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# N-Dimensional Quasipolar Coordinates - Theory and Application

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# N-DIMENSIONAL QUASIPOLAR COORDINATES - THEORY AND APPLICATION

By

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ABSTRACT

N-Dimensional Quasipolar Coordinates - Theory and Application

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In this thesis, various generalizations to the n-dimension of the polar coordinates and spherical coordinates are introduced and compared with each other and the existent ones in the literature. The proof of the Jacobian of these coordinates is very often wrongfully claimed. Currently, prior to our proof, there are only two complete proofs of the Jacobian of these coordinates known to us. A friendlier definition of these coordinates is introduced and an original, direct, short, and elementary proof of the Jacobian of these coordinates is given. A method, which we call a perturbative method, is introduced and applied so that the approach in the general case is also valid in all special cases.

After the proof, the definitions of the n-dimensional quasiballs (hyperballs for $n \geq 4$) and the n-dimensional quasispheres (hyperspheres for $n \geq 4$) are given. The Jacobian is used to calculate the n-dimensional quasivolume of the n-dimensional quasiball and the n-dimensional quasi-surface area of the n-dimensional quasisphere directly. The formulas obtained afterwards are free of any special functions and could be introduced without any advanced mathematical knowledge. Numerical results are provided in a table followed by interpretations of these results.

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## DEDICATION

To Dr. Angel Muleshkov,his wife Mrs. Sonya Muleshkov, my parents and my sister.

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# Chapter 1. Introduction of the N-Dimensional Polar Coordinates and Their Jacobians

## 1.1 Various Definitions of Polar Coordinates and Spherical Coordinates

**Definition 1.** Polar Coordinates

For any point $P(x, y)$ in the $x - y$ plane, the ordered pair of real numbers $[r, \theta]$ is refered to as the polar coordinates of point $P(x, y)$ with respect to the arbitrary real number $\beta$, iff

$$\begin{cases} x & = r \cos\theta \\ y & = r \sin\theta \end{cases} \tag{1}$$

where $r = \sqrt{x^2 + y^2}$ and $\theta \in [\beta, \beta + 2\pi)$ or $\theta \in (\beta, \beta + 2\pi]$.

Our definition for polar coordinates is not what is usually given in textbook and could be considered "non-standard" by many people. A "standard" definition is usually given as above with different intervals for $\theta$. For example, in complex analysis, for principal values usually $\theta \in (-\pi, \pi]$. Also, $\theta \in [0, 2\pi)$ is encountered in textbooks. The "standard" polar coordinates are well-known and given in many textbooks. The history of this type of coordinate system is very interesting and could be found in [1, 2] and thus will not be discussed in details in this paper.

One would think that the definition for $r$ as above would be "standard" and widely accepted, but J. L. Coolide [2] wrote in his paper, in the very first paragraph, that there are "varieties" and some authors allow $r$ to be negative. He called this "an ambiguity." In fact, there are authors who would allow $r$ to be negative in order to explain some mathematical topics in an easier way. One of these topics is to obtain the graphs of certain curves given in polar coordinates.

For example, consider the graph of the equation $r = 2\cos 4\theta$

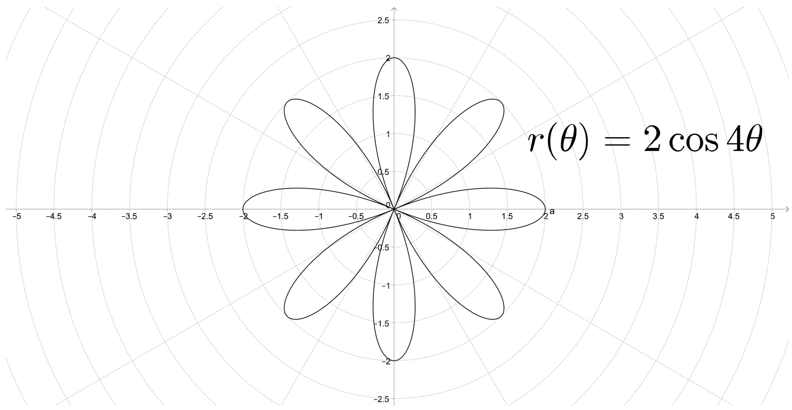


FIGURE 1.

The process of obtaining the graph above would be more difficult if one insists on using only positive values for $r$. That is why $r$ is usually allowed to be negative in order to obtain the graphs of equations in polar coordinates in an easier way. What happens is that when $r < 0$ then $\theta$ is replaced by $\theta + \pi$. Even in such textbooks, where $r$ is allowed to be negative, the authors only allow $r$ to be negative in the sections concerning graphing of polar coordinates and later on $r$ is always understood to take only positive values.

The issue of allowing $r$ to be negative is simply to introduce certain things in an easier way. Mathematically speaking, there is nothing wrong with doing that. However, many people overlook the issue of using polar coordinates at the origin, i.e. $r = 0$. Using polar coordinates at the origin is not only something undesirable, but also leads to contradictions. Mathematical theorems become impossible to be applied at the origin when the polar coordinates are used there. For instance, the theorem on changing the variables requires the Jacobian to be different than 0. However, when one has a change to polar coordinates, and the origin is in the domain where $r = 0 \implies J = 0$ which means that it is impossible to change the coordinates to polar coordinates at the origin.

One of the many problems of using polar coordinates at the origin is the issue of intersecting lines, whose equations are given in polar coordinates. If the point of intersection is at the origin, it is possible that, without making any mistake, the origin is never found to be an intersection of the two lines.

For example, consider the two lines given in polar equations and try to find the points of intersection by solving the system

$$\begin{cases} r &= 2 - 3\cos\theta \\ r &= 1 + 2\sin\theta \end{cases} \tag{2}$$

The graphs of the two equations are given in Figure 2

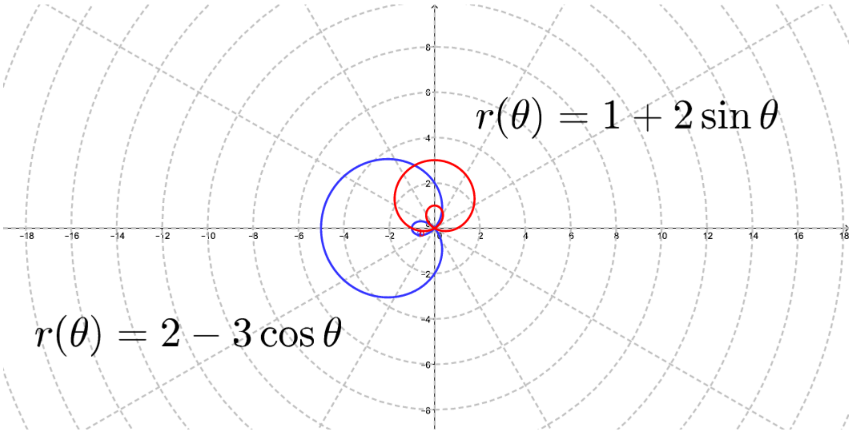


FIGURE 2.

Thus, one gets $1 + 2\sin\theta = 2 - 3\cos\theta \implies 3\cos\theta + 2\sin\theta = 1$. Thus, $\frac{3}{\sqrt{13}}\cos\theta + \frac{2}{\sqrt{13}}\sin\theta = \frac{1}{\sqrt{13}}$.

It is possible to show that there is a real number $\alpha \in (0, \frac{\pi}{2})$ such that $\cos\alpha = \frac{3}{\sqrt{13}}$. Since $\alpha \in (0, \frac{\pi}{2})$, $\sin\alpha = \sqrt{1 - \cos^2\alpha} = \sqrt{1 - \frac{9}{13}} = \frac{2}{\sqrt{13}}$.

Thus, one has $\cos\theta\cos\alpha + \sin\theta\sin\alpha = \frac{1}{\sqrt{13}} \implies \cos(\theta - \alpha) = \frac{1}{\sqrt{13}}$.

Hence, $\theta = \alpha \pm \arccos\frac{1}{\sqrt{13}} + 2k\pi = \arccos\frac{3}{\sqrt{13}} \pm \arccos\frac{1}{\sqrt{13}} + 2k\pi (k \in \mathbb{Z})$.

It follows that $\cos\theta = \cos(\arccos\frac{3}{\sqrt{13}} \pm \arccos\frac{1}{\sqrt{13}}) = \frac{3}{\sqrt{13}}\frac{1}{\sqrt{13}} \mp \sqrt{1 - (\frac{3}{\sqrt{13}})^2}\sqrt{1 - (\frac{1}{\sqrt{13}})^2} = \frac{3}{13} \mp \frac{2}{\sqrt{13}}\frac{2\sqrt{3}}{\sqrt{13}} = \frac{3 \mp 4\sqrt{3}}{13}$.

Also, $\sin\theta = \frac{2}{\sqrt{13}}\frac{1}{\sqrt{13}} \pm \frac{1}{\sqrt{13}}\frac{2\sqrt{13}}{\sqrt{13}} = \frac{2 \pm 2\sqrt{3}}{13}$. Thus, $r = 1 + \frac{4 \pm 4\sqrt{3}}{13} = \frac{17 \pm 4\sqrt{13}}{13}$.

Hence, there are two points of intersection of the two lines given in (2), $A(x_A, y_A)$ and $B(x_B, y_B)$ where

$$x_A = r_A\cos\theta_A = \frac{17 + 4\sqrt{3}}{13}\frac{3 - 4\sqrt{3}}{13} = \frac{3 - 56\sqrt{3}}{169}$$

$$y_A = r_A\sin\theta_A = \frac{17 + 4\sqrt{3}}{13}\frac{2 + 2\sqrt{3}}{13} = \frac{58 + 42\sqrt{3}}{169}$$

$$x_B = r_B \cos\theta_B = \frac{17-4\sqrt{3}}{13}\frac{3+4\sqrt{3}}{13} = \frac{3+56\sqrt{3}}{169}$$

and

$$y_B = r_B \sin\theta_B = \frac{17-4\sqrt{3}}{13}\frac{2-2\sqrt{3}}{13} = \frac{58-42\sqrt{3}}{169}$$

Then one would obtain the WRONG answer that there are only two points of intersection $A(\frac{3-56\sqrt{3}}{169}, \frac{58+42\sqrt{3}}{4})$ and $B(\frac{3+56\sqrt{3}}{169}, \frac{58-42\sqrt{3}}{169})$. In fact $O(0,0)$ is also a point of intersection, but it is not possible to find out that $O(0,0)$ is a point of intersection by algebracially solving the system in (2).

One would think that converting the two polar equations in (2) to two equations in cartesian coordinates and then plugging in the point $(0,0)$ could show that the origin is indeed a point of intersection. However, it is not possible to do that at the origin. In order to convert the equations to cartesian coordinates, one would need to multiply both sides by $r$ which is not possible to be done at the origin, when $r = 0$!

