# Lattice Methods For The Valuation of Options with Regime Switching 

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# LATTICE METHODS FOR THE VALUATION OF OPTIONS WITH REGIME SWITCHING 

by

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A thesis submitted in partial fulfullment of the requirements for the

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May 2014

# ABSTRACT <br> LATTICE METHODS FOR THE VALUATION OF OPTIONS WITH REGIME SWITCHING 

by<br>Atul Sancheti<br>Hongtao Yang, Examination Committee Chair<br>Associate Professor of Mathematical Science<br>University of Nevada, Las Vegas

In this thesis, we have developed two numerical methods for evaluating option prices under the regime switching model of stock price processes: the Finite Difference lattice method and the Monte Carlo lattice method.

The Finite Difference lattice method is based on the explicit finite difference scheme for parabolic problems. The Monte Carlo lattice method is based on the simulation of the Markov chain. The advantage of these methods is their flexibility to compute the option prices for any given stock price at any given time. Numerical examples are presented to examine these methods. It has been shown that the proposed methods provides fast and accurate approximations of option prices. Hence they should be helpful for practitioners working in this field.

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## CHAPTER 1

## INTRODUCTION

### 1.1 Options

An option is a contract that gives the holder the right - but not the obligation, to buy or sell an underlying asset at a contractually specified strike price on a range of future dates. There are two different types of options namely Call Options and Put Options. Call Options give the right to buy the underlying asset, whereas Put Options give the right to sell. The price is known as the strike price or exercise price and the date is known as the expiration date or maturity. There are two major styles of options that are traded at exchanges: the European and American options.

The European options can only be exercised at the end of its life or at the expiration date of the contract. These options stop trading a day before than the third friday of the expiration month. In addition, it is not easy to learn the official closing price or the settlement price for the expiration period for European-style options. Moreover, the settlement price is not published until hours after the market opens for trading. Also, European options sometimes trade at a discount rate than its comparable American Option.

The right to exercise is one of the key differences that set apart American options from European Options. These options can be exercised anytime before the option expires. This allows investors more opportunities to exercise the contract and therefore provides a relatively highly price than European Options. It was also interesting to note that a majority of stocks, options and exchange traded funds (ETFs)
have American-style options. The trading for American options cease at the close of business on the third Friday of the expiration month. Also, the settlement price with American options is the regular closing price, or the last trade before the market closes on the third Friday.

There are the over-the-counter traded options such as Asian Options, Bermuda options, and look-back options, which are referred as the exotic options.

The valuation and optimal exercise of derivatives with American-style exercise features is one of the most important and challenging problems in option pricing theory. These types of derivatives are found in all major financial markets including the equity, commodity, foreign exchange, insurance, energy, sovereign, agency, municipal, mortgage, credit, real estate, convertible, swap, and emerging markets. In spite of the recent developments made in this emerging field, the valuation and optimal exercise of American options remains one of the most difficult problems in derivatives finance. This can be mainly attributed to the fact that finite difference and binomial techniques become impractical when considering multiple factor models which provides a better and more detailed description of practical financial problems [1, 2, 3].

### 1.2 Problems

Besides the classic Black-Scholes model for the underlying assets, various other models have been proposed, for example, jump diffusion models, regime switching models, and stochastic volatility models (see $[4,5,6]$ and references cited therein). As in [6], we suppose that the underlying economy switches among $n$ states $\{1,2, \ldots, n\}$, which is modeled by a finite Markov chain $X(t)$ with the rate matrix $Q=\left(q_{i j}\right)$. Let
constant $r_{i}$ be the interest rate when the economy is in the $i$-th state at time $t$, that is, $X(t)=i$. Assume that the stock pays the continuous dividend at constant rate $d$. The stock price process $S(t)$ is modeled by the following stochastic differential equation (SDE):

$$
\begin{equation*}
d S(t)=S(t)\left(\mu_{X(t)} d t+\sigma_{X(t)} d W(t)\right), \quad t>0 \tag{1.1}
\end{equation*}
$$

where $W(t)$ is a standard Brownian motion under the risk neutral measure, $\mu_{i}=r_{i}-d$, and constant $\sigma_{i}$ is the stock volatility in the $i$-th state of economy.

Consider an American call option with strike price $\$ K$ and expiry date $T$ years. Denote by $C_{i}(S, t)$ the call price in the $i$-th state. Let $C(S, t)=\left(C_{1}(S, t), \ldots, C_{n}(S, t)^{T}\right)$. As usual, we have the following variational inequality problem:

$$
\begin{align*}
& C_{i, t}(S, t)+\mathcal{A}_{i} C(S, t) \leq 0, \quad S>0, \quad 0<t \leq T  \tag{1.2}\\
& C_{i}(S, t) \geq(S-K)^{+}, \quad S \geq 0, \quad 0 \leq t \leq T  \tag{1.3}\\
& \left(C_{i, t}(S, t)+\mathcal{A}_{i} C(S, t)\right)\left(C_{i}(S, t)-(S-K)^{+}\right)=0, \quad S \geq 0,0<t \leq T  \tag{1.4}\\
& C_{i, t}(0, t)=0, \quad 0 \leq t \leq T  \tag{1.5}\\
& C_{i}(S, T)=(S-K)^{+}, \quad S \geq 0 \tag{1.6}
\end{align*}
$$

for $i=1, \ldots, n$, where $\langle\cdot, \cdot\rangle$ is the usual inner product on $R^{2}$, and

$$
\mathcal{A}_{i} C(S, t)=\frac{1}{2} \sigma_{i}^{2} S^{2} C_{i, S S}(S, t)+\mu_{i} S C_{i, S}(S, t)-r_{i} C_{i}(S, t)+\left\langle Q C(S, t), e_{i}\right\rangle .
$$

