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Observability in Traffic Modeling: Eulerian and Lagrangian Coordinates

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OBSERVABILITY IN TRAFFIC MODELING: EULERIAN AND LAGRANGIAN COORDINATES

by

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ABSTRACT

OBSERVABILITY IN TRAFFIC MODELING: EULERIAN AND LAGRANGIAN COORDINATES

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Traditionally, one of the ways traffic flow has been studied is by using the kinematic wave model. This model is studied in the Eulerian framework. Recently, the kinematic wave model has been transformed into Lagrangian coordinates. This model of traffic flow together with the concept of observability for linear time invariant discrete time systems is applied to study the observability of four sections of a freeway in both Eulerian and Lagrangian coordinates. A system with densities in four sections of a freeway is designed, and the observability of the system is studied with different situations for sensors. When the system evolves exactly according to the models, the states of the system could be obtained from measurements from certain situations.

For both, Eulerian and Lagrangian simulations, as long as the fourth section was measured, the states of the system could be obtained. To compare different situations of measurements, the condition number of the observability matrix is used.

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CHAPTER 1

INTRODUCTION

1.1 Motivation

Transportation research is a large and varied field. One of the most important areas in transportation is traffic flow. In order to gain some understanding of traffic phenomena, mathematical models have been proposed and studied. By studying traffic, from a mathematical point of view, better decisions can be made about how to deal with congestion, and how to maximize the flow of traffic. With the limited construction of new roads because of costs, and a projected increase in miles traveled, it is important to use the current transportation networks as efficiently as possible. Sustainability of transportation systems with respect to traffic flow and operations is an extremely important criteria while evaluating transportation improvements, as in [1]. Researchers have used dynamic modeling and non-linear techniques in [2] and [3] to integrate them with policy analysis.

To know which locations or areas need to be addressed in a transportation system, the system must be observed or measured with sensors. There are a variety of sensors including inductive loop detectors, magnetometers, cameras, probe vehicles, etc. According to [4], the most widely used sensor in modern traffic control systems, is by far, the inductive loop detector. These sensors are Eulerian sensors because these sensors are fixed in position. Tracking vehicles with phones, and using aerial vehicles are other ways that traffic can be measured. As mentioned in [5], because recently smartphones have become widespread, smartphones are very useful sensors. When using smartphones as sensors, measurements are in Lagrangian coordinates because the sensors travel with a vehicle.

However, there are limitations with sensors. The number of sensors available to monitor a traffic transportation system is limited by cost. Other times, sensors can fail or have problems, and can be considered unreliable.

1.2 Traffic Flow Modeling

One of the most used models to study traffic flow is the kinematic wave model, which formulates how traffic flows along a road, see [6], [7], and [8]. This model treats many vehicles together similarly to fluids. This is the model that will be used for traffic flow in this work. It is one of the simplest models to study while still showing properties of real vehicle interactions. This model has been recently transformed into a Lagrangian framework, where vehicles are treated individually as particles, see [21] and [22]. Both of these frameworks can be used to study observability of transportation networks.

When using the kinematic wave model, the most important variable in the system is density. The flux and velocity of a traffic stream are functions of density. In Lagrangian coordinates, the main variable in the system is the inverse of density, spacing. The velocity of a vehicle depends specifically on the spacing of the vehicle. As mentioned, sensors exist to obtain measurements in both kinds of coordinate systems.

1.3 Sensors in Traffic Systems

Observability in a transportation network modeled with traffic count sensors has been studied in works such as [9] and [10] to obtain the best locations to put sensors in a network. In [11], a strategy using a switching mode model using the cell transmission model is used to estimate densities in sections of a freeway. In [12], a particle filtering based estimation/prediction method is used to estimate densities on a four-cell freeway segment. In [13], GPS equipped probe vehicles are used to measure spacing data which is used for traffic estimation. Other authors have also studied in several ways how to incorporate data obtained from vehicles using smartphones for traffic estimation, see [14], [15], [16], and [17].

Thus both Eulerian sensors, such as loop detectors, and more recently Lagrangian sensors, such as smartphones, are used as measuring tools for traffic networks. In this work, a section of a freeway will be divided into four sections, and Eulerian sensors are placed so that densities in less than the four sections of the freeway can be measured. What is studied is if the densities in all the sections can be obtained. Similarly, for Lagrangian sensors, a line of vehicles will be divided into four parts, and cars from less than the four parts will be sensed. It is determined if all the spacings in the four parts of the line can be obtained.

1.4 Outline of the Thesis

This thesis is divided into chapters. Chapter 1 presents the motivation, background and arrangement of the thesis. Chapters 2 and 3 present the traffic flow model in Eulerian coordinates and Lagrangian coordinates, respectively. In chapter 4, observability in linear systems is presented. In chapters 5 and 6, observability of densities as states in a system of Eulerian traffic modeling and spacings as states in a system of Lagrangian traffic modeling are studied, respectively. In chapter 7 and 8, examples of simulations are demonstrated. Chapter 8 concludes this thesis and presents future work.

CHAPTER 2

Traffic Modeling In Eulerian Coordinates

2.1 LWR Model in Eulerian Coordinates

One of the most used models for studying traffic is the Lighthill-Whitman-Richards (LWR) model in [6], [7], [8]. This theory describes one-dimensional wave motion for the study of traffic flow. In this theory, there is a relationship between flow, the rate at which vehicles pass some point, and density (the number of vehicles per unit length of the road), [18]. The relationship between flow and density is

$$\text{flow} = \text{density} \times \text{velocity}.$$

The following variables will be used:

x is a variable for position in space,

t is a variable for time,

$\rho(x,t)$ is the density at time t at position x,

ρ_m is maximum density that is possible,

$q(x,t)$ is the flow at time t at position x,

q_m is maximum flow that is possible,

$v(\rho)$ is the velocity as a function of density p,

v_f is free flow velocity,

The velocity of the cars, v, can be written as a function of density and space.

$$v(x,t) = v(\rho(x,t), x)$$

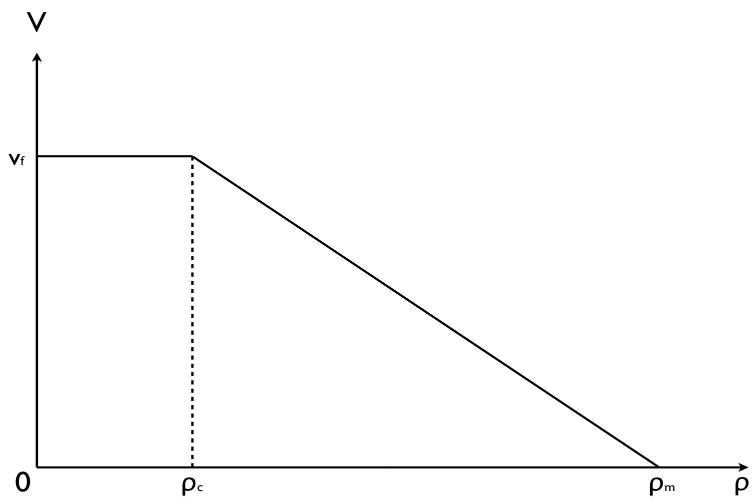

Figure 2.1: Velocity vs. Density

A relationship with v as a function of ρ is shown in Figure 2.1. From zero density until a critical density ρ_c, vehicles will travel at free flow speed. From ρ_c until ρ_m, the velocity of vehicles will depend on density. As density increases the velocity of the vehicles will decrease until the velocity becomes 0 at maximum density.

The flow, q, can be written as a function of density and space instead of a function of space and time.

$$q(x,t) = q^*(\rho(x,t), x)$$

A relationship with q as a function of ρ is shown in Figure 2.2. From zero density until a critical density ρ_c, the flow of vehicles will increase because as free flow speed stays the same, ρ increases. From ρ_c until ρ_m, the velocity of vehicles will depend on density. As density increases the velocity of the vehicles will decrease until the velocity becomes 0 at maximum density. Thus flow of vehicles will decrease until flow is zero at maximum density.

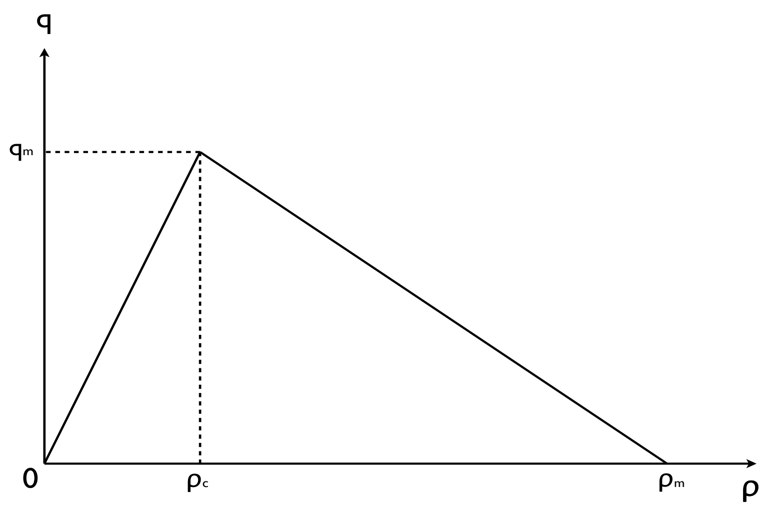

Figure 2.2: Fundamental Diagram, Flow vs. Density

In a similar manner, the density can be written as a function of flow and space if the function from $q(\rho)$ is one-to-one, i.e,

$$\rho(x,t) = \rho^*(q(x,t),x) \tag{2.1}$$

The main result from the LWR theory is used in the partial differential equation of the conservation of the number of cars, which is

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0. \tag{2.2}$$

With the substitution of equation (2.1), the above PDE becomes

$$\frac{\partial \rho^*(q(x,t),x)}{\partial t} + \frac{\partial q(x,t)}{\partial x} = 0,$$

$$\frac{\partial \rho^*(q(x,t),t)}{\partial q} \times \frac{\partial q(x,t)}{\partial t} + \frac{\partial q(x,t)}{\partial x} = 0,$$

i.e.

$$w(q(x,t),x)\frac{\partial q(x,t)}{\partial t} + \frac{\partial q(x,t)}{\partial x} = 0.$$

The above is true by defining $w(q(x,t),x) = \dfrac{\partial \rho^*(q(x,t),t)}{\partial q}$. This function w has units of vehicles/distance divided by vehicles/time. That is, the units of w are time/distance.

The full derivative of q with respect to x is

$$\frac{d}{dx}q = \frac{\partial q(x,t)}{\partial x} + \frac{\partial q(x,t)}{\partial t}\frac{dt}{dx} = 0.$$

Solving the PDE by the method of characteristics,

$$\frac{dt}{dx} = w(q(x,t),x).$$

Therefore, the flow q at some point (x_0, t_0) will remain constant along the characteristic curve described by

$$t(x) = t_0 + \int_{x_0}^{x} w(q(x_0,t_0),z)dz.$$

2.2 Cumulative Flows

Cumulative flows are useful for traffic analysis and we study them next. Let the function $N(x,t)$ be a cumulative flow function as in [19] and [20]. For this function, an Eulerian observer will start counting cars at location x starting with some reference car. The first car that passes the observer would be labeled 1, the second 2, and the nth car that has passed the observer would be labeled n. The output of the function $N(x,t)$ will be the number of the last car that passed position x at time t.

In Figure 2.3, two curves are drawn on the same graph, $N(x_1,t)$ and $N(x_2,t)$ for two locations x_1 and x_2. The vertical difference at time t_0 is the number of vehicles between positions x_1 and x_2. Similarly, the horizontal difference between the curves at the height j is the time it takes the vehicle labelled j to reach x_2 from x_1. The partial derivative of this curve with respect to time has units of number of vehicles/time. These are units of flow, q.

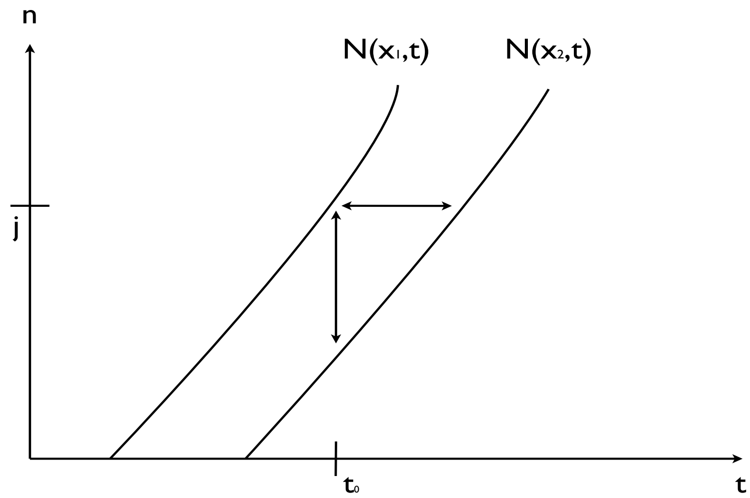

Figure 2.3: Two Cumulative Flow Functions, Car Number vs. Time

In Figure 2.4, two curves are drawn on the same graph, $N(x, t_1)$ and $N(x, t_2)$ for two different times t_1 and t_2. The vertical difference at position x_0 is the number of vehicles that passed position x_0 during the time $t_2 - t_1$. Similarly, the horizontal difference between the curves at the height j is the distance the vehicle labelled j travelled during the time $t_2 - t_1$. The partial derivative of this curve with respect to position has units of number of vehicles/distance. These are units of density, ρ.

The N curves are actually step functions, since counting cars is an increment in integers. However, for N to have a relationship with flow, q, and density, ρ, the N curve must be smoothed.