However, $O(0,0)$ is a point of intersection because $O$ is on the first line since $r = 0 \implies 2 - 3\cos\theta = 0$ which has at least one solution for $\theta$ and $O$ is on the second line since $r = 0 \implies 0 = 1 + 2\sin\theta$ also has at least one solution for $\theta$. Unfortunately, in good textbooks, the authors advised to draw the graphs of the two lines in order to see whether the origin is a point of intersection. There is, of course, no need for that at all.

Note that Figure 2 is simply an illustration of the graphs of the two equations in (2). As illustrated directly above, it is not necessary at all to have the graph to find out that $O(0,0)$ is also a point of intersection of the two lines in (2).

The above example illustrates only one of the many cases that a lot of things could go wrong at the origin and it does not matter what definitions are used at $O(0,0)$, one always reaches contradictions with polar coordinates at the origin. There is no theoretical consideration known to us so that the origin could be included without violating the theories of mathematics. Hence, polar coordinates should not be used at the origin.

If the above seemingly "standard" definition for $r$ could have varieties and misunderstandings, then there is no doubt that our definition for $\theta$ is at least questionable. Firstly, as mentioned above, in some applications, we need to have a much wider range for $\theta$. An example is the motion of runners around a track. They could run many revolutions around the track, and hence the range for $\theta$ should not be limited to the interval $[0, 2\pi)$. One could extend the range of $\theta$ so that $\theta \in [0, 2k\pi]$ for some $k \in \mathbb{R}$, e.g. $k = 2.5$ for distance 1000m.

Note that $\theta \in (-\infty, +\infty)$ is only a theoretical notion. If in some applications $\theta$ is used as a variable for time, then $\theta \in [0, +\infty?]$ is still a valid interval for time is considered starting from The Big Bang.

However, our definition of the polar coordinates does not concern with such motion, where there is no one to one correspondance between the Cartesian coordinates and the polar coordinates. This definition only deals with the cases that one requires a one to one correspondance between the cartesian coordinates system and the polar coordinates system.

Under such requirement, the definition of the polar coordinates as given in (1) is better than the "standard" definition of polar coordinates given in textbooks, where $\theta \in [0, 2\pi)$. Firstly, this definition is more general. One can easily obtain the "standard" definition by choosing $\beta = 0$. Secondly, this definition is friendlier to be used in other mathematical disciplines.

For example, for the issue of having a branch cut in Complex Analysis, consider a hypothetical branch cut illustrated by

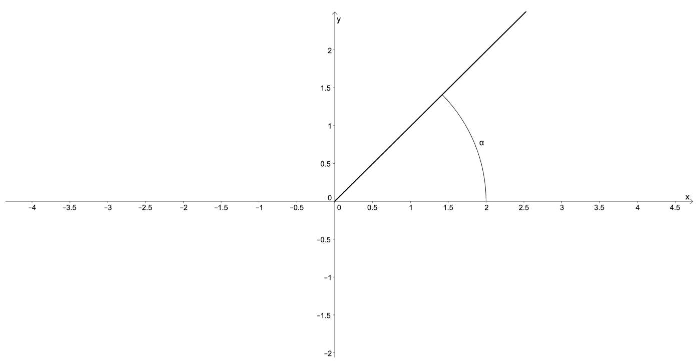


Figure 3.

Using the definition above, letting $\beta = \alpha$ then the branch is naturally defined and continous for $\theta \in (\beta, \beta + 2\pi)$.

Later on in this thesis, it is shown that this flexibility of the first polar angle allows us to facilitate our proof the Jacobian of the n-dimensional quasipolar coordinates.

For the same reasons, the following is a more flexible definition for the spherical coordinates

**Definition 2.** Spherical Coordinates

For any point $P(x, y, z)$ in the $x - y - z$ space, the ordered triples of real numbers $[\rho, \theta, \phi]$, of point $P(x, y, z)$ is refered to as the spherical coordinates of point $P(x, y, z)$ with respect to the arbitrary real number $\beta$, iff

$$\begin{cases} x &= \rho \cos\theta \sin\phi \\ y &= \rho \sin\theta \sin\phi \\ z &= \rho \cos\phi \end{cases} \tag{3}$$

where $\rho = \sqrt{x^2 + y^2 + z^2}$ ,$\theta \in [\beta, \beta + 2\pi)$ or $\theta \in (\beta, \beta + 2\pi]$, and $\phi \in [0, \pi]$.

### 1.2 Definitions of the "N-dimensional Polar Coordinates" in the Literature

In the literature, the definition of the spherical coordinates is consistent and given in one and the same way. It is given as above, but with $\theta \in [0, 2\pi)$. However, the "generalization" of the polar coordinates and the spherical coordinates is usually called the n-dimensional polar coordinates and their definition is varied greatly. It would make sense to think that the n-dimensional polar coordinates would reduce to the polar coordinates for $n = 2$ and the spherical coordinates for $n = 3$, but this is not the case. In some definitions, as given below, one does not get the polar coordinates and spherical coordinates for $n = 2$ and $n = 3$, respectively.

A. P. Lehnen on his website [8], in the appendix, defines $x_1, x_2, ..., x_n$ in terms of $\rho, \phi_1, \phi_2, ..., \phi_{n-1}$ as

$$
\begin{cases}
x_1 & = \rho \sin\phi_1 \sin\phi_2 \sin\phi_3 ... \sin\phi_{n-3} \sin\phi_{n-2} \sin\phi_{n-1} \\
x_2 & = \rho \sin\phi_1 \sin\phi_2 \sin\phi_3 ... \sin\phi_{n-3} \sin\phi_{n-2} \cos\phi_{n-1} \\
x_3 & = \rho \sin\phi_1 \sin\phi_2 \sin\phi_3 ... \sin\phi_{n-3} \cos\phi_{n-2} \\
x_4 & = \rho \sin\phi_1 \sin\phi_2 \sin\phi_3 ... \cos\phi_{n-3} \\
... & ... \\
x_{n-1} & = \rho \sin\phi_1 \cos\phi_2 \\
x_n & = \rho \cos\phi_1
\end{cases} \tag{4}
$$

where $\rho = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$, $\phi_i \in [0, \pi]$ for $i = 1, 2, ..., n-2$, and $\phi_{n-1} \in [0, 2\pi)$. He called this type of coordinates the "Spherical Coordinates in N Dimensions." According to him, the type of coordinates in (4) was developed by Vilenkin in the 1960's [12] in which they call the coordinates as "Polyspherical Coordinates" (many spheres?!) Even if A. P. Lehnen refers to these coordinates as the spherical coordinates in $n$ dimensions, when $n = 3$, (4) does not give the spherical coordinates. Hassani also uses the definition in (4) in his book [5] to describe the n-dimensional polar coordinates.

M. G. Kendall, in his book [7], defines $x_1, x_2, ..., x_n$ in terms of $r, \theta_1, \theta_2, ..., \theta_{n-1}$ as

$$
\begin{cases}
x_1 & = r \cos\theta_1 \cos\theta_2 ... \cos\theta_{n-2} \cos\theta_{n-1} \\
x_2 & = r \cos\theta_1 \cos\theta_2 ... \cos\theta_{n-2} \sin\theta_{n-1} \\
x_3 & = r \cos\theta_1 \cos\theta_2 ... \sin\theta_{n-2} \\
... & ... \\
x_j & = r \cos\theta_1 ... \cos\theta_{n-j} \sin\theta_{n-j+1} \\
... & ... \\
x_n & = r \sin\theta_1
\end{cases} \tag{5}
$$

where $r^2 = \sum_{i=1}^{n} x_i^2$, $\theta_i \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ for $i = 1, 2, ..., n-2$, and $0 \leq \theta_{n-1} \leq 2\pi$. It is obvious that since the above implied definition includes both end points, 0 and $2\pi$

are included, this definition does not give a one to one correspondance between points in the cartesian coordinates and these coordinates. However, except for a change of the order of the angles $\theta_i$ ($i = 1, 2, ..., n-2$), (5) does give the polar coordinates for $n = 2$ and the spherical coordinates for $n = 3$.

These types of coordinates do bring a lot of benefits to many scientific disciplines. For instance, in a paper by L. E. Espinola Lόpez and J. J. Soares Neto [4], a similar type of coordinates system (which they called hyperspherical coordinates) was used for a three-body (three molecules) problem in space. They explained the reason behind this choice of coordinates system. They wrote that the three-body system could be divided into two coordinates, internal and external. The former describes the shape of the triangle formed by the particles while the latter describes the orientation of the triangle in space. According to the authors, this type of coordinates system also allow them to have only one coordinate (which they called the "hyperradius") to have "unlimited range" while the rest of the coordinates have finite ranges.

This is not the first time that this type of coordinates system has received attention from sciences. In fact, L. E. Espinola Lόpez and J. J. Soares Neto [4] quoted an earlier paper by Yngve Öhrn and Jan Linderberg [10], which provides explicit formulas for the kinetic energy functional of a four particle system. Yngve Öhrn and Jan Linderberg wrote in [10] that "Hyperspherical coordinates have recently received attention as a tool in the description of atomic and molecular processes..." They also wrote that the hyperspherical coordinates seems to be used at first by Gronwall [3] for a study of helium atom. Yngve Öhrn and Jan Linderberg [10] also explained that their choice of the coordinates was resulted from the symmetry of the problem.

As seen above, there is not really a "standard" definition for the n-dimensional polar coordinates.

### 1.3 Definition of the Jacobian for the coordinates and Discussion of its Proofs

**Definition 3.** In general, the Jacobian of the n-d polar coordinates could be given in determinant form by

$$J_n = \frac{\partial(x_1, x_2, ..., x_n)}{\partial(r, \theta_1, \theta_2, ..., \theta_{n-1})} = \det[A_{n,1}, B_{n,n-1}] \tag{6}$$

where $A_{n,1} = \begin{bmatrix} \frac{\partial x_1}{\partial r} \\ \frac{\partial x_2}{\partial r} \\ \vdots \\ \frac{\partial x_n}{\partial r} \end{bmatrix}$ and $B_{n,n-1} = \begin{bmatrix} \frac{\partial x_m}{\partial \theta_k} \end{bmatrix}$ for $m = 1, ..., n$ and $k = 2, ..., n-1$.

When Vilenkin [12] wrote about his "Polyspherical Coordinates," he did not provide a proof of their Jacobian. Later on, S. Hassani [5] claims, in his book, that the proof of the Jacobian of the n-dimensional polar coordinates could be done by mathematical induction. This is erroneous. This claim is not only wrong, but it is also a common claim whenever the Jacobian of the n-dimensional polar coordinates is mentioned. In fact, it is not possible to provide a proof of the Jacobian of the n-dimensional polar coordinates in the general case for there is a lack of a recursive relation between the Jacobians of different orders. There is no proof of the Jacobian of the n-dimensional polar coordinates, done by mathematical induction, written thus far and probably there will not ever be any.