Similarly, we have the variational inequality problem for the American put option:

$$
\begin{align*}
& P_{i, t}(S, t)+\mathcal{A}_{i} C(S, t) \leq 0, \quad S>0, \quad 0<t \leq T  \tag{1.7}\\
& P_{i}(S, t) \geq(K-S)^{+}, \quad S \geq 0, \quad 0 \leq t \leq T \tag{1.8}
\end{align*}
$$

$$
\begin{align*}
& \left(P_{i, t}(S, t)+\mathcal{A}_{i} C(S, t)\right)\left(P_{i}(S, t)-(K-S)^{+}\right)=0, \quad S \geq 0,0<t \leq T  \tag{1.9}\\
& P_{i, t}(0, t)=0, \quad 0 \leq t \leq T  \tag{1.10}\\
& P_{i}(S, T)=(K-S)^{+}, \quad S \geq 0 \tag{1.11}
\end{align*}
$$

for $i=1, \ldots, n$.

### 1.3 Thesis Structure

Numerical methods have been extensively investigated for valuation of American options and other path-dependent financial derivatives for more than three decades (see [7], [8], [9], and references cited therein). In this thesis, we shall develop two lattice methods for the above variational inequality problems. One is the generalization of the lattice method proposed in [12] when there are only two states of economy. The other is a lattice method based on the Monte Carlo simulation of the Markov chain. Lattice methods are more attractive to practitioners since they can be easily implemented. Moreover, it is more flexible to compute option prices and hedge ratios at any given point. A favorable feature of our methods is that there is only one set of nodes for stock price for all states.

The remaining of the thesis is outlined as follows. In Chapter 2, we shall review basic theory about linear complementary problems (LCP) since these problems are formed by discretizing variational inequality problems (1.2)-(1.6) and (1.7)-(1.11). Especially, two pivoting algorithms: Chandrasekaran and Lemke methods are described for LCPs with $M$-matrices. In Chapter 3, a Finite Difference lattice method
is proposed to compute the option prices for the given stock prices at given times. In Chaper 4, we shall develop a lattice method based on Monte Carlo simulation of Markov Chain. Numerical examples are presented in Chapter 5 to examine our new methods. The Conclusion remarks are given in the last chapter, Chapter 6.

## CHAPTER 2

## LINEAR COMPLEMENTARY PROBLEMS

### 2.1 Introduction

Let $M$ be a $n \times n$ matrix in $\mathbb{R}^{n \times n}$ and $q$ a column vector in $\mathbb{R}^{n}$. Then the linear complementary problem, denoted by $\operatorname{LCP}(q, M)$, is to find $w, z \in \mathbb{R}^{n}$ such that

$$
\begin{align*}
& w-M z=q  \tag{2.1}\\
& w^{T} z=0  \tag{2.2}\\
& w \geq 0, \quad z \geq 0 \tag{2.3}
\end{align*}
$$

where $w^{T}$ is the transpose of $w$ and $w \geq 0$ means that every component of $w$ is nonnegative.

The linear complementarity problems (LCPs) can be considered as a more general case for linear, quadratic and bimatrix problems. The study of a LCP has led to development of several highly effective algorithms which aids in solving the highly complex problems in an efficient manner. In this Chapter, we introduce the complementary pivot algorithm for solving LCPs, in particular, the Lemke method.

### 2.2 Solution Existence and Uniqueness

A matrix is called a $P$-matrix if its principal minors are positive. In other words, a matrix is a $P$-matrix if and only if the real eigen values of the principal submatrices of $M$ are positive. Thus positive definite matrices are $P$-matrices. Concerning the solution existence and uniqueness of the LCP (2.1)-(2.3), we have the following result ([10]).

Theorem 2.1. (Samelson, Thrall and Wesler) The LCP ( $q, M$ ) has a unique solution for every $q$ if and only $M$ is a $P$-matrix.

A matrix $B$ is called nonnegative (write $B \geq 0$ ) if all element of $B$ are nonnegative numbers. We say that a matrix $A$ is an $M$-matrix if there is a positive number $s$ and a nonnegative matrix $B$ such that $A=s I-B$ and $s>\rho(A)$. A matrix of form $s I-B$ is an $M$-matix if and only if all principal minors of $A$ are positive. Hence, we have the following corollary.

Corollary 2.1. If $M$ is an $M$-matrix, then $L C P(q, M)$ has a unique solution for every $q$.

It should be pointed out that the corresponding matrix $M$ is an $M$-matrix for our lattice method for the regime-switching problems.

### 2.3 An Augmented LCP

Let $d \in \mathbb{R}^{n}$ be a positive vector and $s$ be a positive number. For the LCP $(q, M)$, the corresponding augmented LCP, denoted by $\operatorname{ALCP}(q, M ; d, s)$ is $\operatorname{LCP}(\widetilde{q}, \widetilde{M})$, where

$$
\widetilde{q}=\left[\begin{array}{l}
s \\
q
\end{array}\right], \quad \widetilde{M}=\left[\begin{array}{cc}
0 & -d^{T} \\
d & M
\end{array}\right] .
$$

Here $d$ is called the covering vector. The LCP $(\widetilde{q}, \widetilde{M})$ reads as follows: Find $z \in \mathbb{R}^{n}$ and $t \in \mathbb{R}$ such that

$$
\begin{aligned}
& \sigma=s+0 t-d^{T} z \geq 0, \quad t \geq 0, \quad t \sigma \geq 0 \\
& w=q+M z+t d \geq 0, \quad z \geq 0, \quad z^{T} w=0
\end{aligned}
$$

We can see that a solution $(t, z)$ of the $\operatorname{ALCP}(q, M ; d, s)$ with $t=0$ provides a solution $z$ of the LCP $(q, M)$. Furthermore, we have the following results ([10]).