The partial derivative of $N(x, t)$ with respect to x is density, $\rho(x, t)$.

$$-\frac{\partial N(x,t)}{\partial x} = \rho(x,t).$$

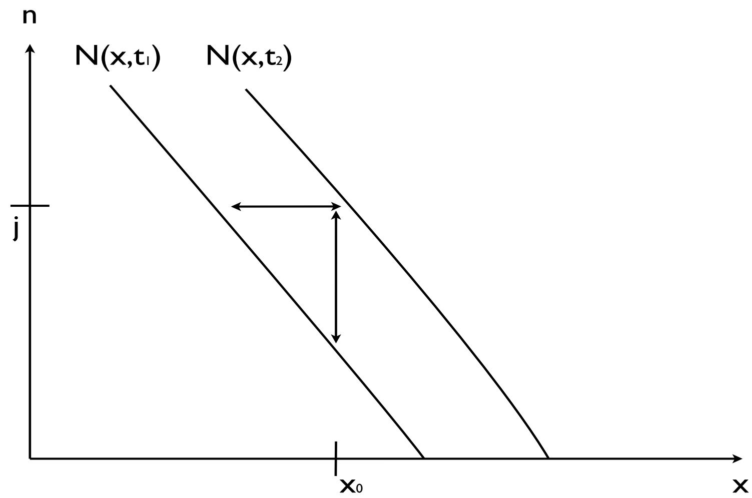

Figure 2.4: Two Cumulative Flow Functions, Car Number vs. Position

The partial derivative of $N(x,t)$ with respect to t is flow, $q(x,t)$.

$$\frac{\partial N(x,t)}{\partial t} = q(x,t).$$

Plugging in these new definitions of q and ρ into equation (2.2), one obtains,

$$\frac{\partial}{\partial t}\left(-\frac{\partial N(x,t)}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{\partial N(x,t)}{\partial t}\right) = 0.$$

or

$$\frac{\partial^2 N(x,t)}{\partial t \partial x} = \frac{\partial^2 N(x,t)}{\partial x \partial t}.$$

This equation is equivalent to equation (2.2) if the second derivatives of $N(x,t)$ exist. When there is discontinuity in the first derivative of N, a shock wave will form. The conservation of the number of cars is satisfied as long as $N(x,t)$ is piecewise continuous.

When k and q are continuous then the relationship

$$q(x,t) = q^*(\rho(x,t), x)$$

or

$$\frac{\partial N(x,t)}{\partial t} = q^*(-\frac{\partial N(x,t)}{\partial x}, x) \tag{2.3}$$

is valid. When there are no shocks, the solution of this equation is found by the method of characteristics. Knowing what q is determines what ρ is by equation (2.1).

2.3 Hamilton-Jacobi Equation in Eulerian Coordinates

The above theory is further extended in [19] and [20]. The cumulative flow function $N(x,t)$ satisfies equation (2.3) where q^* is a differentiable function. It is noted that the above equation has the form of a Hamilton-Jacobi equation. The above equation is satisfied everywhere in its solution domain except on shock curves where the function $N(x,t)$ is not differentiable. Along the shocks, however, the function N must be continuous. When a kinetic wave problem is well posed, it has a unique solution with stable shocks. In these extensions, it is further assumed that q^* is concave with respect to $\frac{-\partial N(x,t)}{\partial x}$, or density, ρ.

CHAPTER 3

Traffic Modeling in Lagrangian Coordinates

3.1 Hamilton-Jacobi Equation in Lagrangian Coordinates

When using the cumulative flow function, $N(x,t)$, defined in the previous chapter, there is a connection to the Lagrangian framework. When $N(x,t) = n$, the resulting curve is the path that car n takes for x and t. This curve is what an observer who is traveling with the vehicle will record. The coordinate transformation between (x,t) and (n,t) is made by inverting the cumulative flow function $N(x,t)$.

In the resulting transformation, it will be assumed that density, ρ, will be strictly positive. If ρ is zero somewhere, then the domain can be made smaller to only regions where ρ is strictly positive. More on this is found in [21].

Fixing the variable t, $N(\cdot,t)$ will be a decreasing function of x. To solve for x, we have some function of n and t, i.e.,

$$x = X(n,t).$$

Herein, $X(n,t)$ defines the position of the vehicle labeled n at time t. We also have the following relationship:

$$\frac{\partial X(n,t)}{\partial t} = v(n,t) \tag{3.1}$$

The instantaneous velocity of the vehicle labelled n is its change in position at time t.

$$\frac{\partial X(n,t)}{\partial n} = -s(n,t) = -\frac{1}{\rho(n,t)} \tag{3.2}$$

The difference in position between vehicles, spacing, or the reciprocal of density, at time t is defined as the variable s.

To simplify the relationship between flow q and density ρ, velocity v is made a function of just ρ, i.e.,

$$q(\rho) = \rho v(\rho). \tag{3.3}$$

Similarly, the relationship between q and v is

$$v = \frac{q(\rho)}{\rho} = q\left(\frac{1}{s}\right) * s. \tag{3.4}$$

From Equations (3.1), (3.4), and (3.2),

$$\frac{\partial X(n,t)}{\partial t} = v = q\left(\frac{1}{s}\right) * s = V^*(s) = V^*\left(-\frac{\partial X(n,t)}{\partial n}\right).$$

or

$$\frac{\partial X(n,t)}{\partial t} - V^*\left(-\frac{\partial X(n,t)}{\partial n}\right) = 0. \tag{3.5}$$

The above equation is the Hamilton-Jacobi equation in Lagrangian coordinates.

In [22], it is shown that if there is a viscosity solution to the Hamilton-Jacobi equation in Eulerian, $N(x,t)$, then the viscosity solution when the problem is transformed into Lagrangian coordinates is $X(n,t)$. The vice-versa is also true.

3.2 Conservation Equation in Lagrangian Coordinates

Summarizing the process started in the previous chapter, everything started with the LWR partial differential equation in Eulerian coordinates. From there, assuming a relationship between q and ρ, we obtained a Hamilton-Jacobi equation in Eulerian coordinates. Using the relationship between q, v, and $\rho = \frac{1}{s}$, the Hamilton-Jacobi equation in Lagrangian coordinates was obtained by using transformations. More details on the Lagrangian coordinates can be found in [23]. Now the LWR PDE will be obtained in Lagrangian coordinates.

Starting with the Hamilton-Jacobi equation in Lagrangian coordinates,

$$\frac{\partial X(n,t)}{\partial t} = V^*(s) = V^*\left(-\frac{\partial X(n,t)}{\partial n}\right),$$

the partial derivative with respect to n will be taken on both sides,

$$\frac{\partial}{\partial n}\left(\frac{\partial X(n,t)}{\partial t}\right) = \frac{\partial}{\partial n}V^*(s).$$

Rearranging the left side of the above equation, when $X(n,t)$ is twice differentiable, one obtains

$$\frac{\partial}{\partial n}\left(\frac{\partial X(n,t)}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial X(n,t)}{\partial n}\right) = -\frac{\partial}{\partial t}s.$$

Finally, plugging the above into the Hamilton-Jacobi equation, one obtains the LWR PDE in Lagrangian coordinates,

$$\frac{\partial}{\partial t}s + \frac{\partial}{\partial n}V^*(s) = 0 \tag{3.6}$$

3.3 LWR Model in Lagrangian Coordinates

As mentioned in the previous section, the LWR PDE in Lagrangian coordinates is equation (3.6). Similarly to how a fundamental diagram (FD) is needed for Eulerian coordinates, a fundamental diagram is also needed in Lagrangian coordinates. This fundamental diagram must relate velocity to spacing. In Eulerian coordinates speed is a function of density. This diagram is shown in Figure 3.1. The relationship is then transformed into a velocity spacing relationship. The transformed FD used in Lagrangian coordinates is shown in Figure 3.2.

The function used for velocity is,

$$V^*(s) = \begin{cases} \dfrac{v_f}{s_c - s_m}(s - s_c) + v_f & s_m \le s \le s_c \\ v_f & s > s_c \end{cases}$$

This function, unlike q^* (used for Eulerian coordinates given in Figure 2.2), only

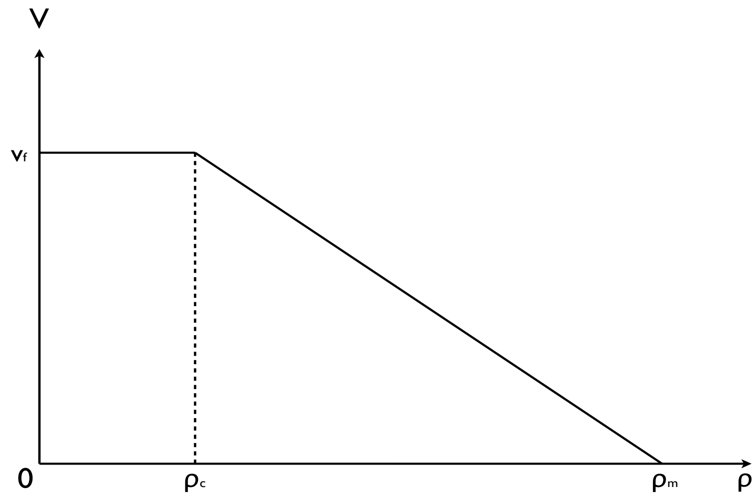

Figure 3.1: Velocity vs. Density

has nonnegative slopes.

$$\frac{d}{ds}V^*(s) = \begin{cases} \dfrac{v_f}{s_c - s_m} & s_m \leq s < s_c \\ 0 & s > s_c \end{cases}$$

The function $V^*(s) = \dfrac{v_f}{s_c - s_m}(s - s_c) + v_f$ can be made simpler.

$$\frac{v_f}{s_c - s_m}(s - s_c) + v_f = \frac{v_f}{s_c - s_m}s - \frac{v_f}{s_c - s_m}s_c + \frac{s_c - s_m}{s_c - s_m}v_f = \frac{v_f}{s_c - s_m}s - \frac{s_m v_f}{s_c - s_m}$$

Defining the variable $w = \dfrac{s_m v_f}{s_c - s_m}$, then

$$V^*(s) = w\rho_m s - w$$

Let us take a closer look at the relationship between $\rho(x, t)$ and $s(n, t)$. As distance

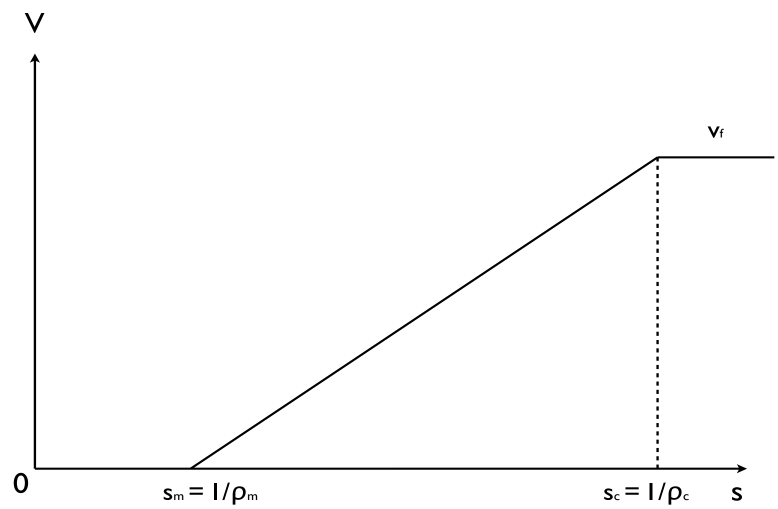

Figure 3.2: Velocity vs. Spacing

x increases, the vehicle number n decreases, because it is closer to the lead in the queue of cars. This is shown in Figure 3.3.

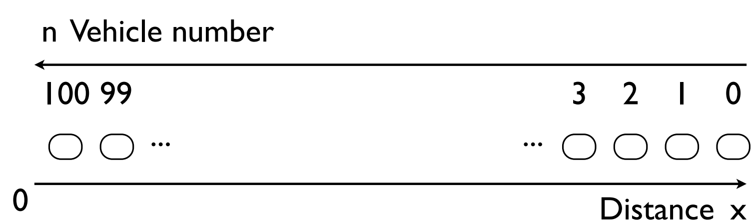

Figure 3.3: Position and Vehicle Number

CHAPTER 4

Observability for LTI Systems

In this chapter, the concept of observability for linear time invariant (LTI) systems from controls theory will be introduced, see [24], [25], and [26].

We will use the following notations:

k is a variable for discrete time,

$\overrightarrow{x(k)}$ is a vector of n states at discrete time k,

$x_i(k)$ is the ith component of $\overrightarrow{x(k)}$ at discrete time k,

A is a n by n matrix of real numbers,

$\overrightarrow{y(k)}$ is a vector of m measurements at discrete time k,

C is a m by n matrix of real numbers.

We will consider a linear, time-invariant, discrete time system,

$$\overrightarrow{x(k+1)} = A\overrightarrow{x(k)} \tag{4.1}$$

We can obtain measurements of the states like below.

$$\overrightarrow{y(k)} = C\overrightarrow{x(k)} \tag{4.2}$$

Equation (4.1) models how our system behaves. Equation (4.2) models how and which states of the system are measured. Sometimes we do not know all values of the states in the system. If we obtain measurements of only some states of the system, with sensors, we want to know if we can obtain the values of all the states in the system.

4.1 Observability Matrix

Taking n measurements according to (4.2) and substituting with (4.1), we have the following:

$$\begin{aligned}
\overrightarrow{y(0)} &= C\overrightarrow{x(0)} \\
\overrightarrow{y(1)} &= C\overrightarrow{x(1)} && = CA\overrightarrow{x(0)} \\
\overrightarrow{y(2)} &= C\overrightarrow{x(2)} && = CA\overrightarrow{x(1)} && = CA^2\overrightarrow{x(0)} \\
&\vdots \\
\overrightarrow{y(n-1)} &= C\overrightarrow{x(n-1)} && = CA^{n-1}\overrightarrow{x(0)}
\end{aligned}$$

In matrix notation we have

$$\begin{bmatrix} \overrightarrow{y(0)} \\ \overrightarrow{y(1)} \\ \overrightarrow{y(2)} \\ \vdots \\ \overrightarrow{y(n-1)} \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} \overrightarrow{x(0)}$$

The matrix on the right hand side of the above system consisting of C and powers of A is called the **observability matrix**, O. To solve for all n states of $\overrightarrow{x(0)}$ in the system, it is expected there should be at least n equations in a linear system of equations. Because $\overrightarrow{y(k)}$ is at least one entry long, we can guarantee that there will be at least n equations by taking n measurements. To obtain the solution $\overrightarrow{x(0)}$ in the above system, the observability matrix must have rank n. Thus, if $\overrightarrow{x(0)}$ can be obtained after a finite amount of discrete time steps, then the system is observable. Knowing the initial states, $\overrightarrow{x(0)}$, and using equation (4.1), we can obtain the states at all instants of discrete time.