Prior to our proof, there are only two proofs of the Jacobian of the n-dimensional polar coordinates, one is from A. P. Lehnen [8] and the other one is from W. D. Richter [11] (he refered to them as simplicial coordinates). In both of these papers, the proof of the Jacobian appears at the very end of their theories. It is because the authors needed a lot of technical preparation and to prove different theorems before the proof of the Jacobian could be provided. As a result, W. D. Richter needed 13 pages for his proof and A. P. Lehnen needed 15 pages. These proofs are not direct at all.

M. G. Kendall [7], in his book, despite using different definition of the Jacobian, does give an attempt for a direct proof, which is similar to our approach, but did not carry it out fully. Note that the approach used in this paper, similar one in Kendall's book [7], is not possible to be done in all possible cases. He did not mention anything about this difficulty. Without addressing the difficulty, his approach actually is not possible to be carried out.

Currently our proof is the only known direct proof of the Jacobian of the n-dimensional quasipolar coordinates known to us.

## Chapter 2. Definition of the N-Dimensional Quasipolar Coordinates and A Direct Proof of Their Jacobian

### 2.1 Definitions of the N-Dimensional Quasipolar Coordinates for $n = 2, 3$ and Premilinary Calculations of Their Jacobians

As mentioned above, the "standard" definitions for the polar coordinates and spherical coordinates do not provide flexibility for them to be used in other disciplines of mathematics. Also, their generalization to the n-dimensional case is not consistent, and some of them are not actually generalizations!! In this section, we shall provide definitions that are very friendly to be used in other branches of mathematics. We will also provide a consistent generalization of the coordinates.

**Definition 4.** 2-d Quasipolar Coordinates

For any point $P(x_1, x_2)$ in the $x_1 - x_2$ plane, the ordered pair of real numbers $[r, \theta_1]$ is refered to as the 2-d quasipolar coordinates of point $P(x_1, y_1)$ with respect to the arbitrary real number $\beta$, iff

$$\begin{cases} x_1 & = r\sin\theta_1 \\ x_2 & = r\cos\theta_1 \end{cases} \tag{7}$$

where $r = \sqrt{x_1^2 + x_2^2}$ and $\theta_1 \in [\beta, \beta + 2\pi)$ or $\theta_1 \in (\beta, \beta + 2\pi]$.

Using Definition 3 for the Jacobian, the Jacobian of the 2-d quasipolar coordinates is

$$J_2 = \frac{\partial(x_1, x_2)}{\partial(r, \theta_1)} = \begin{vmatrix} \frac{\partial x_1}{\partial r} & \frac{\partial x_1}{\partial \theta_1} \\ \frac{\partial x_2}{\partial r} & \frac{\partial x_2}{\partial \theta_1} \end{vmatrix} = \begin{vmatrix} \sin\theta_1 & r\cos\theta_1 \\ \cos\theta_1 & -r\sin\theta \end{vmatrix} = -r \tag{8}$$

In comparision with the polar coordinates given in Definition 1, the Jacobian of the polar coordinates is

$$J_2^* = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r \tag{9}$$

**Definition 5.** 3-d Quasipolar Coordinates

For any point $P(x_1, x_2, x_3)$ in the $x_1 - x_2 - x_3$ space, the ordered triple of real numbers $P(x_1, x_2, x_3)$ is refered to as the 3-d quasipolar coordinates of the point $P(x_1, x_2, x_3)$ with respect to the abitrary number $\beta$, iff

$$\begin{cases} x_1 & = r\sin\theta_1 \sin\theta_2 \\ x_2 & = r\cos\theta_1 \sin\theta_2 \\ x_3 & = r\cos\theta_2 \end{cases} \tag{10}$$

where $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$ , $\theta_1 \in [\beta, \beta + 2\pi)$ or $\theta_1 \in (\beta, \beta + 2\pi]$, and $\theta_2 \in [0, \pi]$.

*Remark.* One could observe that $\lambda = (\theta_1 - \gamma - \pi)\frac{180}{\pi}$ is the longitude and $\delta = (\theta_2 - \frac{\pi}{2})\frac{180}{\pi}$ is the latitude.

Instead of (10), one could also use

$$\begin{cases} x_1 & = r\cos\theta_1 \cos\theta_2 \\ x_2 & = r\sin\theta_1 \cos\theta_2 \\ x_3 & = r\sin\theta_2 \end{cases} \tag{11}$$

to generalize spherical coordinates, which are exactly the geographical coordinates with only a name change and using radians instead of degrees.

Using Definition 3 for the Jacobian, the Jacobian of the 3-d quasipolar coordinates is

$$J_3 = \frac{\partial(x_1, x_2, x_3)}{\partial(r, \theta_1, \theta_2)} = \begin{vmatrix} \frac{\partial x_1}{\partial r} & \frac{\partial x_1}{\partial \theta_1} & \frac{\partial x_1}{\partial \theta_2} \\ \frac{\partial x_2}{\partial r} & \frac{\partial x_2}{\partial \theta_1} & \frac{\partial x_2}{\partial \theta_2} \\ \frac{\partial x_3}{\partial r} & \frac{\partial x_3}{\partial \theta_1} & \frac{\partial x_3}{\partial \theta_2} \end{vmatrix} = r^2 \sin\theta \tag{12}$$

In contrast, the Jacobian of the spherical coordinates is

$$J_3^* = \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = -r^2 \sin\phi \tag{13}$$

### 2.2 Generalization to the N-dimensional Case, Definition of the N-dimensional Quasipolar Coordinates

**Definition 6.** For $n \geq 2$, and for any point $P(x_1, x_2, ..., x_n)$ in the $x_1 - x_2 - ... - x_n$ space, the ordered n-tuple of real numbers $[r, \theta_1, \theta_2, ..., \theta_{n-1}]$ is referred to as n-dimensional quasipolar coordinates of point $P(x_1, x_2, ..., x_n)$ with respect to the abitrary real number $\beta$, iff

$$\begin{cases} x_1 & = r\sin\theta_1 \sin\theta_2 ... \sin\theta_{n-1} \\ x_2 & = r\cos\theta_1 \sin\theta_2 ... \sin\theta_{n-1} \\ x_3 & = r\cos\theta_2 \sin\theta_3 ... \sin\theta_{n-1} \\ ... & \\ x_{n-1} & = r\cos\theta_{n-2} \sin\theta_{n-1} \\ x_n & = r\cos\theta_{n-1} \end{cases} \tag{14}$$

where $r = \sqrt{x_1^2 + x_2^2 + x_3^2 + ... + x_n^2}$, $\theta_1 \in [\beta, \beta + 2\pi)$ or $\beta \in (\beta, \beta + 2\pi]$, and $\theta_i \in [0, \pi]$ for $i = 1, 2, 3, ..., n - 2$.

It is clear that when $n = 2$ (14) gives the definition of the 2-d quasipolar coordinates and when $n = 3$ it gives the definition of the 3-d quasipolar coordinates.

*Remark.* We could use the term hyperpolar coordinates for the n-dimensional quasipolar coordinates in the case that $n \geq 4$.

### 2.3 A Short and Direct Proof of the Jacobian of the N-dimensional Quasipolar Coordinates Using Only Elementary Means

*Remark.* It is easy to see that the definitions we presented as above, from (7) to (14), carries the advantages that were mentioned in Chapter 1 over the "standard" definitions for the n-dimensional polar coordinates. Firstly, the n-dimensional quasipolar coordinates give a unique one to one correspondance of every point from the Cartesian coordinates to these coordinates. Thus, this type of coordinates is suitable to be used in applications that do require one to one correspondance between the points with the two types of coordinates. Secondly, since $\theta_1 \in [\beta, \beta + 2\pi)$ or $\theta_1 \in (\beta, \beta + 2\pi]$ for some real number $\beta$, the n-dimensional quasipolar coordinates give the great flexibility mentioned in chapter 1 and thus are very friendly to be used in many applications.

Thirdly, this definition will facilitate our proof greatly and is illustrated in this section. As mentioned above, initially, there is a great difficulty to use our approach which is similar to the one mentioned in [7]. In that book, Kendall [7] simply introduced his implied definition of the n-dimensional polar coordinates as given in (7) and then simply wrote down the Jacobian as

$$J = r^{n-1} \begin{vmatrix} \cos\theta_1 ... \cos\theta_{n-1} & \cos\theta_1 ... \cos\theta_{n-2}\sin\theta_{n-1} & ... & \sin\theta_1 \\ -\sin\theta_1\cos\theta_2 ... \cos\theta_{n-1} & -\sin\theta_1 ... \cos\theta_{n-2}\sin\theta_{n-1} & ... & \cos\theta_1 \\ -\cos\theta_1\sin\theta_2 ... \cos\theta_{n-1} & -\cos\theta_1\sin\theta_2 ... \cos\theta_{n-2}\sin\theta_{n-1} & ... & 0 \\ ... & ... & ... & ... \\ -\cos\theta_1\cos\theta_2 ... \sin\theta_{n-1} & \cos\theta_1\cos\theta_2 ... \cos\theta_{n-1} & ... & 0 \end{vmatrix} \tag{15}$$

$$J = r^{n-1}\cos^{n-1}\theta_1\cos^{n-2}\theta_2 ... \cos\theta_{n-1}\sin\theta_2 ... \sin\theta_{n-1} * \begin{vmatrix} 1 & 1 & 1 & ... & 1 \\ -\tan\theta_1 & -\tan\theta_1 & -\tan\theta_1 & ... & \frac{1}{\tan\theta_1} \\ -\tan\theta_2 & -\tan\theta_2 & -\tan\theta_2 & ... & 0 \\ ... & ... & ... & ... & ... \\ -\tan\theta_{n-1} & \frac{1}{\tan\theta_{n-1}} & 0 & ... & 0 \end{vmatrix} \tag{16}$$

He then wrote that by "subtracting each column from the preceding one, we find, with a little manipulation,"

$$J = r^{n-1} \cos^{n-2}\theta_1 \cos^{n-3}\theta_2 ... \cos\theta_{n-2} \tag{17}$$

Note that he uses the transpose of the Jacobian matrix to calculate the Jacobian, i.e.

$$J = \frac{\partial(x_1, x_2, ..., x_n)}{\partial(r, \theta_1, \theta_2, ..., \theta_{n-1})} = \begin{vmatrix} \frac{\partial x_1}{\partial r} & \frac{\partial x_2}{\partial r} & ... & \frac{\partial x_n}{\partial r} \\ \frac{\partial x_1}{\partial \theta_1} & \frac{\partial x_2}{\partial \theta_1} & ... & \frac{\partial x_n}{\partial \theta_1} \\ ... & ... & ... & ... \\ \frac{\partial x_1}{\partial \theta_{n-1}} & \frac{\partial x_2}{\partial \theta_{n-1}} & ... & \frac{\partial x_n}{\partial \theta_{n-1}} \end{vmatrix} \tag{18}$$

Unfortunately, Kendall [7] did not give much details and overlooked some very important elementary fact that would not allow him to do the manipulations to go from (15) to (16). From (15), he factored out, or divide by, $\cos\theta_1, \cos\theta_2, ... \cos\theta_{n-1}$ and $\sin\theta_2, \sin\theta_3, ..., \sin\theta_{n-1}$ to obtain (16). Unfortunately, this manipulation is not possible to be done in the case that even if only one of $\theta_i = \frac{l\pi}{2}$ ($l \in \mathbb{Z}$, $i = 1, 2, 3, ..., n-1$) or similarly at least one of the $\cos\theta_i = 0$ ($i = 1, 2, 3, ..., n-1$) or $\sin\theta_k = 0$ ($k = 2, 3, 4, ..., n-1$). Thus, what he suggested as a proof actually is only a suggestion for a portion of the proof and did not suggest what to do in the special cases where at least one of $\tan\theta_i$ or $\cot\theta_i$ ($i = 1, 2, 3..., n-1$) is zero (implies the other one is undefinied.)