Theorem 2.2. (a) For every given $d>0$ and $s>0$, the $\operatorname{ALCP}(q, M ; d, s)$ has a solution.
(b) Suppose that there is a positive number $k$ such that if $x \geq 0$ and $e^{T} x=$ $k$ then $x^{T}(q+M x) \geq 0$, where $e=(1, \ldots, 1)^{T}$. Let $(t, z)$ be a solution of the $\operatorname{ALCP}(q, M ; e, k)$. Then $t=0$ and thus $z$ is a solution of the $\operatorname{LCP}(q, M)$.

### 2.4 Pivoting Methods for the LCP $(q, M)$

From now on, we shall assume that $M$ is an $P$-matrix. Let $w=\left(w_{1}, \ldots, w_{n}\right)^{T}$ and $z=\left(z_{1}, \ldots, z_{n}\right)^{T}$ be a solution of the $\operatorname{LCP}(q, M)$. Notice that equation (2.3) implies that one element of each pair $\left(w_{j}, z_{j}\right)$ must be zero. If one is positive then the other must be zero. Hence the pair $\left(w_{j}, z_{j}\right)$ is called the $j$-th complementary pair of variables.

Denoted by $I_{j}$ and $M_{j}$ the $\mathrm{j}-t h$ columns of the identity matrix $I$ and $M$, repectviely. Then we can rewrite (2.1) as follows

$$
\begin{equation*}
q=I w+(-M) z=\sum_{j=1}^{n} w_{j} I_{j}+\sum_{j=1}^{n} z_{j}\left(-M_{j}\right) \tag{2.1}
\end{equation*}
$$

Thus solving the LCP $(q, M)$ can be interpreted as finding a complementary pair of nonnegative vectors $w$ and $z$ such that $q$ is a linear combination of $n$ vectors consisting of the column vectors of $I$ and $M$. This intepretation leads to pivoting methods for the LCP $(q, M)$.

We shall group the $2 n$ variables $\{w, z\}=\left\{w_{1}, \ldots, w_{n}, z_{1}, \ldots, z_{n}\right\}$ into basic vari-
ables $\left\{y_{1}, \ldots, y_{n}\right\}$ and nonbasic variables $\left\{v_{1}, \ldots, v_{n}\right\}$. It follows from equation (2.1) that

$$
\begin{equation*}
w=q+M z . \tag{2.2}
\end{equation*}
$$

Here variables $\left\{w_{1}, \ldots, w_{n}\right\}$ are basic and $\left\{z_{1}, \ldots, z_{n}\right\}$ are nonbasic. That is, the basic variables are the variables that depends on the nonbasic ones. Consider the $r$-th equation of the system (2.2):

$$
w_{r}=q_{q}+m_{r 1} z_{1}+\ldots+w_{r n} z_{n} .
$$

If $m_{r s} \neq 0$, we can solve for $z_{s}$ in terms of $w_{r}$ and all the other nonbasic varibales $z_{j}$ with $j \neq s$. Then we have

$$
z_{s}=-\frac{q_{r}}{m_{r s}}+\sum_{j \neq s}\left(-\frac{m_{r j}}{m_{r s}}\right) z_{j}+\frac{1}{m_{r s}} w_{r}
$$

After substituting this expression for $z_{s}$ into all the other eqaution in (2.2), we have

$$
w_{i}=q_{i}-q_{r} \frac{m_{i s}}{m_{r s}}+\sum_{j \neq s}\left(m_{i j}-m_{r j} \frac{m_{i s}}{m_{r s}}\right) z_{j}+\frac{m_{i s}}{m_{r s}} w_{r}, \quad i \neq r .
$$

This operation is called simple pivoting, which exchanges the roles of $w_{s}$ and $z_{s}$. Namely, $w_{s}$ and $z_{s}$ becomes nonbasic and basic, respectively. Now the basic variables are $\left\{w_{1}, \ldots, w_{s-1}, z_{s}, w_{s+1}, \ldots, w_{n}\right\}$ and the nonbasic variables are $\left\{z_{1}, \ldots, z_{s-1}, w_{s}\right.$, $\left.z_{s+1}, \ldots, z_{n}\right\}$. The LCP $(q, M)$ can be represented by the following tableau:

| $w$ | $z$ |  |
| :---: | :---: | :---: |
| $I$ | $-M$ | $q$ |
| $w \geq 0, \quad z \geq 0$ |  |  |

A pivoting method consists of a sequences of pivoting steps to transform the above initial tableau. Let the resulting tableau be as follows:

| $v$ | $y$ |  |
| :---: | :---: | :---: |
| $I$ | $-\bar{M}$ | $\widetilde{q}$ |

where $v$ is the vector for the basic variables and $y$ is the vector for the nonbasic ones. If $\widetilde{q} \geq 0$, a solution has been found and it can be obtained by letting all the nonbasic variables be 0 and basic ones be equal to the corresponding elements of $\widetilde{q}$.

A detailed account in pivoting method can be found in Cottle et al [10]. For our purpose, we only need the Chandrasekaran and Lemke methods.