4.2 Observability Index

Observability index, denoted by v, is defined as the smallest natural number which satisfies,

$$rank(O_v) = rank(O_{v+1}),$$

where

$$O_v = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{v-1} \end{bmatrix}.$$

The observability index, v, can be less than the variable n, which represents the number of states in the system. The observability matrix determines if a system is observable or not. It does not give the observability index, which is the minimum number of discrete time steps needed to obtain all states in the system.

CHAPTER 5

Observability of Densities in Four Sections

5.1 The Traffic Equations

From conservation of matter, we know that the number of cars, N, in a lane of length a to b is

$$N = \int_a^b \rho(x,t)dx$$

where $\rho(x,t)$ is function of density at point x at time t.

The rate of change of the number of cars in the lane is

$$\frac{d}{dt}N = q(a,t) - q(b,t)$$

Here $q(x,t)$ is the flow of cars at point x at time t.

Now using both equations together we have

$$\frac{d}{dt}\int_a^b \rho(x,t)dx = q(a,t) - q(b,t)$$

5.2 The Setup of the Problem

We will assume we have a stretch of highway that is divided into four different sections. Section one is from point a to point b and has a constant density $\rho_1(x,t)$, section two from point b to point c has $\rho_2(x,t)$, section three from point c to point d has $\rho_3(x,t)$, and section four from point d to point e has $\rho_4(x,t)$.

We will assume that we know the flow coming into section 1, and call it f_{in}. The flow coming out of section 1 will be $q(b,t) = v_f \cdot \rho_1 \cdot \left(1 - \frac{\rho_1}{\rho_{max}}\right)$. Flows going into or out of the four sections will be labelled similarly.

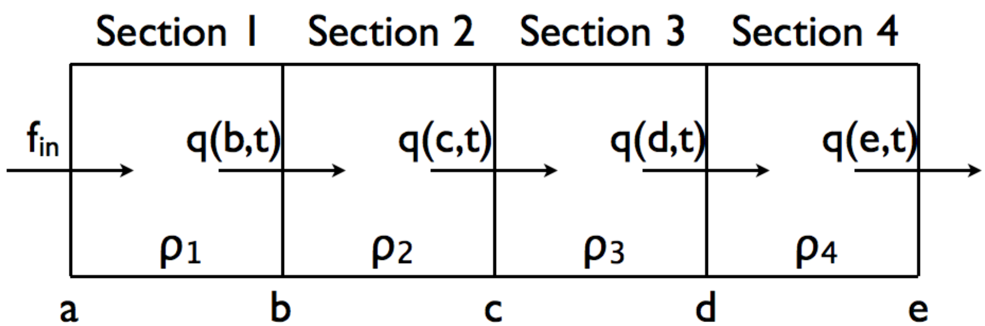

Figure 5.1: Density and Flow of 4 Sections

Since the assumption is that the density of each section is constant, then for section 1,

$$\frac{d}{dt}\int_a^b \rho(x,t)dx = \frac{d}{dt}\rho(t)\int_a^b dx = \frac{d}{dt}\rho(t)(l) = q(a,t) - q(b,t)$$

and

$$\frac{d}{dt}\rho(t) = \frac{1}{l}q(a,t) - \frac{1}{l}q(b,t)$$

	Flow In	Flow Out
Section 1	$q(a,t) = f_{in}$	$q(b,t) = v_f \cdot \rho_1 \cdot \left(1 - \frac{\rho_1}{\rho_{max}}\right)$
Section 2	$q(b,t) = v_f \cdot \rho_1 \cdot \left(1 - \frac{\rho_1}{\rho_{max}}\right)$	$q(c,t) = v_f \cdot \rho_2 \cdot \left(1 - \frac{\rho_2}{\rho_{max}}\right)$
Section 3	$q(c,t) = v_f \cdot \rho_2 \cdot \left(1 - \frac{\rho_2}{\rho_{max}}\right)$	$q(d,t) = v_f \cdot \rho_3 \cdot \left(1 - \frac{\rho_3}{\rho_{max}}\right)$
Section 4	$q(d,t) = v_f \cdot \rho_3 \cdot \left(1 - \frac{\rho_3}{\rho_{max}}\right)$	$q(e,t) = v_f \cdot \rho_4 \cdot \left(1 - \frac{\rho_4}{\rho_{max}}\right)$

Table 5.1: Flow In & Flow Out

5.3 The State Space

We are interested in the four densities of the four sections. The four equations that describe the dynamics are:

$$\begin{aligned}
\dot{\rho}_1 &= f_1(\rho_1, \rho_2, \rho_3, \rho_4) &&= \frac{1}{l_1} f_{in} - \frac{1}{l_1} v_f \rho_1 \left(1 - \frac{\rho_1}{\rho_{max}}\right) \\
\dot{\rho}_2 &= f_2(\rho_1, \rho_2, \rho_3, \rho_4) &&= \frac{1}{l_2} v_f \rho_1 \left(1 - \frac{\rho_1}{\rho_{max}}\right) - \frac{1}{l_2} v_f \rho_2 \left(1 - \frac{\rho_2}{\rho_{max}}\right) \\
\dot{\rho}_3 &= f_3(\rho_1, \rho_2, \rho_3, \rho_4) &&= \frac{1}{l_3} v_f \rho_2 \left(1 - \frac{\rho_2}{\rho_{max}}\right) - \frac{1}{l_3} v_f \rho_3 \left(1 - \frac{\rho_3}{\rho_{max}}\right) \\
\dot{\rho}_4 &= f_4(\rho_1, \rho_2, \rho_3, \rho_4) &&= \frac{1}{l_4} v_f \rho_3 \left(1 - \frac{\rho_3}{\rho_{max}}\right) - \frac{1}{l_4} v_f \rho_4 \left(1 - \frac{\rho_4}{\rho_{max}}\right)
\end{aligned}$$

In vector notation:

$$\dot{\overrightarrow{\rho}} = F(\overrightarrow{\rho})$$

5.3.1 Finding the Equilibrium Point

This system is clearly nonlinear, since there are terms of density to the second power. We will find the equilibrium point and then linearize the system about that equilibrium point. To find the equilibrium point, all functions f_1, f_2, f_3, f_4 must equal zero so that densities $\rho_1, \rho_2, \rho_3, \rho_4$ do not change with respect to time.

$$\begin{aligned}
0 &= \frac{1}{l_1} f_{in} - \frac{1}{l_1} v_f \rho_{1_{eq}} \left(1 - \frac{\rho_{1_{eq}}}{\rho_{max}}\right) \\
0 &= \frac{1}{l_2} v_f \rho_{1_{eq}} \left(1 - \frac{\rho_{1_{eq}}}{\rho_{max}}\right) - \frac{1}{l_2} v_f \rho_{2_{eq}} \left(1 - \frac{\rho_{2_{eq}}}{\rho_{max}}\right) \\
0 &= \frac{1}{l_3} v_f \rho_{2_{eq}} \left(1 - \frac{\rho_{2_{eq}}}{\rho_{max}}\right) - \frac{1}{l_3} v_f \rho_{3_{eq}} \left(1 - \frac{\rho_{3_{eq}}}{\rho_{max}}\right) \\
0 &= \frac{1}{l_4} v_f \rho_{3_{eq}} \left(1 - \frac{\rho_{3_{eq}}}{\rho_{max}}\right) - \frac{1}{l_4} v_f \rho_{4_{eq}} \left(1 - \frac{\rho_{4_{eq}}}{\rho_{max}}\right)
\end{aligned}$$

We will assume that the section length is the same in all sections. This leads to

$$\begin{aligned}
f_{in} &= v_f \rho_{1_{eq}} \left(1 - \frac{\rho_{1_{eq}}}{\rho_{max}}\right) \\
v_f \rho_{1_{eq}} \left(1 - \frac{\rho_{1_{eq}}}{\rho_{max}}\right) &= v_f \rho_{2_{eq}} \left(1 - \frac{\rho_{2_{eq}}}{\rho_{max}}\right) \\
v_f \rho_{2_{eq}} \left(1 - \frac{\rho_{2_{eq}}}{\rho_{max}}\right) &= v_f \rho_{3_{eq}} \left(1 - \frac{\rho_{3_{eq}}}{\rho_{max}}\right) \\
v_f \rho_{3_{eq}} \left(1 - \frac{\rho_{3_{eq}}}{\rho_{max}}\right) &= v_f \rho_{4_{eq}} \left(1 - \frac{\rho_{4_{eq}}}{\rho_{max}}\right)
\end{aligned}$$

If v_f and ρ_{max} are the same for all four sections, then

$$\rho_{4_{eq}} = \rho_{3_{eq}} = \rho_{2_{eq}} = \rho_{1_{eq}} = \frac{\rho_{max} \pm \sqrt{\rho_{max}^2 - 4\frac{\rho_{max} f_{in}}{v_f}}}{2}$$

at steady state.

5.3.2 Linearizing about the Equilibrium Point

The Jacobian matrix, denoted as $\dfrac{\partial F}{\partial \rho}$, of the right hand side of the system would be

$$\begin{bmatrix} \dfrac{\partial f_1}{\partial \rho_1} & \dfrac{\partial f_1}{\partial \rho_2} & \dfrac{\partial f_1}{\partial \rho_3} & \dfrac{\partial f_1}{\partial \rho_4} \\ \\ \dfrac{\partial f_2}{\partial \rho_1} & \dfrac{\partial f_2}{\partial \rho_2} & \dfrac{\partial f_2}{\partial \rho_3} & \dfrac{\partial f_2}{\partial \rho_4} \\ \\ \dfrac{\partial f_3}{\partial \rho_1} & \dfrac{\partial f_3}{\partial \rho_2} & \dfrac{\partial f_3}{\partial \rho_3} & \dfrac{\partial f_3}{\partial \rho_4} \\ \\ \dfrac{\partial f_4}{\partial \rho_1} & \dfrac{\partial f_4}{\partial \rho_2} & \dfrac{\partial f_4}{\partial \rho_3} & \dfrac{\partial f_4}{\partial \rho_4} \end{bmatrix}$$

which is

$$\begin{bmatrix} -\frac{1}{l_1}v_f\left(1-\frac{2\rho_1}{\rho_{max}}\right) & 0 & 0 & 0 \\ \frac{1}{l_2}v_f\left(1-\frac{2\rho_1}{\rho_{max}}\right) & -\frac{1}{l_2}v_f\left(1-\frac{2\rho_2}{\rho_{max}}\right) & 0 & 0 \\ 0 & \frac{1}{l_3}v_f\left(1-\frac{2\rho_2}{\rho_{max}}\right) & -\frac{1}{l_3}v_f\left(1-\frac{2\rho_3}{\rho_{max}}\right) & 0 \\ 0 & 0 & \frac{1}{l_4}v_f\left(1-\frac{2\rho_3}{\rho_{max}}\right) & -\frac{1}{l_4}v_f\left(1-\frac{2\rho_4}{\rho_{max}}\right) \end{bmatrix}$$

The full system with first order Taylor series expansion of F about the equilibrium point ρ_{eq} is

$$\begin{bmatrix} \dot{\rho}_1 \\ \dot{\rho}_2 \\ \dot{\rho}_3 \\ \dot{\rho}_4 \end{bmatrix} = \begin{bmatrix} \dot{\rho}_{eq_1} \\ \dot{\rho}_{eq_2} \\ \dot{\rho}_{eq_3} \\ \dot{\rho}_{eq_4} \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1(\overrightarrow{p_{eq}})}{\partial \rho_1} & \frac{\partial f_1(\overrightarrow{p_{eq}})}{\partial \rho_2} & \frac{\partial f_1(\overrightarrow{p_{eq}})}{\partial \rho_3} & \frac{\partial f_1(\overrightarrow{p_{eq}})}{\partial \rho_4} \\ \frac{\partial f_2(\overrightarrow{p_{eq}})}{\partial \rho_1} & \frac{\partial f_2(\overrightarrow{p_{eq}})}{\partial \rho_2} & \frac{\partial f_2(\overrightarrow{p_{eq}})}{\partial \rho_3} & \frac{\partial f_2(\overrightarrow{p_{eq}})}{\partial \rho_4} \\ \frac{\partial f_3(\overrightarrow{p_{eq}})}{\partial \rho_1} & \frac{\partial f_3(\overrightarrow{p_{eq}})}{\partial \rho_2} & \frac{\partial f_3(\overrightarrow{p_{eq}})}{\partial \rho_3} & \frac{\partial f_3(\overrightarrow{p_{eq}})}{\partial \rho_4} \\ \frac{\partial f_4(\overrightarrow{p_{eq}})}{\partial \rho_1} & \frac{\partial f_4(\overrightarrow{p_{eq}})}{\partial \rho_2} & \frac{\partial f_4(\overrightarrow{p_{eq}})}{\partial \rho_3} & \frac{\partial f_4(\overrightarrow{p_{eq}})}{\partial \rho_4} \end{bmatrix} \begin{bmatrix} \rho_1 - \rho_{eq_1} \\ \rho_2 - \rho_{eq_2} \\ \rho_3 - \rho_{eq_3} \\ \rho_4 - \rho_{eq_4} \end{bmatrix}$$