Our proof also employs similar manipulations and we also have this difficulty of at least one of the $\tan\theta_i$ or $\cot\theta_i$ is zero (or undefined.) However, we see that it is not possible to do this kind of manipulation all the time. This is why we need to use a special technique, which we call a perturbative method, in order to make the manipulation valid in all possible cases.

With this technique, not only that we are able to keep the short and direct approach, but also we are able to extend this approach from being valid only in the general case to being valid in all possible cases, including all the special cases. One would think that the general case of a proof would be more difficult than the special cases. However, the situation is just the opposite in this proof. The special cases are much more difficult than the general case. This is due to the random and chaotic change of the Jacobian when at least one of the $\tan\theta_i$ or $\cot\theta_i$ is zero or undefined. Since the zero could appear in almost any place in the $n$ by $n$ matrix, there is not a simple formula to model this change. After using the perturbative method, there is simply no special cases anymore and the proof has no special cases!!! The pertubative method is described directly below.

Suppose that one wishes to prove or obtain a certain mathematical statement or formula $S(x)$, but there are special values of $x$, let's say $x_n (n \in \mathbb{Z})$, that simply require an entirely different approach than the general approach in the solution. Suppose further that $S(x)$ is suspected to be true for all values of $x$ including $x_n$. Also, the exact lower bound of the set of the distances between any two different points $x_m$ and $x_l$ is not 0 but some positive number. For a particular value of $x$, we consider the distance from $x$ to the nearest $x_n$ that is different than $x$ and the minimum of all of these distances we call $\delta$. Then one could choose $\epsilon$ to be a variable in $(0, \frac{\delta}{2})$. In this case, $\forall x$, $x + \epsilon \neq x_n$ for every integer $n$. Thus, instead of proving $S(x)$, one could prove $S(x+\epsilon)$ treating $x$ as a parameter and $\epsilon$ as a variable. In that sense, $S(x+\epsilon)$ is a function of $\epsilon$. Since $x + \epsilon \neq x_n$, the general approach could be used for all values of $x$. Also, the domain of $S(x+\epsilon)$ would be all $x \in D \setminus \{x_n^\star\}$. In order to cover all possible points $x \in D$, we define a continuous extension of $S(x+\epsilon)$ to be $S(x_n^\star)$ at the points $x_n^\star$. This is equivalent to letting $\epsilon \to +0$.

As a result, one does not need a different approach for providing the proof of the special cases. At the end, by letting $\epsilon \to +0$, one obtains the desired result $S(x)$ at the points $x_n^\star$. We shall employ this idea below to facilitate our proof the Jacobian of the n-dimensional quasipolar coordinates and illustrate the perturbative method.

Firstly, we wish to find general formulas to express the elements of the Jacobian matrix, which are partial derivatives, when applying Definition 3 to the n-dimensional quasipolar coordinates (14).

In fact, for $n \geq 3$, (14) could be written as

$$\begin{cases} x_1 & = r \prod_{k=1}^{n-1} \sin \theta_k \\ x_m & = r \cos \theta_{m-1} \prod_{k=m}^{n-1} \sin \theta_k, \;\; 2 \leq m \leq n-1 \\ x_n & = r \cos \theta_{n-1} \end{cases} \tag{19}$$

For $r > 0$, we also wish to obtain the partial derivatives as

$$\frac{\partial x_m}{\partial r} = \frac{1}{r} x_m \tag{20}$$

$$\frac{\partial x_m}{\partial \theta_k} = x_m \cot \theta_k, \;\; 1 \leq m \leq n, \;\; m \leq k \leq n-1 \tag{21}$$

$$\frac{\partial x_m}{\partial \theta_k} = 0, \;\; 3 \leq m \leq n, \;\; 1 \leq k \leq m-2 \tag{22}$$

$$\frac{\partial x_m}{\partial \theta_{m-1}} = -x_m \tan \theta_{m-1}, \;\; 2 \leq m \leq n \tag{23}$$

There are several things worth noting of the manipulations from (20) to (23). Firstly, because of the good definition of the n-dimensional quasipolar coordinates given in (16), we were able to find only a few set of general formula for the partial derivatives. We had tried the other definitions of the n-dimensional polar coordinates mentioned above, but (14) allows us to have the least number of general formulas to describe the Jacobian.

Secondly, as mentioned above, it is obvious that the manipulations we "wish" to do were invalid if $\tan \theta_i$ or $\cot \theta_i$ is zero (or undefined) ($i = 1, 2, 3..., n-1$). So without fixing this issue, (19) is as far as we can go with the approach! This is when we need the perturbative method in order to advance. This method is illustrated below.

Let $d_k$ ($k = 1, 2, 3, ..., n-1$) be the shortest distance from $\theta_k$ to the numbers of the form $\frac{l\pi}{2}$ ($l \in \mathbb{Z}$) which are different than $\theta_k$ and let

$$\delta = \min\{d_1, d_2, ..., d_{n-1}\} \tag{24}$$

It is clear that $d_k > 0$ $(k = 1, 2, 3, ..., n-1)$ and $\delta > 0$.

Define

$$\theta_k^\star = \theta_k^\star(\epsilon) = \theta_k + \epsilon \ (k = 1, 2, ..., n-1) \tag{25}$$

where $\epsilon$ varies in $(0, \frac{\delta}{2})$.

Thus,

$$\theta_k^\star \neq \frac{l\pi}{2} \ (\forall l \in \mathbb{Z}) \tag{26}$$

and $\tan\theta_k^\star$ and $\cot\theta_k^\star$ are all defined (and different than zero.)

Replacing $\theta_k$ by $\theta_k^\star$ $(k = 1, 2, 3, ..., n-1)$, (19) becomes

$$\begin{cases} x_1^\star & = r\prod_{k=1}^{n-1}\sin\theta_k^\star \\ x_m^\star & = r\cos\theta_{m-1}^\star\prod_{k=m}^{n-1}\sin\theta_k^\star, \quad 2 \leq m \leq n-1 \\ x_n^\star & = r\cos\theta_{n-1}^\star \end{cases} \tag{27}$$

For $r > 0$, one obtains the following:

$$\frac{\partial x_m^\star}{\partial r} = \frac{1}{r}x_m^\star, \quad 1 \leq m \leq n \tag{28}$$

$$\frac{\partial x_m^\star}{\partial \theta_k^\star} = x_m^\star \cot\theta_k^\star, \quad 1 \leq m \leq n, \ m \leq k \leq n-1 \tag{29}$$

$$\frac{\partial x_m^\star}{\partial \theta_k^\star} = 0, \quad 3 \leq m \leq n, \ 1 \leq k \leq m-2 \tag{30}$$

$$\frac{\partial x_m^\star}{\partial \theta_{m-1}^\star} = -x_m \tan\theta_{m-1}^\star, \quad 2 \leq m \leq n \tag{31}$$

Therefore,

$$
\begin{aligned}
J_n^\star &= \frac{\partial(x_1^\star, x_2^\star, ..., x_n^\star)}{\partial(r, \theta_1^\star, \theta_2^\star, ..., \theta_{n-1}^\star)} \\
&= \begin{vmatrix}
\frac{x_1^\star}{r} & x_1^\star \cot\theta_1^\star & x_1^\star \cot\theta_2^\star & ... & x_1^\star \cot\theta_{n-2}^\star & x_1^\star \cot\theta_{n-1}^\star \\
\frac{x_2^\star}{r} & -x_2^\star \tan\theta_1^\star & x_2^\star \cot\theta_2^\star & ... & x_2^\star \cot\theta_{n-2}^\star & x_2^\star \cot\theta_{n-1}^\star \\
\frac{x_3^\star}{r} & 0 & -x_3^\star \tan\theta_2^\star & ... & x_3^\star \cot\theta_{n-2}^\star & x_3^\star \cot\theta_{n-1}^\star \\
... & ... & ... & ... & ... & ... \\
\frac{x_{n-1}^\star}{r} & 0 & 0 & ... & -x_{n-1}^\star \tan\theta_{n-2}^\star & x_{n-1}^\star \cot\theta_{n-1}^\star \\
\frac{x_n^\star}{r} & 0 & 0 & ... & 0 & -x_n^\star \tan\theta_{n-1}^\star
\end{vmatrix}
\end{aligned}
\tag{32}
$$

Pulling out $\frac{1}{r}$ from the $1^{st}$ column, $\cot\theta_k^\star$ from the $k+1^{st}$column ($k = 1, 2, 3, ..., n-1$), and $x_k^\star$ from the $k^{th}$ row ($k = 1, 2, 3, ..., n$), (32) becomes

$$
J_n^\star = \frac{1}{r} x_1^\star x_2^\star ... x_n^\star \cot\theta_1^\star \cot\theta_2^\star ... \cot\theta_{n-1}^\star \Delta_n^\star \tag{33}
$$

where

$$\Delta_n^\star = \begin{vmatrix} 1 & 1 & 1 & ... & 1 & 1 \\ 1 & -\tan^2\theta_1^\star & 1 & ... & 1 & 1 \\ 1 & 0 & -\tan^2\theta_2^\star & ... & 1 & 1 \\ ... & ... & ... & ... & ... & ... \\ 1 & 0 & 0 & ... & -\tan^2\theta_{n-2}^\star & 1 \\ 1 & 0 & 0 & ... & 0 & -\tan^2\theta_{n-1}^\star \end{vmatrix} \tag{34}$$

Subtracting the $1^{st}$ column of $\Delta_n^\star$ from the rest of the columns, yields

$$\Delta_n^\star = \begin{vmatrix} 1 & 0 & 0 & ... & 0 & 0 \\ 1 & -\sec^2\theta_1^\star & 0 & ... & 0 & 0 \\ 1 & -1 & -\sec^2\theta_2^\star & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... \\ 1 & -1 & -1 & ... & -\sec^2\theta_{n-2}^\star & 0 \\ 1 & -1 & -1 & ... & -1 & -\sec^2\theta_{n-1}^\star \end{vmatrix} \tag{35}$$

which is a lower triangular determinant.