### 2.5 Chandrasekaran Method

The following Chandrasekaran's Method is a drect application of the above pivoting method to the LCP $(q, M)$ when $M$ is a $Z$-matrix.

Algorithm 1. Chandrasekaran's Method to solve LCP
Consider the LCP $(q, M)$ as represented by the initial tableau (2.3) with $w=\left(w_{1}, w_{2}, \ldots, w_{n}\right)$ as the initial complementary basic vector.
if $q \geq 0$, i.e. $w$ is a feasible basis, then
$(w, z)=(q, 0)$ (Complementary Basic Feasible Solution);
break;
else do
Display the tableau: $\operatorname{tab}=[\operatorname{eye}(M)-M q]$
$\bar{q}=\operatorname{tab}(:$, end $) ;$
if $\bar{q} \geq 0$;
Present basic vector is a complementary feasible basic vector;
break;
else do
Find $t$, such that $\bar{q}_{t} \leq 0$;
if $-m_{t t} \geq 0$
No nonnegative solution or LCP $(q, M)$ has no solution;
break;
else do
Update the tableau by pivoting at row $t$ and column $t+n$;
end do

A matrix $M=\left(m_{i j}\right)$ is a $Z$-matrix if all its off diagonal entries are nonpositive, that is $m_{i j} \leq 0$ for all $i \neq j$. It can be easily verified that in tableau (2.3) for any $t$ $=1$ to $n$, all the entries in row $t$ are nonnegative except for the entry in column $z_{t}$. Hence, all the pivot elements encountered during the Chandrasekaran's algorithm are strictly negative. In addition, once a pivot has been performed in a row, the value of the updated right hand side constant remains negative for all subsequent steps. Moreoever, once a variable $z_{t}$ has been made a basic variable, it stays as a basic variable and its value remain nonnegative in all subsequent steps. As at most one principal pivot step is performed in each row, hence the algorithm terminates in at most $n$ pivot steps either with the conclusion of infeasibility or with a complementary feasible basis [11].

### 2.6 The Lemke Method

The Lemke method is a pivoting methods for the $\operatorname{ALCP}(q, M ; e, s)$, where $s$ will be determined by the algorithm. The advantage of considering the $\operatorname{ALCP}(q, M ; e, s)$ instead of the LCP $(q, M)$ is that the $\operatorname{ALCP}(q, M ; e, s)$ has a solution (see Theorem 2.2. Also, the Lemke method will either find a solution or indicate no solution for the $\operatorname{LCP}(q, M)$.

The Lemke method uses complementary pivoting schemes and provide a choice of driving variable. One of the major advantages of these complementary pivoting schemes is the very fact that these are relatively easy to state, more versatile and does not depend on the invariance of matrix classes under principal pivoting.

Algorithm 2. Lemke Method to solve LCP
Initialization Step:
if $q \geq 0$, then
$(w, z)=(q, 0)$ (Complementary Basic Feasible Solution);
break;
else do
Display the tableau: tab $=\left[\operatorname{eye}(M)-M-z_{0} q\right]$
let $q_{s}=\min \left\{q_{i}: 1 \leq i \leq n\right\}$
Update the tableau by pivoting at row $s$ and column $z_{0}$
$\operatorname{tab}(s:)=,\operatorname{tab}(s,:) \cdot / \operatorname{tab}\left(s, t_{m}-1\right)$
for $i=1, \ldots, m$, do

$$
\text { if } i \neq s, \operatorname{tab}(i,:)=\operatorname{tab}(i,:)-\operatorname{tab}(s,:) * \operatorname{tab}\left(i, t_{m}-1\right) / \operatorname{tab}\left(s, t_{m}-1\right)
$$

end do

Let $y_{s}=z_{s}$, GOTO Main Step
end do
Main Step
STEP 1: Let $d_{s}$ be the updated column under variable $y_{s}$, while $\left(d_{s}>0\right)$
Determine index $r$ by the minimum ratio test:

$$
\frac{\bar{q}_{r}}{d_{r s}}=\min _{1 \leq i \leq m}\left\{\frac{\bar{q}_{i}}{d_{i s}}: d_{i s}>0\right\}
$$

If the basic variable at row $r$ is $z_{0}$, GOTO STEP 3 else GOTO STEP 2.

STEP 2: Update the tableau by pivoting at row $r$ and column $y_{s}$ if the variable leaving the basis is $w_{l}$, then let $y_{s}=z_{l}$ else if the variable leaving the basis is $z_{l}$, then let $y_{s}=w_{l}$ GOTO STEP 1
STEP 3: Update the tableau by pivoting at row $y_{s}$ column and $z_{0}$ row, break; (Complementary Basic Feasible Solution)
STEP 4: Ray $R=\left\{\left(w, z, z_{0}\right)+\lambda d: \lambda \geq 0\right\}$,
where every point in $R$ satisfies equations (2.1), (2.2), and (2.3)
end do (Almost Complementary Basic Feasible Solution)

We have the following results about the convergence of the Lemke method.

Theorem 2.3. When applied to a nondegenerate instance of $(q, d, M)$, Lemke's Algorithm will terminate in finitely many steps with either a secondary ray or else a complementary feasible solution of $(q, d, M)$ and hence with a solution of $(q, M)$ [10].

When Lemke's algorithm terminates with a secondary ray, it usually requires the strict positivity of the covering vector $d$.

