',x') the trajectory then follows a straight line with slope v_2 until it hits the boundary $x = L$ at the $t = T$. Solving for x', t' and T, we get

$$t' = \frac{X^* - X + v_1 t}{v_1 - v^*},$$

$$x' = X^* + v^* t',$$

$$T = \frac{L - X^*}{v_2} + t'\left(1 - \frac{v^*}{v_2}\right).$$

Thus the travel time $u(t,x)$ given by $T - t$ can be defined by the following expression:

$$u(t,x) = T - t = \frac{L-x}{v_2} + \frac{X^* - x}{v_1 - v^*} + \frac{v^* t}{v_1 - v^*}. \tag{4.11}$$

For the region R_2, the solution can be found to be the same as that for the region R_1.

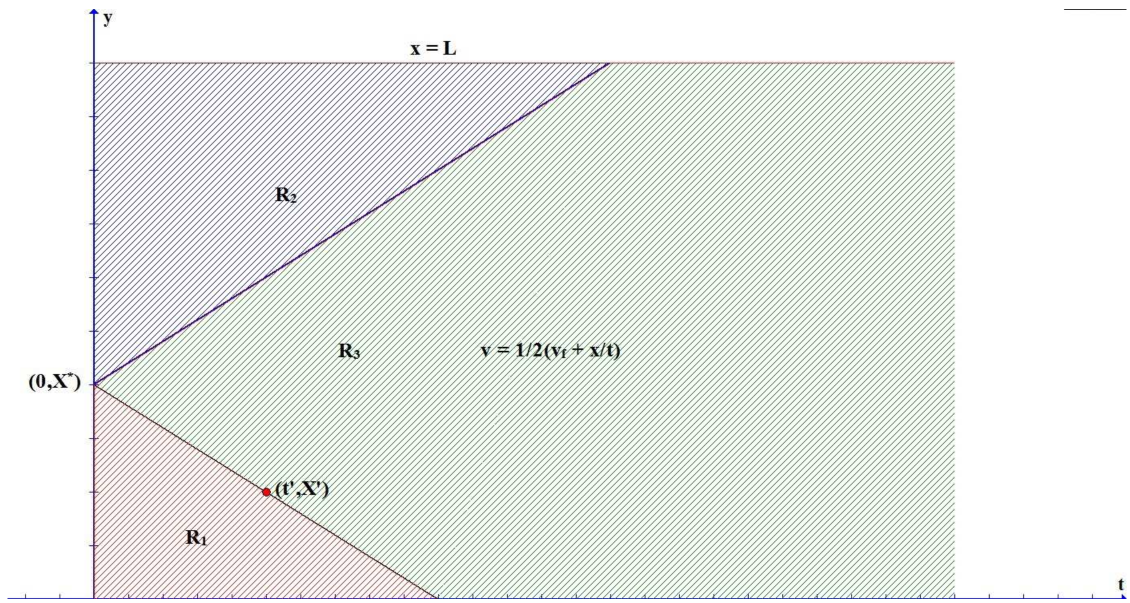

Figure 4.2: The characteristics for a two value initial density case over the domain in (t, x)

4.2.2 $\rho_1 > \frac{\rho_m}{2} > \rho_2$

In the case where $\rho_1 > \frac{\rho_m}{2}$, the solution for $u(t, x)$ cannot be explicitly described as above. For this case, the characteristic curves for the density profile $\rho(t, x)$ can be found to be like in Figure 4.2. Since the initial density value $\rho_1 > \frac{\rho_m}{2}$, the corresponding characteristic curve has a negative slope. If the characteristics for the other density ρ_2 on the other hand have a positive slope, then the characteristics on the two sides do not intersect at all. Thus even in this case, we can find the regions in which the densities remain constant as either ρ_1 or ρ_2. But there is also a region in the (t, x) domain where the solution(the vanishing viscosity solution or the entropy solution) $\rho(t, x)$ takes values between ρ_1 and ρ_2. The unique admissible solution in this region is the vanishing viscosity or the entropy solution. The slope of the characteristic for density ρ is given by

$$\sigma(\rho) = v_f \left(1 - \frac{2\rho}{\rho_m}\right).$$

For the initial density profile described in equation (4.6), the characteristic slopes can be given as

$$\sigma(\rho) = \begin{cases} \sigma_1 \ , \ \rho = \rho_1 \\ \\ \sigma_2 \ , \ \rho = \rho_2. \end{cases} \tag{4.12}$$

where

$$\sigma_i = v_f \left(1 - \frac{2\rho_i}{\rho_m}\right), \quad i = 1, 2$$

With these shock speeds the admissible solution for the density $\rho(t, x)$ is given by

$$\rho(t,x) = \begin{cases} \rho_1 \ , & x - X^* < \sigma_1 t \\ \\ \frac{1}{2}\rho_m \left(1 - \frac{x}{v_f t}\right) \ , & \sigma_1 t < x - X^* < \sigma_2 t \\ \\ \rho_2 \ , & x - X^* > \sigma_2 t \end{cases} \tag{4.13}$$