Therefore,

$$\Delta_n^\star = (-1)^{n-1} \prod_{k=1}^{n-1} \sec^2 \theta_k^\star \tag{36}$$

Thus,

$$J_n^\star = \frac{1}{r} r^n \prod_{k=1}^{n-1} \cos\theta_k^\star \prod_{k=1}^{n-1} \sin^k \theta_k^\star (-1)^{n-1} \prod_{k=1}^{n-1} \frac{1}{\cos^2 \theta_k^\star} \prod_{k=1}^{n-1} \frac{\cos\theta_k^\star}{\sin\theta_k^\star}$$

which simplifies to

$$J_n^\star = (-1)^{n-1} r^{n-1} \prod_{k=2}^{n-1} \sin^{k-1} \theta_k^\star \tag{37}$$

Since $J_n^\star$ is a continous function of $\epsilon$ ($\sin\epsilon$ is an elementary function of $\epsilon$)

$$J_n = \lim_{\epsilon\to+0} J_n^\star(\epsilon) = \lim_{\epsilon\to+0} (-1)^{n-1} r^{n-1} \prod_{k=2}^{n-1} \sin^{k-1} \theta_k^\star = \lim_{\epsilon\to+0} (-1)^{n-1} r^{n-1} \prod_{k=2}^{n-1} \sin^{k-1}(\theta_k + \epsilon) \tag{38}$$

Thus,

$$J_n = (-1)^{n-1} r^{n-1} \prod_{k=2}^{n-1} \sin^{k-1} \theta_k \tag{39}$$

This result remains valid also when $r = 0$, since $J_n = 0$ in that case. This concludes the evaluation/proof of the Jacobian $J_n$.

As mentioned above, the perturbative method helps us to provide the proof in all possible cases, the general case and all the special cases, at once. In the next chapter, the definitions of the n-dimensional quasiballs are given and the Jacobian is used to calculate the n-dimensional quasivolume of the n-dimensional quasiball.

## Chapter 3. Definition of the N-Dimensional Quasiball, its Quasivolume, and the Calculation of its Quasivolume

### 3.1 Preliminary Definitions of the N-Dimensional Quasiball and N-Dimensional Quasivolume

**Definition 7.** Definition of the 1-dimensional Quasiball and its 1-dimensional Quasivolume

A collection of points $P(x_1)$ satisfying $x_1^2 < R^2$ (interior) united with two, one, or none of the points $P(x_1)$ satisfying $x_1^2 = R^2$ (boundary) is called the 1-dimensional quasiball centered at the origin $O(0)$. Centering the 1-dimensional quasiball at the origin does not limit the generality of the definition.

The 1-dimensional quasivolume of a 1-dimensional quasiball is defined as

$$V_1(R) = \int\limits_{D_1} dx_1 = \int\limits_{-R}^{R} dx_1 = 2R \tag{40}$$

where $D_1$ is given by $D_1 : \{x_1 | x_1^2 \leq R^2\}$. Evidently, the 1-dimensional quasivolume of the 1-dimensional quasiball does not depend on the inclusion of two, one, or none of the boundary points.

The 1-dimensional quasiball is nothing else but a line segment and the 1-dimensional quasivolume of the 1-dimensional quasiball is the length of the line segment.

**Definition 8.** Definition of the 2-dimensional Quasiball and its 2-dimensional Quasivolume

The collection of all points $P(x_1, x_2)$ satisfying $x_1^2 + x_2^2 < R^2$ (interior) united with all, or none, or any number of points $P(x_1, x_2)$ satisfying $x_1^2 + x_2^2 = R^2$ (boundary) is called the 2-dimensional quasiball centered at the origin $O(0)$. Centering the 2-dimensional quasiball at the origin does not limit the generality of the definition.

The 2-dimensional quasivolume of a 2-dimensional quasiball with radius $R$ is defined as

$$V_2(R) = \iint\limits_{D_2} dx_1 dx_2 \tag{41}$$

where $D_2 = \{(x_1, x_2) | x_1^2 + x_2^2 \leq R^2\}$. Evidently, the 2-dimensional quasivolume of the 2-dimensional quasiball does not depend on the inclusion of all, none, or any number of the boundary points.

It is well-known that

$$V_2(R) = \pi R^2 \tag{42}$$

The 2-dimensional quasiball is nothing else but a disk and the 2-dimensional quasivolume of the 2-dimensional quasiball is the area of the disk.

**Definition 9.** Definition of the 3-dimensional Quasiball and its 3-dimensional Quasivolume

The collection of all points $P(x_1, x_2, x_3)$ satisfying $x_1^2 + x_2^2 + x_3^2 < R^2$ (interior) united with all, or none, or any number of points $P(x_1, x_2, x_3)$ satisfying $x_1^2 + x_2^2 + x_3^2 = R^2$ (boundary) is called the 3-dimensional Quasiball centered at the origin $O(0)$. Centering the 3-dimensional quasiball at the origin does not limit the generality of the definition.

The 3-dimensional quasivolume of a 3-dimensional quasiball with radius $R$ could be evaluated by

$$V_3(R) = \iiint\limits_{D_3} dx_1 dx_2 dx_3 \tag{43}$$

where $D_3 = \{(x_1, x_2, x_3) | x_1^2 + x_2^2 + x_3^2 \leq R^2\}$. Evidently, the 3-dimensional quasivolume of the 3-dimensional quasiball does not depend on the inclusion of all, none, or any number of the boundary points.

The 3-dimensional quasiball is nothing but a ball and the 3-dimensional quasivolume of the 3-dimensional quasiball is the volume of the ball.

### 3.2 General Definitions of the N-Dimensional Quasiball and the N-Dimensional Quasivolume

**Definition 10.** Definition of the n-dimensional quasiball

A n-dimensional quasiball, also referred to as hyperball for $n > 3$, centered at the origin $O(0, 0, ..., 0) \in \mathbb{R}^n$ with radius $R$ $(R > 0)$ in $\mathbb{R}^n$ is the collection of all points $P(x_1, x_2, ..., x_n)$ satisfying $x_1^2 + x_2^2 + ... + x_n^2 < R^2$ (interior) united with all, or none, or any number of points $P(x_1, x_2, ..., x_n)$ satisfying $x_1^2 + x_2^2 + ... + x_n^2 = R^2$ (boundary). Again, centering the n-dimensional quasiball at the origin does not limit the generality of the definition.

**Definition 11.** Definition of the n-dimensional quasivolume of the n-dimensional quasiball

The n-dimensional quasivolume of the n-dimensional quasiball, also referred to as n-dimensional hypervolume of the n-dimensional hyperball for $n > 3$ is defined as

$$V_n(R) = \int\!\!\cdots\!\!\int_{D_n} dx_1 dx_2 ... dx_n \tag{44}$$

where $D_n := \{(x_1, x_2, ..., x_n) | x_1^2 + x_2^2 + ... + x_n^2 \leq R^2\}$. Evidently, the n-dimensional quasivolume of the n-dimensional quasiball does not depend on the inclusion of all, none, or any number of the boundary points.

### 3.3 Calculation of the N-Dimensional Quasivolume of the N-Dimensional Quasiball

The formula for the n-dimensional quasivolume of the n-dimensional quasiball is available in mathematical handbooks in terms of Gamma function [13]. However, these results are obtained by the use of mathematical induction and not by direct calculations using the definition of volumes (integrals). In fact, Greg Huber wrote a paper [6] and showed the derivation of what he called the “n-sphere volumes” using mathematical induction and Gamma function. To our research, besides our direct calculation (direct calculation of the integrals) of the n-dimensional quasivolume (which is shown below), there is no other direct derivation of the n-dimensional quasivolume of the n-dimensional quasiball.

With the direct calculation of the integrals, we are able derive a formula for the n-dimensional quasivolume of the n-dimensional quasiball that is free of any special function. As a result, our formula could be introduced without any advanced mathematical knowledge. The derivation of the formula will be shown below.

Using the n-dimensional quasipolar coordinates, one gets

$$V_n(R) = \int_0^R r^{n-1} dr \int_0^{2\pi} d\theta_1 \prod_{k=2}^{n-1} \int_0^{\pi} \sin^{k-1}\theta_k d\theta_k = \frac{R^n}{n} 2\pi \prod_{k=1}^{n-2} \int_0^{\pi} \sin^k \theta d\theta. \qquad (45)$$

Let

$$I_k = \int_0^{\pi} \sin^k \theta d\theta. \qquad (46)$$

Thus

$$I_0 = \int_0^{\pi} d\theta = \pi, \qquad (47)$$

and

$$I_1 = \int_0^{\pi} \sin\theta d\theta = -\cos\theta|_0^{\pi} = 2. \qquad (48)$$

For $k \geq 0$,

$$I_{k+2} = -\int_0^{\pi} \sin^{k+1}\theta dcos\theta = -\sin^{k+1}\theta\cos\theta|_0^{\pi} + (k+1)\int_0^{\pi} \cos^2\theta \sin^k \theta d\theta. \qquad (49)$$

It follows that

$$I_{k+2} = (k+1)\int_0^{\pi} \sin^k \theta d\theta - (k+1)\int_0^{\pi} \sin^{k+2}\theta d\theta. \qquad (50)$$

From (46) and (50), one obtains

$$\frac{I_{k+2}}{I_k} = \frac{k+1}{k+2} \tag{51}$$

For odd values of $k$, one gets

$$\frac{I_{2m+1}}{I_{2m-1}} = \frac{2m}{2m+1} \tag{52}$$

And for even values of $k$

$$\frac{I_{2m}}{I_{2m-2}} = \frac{2m-1}{2m} \tag{53}$$

It follows from (47) and (53) that

$$I_{2k} = I_0 \prod_{m=1}^{k} \frac{I_{2m}}{I_{2m-2}} = \pi \prod_{m=1}^{k} \frac{2m-1}{2m} = \pi \frac{(2k-1)!!}{(2k)!!} \tag{54}$$

Where for $n > 1$,

$$(2n-1)!! = (1)(3)(5)...(2n-1) \tag{55}$$

$$(2n)!! = (2)(4)(6)...(2n) \tag{56}$$

Also,

$$1!! = 1,\ \ 2!! = 2\ ,0!! = 1,\ \ (-1)!! = 1 \tag{57}$$

Similarly, from (48) and (52), it follows that

$$I_{2k-1} = I_1 \prod_{m=1}^{k-1} \frac{I_{2m+1}}{I_{2m-1}} = 2 \prod_{m=1}^{k-1} \frac{2m}{2m+1} = 2\frac{(2k-2)!!}{(2k-1)!!} \tag{58}$$

Multiplying (54) and (58), one gets

$$I_{2k-1} I_{2k} = 2\frac{(2k-2)!!}{(2k-1)!!}\pi\frac{(2k-1)!!}{(2k)!!} = \frac{\pi}{k} \tag{59}$$

Thus

$$V_n = 2\pi \frac{R^n}{n} \prod_{k=1}^{n-2} I_k \implies V_{2n} = \pi \frac{R^{2n}}{n} \prod_{k=1}^{2n-2} I_k = \pi \frac{R^{2n}}{n} \prod_{k=1}^{n-1} I_{2k-1} I_{2k}. \tag{60}$$

From (59) and (60), one obtains

$$V_{2n} = \pi \frac{R^{2n}}{n} \prod_{k=1}^{n-1} \frac{\pi}{k} = \pi \frac{R^{2n}}{n} \frac{\pi^{n-1}}{(n-1)!} = \frac{\pi^n R^{2n}}{n!} \quad (n \geq 2) \tag{61}$$

Similarly from (58) and (59), one gets

$$V_{2n+1} = 2\pi \frac{R^{2n+1}}{2n+1} \prod_{k=1}^{2n-1} I_k = 2\pi \frac{R^{2n+1}}{2n+1} I_{2n-1} \prod_{k=1}^{n-1} I_{2k-1} I_{2k}. \tag{62}$$

Finally by plugging in the results from (58) and (59) into (62)

$$V_{2n+1} = 2\pi \frac{R^{2n+1}}{2n+1} 2 \frac{(2n-2)!!}{(2n-1)!!} \frac{\pi^{n-1}}{(n-1)!} \frac{2^n}{2^{n-1}} \frac{1}{2}. \tag{63}$$

Thus

$$V_{2n+1} = \frac{2^{n+1} \pi^n R^{2n+1}}{(2n+1)!!} \quad (n \geq 1) \tag{64}$$

Note that $V_2 = \pi R^2$, which complies with (62) for $n = 1$; and $V_1 = 2R$ which complies with (61) for $n = 0$.