We know that for the equilibrium point, ρ_{eq}, that $\dot{\rho}_{eq} = 0$. We will define a new variable

$$z_i = p_i - \rho_{eq_i} \quad i = 1, 2, 3, 4.$$

Then

$$\dot{z}_i = \dot{p}_i \quad i = 1, 2, 3, 4.$$

because $\dot{p}_{eq_i} = 0$. These new variables would leave us with

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1(\overrightarrow{p_{eq}})}{\partial \rho_1} & \frac{\partial f_1(\overrightarrow{p_{eq}})}{\partial \rho_2} & \frac{\partial f_1(\overrightarrow{p_{eq}})}{\partial \rho_3} & \frac{\partial f_1(\overrightarrow{p_{eq}})}{\partial \rho_4} \\ \frac{\partial f_2(\overrightarrow{p_{eq}})}{\partial \rho_1} & \frac{\partial f_2(\overrightarrow{p_{eq}})}{\partial \rho_2} & \frac{\partial f_2(\overrightarrow{p_{eq}})}{\partial \rho_3} & \frac{\partial f_2(\overrightarrow{p_{eq}})}{\partial \rho_4} \\ \frac{\partial f_3(\overrightarrow{p_{eq}})}{\partial \rho_1} & \frac{\partial f_3(\overrightarrow{p_{eq}})}{\partial \rho_2} & \frac{\partial f_3(\overrightarrow{p_{eq}})}{\partial \rho_3} & \frac{\partial f_3(\overrightarrow{p_{eq}})}{\partial \rho_4} \\ \frac{\partial f_4(\overrightarrow{p_{eq}})}{\partial \rho_1} & \frac{\partial f_4(\overrightarrow{p_{eq}})}{\partial \rho_2} & \frac{\partial f_4(\overrightarrow{p_{eq}})}{\partial \rho_3} & \frac{\partial f_4(\overrightarrow{p_{eq}})}{\partial \rho_4} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}$$

5.3.3 Discretizing

We will discretize the continuous time equations. The four equations that describe the dynamics become:

$$\begin{bmatrix} \frac{z_1(k+1) - z_1(k)}{\Delta t} \\ \frac{z_2(k+1) - z_2(k)}{\Delta t} \\ \frac{z_3(k+1) - z_3(k)}{\Delta t} \\ \frac{z_4(k+1) - z_4(k)}{\Delta t} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1(\overrightarrow{p_{eq}})}{\partial \rho_1} & \frac{\partial f_1(\overrightarrow{p_{eq}})}{\partial \rho_2} & \frac{\partial f_1(\overrightarrow{p_{eq}})}{\partial \rho_3} & \frac{\partial f_1(\overrightarrow{p_{eq}})}{\partial \rho_4} \\ \frac{\partial f_2(\overrightarrow{p_{eq}})}{\partial \rho_1} & \frac{\partial f_2(\overrightarrow{p_{eq}})}{\partial \rho_2} & \frac{\partial f_2(\overrightarrow{p_{eq}})}{\partial \rho_3} & \frac{\partial f_2(\overrightarrow{p_{eq}})}{\partial \rho_4} \\ \frac{\partial f_3(\overrightarrow{p_{eq}})}{\partial \rho_1} & \frac{\partial f_3(\overrightarrow{p_{eq}})}{\partial \rho_2} & \frac{\partial f_3(\overrightarrow{p_{eq}})}{\partial \rho_3} & \frac{\partial f_3(\overrightarrow{p_{eq}})}{\partial \rho_4} \\ \frac{\partial f_4(\overrightarrow{p_{eq}})}{\partial \rho_1} & \frac{\partial f_4(\overrightarrow{p_{eq}})}{\partial \rho_2} & \frac{\partial f_4(\overrightarrow{p_{eq}})}{\partial \rho_3} & \frac{\partial f_4(\overrightarrow{p_{eq}})}{\partial \rho_4} \end{bmatrix} \begin{bmatrix} z_1(k) \\ z_2(k) \\ z_3(k) \\ z_4(k) \end{bmatrix}$$

which is

$$\begin{bmatrix} z_1(k+1) \\ z_2(k+1) \\ z_3(k+1) \\ z_4(k+1) \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1(\overrightarrow{p_{eq}})}{\partial \rho_1} & \frac{\partial f_1(\overrightarrow{p_{eq}})}{\partial \rho_2} & \frac{\partial f_1(\overrightarrow{p_{eq}})}{\partial \rho_3} & \frac{\partial f_1(\overrightarrow{p_{eq}})}{\partial \rho_4} \\ \frac{\partial f_2(\overrightarrow{p_{eq}})}{\partial \rho_1} & \frac{\partial f_2(\overrightarrow{p_{eq}})}{\partial \rho_2} & \frac{\partial f_2(\overrightarrow{p_{eq}})}{\partial \rho_3} & \frac{\partial f_2(\overrightarrow{p_{eq}})}{\partial \rho_4} \\ \frac{\partial f_3(\overrightarrow{p_{eq}})}{\partial \rho_1} & \frac{\partial f_3(\overrightarrow{p_{eq}})}{\partial \rho_2} & \frac{\partial f_3(\overrightarrow{p_{eq}})}{\partial \rho_3} & \frac{\partial f_3(\overrightarrow{p_{eq}})}{\partial \rho_4} \\ \frac{\partial f_4(\overrightarrow{p_{eq}})}{\partial \rho_1} & \frac{\partial f_4(\overrightarrow{p_{eq}})}{\partial \rho_2} & \frac{\partial f_4(\overrightarrow{p_{eq}})}{\partial \rho_3} & \frac{\partial f_4(\overrightarrow{p_{eq}})}{\partial \rho_4} \end{bmatrix} \begin{bmatrix} z_1(k) \\ z_2(k) \\ z_3(k) \\ z_4(k) \end{bmatrix} \Delta t + \begin{bmatrix} z_1(k) \\ z_2(k) \\ z_3(k) \\ z_4(k) \end{bmatrix}$$

i.e.,

$$
\begin{bmatrix} z_1(k+1) \\ z_2(k+1) \\ z_3(k+1) \\ z_4(k+1) \end{bmatrix} =
$$

$$
\begin{bmatrix}
\frac{\partial f_1(\overrightarrow{p_{eq}})}{\partial \rho_1}\Delta t + 1 & \frac{\partial f_1(\overrightarrow{p_{eq}})}{\partial \rho_2}\Delta t & \frac{\partial f_1(\overrightarrow{p_{eq}})}{\partial \rho_3}\Delta t & \frac{\partial f_1(\overrightarrow{p_{eq}})}{\partial \rho_4}\Delta t \\
\frac{\partial f_2(\overrightarrow{p_{eq}})}{\partial \rho_1}\Delta t & \frac{\partial f_2(\overrightarrow{p_{eq}})}{\partial \rho_2}\Delta t + 1 & \frac{\partial f_2(\overrightarrow{p_{eq}})}{\partial \rho_3}\Delta t & \frac{\partial f_2(\overrightarrow{p_{eq}})}{\partial \rho_4}\Delta t \\
\frac{\partial f_3(\overrightarrow{p_{eq}})}{\partial \rho_1}\Delta t & \frac{\partial f_3(\overrightarrow{p_{eq}})}{\partial \rho_2}\Delta t & \frac{\partial f_3(\overrightarrow{p_{eq}})}{\partial \rho_3}\Delta t + 1 & \frac{\partial f_3(\overrightarrow{p_{eq}})}{\partial \rho_4}\Delta t \\
\frac{\partial f_4(\overrightarrow{p_{eq}})}{\partial \rho_1}\Delta t & \frac{\partial f_4(\overrightarrow{p_{eq}})}{\partial \rho_2}\Delta t & \frac{\partial f_4(\overrightarrow{p_{eq}})}{\partial \rho_3}\Delta t & \frac{\partial f_4(\overrightarrow{p_{eq}})}{\partial \rho_4}\Delta t + 1
\end{bmatrix}
\begin{bmatrix} z_1(k) \\ z_2(k) \\ z_3(k) \\ z_4(k) \end{bmatrix}
$$

Let A denote the matrix on the right hand side of the above system. Thus

$$
\overrightarrow{z(k+1)} = A\overrightarrow{z(k)} \tag{5.1}
$$

5.4 Observability of the Linearized State Space

Suppose for our system, equation (5.1), we can obtain measurements in the following form,

$$
\overrightarrow{y(k)} = C\overrightarrow{z(k)}
$$

where $\overrightarrow{y(k)} \in \Re^p$, $C \in \Re^{p \times n}$, and $\overrightarrow{z(k)} \in \Re^4$. The system

$$\begin{aligned} \overrightarrow{z(k+1)} = & \quad A\overrightarrow{z(k)} \\ \overrightarrow{y(k)} = & \quad C\overrightarrow{z(k)} \end{aligned}$$

is observable if the observability matrix

$$\begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} \tag{5.2}$$

has rank 4, because $\overrightarrow{z_k}$ has four variables.

5.4.1 Sensing Density in All Sections

If all the four states $(z_1(k), z_2(k), z_3(k), z_4(k))$ are measured directly, this scenario is represented by the equation

$$\overrightarrow{y(k)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_1(k) \\ z_2(k) \\ z_3(k) \\ z_4(k) \end{bmatrix}$$

Here

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

Here we only need to check the rank of the observability matrix.

5.4.2 Numerical Example

The following values of the corresponding parameters are used:

- $\rho_{max} = 0.14$ vehicles/m
- $v_f = 30$ m/s
- $l_1 = l_2 = l_3 = l_4 = 500$ m
- $f_{in} = 0.3$ vehicles/s, assumed to be constant for different time steps
- $\Delta t = 15$ s, the time interval between two readings of sensors

Then,

$$\rho_{4_{eq}} = \rho_{3_{eq}} = \rho_{2_{eq}} = \rho_{1_{eq}} = \frac{0.14 \pm \sqrt{0.14^2 - 4\frac{0.140.3}{30}}}{2} = 0.1292, 0.0108.$$

Using the equilibrium point 0.1292 and the above values,

$$A = \begin{bmatrix} 1.7606 & 0 & 0 & 0 \\ -0.7606 & 1.7606 & 0 & 0 \\ 0 & -0.7606 & 1.7606 & 0 \\ 0 & 0 & -0.7606 & 1.7606 \end{bmatrix}$$

To determine if the linearized system is observable, we need to check the rank of the observability matrix given by

$$\begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = \begin{bmatrix} I \\ A \\ A^2 \\ A^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1.7606 & 0 & 0 & 0 \\ -0.7606 & 1.7606 & 0 & 0 \\ 0 & -0.7606 & 1.7606 & 0 \\ 0 & 0 & -0.7606 & 1.7606 \\ 3.0998 & 0 & 0 & 0 \\ -2.6784 & 3.0998 & 0 & 0 \\ 0.5786 & -2.6784 & 3.0998 & 0 \\ 0 & 0.5786 & -2.6784 & 3.0998 \\ 5.4577 & 0 & 0 & 0 \\ -7.0736 & 5.4577 & 0 & 0 \\ 3.0560 & -7.0736 & 5.4577 & 0 \\ -0.4401 & 3.0560 & -7.0736 & 5.4577 \end{bmatrix}$$

Since the rank is 4, and this is obvious since $C = I$, then the linearized system with these parameters is observable.

5.4.3 Sensing Density in Three Sections

We will investigate the scenario when only three of the four states $(z_1(k), z_2(k), z_3(k), z_4(k))$ are observed. Different ways to sense three sections are represented with different instances of the matrix C.

When the section that is not sensed is the first section, then

$$\begin{bmatrix} y_2(k) \\ y_3(k) \\ y_4(k) \end{bmatrix} = C \times \begin{bmatrix} z_1(k) \\ z_2(k) \\ z_3(k) \\ z_4(k) \end{bmatrix}$$

where

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

When the section that is not sensed is the second section, then

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

When the section that is not sensed is the third section, then

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Finally, when the section that is not sensed is the fourth section, then

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

After checking the rank of the observability matrix, equation (5.2) for these different cases, we obtain Table 5.2. The system is observable when sensing three different sections, as long as section 4 is included.

Sections Sensed	Rank
ρ_2, ρ_3, ρ_4	4
ρ_1, ρ_3, ρ_4	4
ρ_1, ρ_2, ρ_4	4
ρ_1, ρ_2, ρ_3	3

Table 5.2: Measuring Density in 3 Sections

5.4.4 Sensing Density in Two Sections

The scenario when two of the four states $(z_1(k), z_2(k), z_3(k), z_4(k))$ are measured is analyzed.

We need to check the rank of equation (5.2) for different C matrices representing which of the two sections are being sensed. Out of the scenarios when the system is observable, we can investigate the condition numbers of the observability matrix.

Sections Sensed	Rank
ρ_1, ρ_2	2
ρ_1, ρ_3	3
ρ_1, ρ_4	4
ρ_2, ρ_3	3
ρ_2, ρ_4	4
ρ_3, ρ_4	4

Table 5.3: Measuring Density in 2 Sections

After checking the rank of the observability matrix for these different cases, we obtain Table 5.3. We find that we can obtain all four states of the system, while only measuring 2 states for 3 different cases. Again, the system is observable when sensing two different sections, as long as section 4 is included.

5.4.5 Sensing Density in Only One Section

The scenario when only one of the four states $(z_1(k), z_2(k), z_3(k), z_4(k))$ is measured is now analyzed. This is analyzed in the same way as before, by changing the C matrix to match the measurement situation, and checking the rank of equation (5.2). The rank of the observability matrix for different measurements is obtained in Table 5.4.

Sections Measured	Rank
ρ_1	1
ρ_2	2
ρ_3	3
ρ_4	4

Table 5.4: Measuring Density in 1 Section

We can look at the rank of the observability matrix for these different cases, on Table 5.4. We find that we can obtain all four states of the system only if we measure section 4. Once again, the system is observable as long as section 4 is included.

5.5 Stability Investigations

From the previous sections we found that, in some cases, we can obtain all four states of the system, while measuring less than all states of the system. The system is observable as long as section 4 is included. We will investigate the different cases of measuring states by using the condition number at different time steps. The observability matrix and its corresponding condition number are computed for different Δt.

5.5.1 Condition Number of Matrix

The condition number of a matrix, as explained in [27], is some measure of how much precision is lost when solving a system with the inverse of that matrix. When the condition number is 1, that means the system can be solved without loss of precision. When the condition number of a matrix is very large, this situation tends to go to when a matrix is not invertible and there is a great loss in precision.

Our studies show that the condition number of the matrix is affected by changes in $\Delta t, l, v_f, f_{in}$, and p_{max}. However, the change in condition number caused by changing Δt greatly outweighs the change caused by other variables. Therefore, the change in Δt is presented next.

5.5.2 Stability for Measuring Three Sections

When sensing three sections out of four, the system is observable for three different cases. Fig 5.2 shows the condition number as a function of time intervals for the three different cases.