The corresponding vehicle velocity profile $v(t, x)$ can be given by

$$v(t,x) = \begin{cases} v_1 \ , & x - X^* < \sigma_1 t \\ \\ \frac{1}{2}\left(v_f + \frac{x}{t}\right) \ , & \sigma_1 t < x - X^* < \sigma_2 t \\ \\ v_2 \ , & x - X^* > \sigma_2 t \end{cases} \tag{4.14}$$

where

$$v_i = v_f \left(1 - \frac{\rho_i}{\rho_m}\right), \quad i = 1, 2$$

Since $\rho_1 > \rho_2$, we get $v_2 > v_1$. As in the previous case, the domain $\mathbb{D}$ can be divided into various sections. The region with the characteristic speed σ_1 is called R_1 and the one with σ_2 is taken to be R_2. The region where the fan -like characteristics are obtained is labeled as R_3. In the region given by R_2, $u(t, x)$ can be evaluated as before as any vehicle in there follows a trajectory with a constant velocity. So $u(t, x)$ in region R_2 can be given by

$$u(t, x) = \frac{L - x}{v_f \left(1 - \frac{\rho_2}{\rho_m}\right)}.$$

For a vehicle starting in the region R_1, the vehicle follows a straight line where the velocity is v_1 till it hits the straight line given by (4.15)

$$x = X^* + \sigma_1 t \tag{4.15}$$

Solving for X' and t', we get the values as

$$X' = \frac{v_1(X^* - \sigma_1 t) - \sigma_1 x}{v_1 - \sigma_1},$$

$$t' = \frac{X^* - v_1 t - x}{v_1 - \sigma_1}$$

Once the vehicle reaches the point (t', X') on the line given by equation (4.15), it moves into region R_3, where the velocity is no longer constant. In this region the densities vary continually from ρ_1 to ρ_2, and the velocity at (t, x) can be given by

$$v(t, x) = \frac{1}{2}\left(v_f + \frac{x}{t}\right) \tag{4.16}$$

The solution for vehicle trajectory in this region with the velocity profile as given in equation (4.16) is given by

$$\eta(t) = v_f t + c\sqrt{t}. \tag{4.17}$$

where c is a constant of integration to be determined by the boundary condition. Using the fact that $\eta(t') = x'$, we get the constant c as

$$c = \frac{X' - v_f t'}{\sqrt{t'}}$$

Once the vehicle is in region R_3, there are two possibilities for the trajectory followed by the vehicle.

1. it reaches $x = L$ while staying in the region R_3, or

2. crosses into region R_2 before it reaches L

In order to study both the scenarios, we need to define the boundary between the two cases. Following the trajectory η defined in equation (4.17), we calculate the time t_L at which it reaches $\eta = L$. Using the fact that $t > 0$, the only possible solution can be evaluated to be:

$$\sqrt{t_L} = \frac{-c + \sqrt{c^2 + 4v_f L}}{2v_f}$$

which gives us

$$t_L = \left(\frac{-c + \sqrt{c^2 + 4v_f L}}{2v_f} \right)^2 \tag{4.18}$$

To see the intersection of this trajectory with the characteristic boundary of R_2 and R_3 and $x = L$, t in equation (4.18), must be equal to $\frac{L-X^*}{\sigma_2}$, i.e.

$$\frac{-c + \sqrt{c^2 + 4v_f L}}{2v_f} = \sqrt{\frac{L - X^*}{\sigma_2}}$$

Squaring both sides and after simplification, we get c, which we call as c^* as

$$c^* = \frac{L - v_f \left(\frac{L-X^*}{\sigma_2} \right)}{\sqrt{\frac{L-X^*}{\sigma_2}}} \tag{4.19}$$

Using the relation for the boundary of characteristic between R_1 and R_2, the constant c^* can also be given by

$$c^* = \sqrt{t'} \left(\frac{X^*}{t'} + \sigma_1 - v_f \right). \tag{4.20}$$

On equating (4.19) and (4.20), we get

$$\frac{c^*}{\sqrt{t'}} = \frac{X^*}{t'} - (v_f - \sigma_1).$$

Solving for t', we get

$$\frac{1}{\sqrt{t'}} = \frac{c^* + \sqrt{c^{*2} + 4(v_f - \sigma_1 X^*}}{2X^*}.$$

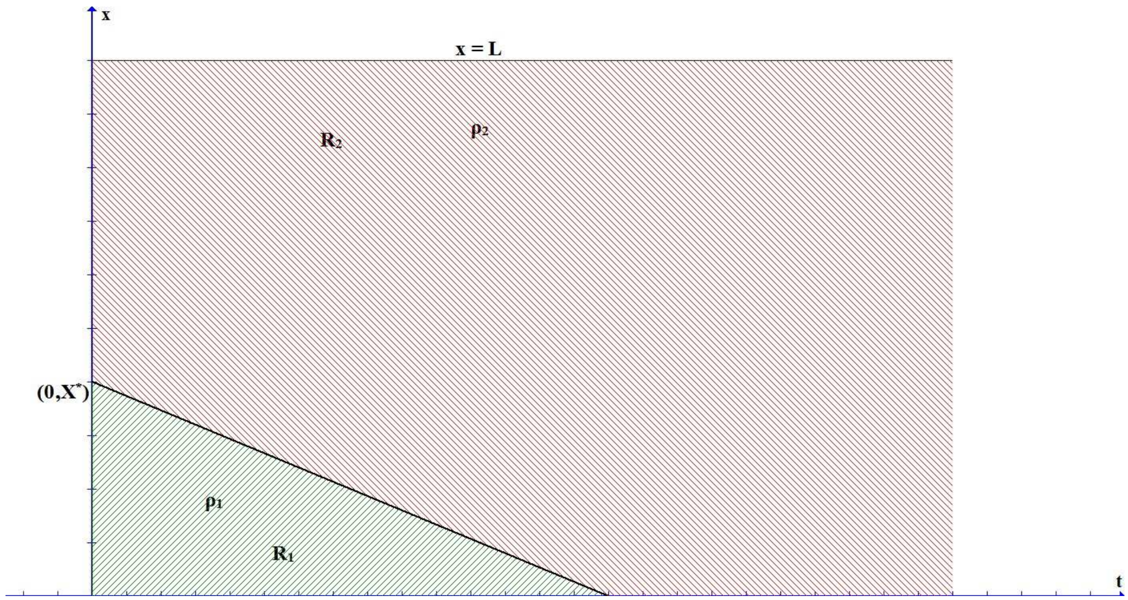

Figure 4.3: The characteristics for a two value initial density case over the domain in (t, x)

i.e.