### 3.4 General Formula for $V_n(R)$ without the use of any Special Function

In Section 3.3, the formulas for the n-dimensional quasivolume of the n-dimensional quasiball for the even and odd cases are derived. In this section, these formulas will be combined into only one formula that is valid for all possible cases.

From (61), one gets:

$$\frac{V_{2n}}{R^{2n}} (2n)!! = 2^n \pi^n. \tag{65}$$

And from (64), it follows that:

$$\frac{V_{2n+1}}{R^{2n+1}} (2n+1)!! = 2^{n+1} \pi^n \tag{66}$$

Note that:

$$\left[\frac{n}{2}\right] = \begin{cases} \frac{n}{2} & = k, \quad n = 2k \\ \frac{n-1}{2} & = k, \quad n = 2k+1 \end{cases} \quad (k \in \mathbb{N}), \tag{67}$$

and

$$\left[\frac{n+1}{2}\right] = \begin{cases} \frac{n+1}{2} & = k, \quad n = 2k-1 \\ \frac{n+2}{2} & = k+1, \quad n = 2k \end{cases} \quad (k \in \mathbb{N}). \tag{68}$$

where $[x]$ is the greatest integer that is less than or equal to $x$.

Thus,

$$V_n(R) = \frac{2^{\left[\frac{n+1}{2}\right]}\pi^{\left[\frac{n}{2}\right]}}{n!!}R^n. \tag{69}$$

The formula for the n-dimensional quasivolume of the n-dimensional quasiball given in (69) does not contain any special function and thus could be introduced without any advanced mathematical knowledge.

**3.5 Definition of the N-dimensional Quasisphere and its N-dimensional Quasi-surface Area**

**Definition 12.** Definition of n-dimensional sphere and its n-dimensional quasi-surface area

A n-dimensional quasisphere, also referred to as hypersphere for $n > 3$, centered at the origin $O(0, 0, ..., 0) \in \mathbb{R}^n$ with radius $R$ $(R > 0)$ in $\mathbb{R}^n$ is the collection of all points $P(x_1, x_2, ..., x_n)$ satisfying $x_1^2 + x_2^2 + ... + x_n^2 = R^2$. Using symmetry, the n-dimensional quasi-surface area of the n-dimensional quasisphere is defined to be the $n-1$- tuple integral

$$\begin{aligned} SA_n(R) &= 2 \int \cdots \int_{x_1^2+x_2^2+...+x_{n-1}^2 \leq R^2} ||\vec{grad}(x_n)|| dx_1 dx_2 ... dx_{n-1} \\ &= 2 \int \cdots \int_{x_1^2+x_2^2+...+x_{n-1}^2 \leq R^2} \sqrt{1 + \left(\frac{\partial x_n}{\partial x_1}\right)^2 + \left(\frac{\partial x_n}{\partial x_2}\right)^2 + ... + \left(\frac{\partial x_n}{\partial x_{n-1}}\right)^2} dx_1 dx_2 ... dx_{n-1} \end{aligned} \tag{70}$$

Note that the n-dimensional quasisphere is actually an n-1 dimensional object seen in n-dimensional space.

**3.6 Derivation of an Elementary Formula of the N-dimensional Quasi-surface Area of the N-dimensional Quasisphere Using the Jacobian**

For simplification purposes, we shall calculate the n-dimensional quasi-surface area of the n+1-dimensional quasisphere.

$$SA_{n+1}(R) = 2 \int \cdots \int_{x_1^2+x_2^2+...+x_n^2 \leq R^2} \sqrt{1 + \left(\frac{\partial x_{n+1}}{\partial x_1}\right)^2 + \left(\frac{\partial x_{n+1}}{\partial x_2}\right)^2 + ... + \left(\frac{\partial x_{n+1}}{\partial x_n}\right)^2} dx_1 dx_2 ... dx_n \tag{71}$$

and the n-dimensional quasisphere is given as the set of all points $x_1, x_2, ..., x_{n+1}$ satisfying

$$x_1^2 + x_2^2 + ... + x_n^2 + x_{n+1}^2 = R^2 \tag{72}$$

$$\Longrightarrow x_{n+1} = \pm\sqrt{R^2 - x_1^2 - x_2^2 - ... - x_n^2} \tag{73}$$

Differentiating both sides of (72) with respect to $x_j$ $(j = 1, 2, 3, ..., n+1)$, one obtains

$$2x_j + 2x_{n+1}\frac{\partial x_{n+1}}{\partial x_j} = 0 \tag{74}$$

$$\Longrightarrow \frac{\partial x_{n+1}}{\partial x_j} = -\frac{x_j}{x_{n+1}} \tag{75}$$

Thus,

$$||gr\vec{a}d(x)|| = \sqrt{1 + \sum_{j=1}^{n} \left(\frac{x_j}{x_{n+1}}\right)^2} = \frac{R}{x_{n+1}} \tag{76}$$

Rewritting (71), one has

$$SA_{n+1}(R) = 2R \underset{x_1^2+x_2^2+...+x_n^2 \le R^2}{\int \cdots \int} \frac{dx_1 dx_2 ... dx_n}{\sqrt{R^2 - \sum\limits_{j=1}^{n} x_j^2}} \tag{77}$$

Using the n-dimensional quasipolar coordinates and the Jacobian, (77) becomes

$$SA_{n+1}(R) = 2R \int_0^R dr \int_0^{2\pi} d\theta_1 \int_0^{\pi} d\theta_2 ... \int_0^{\pi} d\theta_{n-1} \frac{r^{n-1} \prod_{k=2}^{n-1} \sin^{k-1} \theta_k}{\sqrt{R^2 - r^2}} \tag{78}$$

$$SA_{n+1}(R) = 2R \int_0^R \frac{r^{n-1}}{\sqrt{R^2 - r^2}} dr \int_0^{2\pi} d\theta_1 \prod_{k=2}^{n-1} \int_0^{\pi} \sin^{k-1} \theta_k d\theta_k \tag{79}$$

For $n \ge 4$, define

$$K_{n-1} = \int_0^R \frac{r^{n-1}}{\sqrt{R^2 - r^2}} dr = -\int_0^R r^{n-2} d\sqrt{R^2 - r^2} \tag{80}$$

Thus,

$$K_{n-1} = -r^{n-2}\sqrt{R^2 - r^2}\Big|_0^R + \int_0^R (n-2)\sqrt{R^2 - r^2} r^{n-3} dr$$

$$= (n-2)R^2 \int_0^R \frac{r^{n-3}}{\sqrt{R^2 - r^2}} dr - (n-2) \int_0^R \frac{r^{n-1}}{\sqrt{R^2 - r^2}} dr \tag{81}$$

Hence,

$$K_{n-1} = (n-2)R^2 K_{n-3} - (n-2)K_{n-1} \tag{82}$$

$$\implies (n-1)K_{n-1} = (n-2)R^2 K_{n-3} \tag{83}$$

$$\implies nK_n = (n-1)R^2 K_{n-2} \tag{84}$$

Replace $n$ by $2j$ in (84), one gets

$$2jK_{2j} = (2j-1)R^2 K_{2j-2} \tag{85}$$

$$\implies K_{2k} = K_0 \prod_{j=1}^{k} \frac{K_{2j}}{K_{2j-2}} = J_0 \prod_{j=1}^{k} \frac{(2j-1)R^2}{2j} = K_0 \frac{(2k-1)!!R^{2k}}{(2k)!!} \tag{86}$$

where

$$K_0 = \int_0^R \frac{dr}{\sqrt{R^2 - r^2}} = \lim_{\epsilon \to +0} \int_0^{R-\epsilon} \frac{dr}{\sqrt{R^2 - r^2}} = \lim_{\epsilon \to +0} \arcsin \frac{r}{R}\Big|_0^{R-\epsilon} = \frac{\pi}{2} \tag{87}$$

Thus

$$K_{2k} = \frac{\pi}{2} \frac{(2k-1)!!R^{2k}}{(2k)!!} \tag{88}$$

Replacing $n$ by $2j+1$ in (84), one gets

$$(2j+1)K_{2j+1} = 2jR^2 K_{2j-1} \tag{89}$$

Similarly done as above,

$$K_{2k+1} = K_1 \prod_{j=1}^{k} \frac{J_{2j+1}}{J_{2j-1}} = K_1 \prod_{J=1}^{k} \frac{2jR^2}{(2j+1)} \tag{90}$$

$$K_{2k+1} = K_1 \frac{(2k)!!R^{2k}}{(2k+1)!!} \tag{91}$$

where

$$K_1 = -\int_0^R \frac{-2r}{2\sqrt{R^2 - r^2}} dr = -\lim_{\epsilon \to +0} \int_0^{R-\epsilon} \frac{-2r}{2\sqrt{R^2 - r^2}} dr = -\lim_{\epsilon \to +0} \sqrt{R^2 - r^2}\Big|_0^R = R \tag{92}$$

Hence,

$$K_{2k+1} = \frac{(2k)!!R^{2k+1}}{(2k+1)!!} \tag{93}$$

Note that (88) and (93) are very similar. Using this similarity, one could write

$$K_n = \alpha_n \frac{(n-1)!!R^n}{n!!} \tag{94}$$

where

$$\alpha_n = \begin{cases} 1 & , \; n \,\text{is odd} \\ \frac{\pi}{2} & , \; n \,\text{is even} \end{cases} \tag{95}$$

$$\implies \alpha_n - 1 = \begin{cases} 0 & , \; n \,\text{is odd} \\ \frac{\pi}{2} - 1 & , \; n \,\text{is even} \end{cases} \tag{96}$$