Theorem 2.4. If Lemke's Algorithm applied to ( $q, d, M$ ) terminates with a secondary ray, then $M$ reverses the sign of some nonzero nonnegative vecotr $\overline{(z)}$ [10], that is

$$
\begin{equation*}
\bar{z}_{i}\left(M \bar{z}_{i}\right) \leq 0 \quad i=1, \ldots, n \tag{2.1}
\end{equation*}
$$

Hence the above theorem implies that the Lemke's Algorithm cannot terminate in a secondary ray when $M \in P$, as in a $P$ the sign of a nonzero vector is never reversed [10]. Thus for any nondegenrate linear complimentarity problem of the $P$-matrix type, Lemke's Algorithm will obtain its solution.

## CHAPTER 3

## A FINITE DIFFERENCE LATTICE METHOD

In this chapter, we extend the simple lattice method proposed in [12] to compute the option prices for the given stock price $S_{0}$ and time to the expiration date $T_{0}$. Since the method is based on the forward Euler scheme for parabolic problems, we call it the Finite Difference lattice method. We only consider the call option problem since the put option problem can be treated in the same fashion.

Consider the variable transforms

$$
S=K e^{x}, \quad C_{i}(S, T-t)=K u_{i}(x, t), \quad i=1, \ldots, n
$$

The variational inequality problem (1.2)-(1.6) can be reformulated into

$$
\begin{align*}
& \frac{\partial u_{i}}{\partial t}+\mathcal{B}_{i} u_{i}-\sum_{j=1}^{n} \xi_{i j} u_{i} \geq 0, \quad-\infty<x<\infty, 0 \leq t<T  \tag{3.1}\\
& u_{i}(x, t) \geq f(x), \quad-\infty<x<\infty, 0 \leq t<T  \tag{3.2}\\
& \left(\frac{\partial u_{i}}{\partial t}+\mathcal{B}_{i} u_{i}-\sum_{j=1}^{n} q_{i j} u_{i}\right)\left(u_{i}(x, t)-f_{i}(x)\right)=0, \quad-\infty<x<\infty, 0 \leq t<T  \tag{3.3}\\
& u_{i}(-\infty, t)=0, \quad 0 \leq t \leq T  \tag{3.4}\\
& u_{i}(x, 0)=f_{i}(x), \quad-\infty<x<\infty \tag{3.5}
\end{align*}
$$

for $i=1, \ldots, n$, where

$$
\begin{aligned}
& \mathcal{B}_{i} u_{i}=-\gamma_{i} \frac{\partial^{2} u_{i}}{\partial x^{2}}+\nu_{i} \frac{\partial u_{i}}{\partial x}+r_{i} u_{i} \\
& \gamma_{i}=\frac{1}{2} \sigma_{i}^{2}, \quad \nu_{i}=\gamma_{i}-\mu_{i}, \quad f_{i}(x, t)=\left(e^{x}-1\right)^{+}
\end{aligned}
$$

For a given positive integer $M$, let $k=T_{0} / M$ and $t_{m}=m k$ for $m=0,1, \ldots, M$.

For a positive number $\sigma$, let $h=\sigma \sqrt{k}$ be the mesh size in $x$, and let

$$
x_{j}=\log \left(\frac{S_{0}}{K}\right)+j h, \quad j=-M, \ldots, M
$$

Denote by $u_{i, j}^{m}$ be the approximation of $u\left(x_{j}, t_{m}\right)$. Discretizing (3.1)-(3.3) using the finite difference methods, we have the following LCP:

$$
\begin{aligned}
& \frac{u_{i, j}^{m}-u_{i, j}^{m-1}}{k}+\mathcal{L}_{i} u_{i, j}^{m-1}+r_{i} u_{i, j}^{m}-\sum_{j=1}^{n} \xi_{i j} u_{i, j}^{m} \geq 0, \quad u_{i, j}^{m} \geq f_{i, j} \\
& \left(\frac{u_{i, j}^{m}-u_{i, j}^{m-1}}{k}+L_{i} u_{i, j}^{m-1}+r_{i} u_{i, j}^{m}-\sum_{j=1}^{n} \xi_{i j} u_{i, j}^{m}\right)\left(u_{i, j}^{m}-f_{n, j}\right)=0
\end{aligned}
$$

for $i=1,2, \ldots, n$, where $f_{i, j}=f_{i}\left(x_{j}\right)$ and

$$
L_{i} u_{i, j}^{m-1}=-\gamma_{i} \frac{u_{i, j+1}^{m-1}-2 u_{i, j}^{m-1}+u_{i, j-1}^{m-1}}{h^{2}}+\nu_{i} \frac{u_{i, j+1}^{m-1}-u_{i, j-1}^{m-1}}{2 h} .
$$

The above LCP can be rewritten into the following matrix form:

$$
\begin{align*}
& A U_{j}^{m} \geq G_{j}^{m}, \quad U_{j}^{m} \geq F_{j}  \tag{3.6}\\
& \left(A U_{j}^{m}-G_{j}^{m}\right)\left(U_{j}^{m}-F_{j}\right)=0 \tag{3.7}
\end{align*}
$$