$$t' = \frac{4X^{*2}}{(c^* + \sqrt{c^{*2} + 4(v_f - \sigma_1)X^*})^2}.$$

If t' satisfies:

$$t' \geq \frac{4X^{*2}}{(c^* + \sqrt{c^{*2} + 4(v_f - \sigma_1)X^*})^2}.$$

the vehicle trajectory from the point (t', X') passes through R_3 without getting into R_2,

4.2.3 $\rho_2 > \frac{\rho_m}{2} > \rho_1$

In this section we consider the other case where the characteristics for densities ρ_1 and ρ_2 still have slopes of opposite signs but ρ_1 and ρ_2 are reversed. The characteristics for this scenario would lead to a region as shown in Figure 4.3 with the regions divided by the shock wave line with a slope of

$$\frac{dx}{dt} = v_f \left(1 - \frac{\rho_1 + \rho_2}{\rho_m}\right) = v^*.$$

Without loss of generality we can assume the shock speed to be negative. The point where the line of shock given below meets the t-axis is $(t^*, 0)$.

$$x = X^* + v^* t.$$

which gives $t^* = -\frac{X^*}{v^*}$ The velocities for the two regions can be found as

$$v(t,x) = \begin{cases} v_1 \ , & x - X^* \leq v^* t, \\ v_2 \ , & x - X^* > v^* t \end{cases}$$

where

$$v_i = v_f \left(1 - \frac{\rho_i}{\rho_m}\right), \quad i = 1, 2$$

In this case the results for the travel time function $u(t, x)$ are the same as for the case discussed in section 4.2.1

4.2.4 $\rho_1 < \rho_2$

In the case of $\rho_1 < \rho_2$, the (t, x) domain can be divided into areas as depicted below in Figure 4.4 The corresponding velocity profiles for the two regions can then

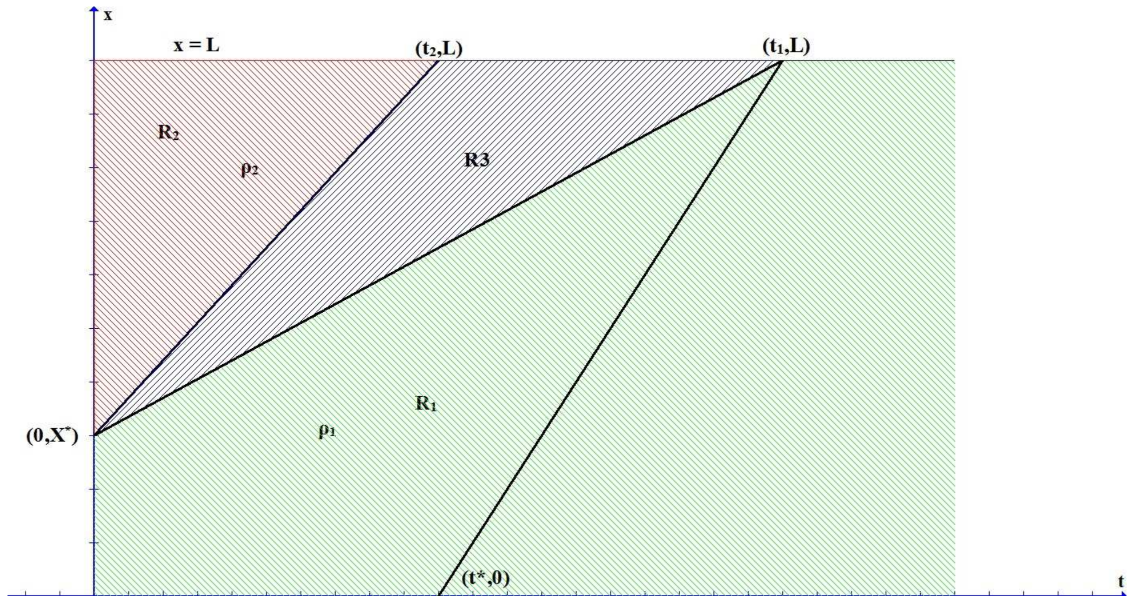

Figure 4.4: The characteristics for a two value initial density case over the domain in (t, x)

be listed as

$$v(t,x) = \begin{cases} v_1\ , & (t,x) \in R_1, \\ v_2\ , & (t,x) \in R_2 \end{cases} \tag{4.21}$$

where

$$v_i = v_f\left(1 - \frac{\rho_i}{\rho_m}\right), \quad i = 1, 2$$

The lines between $(0, X^*)$ and (t_1, L) given by L_1 and that between $(0, X^*)$ and (t_2, L) given by L_2 define the boundaries between the regions R_1, R_2 and R_3. These lines can be defined as:

$$x_i = X^* + \sigma_i t, \quad i = 1, 2$$

$$\sigma_i = v_f\left(1 - \frac{2\rho_i}{\rho_m}\right), \quad i = 1, 2$$

where σ_i are the slopes of the characteristics for the two densities. From a point (t, x) in the region R_2, a vehicle follows a straight line with the velocity v_2. Hence the function $u(t, x)$ for region R_2 is given by:

$$u(t, x) = \frac{L - x}{v_2}.$$

For the region R_1, the solution $u(t, x)$ cannot be explicitly found as in the case of R_2. From a point (t, x) in R_1, the vehicle may reach $x = L$ with or without crossing into R_3. Consider a point on t axis as $(0, t^*)$. For a vehicle starting from this point to reach $x = L$ at t_1, the following must hold true

$$L = v_1(t_1 - t^*).$$

and using the fact that $t_1 = \frac{L-X^*}{\sigma_1}$, we get t^* as

$$t^* = L\left(\frac{1}{\sigma_1} - \frac{1}{v_1}\right) - \frac{X^*}{\sigma_1}. \tag{4.22}$$