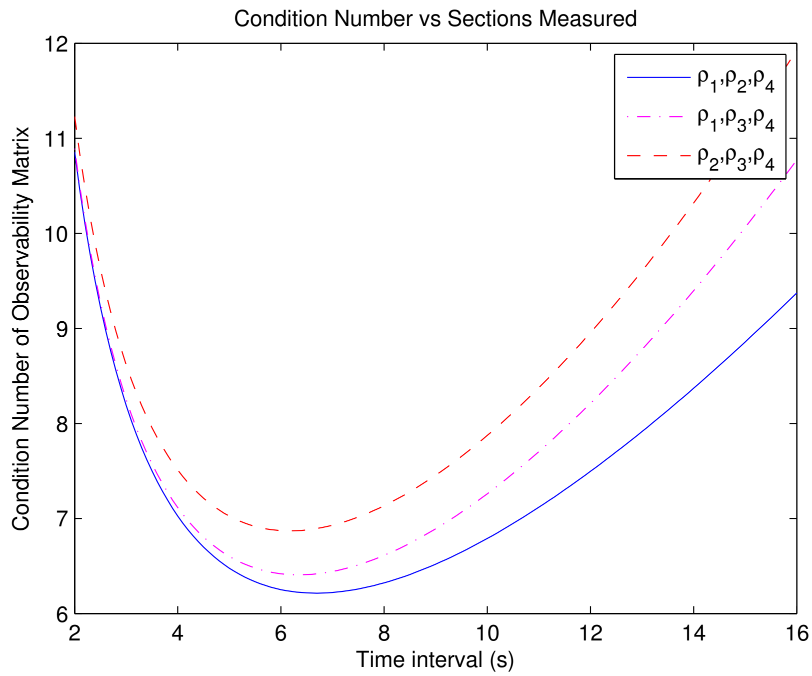

Figure 5.2: Condition Number vs. Sections Sensed: 3 Sections

From the figure, we can conclude that for the time intervals shown, measuring sections 1, 2, and 4 always resulted in the best condition numbers. The lowest condition number for this situation happens when the time interval is around 6.5 seconds. The second best situation is when measuring sections 1, 3 , and 4. The worst of the three cases is when measuring 2, 3, and 4. If 10 is taken to be an acceptable condition number, then all the three cases of measuring can be used.

5.5.3 Stability for Measuring Two Sections

When sensing two sections out of four, the system is observable for three different cases. Fig 5.3 shows the condition number as a function of time intervals for the three different cases.

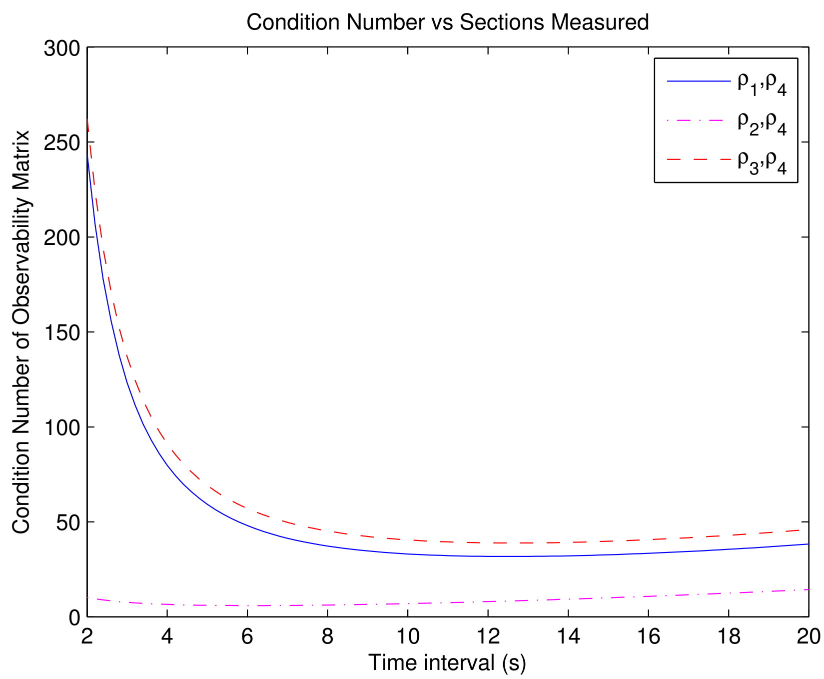

Figure 5.3: Condition Number vs. Sections Sensed: 2 Sections

From the figure, we can conclude that for the time intervals shown, measuring sections 2 and 4 always resulted in the best condition numbers. The other two cases (measuring sections 1 and 4, and measuring sections 3 and 4) show significantly higher condition numbers. If 10 is taken to be an acceptable condition number, then only the first case of measuring should be used.

Fig 5.4 shows a more detailed graph of the case when measuring sections 2 and 4.

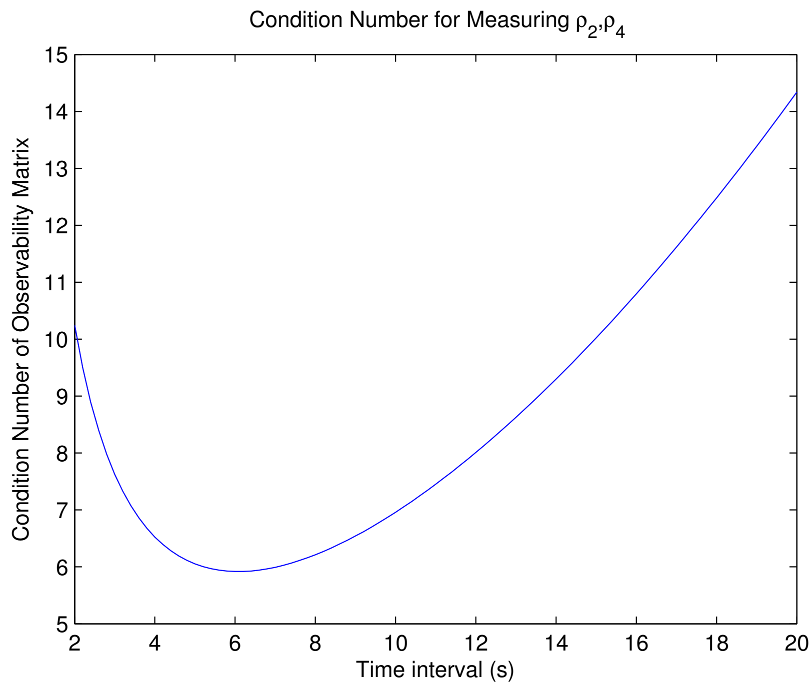

Figure 5.4: Condition Number vs. Sections Sensed: 2 Sections Detailed

We can conclude that the lowest condition number for this situation happens when the time interval is around 6 seconds.

5.5.4 Stability for Measuring One Section

Even with only one section being measured, a situation where the system is observable is obtained. Fig 5.5 shows the condition number as a function of time intervals.

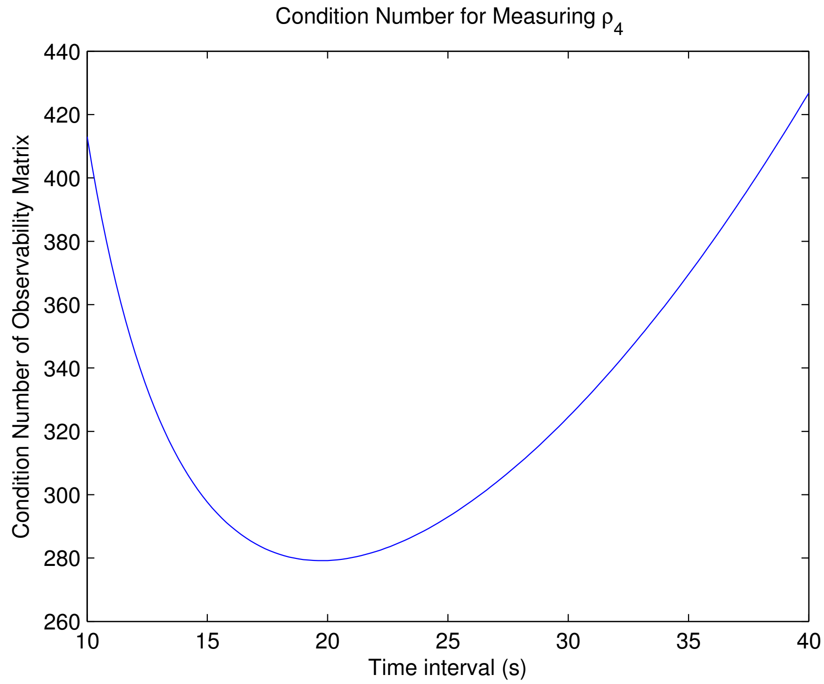

Figure 5.5: Condition Number vs. Sections Sensed: 1 Section

The lowest condition number for this situation happens when the time interval is around 20 seconds. If 10 is taken to be an acceptable condition number, then this measurement case should not be used.

5.6 Investigation of Observability Index

The observability matrix (5.2) informs whether a system is observable or not. It uses n number of discrete steps, where n is the number of states in the system. However, when the observability index is less than n, the states in the system can be obtained with less than n discrete steps, and that number is denoted by v herein.

Next, investigations for different cases are presented.

5.6.1 Observability Index for 3 Sections Case

In this section, the effect of different number of steps for measuring only three out of four sections is presented. There are three situations for which measuring only three out of four sections results in an observable system. For those cases, listed below in Tables 5.5-5.7, different number of discrete steps are used. The time step Δt that gave the lowest condition number was presented in the table.

The three situations for C have similar total times and condition numbers for finding all states of the system for different numbers of discrete steps. Though less discrete steps than n steps can be used for obtaining all states in the system, it takes more total time than using n steps. Increasing the discrete steps beyond n lowers the total time by few seconds.

# of Steps	Time Step (Δt)	Lowest Condition Number	Total Time (# * Δt)
$v = 2$	18.6	6.6309	37.2
3	10	6.2272	30
$n = 4$	6.6	6.2147	26.4
5	5	6.2546	25
6	4	6.3003	24

Table 5.5: Case: C=[1 0 0 0; 0 1 0 0; 0 0 0 1]

# of Steps	Time Step (Δt)	Lowest Condition Number	Total Time (# * Δt)
$v = 2$	20	6.3721	40
3	9.8	6.2915	29.4
$n = 4$	6.4	6.4073	25.6
5	4.6	6.5118	23
6	3.6	6.5950	21.6

Table 5.6: Case: C=[1 0 0 0; 0 0 1 0; 0 0 0 1]

# of Steps	Time Step (Δt)	Lowest Condition Number	Total Time (# * Δt)
$v = 2$	20.2	6.6447	40.4
3	9.6	6.7051	28.8
$n = 4$	6.2	6.8691	24.8
5	4.6	6.9982	23
6	3.6	7.0942	21.6

Table 5.7: Case: C=[0 1 0 0; 0 0 1 0; 0 0 0 1]

5.6.2 Observability Index for 2 Sections Case

In this section the effect of different number of steps for measuring two sections is presented. There are three situations for which measuring only two out of four sections results in an observable system. The results are presented below in Tables 5.8-5.10.

The case on Table 5.9 is clearly better than the other two cases. The total time to obtain all states in the system and the condition numbers are considerably lower. If only two sections out of four can be measured, these are the two sections to measure. In this case, increasing the discrete steps beyond n lowers the total time by few seconds.

# of Steps	Time Step (Δt)	Lowest Condition Number	Total Time (# * Δt)
$v = 3$	19.8	37.9741	59.4
$n = 4$	12.6	31.7690	50.4
5	9	30.4494	45
6	6.8	30.1839	40.8

Table 5.8: Case: C=[1 0 0 0; 0 0 0 1]

# of Steps	Time Step (Δt)	Lowest Condition Number	Total Time (# * Δt)
$v = 2$	19.8	5.8285	39.6
3	9.4	5.8065	28.2
$n = 4$	6	5.9186	24
5	4.4	6.0118	22
6	3.6	6.0830	21.6

Table 5.9: Case: C=[0 1 0 0; 0 0 0 1]

# of Steps	Time Step (Δt)	Lowest Condition Number	Total Time (# * Δt)
$v = 3$	20.4	44.8649	61.2
$n = 4$	12.6	38.8827	50.4
5	9	37.9236	45
6	6.8	37.9437	40.8

Table 5.10: Case: C=[0 0 1 0; 0 0 0 1]

5.6.3 Observability Index for 1 Section Case

In this section the effect of different number of steps for measuring only one section is presented. There is only one situation for which measuring only one section results in an observable system.

In this case, using more than n discrete steps decreases the total time, and condition number. However, this condition number is still too large.

# of Steps	Time Step (Δt)	Lowest Condition Number	Total Time (# * Δt)
$v = n = 4$	19.8	279.1717	79.2
5	14	212.5518	70
6	10.4	194.7532	62.4

Table 5.11: Case: C=[0 0 0 1]

CHAPTER 6

Observability of Spacings in Four Sections

6.1 The Traffic Equations

From conservation of matter, the length L, of a road segment with N number of cars, each with a spacing of $s(n,t)$ is

$$L = \int_1^N s(n,t)dn$$

where $s(n,t)$ is a function of spacing for vehicle n at time t.

The rate of change of the length of the road segment is

$$\frac{d}{dt}L = v(0,t) - v(N,t)$$

Here $v(n,t)$ is the velocity of the vehicle n at time t.

Now using both equations together we have

$$\frac{d}{dt}\int_1^N s(n,t)dn = v(0,t) - v(N,t)$$

6.2 The Setup of the Problem

We will assume we have a line of vehicles. The line of vehicles is then discretized and divided into four different sections each with constant spacing. Section one is from vehicle 1 to vehicle $N/4$ and has a constant spacing $s_1(n,t)$, section two from vehicle $N/4+1$ to vehicle $N/2$ has $s_2(n,t)$, section three from vehicle $N/2+1$ to vehicle $3N/4$ has $s_3(n,t)$, and section four from vehicle $3N/4+1$ to vehicle $N/4$ has $s_4(n,t)$.

We will assume that we know the velocity of the vehicle in front of vehicle 1, and call it v_{-}. The velocity of a vehicle in section 1 will be $V(s_1,t)$. Velocities in front of each section and velocities of each of the four sections will be labelled similarly.