where

$$
\begin{aligned}
& A=\left[\begin{array}{cccc}
1+k\left(r_{1}+q_{11}\right) & -k q_{12} & \cdots & -k q_{1 n} \\
-k q_{21} & 1+k\left(r_{2}+q_{22}\right) & \cdots & -k q_{2 n} \\
\vdots & \vdots & & \vdots \\
-k q_{n 1} & -k q_{n 2} & \cdots & 1+k\left(r_{n}+q_{n n}\right)
\end{array}\right], \\
& U_{j}^{m}=\left[\begin{array}{c}
u_{1, j}^{m} \\
u_{2, j}^{m} \\
\cdots \\
u_{n, j}^{m}
\end{array}\right], \quad F_{j}=\left[\begin{array}{c}
f_{1, j} \\
f_{2, j} \\
\cdots \\
f_{n, j}
\end{array}\right], \quad G_{j}^{m-1}=\left[\begin{array}{c}
g_{1, j}^{m-1} \\
g_{2, j}^{m-1} \\
\cdots \\
g_{n, j}^{m-1}
\end{array}\right], \\
& g_{i, j}^{m-1}=P_{i}^{+} u_{i, j+1}^{m-1}+P_{i}^{0} u_{i, j}^{m-1}+P_{i}^{-} u_{i, j-1}^{m-1}, \\
& P_{i}^{+}=\frac{\gamma_{i}}{\sigma^{2}}-\frac{\sqrt{k} \nu_{i}}{2 \sigma}, \quad P_{i}^{0}=1-\frac{2 \gamma_{i}}{\sigma^{2}}, \quad P_{i}^{-}=\frac{\gamma_{i}}{\sigma^{2}}+\frac{\sqrt{k} \nu_{i}}{2 \sigma} .
\end{aligned}
$$

Notice that

$$
\begin{equation*}
P_{i}^{+}+P_{i}^{0}+P_{i}^{-}=1, \quad \forall i=1,2, \ldots, n . \tag{3.8}
\end{equation*}
$$

We can regard $P_{i}^{-}$and $P_{i}^{+}$as the probabilities for which the stock price goes down and up and $P_{i}^{0}$ as the probability for which the stock price does not change when the underlying economy is in the $i$-th state. To this end, we shall choose $\sigma$ and $M$ such that

$$
P_{i}^{-} \geq 0, \quad P_{i}^{0} \geq 0, \quad P_{i}^{+} \geq 0
$$

which are equivalent to the following constraints on $\sigma$ and $M$ :

$$
\begin{equation*}
\sigma \geq \max _{1 \leq i \leq n} \sigma_{i}, \quad M \geq \sigma^{2} T_{0} \max _{1 \leq i \leq n} \frac{\nu_{i}^{2}}{\sigma_{i}^{4}} . \tag{3.9}
\end{equation*}
$$

Let $S_{j}=S_{0} e^{x_{j}}$ for $j=-M, \ldots, M$. Denote by $C_{i, j}^{m}$ the approximation of $C_{i}\left(S_{j}, T-t_{m}\right)$. Let

$$
\begin{equation*}
\widetilde{C}_{i, j}^{m}=P_{i}^{+} C_{i, j+1}^{m-1}+P_{i}^{0} C_{i, j}^{m-1}+P_{i}^{-} C_{i, j-1}^{m-1}, \quad i=1, \ldots, n \tag{3.10}
\end{equation*}
$$

Recall that $C\left(S, T-t ; e_{i}\right)=K u_{i}(x, t)$ for $x=\log (S / K)$. The LCP for $U_{j}^{m}$ becomes the following LCP for $C_{j}^{m}=\left(C_{1, j}^{m}, \ldots, C_{n, j}^{m}\right)^{T}$ :

$$
\begin{align*}
& A C_{j}^{m} \geq \widetilde{C}_{j}^{m}, \quad C_{j}^{m} \geq \Phi_{j}  \tag{3.11}\\
& \left(A U_{j}^{m}-\widetilde{C}_{j}^{m}\right)\left(U_{j}^{m}-\Phi_{j}\right)=0, \tag{3.12}
\end{align*}
$$

where

$$
\widetilde{C}_{j}^{m}=\left(\widetilde{C}_{1, j}^{m}, \ldots, \widetilde{C}_{n, j}^{m}\right)^{T}, \quad \Phi_{j}=\left(\left(S_{j}-K\right)^{+}, \ldots,\left(S_{j}-K\right)^{+}\right)^{T}
$$

We have the following algorithm to compute $C_{i, 0}^{M}$, the approximation of $C_{i}\left(S_{0}, T-T_{0}\right)$ :

Algorithm 3. A Finite Difference lattice algorithm for the American call

1. Set

$$
C_{i, j}^{M}=\left(S_{j}-K\right)^{+}, \quad j=-M, \ldots, M, \quad i=1, \ldots, n
$$

2. For $m=1,2, \ldots, M$, do

For $j=-(M-m), \ldots, M-m$, do
(1) Compute $\widetilde{C}_{j}^{m}$ by (3.10).
(2) Solve the LCP (3.11)-(3.12) for $C_{j}^{m}$ by Algorithm 1 or 2 .

End do
End do

The inequalities in (3.1) and (3.2) become equalities for the European call option problem. Then we have the following algorithm to compute $c_{i, 0}^{M}$, the approximation of the European call price $c_{i}\left(S_{0}, T-T_{0}\right)$ :

Algorithm 4. A Finite Difference lattice algorithm for the European call

1. Set

$$
c_{i, j}^{M}=\left(S_{j}-K\right)^{+}, \quad j=-M, \ldots, M, \quad i=1, \ldots, n
$$

2. For $m=1,2, \ldots, M$, do

For $j=-(M-m), \ldots, M-m$, do
(i) Compute $\tilde{c}_{j}^{m}$ by

$$
\tilde{c}_{i, j}^{m}=P_{i}^{+} c_{i, j+1}^{m-1}+P_{i}^{0} c_{i, j}^{m-1}+P_{i}^{-} c_{i, j-1}^{m-1}, \quad i=1, \ldots, n .
$$

(ii) Solve the following equation for $c_{j}^{m}$ :

$$
A c_{j}^{m}=\tilde{c}_{j}^{m} .
$$

End do
End do

## CHAPTER 4

## MONTE CARLO METHODS

In this chapter, we develop two new methods that are based on the Monte Carlo simulation of the markov chain. In particular, the method will be named as the Monte Carlo lattice method (the MC lattice method for simplicity) for American options. Again, we only consider the call option problem since the put option problem can be treated in the same fashion.