Since t^* must be non negative, we must have

$$X^* \leq L\left(1 - \frac{\sigma_1}{v_1}\right) \tag{4.23}$$

If the condition in equation (4.23) is met, then there exists a line from $(t^*, 0)$ with t^* given by equation (4.22) that intersects $x = L$ at (t_1, L). For any point (t, x) on this line or to the right of it, the function $u(t,x)$ can be given by

$$u(t,x) = \frac{L-x}{v_1} \tag{4.24}$$

The overall solution for the density profile can be given by

$$\rho(t,x) = \begin{cases} \rho_1 \ , & x - X^* < \sigma_1 t, \\ \frac{1}{2}\rho_m \left(1 - \frac{x}{v_f t}\right) \ , & \sigma_1 t < x - X^* < \sigma_2 t, \\ \rho_2 \ , & x - X^* > \sigma_2 t \end{cases} \tag{4.25}$$

which on using the Greenshields Model, gives the velocity profile as:

$$v(t,x) = \begin{cases} v_1 \ , & x - X^* < \sigma_1 t, \\ \frac{1}{2}\left(v_f + \frac{x}{t}\right) \ , & \sigma_1 t < x - X^* < \sigma_2 t, \\ v_2 \ , & x - X^* > \sigma_2 t \end{cases} \tag{4.26}$$

For a point (t,x) in R_3, the trajectory of a vehicle may cross into R_2 or reach $x = L$ without crossing into R_2. The trajectory of a vehicle from point $(\hat{t},\hat{x})$ as long as it stays in R_3 can be given by

$$\eta(t) = v_f t + c\sqrt{t} \tag{4.27}$$

Another way of denoting the trajectory $\eta(t)$ is $\eta(t,\hat{t},\hat{x})$, as the starting point determines the trajectory of the vehicle. Because of this, the constant c depends on the starting point $(\hat{t},\hat{x})$ as well. A limiting point of the trajectory can be found which intersects $x = L$ at (t_2, L). We try and find a point (t', x') on the straight line that is the boundary between R_1 and R_3. Since (t_2, L) must satisfy the trajectory given in equation (4.27), we get the constant c as:

$$c = \frac{L - v_f t_2}{\sqrt{\hat{t}}}.$$

Since $(\hat{t}, \hat{X})$ lies on the trajectory defined in equation (4.27), c can be given as:

$$c = \frac{\hat{X} - v_f \hat{t}}{\sqrt{\hat{t}}} = \frac{L - v_f t_2}{\sqrt{t_2}}.$$

along with the condition that $(\hat{t}, \hat{X})$ lies on the line given by $x = X^* + \frac{L - X^*}{t_1} t$, allows us to solve for $(\hat{t}, \hat{X})$ and subsequently c.

$$(v_f - \sigma_1)\hat{t} + c\sqrt{\hat{t}} - X^* = 0$$

On solving for $\hat{t}$ and including only the positive value for $\hat{t}$, we get

$$\hat{t} = \left(\frac{-c + \sqrt{c^2 + 4(v_f - \sigma_1)X^*}}{2(v_f - \sigma_1)} \right)^2.$$

which gives us $\hat{X}^*$ from the relation

$$\hat{X} = X^* + \sigma_1 \hat{t}.$$

So any trajectory starting from a point to the right of the point $\hat{t}\hat{X}$, the vehicle does not cross into R_2 before reaching $x = L$.

CHAPTER 5

NUMERICAL METHODS FOR TRAVEL TIME CALCULATION

This chapter highlights the numerical schemes that are used for solving the density equation. Based on them, we devise numerical schemes for solving for travel time. The numerical results obtained are presented along with the errors obtained, and the convergence of the schemes are discussed as well.

5.1 Numerical Schemes for Density Equation

The Initial value problem described for traffic density can be transformed into an equivalent inviscid Burgers' problem as shown in §3.1. Thus this section details the numerical schemes used to solve the inviscid Burgers' equation.

The fundamental convergence theorem for linear difference methods states that for a consistent , linear method, stability is necessary and sufficient for convergence [2] [3]. The numerical schemes discussed in this section, and developed in the subsequent ones will be based on this fundamental theorem.