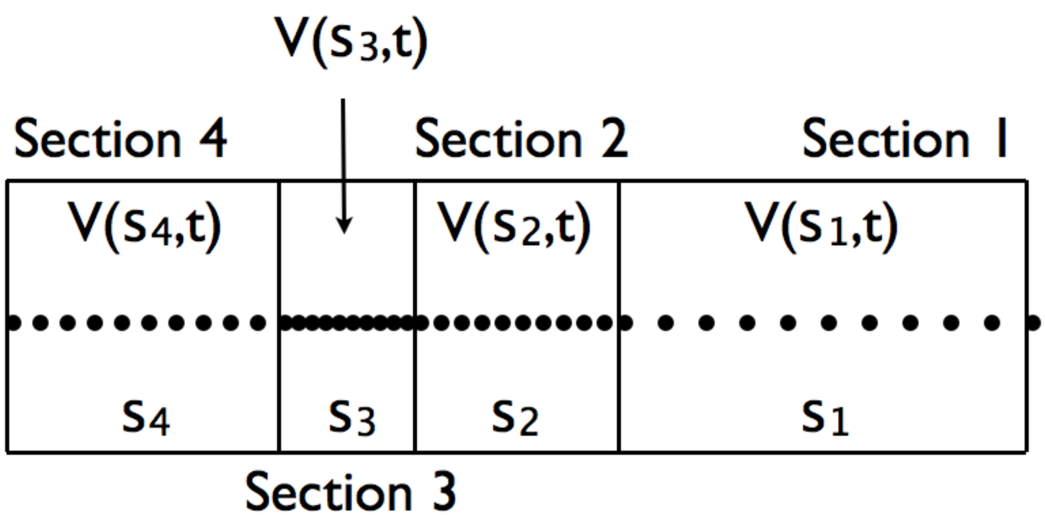

Figure 6.1: Spacing and Velocity of 4 Sections

Since the assumption is that the spacing of each section is constant, then for

	Velocity in Front	Velocity of
Section 1	v_-	$V(s_1, t)$
Section 2	$V(s_1, t)$	$V(s_2, t)$
Section 3	$V(s_2, t)$	$V(s_3, t)$
Section 4	$V(s_3, t)$	$V(s_4, t)$

Table 6.1: Velocity in Front of 4 Sections & Each of the 4 Sections

section 4,

$$\frac{d}{dt}\int_{3N/4}^{N} s(n,t)dn = \frac{d}{dt}s(t)\int_{3N/4}^{N} dn = \frac{d}{dt}s(t)N/4 = v(s_3,t) - v(s_4,t)$$

and

$$\frac{d}{dt}s(t) = \frac{4}{N}v(s_3,t) - \frac{4}{N}v(s_4,t)$$

6.3 The State Space

We are interested in the four spacings of the four sections. The four equations that describe the dynamics are:

$$
\begin{aligned}
\dot{s}_1 &= f_1\left(s_1, s_2, s_3, s_4\right) &&= \tfrac{4}{N}(v_- - V(s_1,t)) \\
\dot{s}_2 &= f_2\left(s_1, s_2, s_3, s_4\right) &&= \tfrac{4}{N}(V(s_1,t) - V(s_2,t)) \\
\dot{s}_3 &= f_3\left(s_1, s_2, s_3, s_4\right) &&= \tfrac{4}{N}(V(s_2,t) - V(s_3,t)) \\
\dot{s}_4 &= f_4\left(s_1, s_2, s_3, s_4\right) &&= \tfrac{4}{N}(V(s_3,t) - V(s_4,t))
\end{aligned}
$$

From an earlier chapter, $V(s,t) = w\rho_m s - w$ where $w = \dfrac{v_m s_m}{s_c - s_m}$. Using this, the equations are

$$
\begin{aligned}
\dot{s}_1 = &\quad \tfrac{4}{N}\left(v_- - (w\rho_{m_1}s_1 - w)\right) &&= -\tfrac{4}{N}w\rho_{m_1}s_1 + (\tfrac{4}{N}w + \tfrac{4}{N}v_-) \\
\dot{s}_2 = &\quad \tfrac{4}{N}\left((w\rho_{m_1}s_1 - w) - (w\rho_{m_2}s_2 - w)\right) &&= \tfrac{4}{N}w\rho_{m_1}s_1 - \tfrac{4}{N}w\rho_{m_2}s_2 \\
\dot{s}_3 = &\quad \tfrac{4}{N}\left((w\rho_{m_2}s_2 - w) - (w\rho_{m_3}s_3 - w)\right) &&= \tfrac{4}{N}w\rho_{m_2}s_2 - \tfrac{4}{N}w\rho_{m_3}s_3 \\
\dot{s}_4 = &\quad \tfrac{4}{N}\left((w\rho_{m_3}s_3 - w) - (w\rho_{m_4}s_4 - w)\right) &&= \tfrac{4}{N}w\rho_{m_3}s_3 - \tfrac{4}{N}w\rho_{m_4}s_4
\end{aligned}
$$

In matrix form,

$$
\begin{bmatrix} \dot{s}_1 \\ \dot{s}_2 \\ \dot{s}_3 \\ \dot{s}_4 \end{bmatrix}
=
\begin{bmatrix}
-\dfrac{4}{N}w\rho_{m_1} & 0 & 0 & 0 \\
\dfrac{4}{N}w\rho_{m_1} & -\dfrac{4}{N}w\rho_{m_2} & 0 & 0 \\
0 & \dfrac{4}{N}w\rho_{m_2} & -\dfrac{4}{N}w\rho_{m_3} & 0 \\
0 & 0 & \dfrac{4}{N}w\rho_{m_3} & -\dfrac{4}{N}w\rho_{m_4}
\end{bmatrix}
\begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix}
+
\begin{bmatrix} (\tfrac{4}{N}w + \tfrac{4}{N}v_-) \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

This system can be simplified further so that it can be linear in z_i's.

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} -\frac{4}{N}w\rho_{m_1} & 0 & 0 & 0 \\ \frac{4}{N}w\rho_{m_1} & -\frac{4}{N}w\rho_{m_2} & 0 & 0 \\ 0 & \frac{4}{N}w\rho_{m_2} & -\frac{4}{N}w\rho_{m_3} & 0 \\ 0 & 0 & \frac{4}{N}w\rho_{m_3} & -\frac{4}{N}w\rho_{m_4} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}$$

In vector notation:

$$\dot{\overrightarrow{z}} = F\overrightarrow{z} \tag{6.1}$$

where

$$F = \begin{bmatrix} -\frac{4}{N}w\rho_{m_1} & 0 & 0 & 0 \\ \frac{4}{N}w\rho_{m_1} & -\frac{4}{N}w\rho_{m_2} & 0 & 0 \\ 0 & \frac{4}{N}w\rho_{m_2} & -\frac{4}{N}w\rho_{m_3} & 0 \\ 0 & 0 & \frac{4}{N}w\rho_{m_3} & -\frac{4}{N}w\rho_{m_4} \end{bmatrix}$$

6.3.1 Equilibrium Point and Linearization

The system for lagrangian coordinates is linear, as opposed to the system derived in the previous chapter for eulerian coordinates. The equilibrium point for linear systems is the zero vector since F is invertible. Since this system is already linear, there is no need to linearize about an equilibrium point.

6.3.2 Discretizing

We will discretize the continuous time equations. The four equations that describe the dynamics become:

$$\begin{bmatrix} \frac{z_1(k+1)-z_1(k)}{\Delta t} \\ \frac{z_2(k+1)-z_2(k)}{\Delta t} \\ \frac{z_3(k+1)-z_3(k)}{\Delta t} \\ \frac{z_4(k+1)-z_4(k)}{\Delta t} \end{bmatrix} = \begin{bmatrix} -\frac{4}{N}w\rho_{m_1} & 0 & 0 & 0 \\ \frac{4}{N}w\rho_{m_1} & -\frac{4}{N}w\rho_{m_2} & 0 & 0 \\ 0 & \frac{4}{N}w\rho_{m_2} & -\frac{4}{N}w\rho_{m_3} & 0 \\ 0 & 0 & \frac{4}{N}w\rho_{m_3} & -\frac{4}{N}w\rho_{m_4} \end{bmatrix} \begin{bmatrix} z_1(k) \\ z_2(k) \\ z_3(k) \\ z_4(k) \end{bmatrix}$$

which is

$$\begin{bmatrix} z_1(k+1) \\ z_2(k+1) \\ z_3(k+1) \\ z_4(k+1) \end{bmatrix} = \begin{bmatrix} -\frac{4}{N}w\rho_{m_1} & 0 & 0 & 0 \\ \frac{4}{N}w\rho_{m_1} & -\frac{4}{N}w\rho_{m_2} & 0 & 0 \\ 0 & \frac{4}{N}w\rho_{m_2} & -\frac{4}{N}w\rho_{m_3} & 0 \\ 0 & 0 & \frac{4}{N}w\rho_{m_3} & -\frac{4}{N}w\rho_{m_4} \end{bmatrix} \begin{bmatrix} z_1(k) \\ z_2(k) \\ z_3(k) \\ z_4(k) \end{bmatrix} \Delta t + \begin{bmatrix} z_1(k) \\ z_2(k) \\ z_3(k) \\ z_4(k) \end{bmatrix}$$

or

$$\begin{bmatrix} z_1(k+1) \\ z_2(k+1) \\ z_3(k+1) \\ z_4(k+1) \end{bmatrix} =$$

$$\begin{bmatrix} -\frac{4}{N}w\rho_{m_1}\Delta t + 1 & 0 & 0 & 0 \\ \frac{4}{N}w\rho_{m_1}\Delta t & -\frac{4}{N}w\rho_{m_2}\Delta t + 1 & 0 & 0 \\ 0 & \frac{4}{N}w\rho_{m_2}\Delta t & -\frac{4}{N}w\rho_{m_3}\Delta t + 1 & 0 \\ 0 & 0 & \frac{4}{N}w\rho_{m_3}\Delta t & -\frac{4}{N}w\rho_{m_4}\Delta t + 1 \end{bmatrix} \begin{bmatrix} z_1(k) \\ z_2(k) \\ z_3(k) \\ z_4(k) \end{bmatrix}$$

Let A denote the matrix on the right hand side of the above system. Thus

$$\overrightarrow{z(k+1)} = A\overrightarrow{z(k)} \tag{6.2}$$

6.4 Observability of the State Space

Suppose for our system, equation (6.2), we can obtain measurements in the following form,

$$\overrightarrow{y(k)} = C\overrightarrow{z(k)}$$

where $\overrightarrow{y(k)} \in \Re^p$, $C \in \Re^{p \times n}$, and $\overrightarrow{z(k)} \in \Re^4$. The system

$$\begin{aligned} \overrightarrow{z(k+1)} &= A\overrightarrow{z(k)} \\ \overrightarrow{y(k)} &= C\overrightarrow{z(k)} \end{aligned}$$

is observable if the observability matrix

$$\begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} \tag{6.3}$$

has rank 4, because $\overrightarrow{z(k)} \in \Re^4$.

6.4.1 Sensing Spacing in All Sections

If all the four states $(z_1(k), z_2(k), z_3(k), z_4(k))$ are measured directly, this scenario is represented by the equation

$$\overrightarrow{y(k)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_1(k) \\ z_2(k) \\ z_3(k) \\ z_4(k) \end{bmatrix}$$

Here

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

Here we only need to check the rank of

$$\begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = \begin{bmatrix} I \\ A \\ A^2 \\ A^3 \end{bmatrix}$$

6.4.2 Numerical Example

The following values of the corresponding parameters are used:

- $\rho_{m_1} = \rho_{m_2} = \rho_{m_3} = \rho_{m_4} = 0.14$ vehicles/m,
- $s_c = 50$ m/vehicle,
- $v_f = 30$ m/s,
- $N = 40$ vehicles,
- $\Delta t = 15$ s, the time step between two readings of sensors.

Then,

$$A = \begin{bmatrix} -0.07 & 0 & 0 & 0 \\ 0.07 & -0.07 & 0 & 0 \\ 0 & 0.07 & -0.07 & 0 \\ 0 & 0 & 0.07 & -0.07 \end{bmatrix}$$

To determine if the system is observable, we need to check the rank of

$$\begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = \begin{bmatrix} I \\ A \\ A^2 \\ A^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -0.0700 & 0 & 0 & 0 \\ 0.0700 & -0.0700 & 0 & 0 \\ 0 & 0.0700 & -0.0700 & 0 \\ 0 & 0 & 0.0700 & -0.0700 \\ 0.0049 & 0 & 0 & 0 \\ -0.0098 & 0.0049 & 0 & 0 \\ 0.0049 & -0.0098 & 0.0049 & 0 \\ 0 & 0.0049 & -0.0098 & 0.0049 \\ -0.0003 & 0 & 0 & 0 \\ 0.0010 & -0.0003 & 0 & 0 \\ -0.0010 & 0.0010 & -0.0003 & 0 \\ 0.0003 & -0.0010 & 0.0010 & -0.0003 \end{bmatrix}$$

The system with these parameters is observable since the rank is 4, which is obvious since $C = I$.

6.4.3 Sensing Spacing in Three Sections

We will investigate the scenario when three of the four states $(z_1(k), z_2(k), z_3(k), z_4(k))$ are observed. Different ways to sense three sections are represented with different instances of the matrix C.

After checking the rank of the observability matrix (6.3), for these different cases, we obtain Table 6.2. The system is observable when sensing three different sections, as long as section 4 is included.

Sections Sensed	Rank
s_1, s_2, s_3	3
s_1, s_2, s_4	4
s_1, s_3, s_4	4
s_2, s_3, s_4	4

Table 6.2: Measuring Spacing in 3 Sections

6.4.4 Sensing Spacing in Two Sections

The scenario when two of the four states $(z_1(k), z_2(k), z_3(k), z_4(k))$ are measured is analyzed.

We need to check the rank of (6.3), for different C matrices representing which of the two sections are being sensed. Out of the scenarios when the system is observable, we can investigate the condition numbers of the observability matrix.