### 4.1 American call options

Consider the American call options with strike price $\$ K$ and expiration date $T$ years. Its price is denote by $C\left(S_{0}, t_{0}\right)$ when the stock price is equal to $S_{0}$ at time $t_{0}$.

We shall follow the idea in the introduction section of [6]. For a given sample path $X(t)$ of the Markov chain, we let

$$
\sigma(t)=\sigma_{X(t)}, \quad \mu(t)=\mu_{X(t)} .
$$

Solving the following SDE

$$
d S(t)=S(t)(\mu(t) d t+\sigma(t) d W(t))
$$

we get

$$
S(T)=S(t) \exp \left(\int_{t}^{T}\left(\mu(s)-\frac{1}{2} \sigma(s)^{2}\right) d s+\int_{t}^{T} \sigma(s) d W(t)\right)
$$

Then the American call price at time $t$ when $X(t)=i$ and $S(t)=S$ is given by

$$
C_{i}(S, t)=\mathrm{E}\left[C(S, t, X(\cdot)) \mid \mathcal{G}_{T}\right]
$$

where the $c(S, t, X(\cdot)$ is the American call price with given sample path $X(\cdot)$ and $\mathcal{G}_{T}=\sigma\{X(s): t \leq s \leq T\}$. As usual, we have

$$
\begin{equation*}
C(S, t, X(\cdot))=\max _{t \leq \tau \leq T} \mathrm{E}\left[\exp \left(-\int_{t}^{\tau} r(s) d s\right)(S(\tau)-K)^{+} \mid \mathcal{F}_{t}\right] \tag{4.1}
\end{equation*}
$$

where $\tau$ is a stopping time taking value in interval $[t, T]$.
Now let us show how to compute $C(S, t, X(\cdot))$ by a lattice method. Let $Y(t)=$ $\log (S(x))$. It follows from Itō's Lemma that

$$
\begin{equation*}
d Y(t)=\nu(t) d x+\sigma(t) d W(t) \tag{4.2}
\end{equation*}
$$

where

$$
\nu(t)=\mu(t)-\frac{1}{2} \sigma(t)^{2} .
$$

Recall that the sample path $X(t)$ is a piecewise right-continuous functon with values in the set $\{1, \ldots, n\}$. Let

$$
t_{0}<t_{1}<\ldots<t_{m}=T
$$

be a partion of the interval $\left[t_{0}, T\right]$, where $M$ is a positive integer. Here we have assumed that the discontinuity of $X(t)$ occurs at the partition nodes. Discretizing the SDE (4.2) by the Euler-Maruyama scheme, we have

$$
Y_{m}-Y_{m-1}=\nu\left(t_{m-1}\right) \Delta t+\sigma\left(t_{m-1}\right) \sqrt{\Delta t} \xi_{m-1}, \quad m=1,2, \ldots, M
$$

where $Y_{m}$ is the approximation of $Y\left(t_{m}\right)$ and $\xi_{m} \sim N(0,1)$. Let $\Delta y$ be positive number. Assume that $P_{m}^{+}, P_{m}^{0}$ and $P_{m}^{-}$are the probabilities under which $Y_{m}$ takes values $Y_{m-1}+\Delta y, Y_{m-1}$ and $Y_{m-1}-\Delta y$, repectively. Then we have by matching the
mean and variance of the change $Y_{m}-Y_{m-1}$ :

$$
\begin{aligned}
& P_{m}^{+}+P_{m}^{0}+P_{m}^{-}=1 \\
& (\Delta y) P_{m}^{+}+(0) P_{m}^{0}+(-d y) P_{m}^{-}=\nu\left(t_{m-1}\right) \Delta t \\
& (\Delta y)^{2} P_{m}^{+}+(0)^{2} P_{m}^{0}+(-d y)^{2} P_{m}^{-}=\sigma\left(t_{m-1}\right)^{2} \Delta t .
\end{aligned}
$$

Sovling the above system for $P_{m}^{+}, P_{m}^{0}$ and $P_{m}^{-}$, we obtain

$$
\begin{aligned}
& P_{m}^{+}=\frac{\sigma\left(t_{m-1}\right)^{2} \Delta t}{2 \Delta y^{2}}+\frac{\nu\left(t_{m-1}\right) \Delta t}{2 \Delta y}, \\
& P_{m}^{0}=1-\frac{\sigma\left(t_{m-1}\right)^{2} \Delta t}{\Delta y^{2}} \\
& P_{m}^{-}=\frac{\sigma\left(t_{m-1}\right)^{2} \Delta t}{2 \Delta y^{2}}-\frac{\nu\left(t_{m-1}\right) \Delta t}{2 \Delta y} .
\end{aligned}
$$