We divide the domain on which the solution needs to be obtained into a mesh, with the mesh size in x defined as Δx, and in t as Δt. A classical linear numerical scheme used to solve the Burgers' equation is the Lax-Friedrichs method. We use the notation ϕ_i^n for $\phi(t_n, x_i)$. Writing the differential equation in difference form:

$$\frac{\phi_i^{n+1} - \frac{1}{2}\left(\phi_{i+1}^n + \phi_{i-1}^n\right)}{\Delta t} + \frac{\Delta t}{2\Delta x}\left(\frac{1}{2}\left(\phi_{i+1}^n\right)^2 - \frac{1}{2}\left(\phi_{i-1}^n\right)^2\right) = 0.$$

On rearranging the terms, we get:

$$\phi_i^{n+1} = \phi_i^n + \frac{1}{2}\left(\phi_{i+1}^n + \phi_{i-1}^n\right) - \frac{\Delta t}{4\Delta x}\left({\phi_{i+1}^n}^2 - {\phi_{i-1}^n}^2\right). \tag{5.1}$$

The condition of stability, and hence convergence for the Lax-Friedrichs scheme is known as the Courant-Friedrichs-Lax (CFL) condition, and it states that

$$\phi \frac{\Delta t}{\Delta x} \leq 1.$$

Since $\phi(t,x) = v_f\left(1 - \frac{2\rho}{\rho_m}\right)$, and using the fact $\max\left(v_f\left(1 - \frac{2\rho}{\rho_m}\right)\right) = v_f$, the CFL condition can be simplified to yield

$$v_f \frac{\Delta t}{\Delta x} \leq 1. \tag{5.2}$$

The condition stated in (5.2) can be used to select the mesh size Δt and Δx, and solve for $\phi(t,x)$. Once the initial-boundary value problem for $\phi(t,x)$ is solved, the density profile can be obtained using the inverse substitution:

$$\rho(t,x) = \frac{\rho_m}{2} - \frac{\rho_m}{2v_f}\phi(t,x).$$

The velocity profile $v(t,x)$ can be subsequently obtained using the Greenshield's Model discussed in Chapter 2.

5.2 Numerical Schemes for Travel Time Evaluation

In this section we discuss and devise the numerical schemes required for solving for $w(t,x)$. The conditions for convergence and the stability of the scheme are also discussed.

Since as discussed in §3.2, the initial-boundary value problem can be converted into a homogeneous equation in $p(\tau,y)$ with its corresponding initial value functions, this section deals with devising numerical schemes for solving for $p(\tau,y)$. We develop a forward numerical scheme for solving the equivalent initial boundary value problem in $p(\tau,y)$. For this let $\Delta\tau = \frac{L}{M}$ for a positive integer M. Let Δy be the mesh size in y. We define:

$$\tau_m = m\Delta\tau, m = 0, 1, 2, ..., M$$

$$y_j = j\Delta y, j = 0, 1, 2, ...$$

Denote by p_j^m the approximation of $p(\tau_m, y_j)$ The system can be discretized about the point (τ_m, y_j) to get:

$$\frac{p(\tau_{m+1}, y_j) - p(\tau_m y_j)}{\Delta\tau} - \theta(\tau_m, y_j)\frac{p(\tau_m, y_{j+1}) - p(\tau_m, y_j)}{\Delta y} = 0.$$

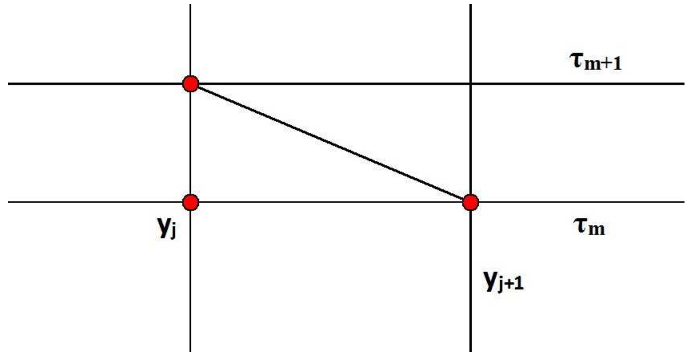

Figure 5.1: The lattice used in the forward one-sided numerical scheme

Using p_j^m for $p(\tau_m, y_j)$, we get the scheme as

$$\frac{p_j^{m+1} - p_j^m}{\Delta\tau} - \theta_j^m \frac{p_{j+1}^m - p_j^m}{\Delta y} = 0.$$

which gives p_j^{m+1} as:

$$\begin{aligned} p_j^{m+1} &= \quad p_j^m + \theta_j^m r(p_{j+1}^m - p_j^m) \\ &= (1 - \theta_j^m r)p_j^m + \theta_j^m r p_{j+1}^m. \end{aligned}$$

where $r = \frac{\Delta\tau}{\Delta y}$ is the mesh ratio, $m = 0, 1, 2, ...$ and $j = 0, 1, 2, ..., M - 1$. The initial value function $p(0, y) = y$ and the fact that θ_j^n is always non-negative allow the numerical scheme to be consistent, and thus $p(\tau, y)$ computed is convergent to the actual solution.

Using the Courant, Friedrichs and Lewy [11]condition, the scheme must have

$$\theta_j^m r \leq 1$$

Using the fact $\theta(\tau, y) = \frac{1}{v(t,L-x)}$, for all (t, x) the following condition must be true

$$r \leq v(t, x)$$

Also since $v(t, x)$ has an upper bound as v_f, the condition can be further updated to

$$\Delta\tau \leq \Delta y v_f \tag{5.3}$$

This equivalent CFL condition from (5.3) can be used to select the optimum mesh size for the numerical scheme. The inequality in the scheme can be replaced by equality in an explicit scheme for the case of simplicity where the mesh size remains the same.

The CFL conditions for Burgers' equation (for $\phi(t, x)$) and that for $p(\tau, y)$, are stated in equations (5.2) and (5.3). We plan to devise a generic CFL scheme, which can be used for the overall system. Since Δt is synonymous with Δy and Δx is synonymous with $\Delta\tau$, the two equations state

$$v_f \frac{\Delta t}{\Delta x} \leq 1.$$

$$\frac{1}{v} \cdot \frac{\Delta x}{\Delta t} \leq 1.$$

The two inequalities can be satisfied simultaneously if and only if

$$\frac{\Delta x}{\Delta t} = v_f \tag{5.4}$$

On choosing the optimum mesh size r base on (5.4), the scheme is employed to solve for $p(\tau, y)$. Once the solution is obtained, a back transformation is applied to obtain $u(t, x)$ from $p(\tau, y)$.