Sections Sensed	Rank of Matrix
s_1, s_2	2
s_1, s_3	3
s_1, s_4	4
s_2, s_3	3
s_2, s_4	4
s_3, s_4	4

Table 6.3: Measuring Spacing in 2 Sections

After checking the rank of the observability matrix for these different cases, we obtain Table 6.3. We find that we can obtain all four states of the system, while only measuring 2 states for 3 different cases. Again, the system is observable when sensing two different sections, as long as section 4 is included.

6.4.5 Sensing Spacing in Only One Section

The scenario when only one of the four states $(z_1(k), z_2(k), z_3(k), z_4(k))$ is measured is now analyzed. This is analyzed in the same way as before, by changing the

C matrix to match the measurement situation, and checking the rank of (6.3). The rank of the observability matrix for different measurements is obtained in Table 6.4.

Sections Measured	Rank of Matrix
s_1	1
s_2	2
s_3	3
s_4	4

Table 6.4: Measuring Spacing in 1 Section

We can look at the rank of the observability matrix for these different cases on Table 6.4. We find that we can obtain all four states of the system only if we measure section 4. Once again, the system is observable as long as section 4 is included.

6.5 Stability Investigations

From the previous sections we found that, in some cases, we can obtain all four states of the system, while measuring less than all states of the system. The system is observable as long as section 4 is included. We will investigate the different cases of measuring states by using the condition number for different time steps Δt. Our studies show that the condition number of the matrix is affected by changes in

$\Delta t, s_c, v_f, N$, and p_{max}. However, the change in condition number caused by changing Δt greatly outweighs the change caused by other variables. Therefore, the change in Δt is presented exclusively. The observability matrix and its corresponding condition number are computed for different Δt.

6.5.1 Stability for Measuring Three Sections

When sensing three sections out of four, the system is observable for three different cases. Fig 6.2 shows the condition number as a function of time intervals for the three different cases.

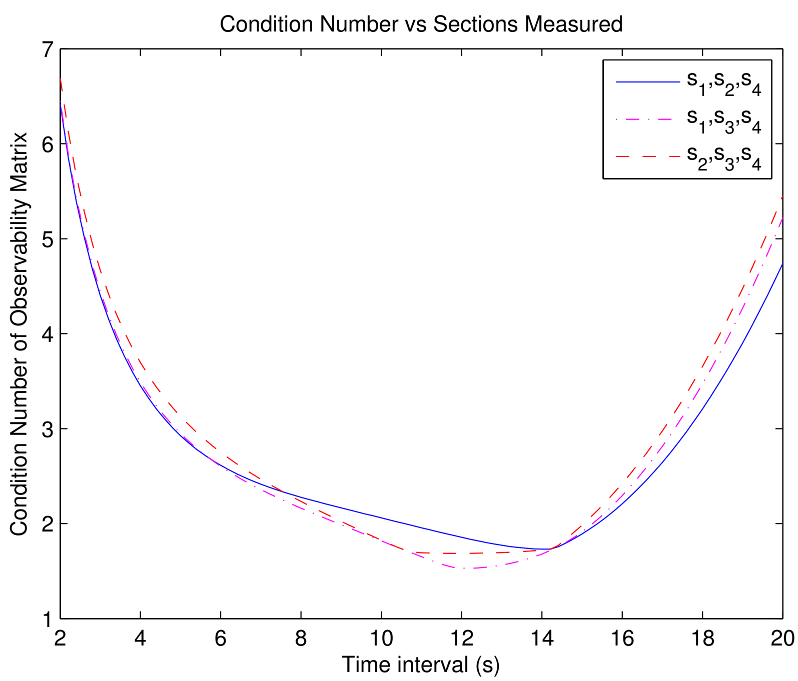

Figure 6.2: Condition Number vs. Sections Sensed: 3 Sections

From the figure, we can conclude that for the time steps shown, the three different cases are very similar. If 10 is taken to be an acceptable condition number, then all the three cases of measuring can be used for the time steps shown. The lowest condition number for these situations happen when the time step is around 12 seconds.

6.5.2 Stability for Measuring Two Sections

When sensing two sections out of four, the system is observable for three different cases. Fig 6.3 shows the condition number as a function of time intervals for the three different cases.

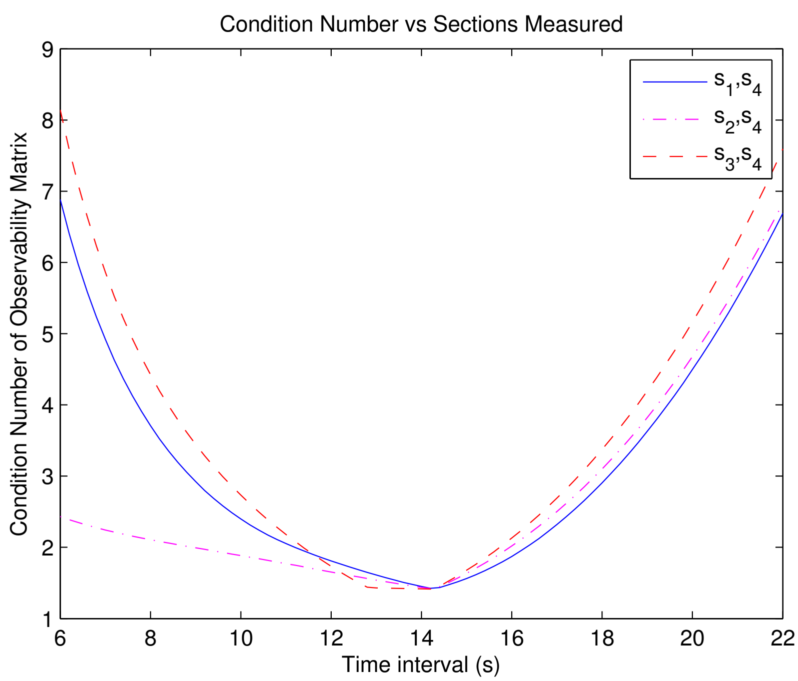

Figure 6.3: Condition Number vs. Sections Sensed: 2 Sections

From the figure, we can conclude that measuring sections 2 and 4 result in the best condition numbers for time steps less than 12 seconds. For time steps higher than 14 seconds, the best situation is to measure sections 1 and 4. If 10 is taken to be an acceptable condition number, then all cases of measuring can be used.

6.5.3 Stability for Measuring One Section

Even with only one section being measured, a situation where the system is observable is obtained. Fig 6.4 shows the condition number as a function of time steps.

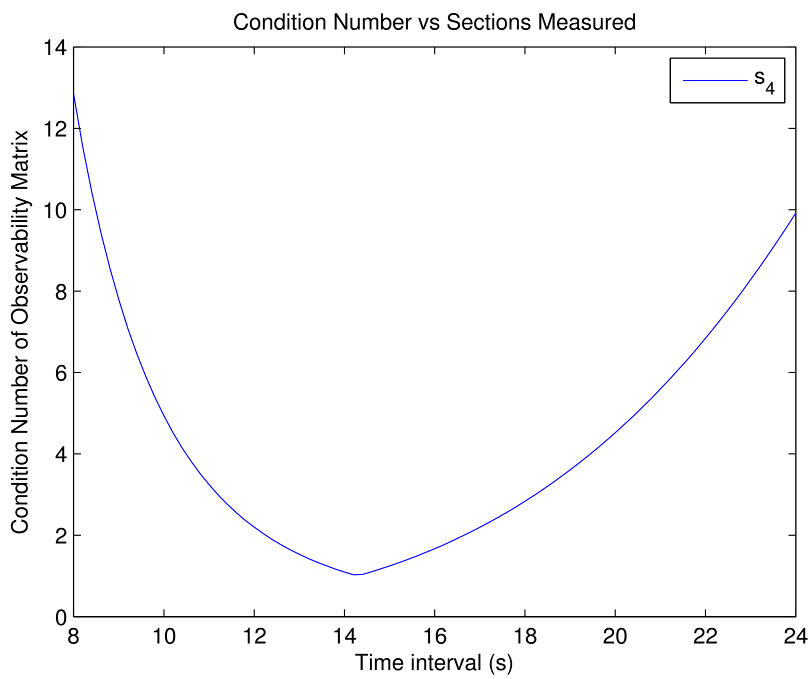

Figure 6.4: Condition Number vs. Sections Sensed: 1 Section

The lowest condition number for this situation happens when the time step is around 14.5 seconds. If 10 is taken to be an acceptable condition number, then this measurement case can be used for time intervals of around 10 through 24 seconds.

6.6 Investigation of Observability Index

Herein, we investigate the observability index similarly to what has been done in Section 5.6.

6.6.1 Observability Index for 3 Sections Case

In this section, the effect of different number of steps for measuring only three out of four sections is presented. There are three situations for which measuring only three out of four sections results in an observable system. For those cases, listed below in Tables 6.5-6.7, different number of discrete steps are used. The time step Δt that gave the lowest condition number was presented in the table.

The three situations for C have similar total times and condition numbers for finding all states of the system for different numbers of discrete steps. For all three cases, using three discrete time steps results in the lowest condition number and lowest total time to obtain all states in the system.

# of Steps	Time Step (Δt)	Condition Number	Total Time (# * Δt)
$v = 2$	26.8	6.6310	53.6
3	14.2	1.4212	42.6
$n = 4$	14	1.7310	56
5	14.2	1.7312	71
6	14.2	1.7312	85.2

Table 6.5: Case: C=[1 0 0 0; 0 1 0 0; 0 0 0 1]

# of Steps	Time Step (Δt)	Condition Number	Total Time (# * Δt)
$v = 2$	28.8	6.3720	57.6
3	13.4	1.4126	40.2
$n = 4$	12.2	1.5292	48.8
5	12.2	1.5686	61
6	12.4	1.5726	74.4

Table 6.6: Case: C=[1 0 0 0; 0 0 1 0; 0 0 0 1]

# of Steps	Time Step (Δt)	Condition Number	Total Time (# * Δt)
$v = 2$	29	6.6447	58
3	9.6	1.5891	34.8
$n = 4$	11.8	1.6865	47.2
5	12.8	1.7029	64
6	13	1.7035	78

Table 6.7: Case: C=[0 1 0 0; 0 0 1 0; 0 0 0 1]

6.6.2 Observability Index for 2 Sections Case

In this section the effect of different number of steps for measuring two sections is presented. There are three situations for which measuring only two out of four sections results in an observable system. The obtained results are presented below in Tables 6.8-6.10.

All three cases are very similar in the condition numbers and time steps. Since the time steps for the lowest condition number are very similar, it makes sense to use the least number of discrete steps, which is 3. This gives the least total time.

# of Steps	Time Step (Δt)	Condition Number	Total Time (# * Δt)
$v = 3$	14.2	1.0178	42.6
$n = 4$	14.2	1.4242	56.8
5	14.2	1.4244	71
6	14.2	1.4244	85.2

Table 6.8: Case: C=[1 0 0 0; 0 0 0 1]

# of Steps	Time Step (Δt)	Condition Number	Total Time (# * Δt)
$v = 2$	28.6	5.8284	57.2
3	14.2	1.4161	42.6
$n = 4$	14.2	1.4213	56.8
5	14.2	1.4214	71
6	14.2	1.4214	85.2

Table 6.9: Case: C=[0 1 0 0; 0 0 0 1]

# of Steps	Time Step (Δt)	Condition Number	Total Time (# * Δt)
$v = 3$	14.2	1.4306	42.6
$n = 4$	14.2	1.4149	56.8
5	14.2	1.4149	71
6	14.2	1.4149	85.2

Table 6.10: Case: C=[0 0 1 0; 0 0 0 1]

6.6.3 Observability Index for 1 Section Case

In this section the effect of different number of steps for measuring only one section is presented. There is only one situation for which measuring only one section results in an observable system.

In this case, n discrete steps should be used to obtain the least total time.

# of Steps	Time Step (Δt)	Condition Number	Total Time (# * Δt)
$v = n = 4$	14.2	1.2619	56.8
5	14.2	1.0275	71
6	14.2	1.0275	85.2

Table 6.11: Case: C=[0 0 0 1]

CHAPTER 7

Eulerian Simulations: Obtaining All States

For real situations, we are interested in cases where sensors can fail, or there are constraints on sensors because of cost considerations. We will consider cases where less than all sections available are being measured. In these simulations, ρ_{max}, v_f, and l will be constant for the 4 sections. We will use the following values.

- $\rho_{max} = 0.14$ vehicles/m
- $v_f = 30$ m/s
- $l_1 = l_2 = l_3 = l_4 = 500$ m
- $f_{in} = 0.3$ vehicles/s, assumed to be constant for different time steps
- $\Delta t = 6.5$s (when measuring 1 section) or 20s (when measuring 3 and 2 sections), the time interval between two readings of sensors

By fixing f_{in}, we also fix $\rho_{eq} = 0.1292$.

For the initial states of the system, we will use 60 cars in section 1, 61 cars in section 2, 62 cars in section 3, and 64 cars in section 4. In terms of density, $\rho_1 = 60/500 = 0.1200, \rho_2 = 0.1220, \rho_3 = 0.1240, \rho_4 = 0.1260$.

7.1 Measuring Density in 3 Sections Close to the Equilibrium Point

From Chapter 3, there are 3 situations where measuring only 3 out of 4 sections resulted in an observable system. Sensing sections 1, 2, and 4 resulted in the best precision for obtaining all the states. We will see how all four states of the system are obtained by only measuring three states of the system.

In cases where measuring 3 out of 4 sections is not observable, a pseudo inverse to a rank-deficient matrix can be applied.

7.1.1 Sections 1, 2, and 4 Measured

We will simulate the case when sections 1, 2, and 4 are measured. We will use 3 discrete time steps after initialization to obtain all 4 states. The system we have to check is

$$\begin{bmatrix} y_1(0) \\ y_2(0) \\ y_4(0) \\ y_1(1) \\ y_2(1) \\ y_4(1) \\ y_1(2) \\ y_2(2) \\ y_4(2) \\ y_1(3) \\ y_2(3) \\ y_4(3) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} \begin{bmatrix} z_1(0) \\ z_2(0) \\ z_3(0) \\ z_4(0) \end{bmatrix}$$

where

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We will take measurements of sections 1, 2, and 4 only during the 3 discrete time steps. After this, we will solve for $\overrightarrow{\rho(0)}$, and see how close the obtained values are to the actual values.