Hence,

$$\alpha_{n-1} = (\frac{\pi}{2} - 1)\frac{(-1)^n + 1}{2} \tag{97}$$

$$\implies \alpha_n = 1 + \left(\frac{\pi}{2} - 1\right)\frac{(-1)^n + 1}{2} \tag{98}$$

Thus,

$$K_n = \left[1 + \left(\frac{\pi}{2} - 1\right)\frac{(-1)^n + 1}{2}\right]\frac{(n-1)!!R^n}{n!!} \tag{99}$$

Recall that $I_k = \int\limits_0^\pi \sin^k \theta_k d\theta_k$ from (46). We can now write (79) as

$$SA_{n+1}(R) = 2R * 2\pi * K_{n-1}\prod_{k=1}^{n-2} I_k \tag{100}$$

$$\implies SA_{n+1}(R) = 4\pi R K_{n-1}\prod_{k=1}^{n-2} I_k \tag{101}$$

Replace $n$ by $2n$ in (101), one gets

$$SA_{2n+1} = 4\pi R K_{2n-1}\prod_{k=1}^{2n-2} I_k = 4\pi R K_{2n-1}\prod_{k=1}^{n-1} I_{2k-1}I_{2k} \tag{102}$$

Following from (59) where $I_{2k-1}I_{2k} = \frac{\pi}{k}$,

$$SA_{2n+1} = 4\pi R K_{2n-1}\prod_{k=1}^{n-1}\frac{\pi}{k} = 4\pi R K_{2n-1}\frac{\pi^{n-1}}{(n-1)!} = \frac{4\pi^n R}{(n-1)!}K_{2n-1} \tag{103}$$

Recall from (93) that $K_{2k+1} = \frac{(2k)!!R^{2k+1}}{(2k+1)!!}$. Replacing $k$ by $n-1$ in (93), one gets

$$K_{2n-1} = \frac{(2n-2)!!R^{2n-1}}{(2n-1)!!} \tag{104}$$

Thus,

$$SA_{2n+1} = \frac{4\pi^n R}{(n-1)!}\frac{(2n-2)!!}{(2n-1)!!}R^{2n-1} = \frac{4\pi^n(2n-2)!!}{(n-1)!(2n-1)!!}R^{2n} = \frac{2^{n+1}\pi^n R}{(2n-1)!!}R^{2n-1} \tag{105}$$

$$SA_{2n+1} = \frac{2^{n+1}\pi^n R^{2n}}{(2n-1)!!} \tag{106}$$

Similarly, by replacing $n$ by $2n+1$ in (101), one obtains

$$SA_{2n+2} = 4\pi RK_{2n}\prod_{k=1}^{2n-1} I_k = 4\pi RK_{2n}I_{2n-1}\prod_{k=1}^{n-2} I_{2k-1}I_{2k} = 4\pi RK_{2n}I_{2n-1}\frac{\pi^{n-1}}{(n-1)!} \tag{107}$$

From (58), $I_{2k-1} = 2\frac{(2k-2)!!}{(2k-1)!!}$. Hence,

$$SA_{2n+2} = 4\pi RK_{2n}2\frac{(2n-2)!!}{(2n-1)!!}\frac{\pi^{n-1}}{(n-1)!} = \pi RK_{2n}\frac{2^{n+2}\pi^{n-1}}{(2n-1)!!} \tag{108}$$

From (88), we have $K_{2n} = \frac{\pi}{2}\frac{(2n-1)!!}{(2n)!!}R^{2n}$.

Thus,

$$SA_{2n+2} = \frac{2^{n+2}\pi^n}{(2n-1)!!}\frac{\pi}{2}\frac{(2n-1)!!}{(2n)!!}R^{2n+1} \tag{109}$$

$$SA_{2n+2} = \frac{2\pi^{n+1}}{n!}R^{2n+1} \tag{110}$$

$$\implies SA_{2n} = \frac{2\pi^n}{(n-1)!}R^{2n-1} \tag{111}$$

One could write (106) and (107) as

$$\frac{SA_{2n+1}}{\pi^n R^{2n+1}} = \frac{2^{n+1}}{R(2n-1)!!} \tag{112}$$

and

$$\frac{SA_{2n}}{\pi^n R^{2n}} = \frac{2}{R(n-1)!} = \frac{2^n}{R(2n-2)!!} \tag{113}$$

Using the results from (67) and (68), one has

$$SA_n(R) = \frac{2^{\left[\frac{n+1}{2}\right]}\pi^{\left[\frac{n}{2}\right]}}{(n-2)!!}R^{n-1}. \tag{114}$$

One could see a very interesting fact that the n-dimensional quasi-surface area of an n-dimensional quasisphere is the derivative of the n-dimensional quasivolume of the n-dimensional quasiball. This fact is readily seen from (69) and (114). Usually, taking the derivative of the n-dimensional quasivolume of the n-dimensional quasiball is the way that the n-dimensional quasi-surface area of the n-dimensional quasisphere is calculated.

However, this relation is not very obvious at first and requires some justification. The direct approach that is shown above does not require the knowledge of this theorem at all. The theorem comes as the result of the direct calculations. Calculating the surface areas by calculating the derivatives of the volumes is not a natural way to go. Our calculations above are done in a much more natural way with the classical definition of the surface area (by surface integrals.)

### 3.7 Definition of the Generalized N-dimensional Quasipolar Coordinates, Calculation of their Jacobian and the N-dimensional Quasivolume of the N-dimensional Quasi-ellipsoid

**Definition.** For $n \geq 2$, and for any point $P(x_1, x_2, ..., x_n)$ in the $x_1 - x_2 - ... - x_n$ space, the ordered n-tuple of real numbers $[r, \theta_1, \theta_2, ..., \theta_{n-1}]$ is referred to as the generalized n-dimensional quasipolar coordinates of point $P(x_1, x_2, ..., x_n)$ with respect to the abitrary real number $\beta$, iff

$$\begin{cases} x_1 & = a_1\bar{r}\sin\theta_1\sin\theta_2...\sin\theta_{n-1} \\ x_2 & = a_2\bar{r}\cos\theta_1\sin\theta_2...\sin\theta_{n-1} \\ x_3 & = a_3\bar{r}\cos\theta_2\sin\theta_3...\sin\theta_{n-1} \\ ... & \\ x_{n-1} & = a_{n-1}\bar{r}\cos\theta_{n-2}\sin\theta_{n-1} \\ x_n & = a_n\bar{r}\cos\theta_{n-1} \end{cases} \tag{115}$$

where $\bar{r}^2 = x_1^2 + x_2^2 + x_3^2 + ... + x_n^2$, $\theta_1 \in [\beta, \beta + 2\pi)$ or $\beta \in (\beta, \beta + 2\pi]$, $\theta_i \in [0, \pi]$ for $i = 1, 2, 3, ..., n-2$ and $a_i \in \mathbb{R}^+$.

Note that there is no geometrical interpretation for $r^\star$.

The proof of the Jacobian of the generalized n-dimensional quasipolar coordinates could be done in a very similar fashion as the proof of the Jacobiann of the n-dimensional quasipolar coordinates. One simply needs to factor out the constants $a_i$ $(i = 1, 2, ..., n)$ from the $i^{th}$ row of the Jacobian matrix.

Thus, it is easy to verify that the Jacobian of the generalized n-dimensional quasipolar coordinates is

$$\bar{J}_n = (-1)^{n-1}\bar{r}^{n-1}\prod_{i=1}^{n} a_i \prod_{k=2}^{n-1}\sin^{k-1}\theta_k \tag{116}$$

It follows that the n-dimensional quasivolume of the n-dimensional quasi-ellipsoid with equation

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + ... + \frac{x_n^2}{a_n^2} \leq \bar{R}^2 \tag{117}$$

is

$$V_n(\bar{R}) = \frac{2^{\left[\frac{n+1}{2}\right]}\pi^{\left[\frac{n}{2}\right]}}{n!!}\bar{R}^n\prod_{i=1}^{n} a_i \tag{118}$$

Table 1 gives the numerical result of the coefficients of $R^n$ of the n-dimensional quasivolume of the n-dimensional quasisball in (69), the coefficients of $R^{n-1}$ of the n-dimensional quasi-surface area of the n-dimensional quasisphere in (114).

Note that if one differentiates (69) (formula for $V_n(R)$) with respect to $R$, one would get (114) (formula for $SA_n(R)$). In fact, in the literature, this relation is sometimes claimed as a short cut for deriving the formula for the n-dimensional quasi-surface area of the n-dimensional quasisphere. This relation is not at all obvious and it is very unusual to calculate the surface area as the derivative of the volume. In this thesis, the relation simply comes as the result of our direct calculations using the natural definitions by integrals. Thus

$$SA_n(R) = \frac{n}{R}V_n(R) \implies \frac{V_n}{2^n R^n} = \frac{SA_n}{n2^n R^{n-1}} \tag{119}$$

In the n-dimensional space, the n-dimensional quasivolume of the n-dimensional cube is $2^n R^n$ and the n-dimensional quasi-surface of the n-dimensional cube is $n2^n R^{n-1}$. The ratios $\frac{V_n}{2^n R^n} = \frac{SA_n}{n2^n R^{n-1}}$ are also included in Table 1.

TABLE 1. Numerical results of the coefficients of $V_n$, coefficients of $SA_n$, and $\frac{V_n}{2^n R^n} = \frac{SA_n}{n 2^n R^{n-1}}$

| Dimension | Coefficients of $V_n$ | Coefficients of $SA_n$ | $\frac{V_n}{2^n R^n} = \frac{SA_n}{n 2^n R^{n-1}}$ |
|---|---|---|---|
| 2 | 3.141592654 | 6.283185307 | 0.785398163 |
| 3 | 4.188790205 | 12.56637061 | 0.523598776 |
| 4 | 4.934802201 | 19.7392088 | 0.308425138 |
| 5 | **5.263789014** | 26.31894507 | 0.164493407 |
| 6 | 5.16771278 | 31.00627668 | 0.080745512 |
| 7 | 4.72476597 | **33.07336179** | 0.036912234 |
| 8 | 4.058712126 | 32.46969701 | 0.015854344 |
| 9 | 3.298508903 | 29.68658012 | 0.0064424 |
| 10 | 2.55016404 | 25.5016404 | 0.002490395 |
| 11 | 1.884103879 | 20.72514267 | 0.000919973 |
| 12 | 1.335262769 | 16.02315323 | 0.000325992 |
| 13 | 0.910628755 | 11.83817381 | 0.000111161 |
| 14 | 0.599264529 | 8.38970341 | 3.65762E-05 |
| 15 | 0.381443281 | 5.721649212 | 1.16407E-05 |
| 16 | 0.23533063 | 3.765290086 | 3.59086E-06 |
| 17 | 0.140981107 | 2.396678818 | 1.0756E-06 |
| 18 | 0.082145887 | 1.478625959 | 3.13362E-07 |
| 19 | 0.046621601 | 0.88581042 | 8.89236E-08 |
| 20 | 0.025806891 | 0.516137828 | 2.46114E-08 |