5.3 Numerical Results

The above discussed numerical schemes are used to evaluate $p(t, x)$ for various velocity profiles. The cases of both linear and non-linear continuous functions are analyzed, and the errors are evaluated and analyzed.

The algorithm works with setting a final time T till where the function needs to be evaluated. Because of the triangular lattice used in the numerical scheme, for N values at $t = 0$, the final time value needs to evaluated for a larger number M, i.e. $M > N$. The boundary value functions can be calculated using methods discussed in Appendix §A.1. The following subsections discuss the results obtained for various velocity profiles and highlight the errors and the convergence of the numerical scheme.

Example 5.1. In this example, we assume that the velocity is a function of t only, given by $v(t, x) = 1 + \frac{t}{20}$. Using the results from Appendix §A.1, we have

$$p(\tau, 0) = \sqrt{400 + 40(L - \tau)} - 20$$

Let $p_h(\tau_m, 0)$ be the approximate values of $p(\tau, 0)$ computed by Appendix §A.1. We want to examine the following error:

$$e_h = \max_{1 \le m \le M} |p_h(\tau_m, 0) - p(\tau, 0)| \, .$$

In Fig. 5.2, we display the log-log plot obtained between the error e_h and step size h for various step sizes. The corresponding plot depicting the deviation of obtained data from calculated data can be shown in 5.2 The solid line in the Figure represents the

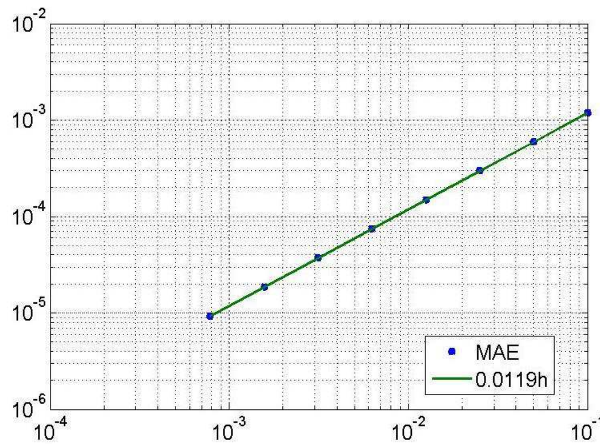

Figure 5.2: The loglog plot between h and e

line obtained by linear regression between $\log(e_h)$ and $\log(h)$, and the markers show the actual data obtained. The Fig. 5.2 shows an almost linear relationship between $\log(e)$ and $\log(h)$ with the slope $\nu = 1$ and the vertical intercept as $c = 0.011909$. As expected, we can observe that the rate of convergence is 1.

Example 5.2. In case the velocity function is not only a function of t, the boundary value function $p(\tau, y)$ cannot be computed and hence the maximum error as described in previous section cannot be calculated. For this section the velocity function is $v(t,x) = (1 + L - x)(1 + \frac{t}{20})$.

Since the exact value $p(\tau, 0)$ is unknown, we shall examine the error between its two approximations $p_h(\tau_m, 0)$ $m = 1, 2, 3..., M$ and $p_{\frac{h}{2}}(\tau_k, 0)$ $k = 2m - 1$. We define the error e_h as follows:

$$e_h = \max_{1 \le m \le M} |p_h(\tau_m, 0) - p_{2h}(\tau_k, 0)|\,,$$

Once the errors are obtained, same analysis as described in the previous section

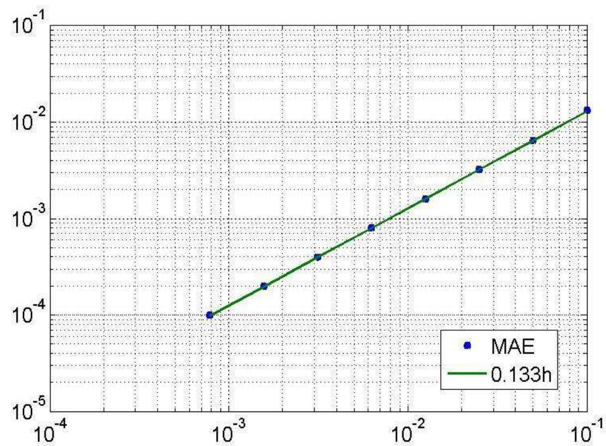

Figure 5.3: The loglog plot between h and e

is used to compute the relationship between $\log(e_h)$ and $\log(h)$ to obtain the slope $\nu = 1$ and the vertical intercept as $c = 0.132530$ The corresponding plot depicting the deviation of obtained data from calculated data can be shown in 5.3

The Fig. 5.3 shows an almost linear relationship between $\log(e)$ and $\log(h)$ with the slope $\nu = 1$ and the vertical intercept as $c = 0.132530$. As expected, in this case as well the rate of convergence is 1.