The values obtained for the densities in the four sections are within 1% of the

	Calculated ρ	Actual ρ	Relative Error
ρ_1	0.1205	0.1200	0.4135 E-03
ρ_2	0.1219	0.1220	0.9775 E-04
ρ_3	0.1240	0.1240	0.9662 E-05
ρ_4	0.1260	0.1260	-0.2212 E-04

Table 7.1: Measuring Density in 3 Sections: Full Rank

actual values.

7.1.2 Sections 1, 2, and 3 Measured

We will simulate the case when sections 1, 2, and 3 are measured. This was a case that was not observable. We will use least squares to find a solution. We will use 3 discrete time steps after initialization to obtain all 4 states. The system we have to check is

$$\begin{bmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \\ y_1(1) \\ y_2(1) \\ y_3(1) \\ y_1(2) \\ y_2(2) \\ y_3(2) \\ y_1(3) \\ y_2(3) \\ y_3(3) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} \begin{bmatrix} z_1(0) \\ z_2(0) \\ z_3(0) \\ z_4(0) \end{bmatrix}$$

where

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

We will take measurements of sections 1, 2, and 3 only during the 3 discrete time steps. After this, we will solve for $\overrightarrow{\rho(0)}$, and see how close the obtained values are to the actual values.

The values obtained for the densities in sections 1, 2, and 3 are within 1% of the

	Calculated ρ	Actual ρ	Relative Error
ρ_1	0.1205	0.1200	0.4162 E-02
ρ_2	0.1219	0.1220	-0.8535 E-03
ρ_3	0.1239	0.1240	-0.1157 E-02
ρ_4	0.1292	0.1260	0.2509 E-01

Table 7.2: Measuring Density in 3 Sections: Rank Deficient

actual values.

7.2 Measuring Density in Two Sections Close to the Equilibrium Point

From Chapter 3, there are 3 situations where measuring only 2 out of 4 sections resulted in an observable system. The best situation is when sensing sections 2 and 4. We will see how all four states of the system are obtained by only measuring two states of the system.

In cases where measuring 2 out of 4 sections is not observable, a pseudo inverse to a rank-deficient matrix can be applied.

7.2.1 Sections 2 and 4 Measured

We will simulate the case when sections 2 and 4 are measured. The system we have to check is

$$\begin{bmatrix} y_2(0) \\ y_4(0) \\ y_2(1) \\ y_4(1) \\ y_2(2) \\ y_4(2) \\ y_2(3) \\ y_4(3) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} \begin{bmatrix} z_1(0) \\ z_2(0) \\ z_3(0) \\ z_4(0) \end{bmatrix}$$

where

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The values obtained for the densities in the four sections are within 1% of the actual values.

	Calculated ρ	Actual ρ	Relative Error
ρ_1	0.1210	0.1200	0.8048 E-02
ρ_2	0.1221	0.1220	0.1076 E-02
ρ_3	0.1241	0.1240	0.4201 E-03
ρ_4	0.1260	0.1260	0.1331 E-03

Table 7.3: Measuring Density in 2 Sections: Full Rank

7.2.2 Sections 1 and 3 Measured

We will simulate the case when sections 1 and 3 are measured. This was a case that was not observable. The system we have to check is

$$\begin{bmatrix} y_1(0) \\ y_3(0) \\ y_1(1) \\ y_3(1) \\ y_1(2) \\ y_3(2) \\ y_1(3) \\ y_3(3) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} \begin{bmatrix} z_1(0) \\ z_2(0) \\ z_3(0) \\ z_4(0) \end{bmatrix}$$

where

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

	Calculated ρ	Actual ρ	Relative Error
ρ_1	0.1205	0.1200	0.3906 E-02
ρ_2	0.1221	0.1220	0.9764 E-03
ρ_3	0.1240	0.1240	-0.1695 E-03
ρ_4	0.1292	0.1260	0.2509 E-01

Table 7.4: Measuring Density in 2 Sections: Rank Deficient

The values obtained for the densities in sections 1, 2, and 3 are within 1% of the actual values.

7.3 Measuring Density in One Section Close to the Equilibrium Point

From Chapter 3, there is only 1 situation where measuring only 1 out of 4 sections resulted in an observable system. Sensing only section 4 resulted in an observable system. Four time steps are required. We will see how all four states of the system are obtained by only measuring one state of the system.

In cases where measuring 1 out of 4 sections is not observable, a pseudo inverse to a rank-deficient matrix can be applied.

7.3.1 Section 4 Measured

We will simulate the case when section 4 only is measured. The system we have to check is

$$\begin{bmatrix} y_4(0) \\ y_4(1) \\ y_4(2) \\ y_4(3) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} \begin{bmatrix} z_1(0) \\ z_2(0) \\ z_3(0) \\ z_4(0) \end{bmatrix}$$

where

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$$

	Calculated ρ	Actual ρ	Relative Error
ρ_1	0.1208	0.1200	0.6573 E-2
ρ_2	0.1224	0.1220	0.3344 E-2
ρ_3	0.1241	0.1240	0.1134 E-2
ρ_4	0.1260	0.1260	0

Table 7.5: Measuring Density in 1 Section: Full Rank

The values obtained for the densities in the four sections are within 1% of the actual values.

7.3.2 Section 3 Measured

We will simulate the case when section 3 only is measured. This is a case that is not observable. The system we have to check is

$$\begin{bmatrix} y_3(0) \\ y_3(1) \\ y_3(2) \\ y_3(3) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} \begin{bmatrix} z_1(0) \\ z_2(0) \\ z_3(0) \\ z_4(0) \end{bmatrix}$$

where

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$$

The values obtained for the densities in sections 1, 2, and 3 are within 1% of the actual values.

	Calculated ρ	Actual ρ	Relative Error
ρ_1	0.1213	0.1200	0.1085 E-1
ρ_2	0.1225	0.1220	0.3768 E-2
ρ_3	0.1241	0.1240	0.5869 E-3
ρ_4	0.1292	0.1260	0.2506 E-1

Table 7.6: Measuring Density in 1 Section: Rank Deficient

CHAPTER 8

Lagrangian Simulations: Obtaining All States

For real situations, we are interested in cases where sensors can fail, or there are constraints on sensors because of cost considerations. We will consider cases where less than all sections available are being measured. In these simulations, ρ_{max}, s_c, v_f, and N will be constant for the 4 sections. We will use the following values.

- $s_c = 50$ m/vehicle
- $\rho_{max} = 0.14$ vehicles/m
- $v_f = 30$ m/s
- $N = 40$ vehicles
- $\Delta t = 15s$, the time step between two readings of sensors

For the initial states of the system, we will use the following spacings for the four sections: $s_1 = 40, s_2 = 28, s_3 = 14, s_4 = 7$.

8.1 Measuring Spacing in 3 Sections

From Chapter 6, there are 3 situations where measuring only 3 out of 4 sections resulted in an observable system. We will simulate sensing sections 1, 2, and 4 for obtaining all the states. We will see how all four states of the system are obtained by only measuring three states of the system.

8.1.1 Sections 1, 2, and 4 Measured

We will simulate the case when sections 1, 2, and 4 are measured. We will use 3 discrete time steps after initialization to obtain all 4 states. The system we have to check is

$$\begin{bmatrix} y_1(0) \\ y_2(0) \\ y_4(0) \\ y_1(1) \\ y_2(1) \\ y_4(1) \\ y_1(2) \\ y_2(2) \\ y_4(2) \\ y_1(3) \\ y_2(3) \\ y_4(3) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} \begin{bmatrix} z_1(0) \\ z_2(0) \\ z_3(0) \\ z_4(0) \end{bmatrix}$$

where

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We will take measurements of sections 1, 2, and 4 only during the 3 discrete time steps. After this, we will solve for $\overrightarrow{s(0)}$, and see how close the obtained values are to the actual values.

	Calculated s	Actual s	Relative Error
s_1	40	40	-0.0178 E-14
s_2	28	28	0.0127 E-14
s_3	14	14	-0.1015 E-14
s_4	7	7	-0.7105 E-14

Table 8.1: Measuring Spacing in 3 Sections: Full Rank

The values obtained for the densities in the four sections are within 1% of the actual values.

8.1.2 Sections 1, 2, and 3 Measured

We will simulate the case when sections 1, 2, and 3 are measured. This was a case that was not observable. We will use least squares to find a solution. We will use 3 discrete time steps after initialization to obtain all 4 states. The system we have to check is

$$\begin{bmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \\ y_1(1) \\ y_2(1) \\ y_3(1) \\ y_1(2) \\ y_2(2) \\ y_3(2) \\ y_1(3) \\ y_2(3) \\ y_3(3) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} \begin{bmatrix} z_1(0) \\ z_2(0) \\ z_3(0) \\ z_4(0) \end{bmatrix}$$

where

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

We will take measurements of sections 1, 2, and 3 only during the 3 discrete time steps. After this, we will solve for $\overrightarrow{s(0)}$, and see how close the obtained values are to the actual values.

	Calculated s	Actual s	Relative Error
s_1	0.1205	0.1200	0
s_2	0.1221	0.1220	0
s_3	0.1240	0.1240	0.1 E-15
s_4	0.1292	0.1260	6.1429

Table 8.2: Measuring Spacing in 3 Sections: Rank Deficient

The values obtained for the spacings in sections 1, 2, and 3 are within 1% of the actual values.

8.2 Measuring Spacing in Two Sections

From Chapter 6, there are 3 situations where measuring only 2 out of 4 sections resulted in an observable system. We will see how all four states of the system are obtained by only measuring sections 2 and 4.

8.2.1 Sections 2 and 4 Measured

We will simulate the case when sections 2 and 4 are measured. The system we have to check is

$$\begin{bmatrix} y_2(0) \\ y_4(0) \\ y_2(1) \\ y_4(1) \\ y_2(2) \\ y_4(2) \\ y_2(3) \\ y_4(3) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} \begin{bmatrix} z_1(0) \\ z_2(0) \\ z_3(0) \\ z_4(0) \end{bmatrix}$$

where

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The values obtained for the densities in the four sections are within 1% of the actual values.

	Calculated s	Actual s	Relative Error
s_1	40	40	-0.0355 E-14
s_2	28	28	0.0381 E-14
s_3	14	14	-0.2030 E-14
s_4	7	7	0.1015 E-14

Table 8.3: Measuring Spacing in 2 Sections: Full Rank

8.2.2 Sections 2 and 3 Measured

We will simulate the case when sections 2 and 3 are measured. This was a case that was not observable. The system we have to check is

$$\begin{bmatrix} y_1(0) \\ y_3(0) \\ y_1(1) \\ y_3(1) \\ y_1(2) \\ y_3(2) \\ y_1(3) \\ y_3(3) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} \begin{bmatrix} z_1(0) \\ z_2(0) \\ z_3(0) \\ z_4(0) \end{bmatrix}$$

where

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

	Calculated s	Actual s	Relative Error
s_1	0.1205	0.1200	0
s_2	0.1221	0.1220	0
s_3	0.1240	0.1240	0.1 E-15
s_4	0.1292	0.1260	6.1429

Table 8.4: Measuring Spacing in 2 Sections: Rank Deficient

The values obtained for the spacings in sections 1, 2, and 3 are within 1% of the actual values.

8.3 Measuring Spacing in One Section

From Chapter 6, there is only 1 situation where measuring only 1 out of 4 sections resulted in an observable system. Sensing only section 4 resulted in an observable system. Four time steps are required. We will see how all four states of the system are obtained by only measuring one state of the system.

8.3.1 Section 4 Measured

We will simulate the case when section 4 only is measured. The system we have to check is

$$\begin{bmatrix} y_4(0) \\ y_4(1) \\ y_4(2) \\ y_4(3) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} \begin{bmatrix} z_1(0) \\ z_2(0) \\ z_3(0) \\ z_4(0) \end{bmatrix}$$

where

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$$

	Calculated s	Actual s	Relative Error
s_1	40	40	-0.1776 E-15
s_2	28	28	0.2538 E-15
s_3	14	14	0
s_4	7	7	0

Table 8.5: Measuring Spacing in 1 Section: Full Rank

The values obtained for the densities in the four sections are within 1% of the actual values.

8.3.2 Section 3 Measured

We will simulate the case when section 3 only is measured. This is a case that is not observable. The system we have to check is

$$\begin{bmatrix} y_3(0) \\ y_3(1) \\ y_3(2) \\ y_3(3) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} \begin{bmatrix} z_1(0) \\ z_2(0) \\ z_3(0) \\ z_4(0) \end{bmatrix}$$

where

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$$

	Calculated s	Actual s	Relative Error
s_1	0.1205	0.1200	0
s_2	0.1221	0.1220	0
s_3	0.1240	0.1240	0.1 E-15
s_4	0.1292	0.1260	6.1429

Table 8.6: Measuring Density in 1 Section: Rank Deficient

The values obtained for the spacings in sections 1, 2, and 3 are within 1% of the actual values.

CHAPTER 9

Conclusion and Future Work

The concept of observability for linear time invariant discrete time systems was applied to study the observability of four sections of a freeway. The kinematic wave model was used for traffic modeling in Eulerian and Lagrangian coordinates. The Lagrangian framework was introduced, and the transformation from the traditional Eulerian coordinates was presented. A system with densities in four sections of a freeway was designed, and the observability of the system was studied with different situations for sensors.

When the system evolves exactly according to the models, the states of the system could be obtained from measurements from certain situations. For both, Eulerian and Lagrangian simulations, as long as the fourth section was measured, the states of the system could be obtained. Some situations took fewer time steps, and when different situations took the same number of steps, the condition number of the observability matrix was used for comparison.

The modeling used for simulations in both coordinates systems can be improved by a two level or higher level model. The current formulation of the kinematic wave model assumes that vehicles cannot pass one another. This can be generalized to take into account that vehicles do pass each other. A mixture of Eulerian and Lagrangian data can be utilized in future studies for observability.

The flow into the system was assumed to be constant during the time interval measurements that were made. This is not always true in real situations. Different flows into the system can be used to describe the system at different capacities of densities.

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