Table 1 continued

| | | | |
|---|---|---|---|
| 21 | 0.01394915 | 0.292932159 | 6.65147E-09 |
| 22 | 0.007370431 | 0.162149481 | 1.75725E-09 |
| 23 | 0.003810656 | 0.087645097 | 4.54266E-10 |
| 24 | 0.001929574 | 0.046309783 | 1.15012E-10 |
| 25 | 0.000957722 | 0.02394306 | 2.85424E-11 |
| 26 | 0.000466303 | 0.012123873 | 6.94845E-12 |
| 27 | 0.000222872 | 0.006017547 | 1.66053E-12 |
| 28 | 0.000104638 | 0.002929867 | 3.89807E-13 |
| 29 | 4.82878E-05 | 0.001400347 | 8.99431E-14 |
| 30 | 2.19154E-05 | 0.000657461 | 2.04103E-14 |
| 31 | 9.78714E-06 | 0.000303401 | 4.55749E-15 |
| 32 | 4.30307E-06 | 0.000137698 | 1.00189E-15 |
| 33 | 1.86347E-06 | 6.14944E-05 | 2.16936E-16 |
| 34 | 7.95205E-07 | 2.70370E-005 | 4.6287E-17 |
| 35 | 3.34529E-07 | 1.17085E-05 | 9.73607E-18 |
| 36 | 1.3879E-07 | 4.99642E-06 | 2.01965E-18 |
| 37 | 5.68083E-08 | 2.10191E-06 | 4.13335E-19 |
| 38 | 2.29484E-08 | 8.72040E-007 | 8.34859E-20 |
| 39 | 9.15223E-09 | 3.56937E-07 | 1.66478E-20 |
| 40 | 3.60473E-09 | 1.44189E-07 | 3.27848E-21 |
| 41 | 1.40256E-09 | 5.75052E-08 | 6.37813E-22 |
| 42 | 5.39266E-10 | 2.26492E-08 | 1.22615E-22 |

Table 1 continued

| | | | |
|---|---|---|---|
| 43 | 2.04944E-10 | 8.81258E-09 | 2.32994E-23 |
| 44 | 7.70071E-11 | 3.38831E-09 | 4.37735E-24 |
| 45 | 2.86155E-11 | 1.2877E-09 | 8.13302E-25 |
| 46 | 1.05185E-11 | 4.8385E-10 | 1.49476E-25 |
| 47 | 3.82546E-12 | 1.79797E-10 | 2.71815E-26 |
| 48 | 1.37686E-12 | 6.60895E-11 | 4.89161E-27 |
| 49 | 4.90532E-13 | 2.40361E-11 | 8.7136E-28 |
| 50 | 1.73022E-13 | 8.6511E-12 | 1.53674E-28 |
| 51 | 6.04334E-14 | 3.0821E-12 | 2.68378E-29 |
| 52 | 2.09063E-14 | 1.08713E-12 | 4.64214E-30 |
| 53 | 7.16442E-15 | 3.79714E-13 | 7.95411E-31 |
| 54 | 2.43256E-15 | 1.31358E-13 | 1.35034E-31 |
| 55 | 8.18462E-16 | 4.50154E-14 | 2.27169E-32 |
| 56 | 2.72933E-16 | 1.52842E-14 | 3.7877E-33 |
| 57 | 9.02201E-17 | 5.14255E-15 | 6.26028E-34 |
| 58 | 2.9567E-17 | 1.71489E-15 | 1.02581E-34 |
| 59 | 9.60796E-18 | 5.6687E-16 | 1.66672E-35 |
| 60 | 3.09625E-18 | 1.85775E-16 | 2.68557E-36 |
| 61 | 9.89649E-19 | 6.03686E-17 | 4.29192E-37 |
| 62 | 3.13779E-19 | 1.94543E-17 | 6.804E-38 |
| 64 | 3.08052E-20 | 1.97153E-18 | 1.66995E-39 |

Table 1 continued

| | | | |
|---|---|---|---|
| 65 | 9.54085E-21 | 6.20155E-19 | 2.58605E-40 |
| 66 | 2.93265E-21 | 1.93555E-19 | 3.97448E-41 |
| 67 | 8.9473E-22 | 5.99469E-20 | 6.06293E-42 |
| 68 | 2.70976E-22 | 1.84264E-20 | 9.18103E-43 |
| 69 | 8.14747E-23 | 5.62176E-21 | 1.38024E-43 |
| 70 | 2.43228E-23 | 1.70259E-21 | 2.06022E-44 |
| 71 | 7.21015E-24 | 5.11921E-22 | 3.05362E-45 |
| 72 | 2.12256E-24 | 1.52824E-22 | 4.4947E-46 |
| 73 | 6.20585E-25 | 4.53027E-23 | 6.5707E-47 |
| 74 | 1.80222E-25 | 1.33364E-23 | 9.54089E-48 |
| 75 | 5.199E-26 | 3.89925E-24 | 1.37616E-48 |
| 76 | 1.48996E-26 | 1.13237E-24 | 1.97195E-49 |
| 77 | 4.24238E-27 | 3.26663E-25 | 2.80737E-50 |
| 78 | 1.20022E-27 | 9.3617E-26 | 3.97119E-51 |
| 79 | 3.37413E-28 | 2.66556E-26 | 5.58203E-52 |
| 80 | 9.42649E-29 | 7.54119E-27 | 7.79741E-53 |
| 81 | 2.61732E-29 | 2.12003E-27 | 1.0825E-53 |
| 82 | 7.22297E-30 | 5.92284E-28 | 1.49368E-54 |
| 83 | 1.98134E-30 | 1.64451E-28 | 2.04866E-55 |
| 84 | 5.40277E-31 | 4.53833E-29 | 2.79317E-56 |
| 86 | 3.94728E-32 | 3.39466E-30 | 5.10174E-58 |

TABLE 1 CONTINUED

| | | | |
|---|---|---|---|
| 87 | 1.05774E-32 | 9.20237E-31 | 6.8355E-59 |
| 88 | 2.81835E-33 | 2.48015E-31 | 9.10658E-60 |
| 89 | 7.46741E-34 | 6.646E-32 | 1.20643E-60 |
| 90 | 1.96758E-34 | 1.77082E-32 | 1.5894E-61 |
| 91 | 5.15595E-35 | 4.69191E-33 | 2.08247E-62 |
| 92 | 1.34377E-35 | 1.23627E-33 | 2.71372E-63 |
| 93 | 3.48342E-36 | 3.23958E-34 | 3.51735E-64 |
| 94 | 8.98207E-37 | 8.44315E-35 | 4.53479E-65 |
| 95 | 2.30389E-37 | 2.1887E-35 | 5.81584E-66 |
| 96 | 5.87875E-38 | 5.6436E-36 | 7.42003E-67 |
| 97 | 1.49235E-38 | 1.44758E-36 | 9.41803E-68 |
| 98 | 3.76911E-39 | 3.69373E-37 | 1.18932E-68 |
| 99 | 9.47141E-40 | 9.37669E-38 | 1.49432E-69 |
| 100 | 2.3682E-40 | 2.3682E-38 | 1.86818E-70 |

It can be easily seen from (69) (formula for the n-dimensional quasivolume of the n-dimensional quasiball), (114) (formula for the n-dimensional quasi-surface area of the n-dimensional quasiball), and Table 1 that for a fixed $R$, as the dimension, $n$, increases to $+\infty$, the n-dimensional quasivolume and the n-dimensional quasi-surface area decreases to 0.

The coefficient of the n-dimensional quasivolume actually increases at first from $n = 2$ to $n = 5$, where it obtains the maximum value. Thus, the 5-dimensional hyperball has the greatest coefficient for its hypervolume. The situation is similar for the coefficient of n-dimensional quasi-surface area. It increases from $n = 2$ to $n = 7$, where it has the greatest value. Thus, the 7-dimensional hypersphere has the greatest coefficient for its hyper-surface area.

The last column of Table 1 gives some very interesting results. Firstly, it is clear that the n-dimensional quasivolume and the n-dimensional quasi-surface area of the n-dimensional circumscribed quasicube go to $+\infty$ as $n \to +\infty$. Since the n-dimensional quasivolume of the n-dimensional quasiball and the n-dimensional quasi-surface area of the n-dimensional quasisphere both go to 0 as the dimension increases, the ratios $\frac{V_n}{2^n R^n} = \frac{SA_n}{n2^n R^{n-1}}$ tend to 0 even more rapidly than the coefficients of $V_n$ and $SA_n$. That is, in the n-dimensional space, the n-dimensional quasiball (despite being inscribed in the n-dimensional quasicube and being tangent to the sides of the n-dimensional quasicube) becomes so small and negligible comparing to its circumscribed n-dimensional cube. This phenomenon is counter intuitive when one looks at the cases for $n = 2$ and 3. In the 2 and 3 dimensional cases, the inscribed disk has most of the area of its circumscribed square and the inscribed ball has most of its circumscribed cube. In higher dimensions, the behavior is just the opposite.

In this thesis, the generalizations of polar coordinates, the so called n-dimensional polar coordinates (as written in the literature), are quoted and discussed. It appears that these types of coordinates are very helpful for many scientific disciplines; e.g. the two papers from theoretical physics and molecular physics are quoted. However, there are limitations in using them. For example, it is illustrated that when two lines (given in polar coordinates) intersect at the origin, it is not possible to find out that the origin is a point of intersection by solving the system of their polar equations. Also, the Jacobians of such coordinate systems are zero at the origin which violates the theorem for the change of variables in integrals.

Some of the "generalizations" of the polar coordinates actually do not yield the polar coordinates and spherical coordinates for the 2-dimensional and 3-dimensional case, respectively. This thesis suggests a friendlier definition of the polar coordinates, which is called the 2-dimensional quasipolar coordinates, and a consistent generalization, which is called the n-dimensional quasipolar coordinates.

Not only there are various definitions, but also there are various proofs of the Jacobian of these coordinates. Most of the time, the proof of such a Jacobian is wrongfully claimed to be done by mathematical induction, but such a proof has never been found and is probably not possible to be found. Prior to the proof given in this thesis, we found only two proofs in the literature. Both of these proofs are very long and not direct.

A method, which we called a perturbative method is also introduced and described in details. This method allows the original direct approach to be valid in all possible special cases. As a result, the proof is done with only one case.

After this proof, the Jacobian of the n-dimensional quasipolar coordinates is used to calculate directly the n-dimensional quasivolume of the n-dimensional quasiball and the n-dimensional quasi-surface area of the n-dimensional quasisphere. The definitions used for the quasivolumes and the quasi-surface areas are given (as they were classically) with integrals of various types. The calculations are direct and without any use of mathematical induction. The formulas obtained are free of any special functions and could be introduced without any advanced mathematical knowledge.

At the end of the thesis, numerical results are presented together with interpretations of these results.

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