Example 5.3. In this example, we assume that velocity profile to be $v(t, x) = 1 + \frac{1}{2}\sin(\pi(L - x))$. The numerical scheme is employed to solve for $p(t, x)$ and the error values are computed and compared. The constants c and ν are computed as described in the previous example. The corresponding plot between the errors and step sizes can be shown in Figure 5.4 The figure shows an almost linear relationship between

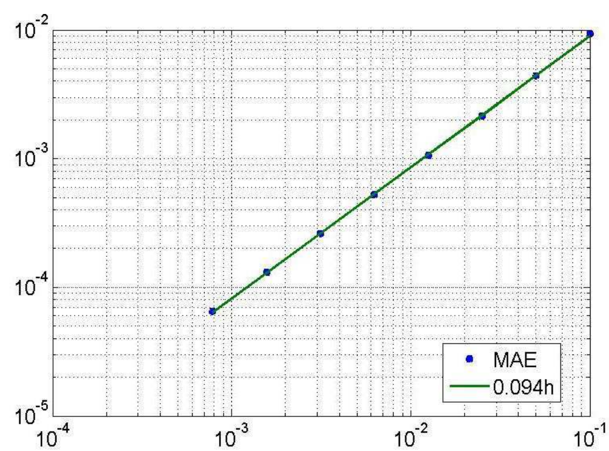

Figure 5.4: The loglog plot between h and e

$\log(e)$ and $\log(h)$ with the slope $\nu = 1$ and the vertical intercept as $c = 0.093978$. As expected, the rate of convergence in this case is 1.

CHAPTER 6

CONCLUSIONS

The current work mainly focused on developing a theory for travel time estimation based on the physical traffic model that is available and is widely used. The work done can be broadly separated into chapters which follow. The various aspects of the study can be briefly outlined as follows.

The entire traffic system was analyzed on which the entire study is based. The underlying traffic density equation is discussed in detail. From the traffic density model, the travel time equation is formulated and the corresponding system is discussed. We present a detailed analysis of equivalence of the traffic density equation to the inviscid Burgers' equation under suitable substitutions. The properties and conditions under which the system holds a unique solution are outlined. The initial value and the boundary value conditions are discussed, which allow the system to have a unique solution in the domain of its definition.

The exact solutions for the travel time problems are mathematically evaluated for some special cases of initial density value profile. The shock waves that originate due to various density profiles, and the travel time solutions in those respective cases are evaluated and studied. The techniques to devise numerical solutions to evaluate travel time solutions for other cases are discussed. The stability and convergence of the scheme are discussed by studying the errors obtained. Numerical schemes are presented for various initial velocity profiles, and the corresponding errors are examined. Numerical results show that the maximum absolute errors decrease linearly with respect to the step size as expected.

6.1 Future Work

The travel time model developed can be used creating predictive travel time systems for use in transport planning. The current limitations imposed by fixed sensors for collecting data can be overcome by collecting data once to obtain $\rho(0, x)$. It can then be used to compute $\rho(t, x)$ and eventually $u(t, x)$. This allows in saving the huge costs that is involved in installing the road sensors. Also the limitation of the current travel time evaluation systems can be reduced by using the discussed model for travel time prediction. Higher order numerical schemes can be proposed computing $p(\tau, y)$ and eventually $u(t, x)$.

APPENDIX A

APPENDIX

A.1 Initial and Boundary Value Functions for $w(t,x)$

As discussed in Chapter 3§3.2, the Initial-Boundary Value Problem in $u(t,x)$ can be converted into an equivalent problem in $w(t,x)$ using the substitution:

$$w(t,x) = u(t,x) + t.$$

We analyze the initial value $w(0,x) = w_0(x)$ and boundary value $w(t,0) = \alpha(t)$ for some special cases. For some x_0 such that $0 < x_0 < L$, consider the characteristic equation:

$$x'(t,x_0) = v(t,x(t;x_0)); \ t \geq 0 \tag{A.1}$$

$$x(0;x_0) = x_0. \tag{A.2}$$

Then

$$w(0,x_0) = w_0(x_0) = w(t,x(t;x_0)).$$

Considering the full derivative of $w(t, (x(t; x_0)$

$$
\begin{aligned}
\frac{d}{dt} w(t, x(t; x_0)) &= w_t + w_x \frac{dx(t; x_0)}{dt} \\
&= w_t + v w_x \\
&= 0.
\end{aligned}
$$

Since the velocity v is always non-negative , so $\exists\, w_0(x_0)$, such that $x(w_0(x_0); x_0) = L$. On integrating the equation (A.1) from 0 to $w_0(x_0)$

$$
\int_0^{w_0(x_0)} x'(t; x_0) dt = \int_0^{w_0(x_0)} v(t, x(t; x_0)) dt.
$$

which gives

$$
L - x_0 = \int_0^{w_0(x_0)} v(t, x(t; x_0)) dt. \tag{A.3}
$$

Now for some time $t_0 \geq 0$, consider the characteristic equation:

$$
\begin{aligned}
&\frac{d}{dt} x(t; t_0) = v(t, x(t; t_0)),\ \ t \geq 0 \\
&x(t_0; t_0) = 0.
\end{aligned}
$$

Let $\alpha(t_0)$ be the time such that the trajectory $x(t; t_0)$ reaches $x = L$. Thus on integrating the above characteristic equation from 0 to $\alpha(t_0)$, we get

$$
L = \int_{t_0}^{\alpha(t_0)} v(t, x(t; t_0)) dt \tag{A.4}
$$

In case $v(t,x) = v(t)$ equations (A.3) and (A.4) become:

$$L - x = \int_0^{w_0(x)} v(s)ds,$$

$$L = \int_t^{\alpha(t)} v(s)ds.$$

Differentiating these equations, we get:

$$-1 = w'_0(x)v(w_0(x))$$

$$0 = \alpha'(t)v(\alpha(t)) - v(t).$$

which give $w'_0(x)$ and $\alpha'(t)$ as

$$w'_0(x) = \frac{-1}{v(w_0(x))};\ 0 \leq x \leq L \tag{A.5}$$

$$w_0(L) = 0. \tag{A.6}$$

$$\alpha'(t) = \frac{v(t)}{v(\alpha(t))};\ t \geq 0 \tag{A.7}$$

$$\alpha(0) = w_0(0) \tag{A.8}$$

The ordinary differential equations in (A.5) and (A.7) can be used to solve for the inital value function $w_0(x)$ and the boundary value function $\alpha(t)$ respectively.

A.2 MATLAB Code for Solving for Traffic Density

A.2.1 One Sided Backward Scheme

```
function traveltime_os(N,varargin)
% Usage: (1) traveltime(N,example) with example = 1, 2, or 3
%             for Example 1, 2, or 3.
%
%        (2) traveltime(N,v) with a matlab function of (t,x).

if nargin ~= 2
    help traveltime;
    error('wrong number of inputs!');
end

L = 1;

vflag = 0;
if isnumeric(varargin{1})
    example = varargin{1};
    switch example
        case 1
            v = @(t,x) 1;
            w0 = @(x) L-x;
        case 2
            v = @(t,x) 1+t./20;
```

```
            w0 = @(x) sqrt(400+40*(L-x))-20;
        case 3
            v = @(t,x) (1+L-x).*(1+t/20);
            vflag = 1;
        otherwise
            help traveltime;
            error('only three examples');
    end
else
    vflag = 1;
    v = varargin{1};
end

M = 2*N;
dx = L/N;
dt = dx;
nu = dt/dx;

x = linspace(0,1,N+1);
t = dt*(0:M);

if vflag == 0
    es = w0(x);
else
    % figure;
    es = zeros(1,N+1);
```

```
    s = linspace(0,4,10*M+1);
    options = odeset('AbsTol', eps);
    for j=1:N
        [s y] = ode45(v,s, x(j), options);
        % plot(s,y); hold on; box on;
        es(j) = spline(y,s,L);
    end
end

%% backward one-sided scheme
w = zeros(M+1,N+1);
w(:,N+1) = t;
for j=N:-1:1
    for n =1:(M-N-1+j)
        w(n,j) = w(n,j+1) + (w(n+1,j+1) - w(n,j+1))/(nu*v(t(n),x(j+1)));
    end
end

as = w(1,:);

A=[x; es; as; es-as]';
fprintf('%8.4f %18.12f %18.12f %16.5e\n', (A(1:floor(N/10):N+1,:))');

fprintf('\nmaximumx error: %-12.5e\n\n', norm(es-as,inf));

% figure;
```

```
% plot(x,es,'r', x,as,'g'); legend('Exact', 'Numerical');
% plot(x,es-as,'r');
% grid on;
% box on;

end
```

A.2.2 One Sided Forward Scheme

```
function traveltime(M, varargin)

% Usage: (1) traveltime_forward(M,example) with example = 1, 2, or 3
%            for Example 1, 2, or 3.
%
%        (2) traveltime(M,v) with a matlab function of (t,x).

if nargin ~= 2
    help traveltime;
    error('wrong number of inputs!');
end

%%

L = 1;
```

```
vflag = 0;

if isnumeric(varargin{1})
    example = varargin{1};
    switch example
        case 1
            theta = @(y,tau) 1;
            % p0 = @(tau) L-tau;
            w0 = @(x) L-x;
        case 2
            theta = @(y,tau) 20./(y + 20);
            % p0 = @(tau) sqrt(400+40*(L-tau)) - 20;
            w0 = @(x) sqrt(400+40*(L-x))-20;
        case 3
            theta = @(y,tau) 20./((1+tau).*(20+y));
            %p0 = 1;
            vflag = 1;
        otherwise
            help traveltime_forward;
            error('only three examples');
    end
else
    vflag = 1;
    v = vargin{1};
end
```

```
%% assign parameters

for k = 1:9

    M = M*(2^(k-1));
    N = 2*M;

    dtau = L/M;
    dy = 2*dtau;

    r = dtau/dy;

    tau = linspace(0,1,M+1);
    y = dy*(0:N+M);

    p = zeros(M+1, N+M+1);

    p(1,:) = y;

    %% forward one sided scheme

    for m = 1:M
        for j = 1:(N+M-m-1)
            p(m+1,j) = (1-r*theta(y(j), tau(m)))*p(m,j) + theta(y(j),
            tau(m))*r*p(m,j+1);
        end
```

```
end

%% convert p to w

% w = zeros(M+1, N+M+1);
%
% for i = 1:N+M+1
%     w(:,N+M+2-i) = p(:,i);
% end

w = p';

temp = w;

for i = 1:M+1
    w(:,i) = temp(:,M+2-i);
end

%% compute error between estimated and actual solutions

if vflag == 0
    es = w0(tau);
else
    % figure;
    es = zeros(1,M+1);
    s = linspace(0,4,10*N+1);
```

```
        options = odeset('AbsTol', eps);

        for j=1:N

            [s y] = ode45(v,s, x(j), options);

            % plot(s,y); hold on; box on;

            es(j) = spline(y,s,L);

        end

    end


    as = w(1,:);


    A=[tau; es; as; as-es]';


    L/M

    fprintf('%8.4f %18.12f %18.12f %16.5e\n', (A(1:floor(M/10):M+1,:))');

    fprintf('\nmaximumx error: %-12.5e\n\n', norm(as-es,inf));


    M = M/(2^(k-1));

end


end
```

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