If $\Delta y$ is chosen such that

$$
\begin{equation*}
\Delta y \geq \bar{\sigma} \sqrt{\Delta t} \quad \text { and } \quad \bar{\sigma}^{2} \leq \bar{\nu} \Delta y \tag{4.3}
\end{equation*}
$$

where $\bar{\sigma}=\max _{t_{0} \leq t \leq T} \sigma(t)$ and $\bar{\nu}=\max _{t_{0} \leq t \leq T}|\nu(t)|$. Then $P_{m}^{+}, P_{m}^{0}$ and $P_{m}^{-}$are nonnegative.
Let $S_{j}=S_{0} e^{j \Delta y}$ for $j=-M, \ldots, M$. Denote by $C_{j}^{m}$ the approximations of option price $C\left(S_{j}, t_{m}, X\right)$. Let $r_{m}=r\left(t_{m}\right)$, where $r(t)$ is the interest rate a time $t$. We have Algorithm 5 to compute $C_{0}^{M}$, the approximation of $C\left(S_{0}, t_{0}, X\right)$. Furthermore, we have Algorithm 6 to compute the approximation of call price $C_{i}\left(S_{0}, t_{0}\right)$. We should point out that these algorithms can be applied to European options. Indeed, we just need to remove step (ii) in Algorithm 5.

Algorithm 5. A lattice algorithm to compute $C\left(S_{0}, t_{0}, X\right)$

1. Set

$$
C_{j}^{M}=\left(S_{j}-K\right)^{+}, \quad j=-M, \ldots, M .
$$

2. For $m=1,2, \ldots, M$, do

For $j=-(M-m), \ldots, M-m$, do
(i) $\tilde{C}_{j}^{m}=e^{-r_{j} \Delta t}\left(P_{m}^{-} C_{j-1}^{m-1}+P_{m}^{0} C_{j}^{m-1}+P_{m}^{+} C_{j+1}^{m-1}\right)$,
(ii) $C_{j}^{m}=\max \left(\tilde{C}_{j}^{m}, S_{j}-K\right)$.

End do
End do

Algorithm 6. A MC lattice algorithm for the American call

1. Input a positive integer $N$.
2. Set $C=0$.
3. For $m=1,2, \ldots, N$, do
(i) Generate a sample path $\left\{X(s): t_{0} \leq s \leq T\right\}$ with $X\left(t_{0}\right)=i$.
(ii) Compute $C\left(S_{0}, t_{0}, X(\cdot)\right)$ by Algorithm 5 .
(iii) Let $C=C+C\left(S_{0}, t_{0}, X(\cdot)\right)$.

End do
4. The American call price at state $i$ is given by $\frac{C}{N}$.

### 4.2 European call options

Consider the European with strike price $\$ K$ and expiration date $T$ years. Denote by $c(S, t)$ the European call price when the stock price is equal to $S$ at time $t$. For a given sample path $X(t)$ of the Markov chain, the European call price at time $t$ is ([6]):

$$
\begin{align*}
c(S(t), t, X(\cdot)) & =\mathrm{E}\left[\exp \left(-\int_{t}^{T} r(s) d s\right)(S(T)-K)^{+} \mid \mathcal{F}_{t}\right]  \tag{4.1}\\
& =S(t) N\left(d_{1}(t, T)\right)-\exp (-R(t, T)) N\left(d_{2}(t, T)\right)
\end{align*}
$$

where

$$
\begin{aligned}
& \mathcal{F}_{t}=\sigma\{W(s): 0 \leq s \leq t\} \\
& r(t)=r_{X(t)}, \quad R(t, T)=\int_{t}^{T} r(s) d s \\
& \Theta(t, T)=\int_{t}^{T} \mu(s) d s, \quad V(t, T)=\int_{t}^{T} \sigma(s)^{2} d s \\
& d_{1}(t, T)=\frac{\log (S(t) / K)+\Theta(t, T)+\frac{1}{2} V(t, T)}{\sqrt{V(t, T)}} \\
& d_{2}(t, T)=d_{1}(t, T)-\sqrt{V(t, T)}
\end{aligned}
$$

Hence, the European call price at time $t$ when $X(t)=i$ and $S(t)=S$ is given by

$$
c_{i}(S, t)=\mathrm{E}\left[c(S, t, X(\cdot)) \mid \mathcal{G}_{T}\right]
$$

We have the following Monte Carlo algorithm for the European option price $c_{i}(S, t)$.

Algorithm 7. An Monte Carlo algorithm for the European call

1. Input a positive integer $N$.
2. Set $c=0$.
3. For $m=1,2, \ldots, N$, do
(i) Generate a sample path $\{X(s): t \leq s \leq T\}$ with $X(t)=i$.
(ii) Compute $c(S, t, X(\cdot))$ by formula (4.1).
(iii) Let $c=c+c\left(S_{0}, t, X(\cdot)\right.$.

End do
4. The European call price is given by $\frac{c}{N}$.

## CHAPTER 5

## NUMERICAL RESULTS

In this section, we examine our Finite Difference lattice method (FDLM), Monte Carlo lattice method (MCLM) amd Monte Carlo method (MCM) developed in the previous chapters. Again, we only consider call options as the put options follow a similar trend. The option expiration date is 1 year and the strike price is $\$ 100$. Numerical results are presented when the number of states of economy is 2 and 4 .

Since no exact solutions are available, we use the numerical soltuions obtained by the FDLM with 10000 steps as "exact values". The accuracy of our FDLM has been checked by using the finite element methods of [12].

For the MC lattice method, we set $N=10 M$ for the given positive integer $M$, the number of steps for the lattice method in Algorithm 5.

Example 5.1. In this example, we assume that there are two states of economy. The rate matrix for the Markov chain is assumed to be

$$
Q=\left[\begin{array}{cc}
-2 & 2 \\
3 & -3
\end{array}\right] \quad \text { and }
$$

The other parameters are as follows:

$$
\sigma=\left[\begin{array}{l}
0.3 \\
0.2
\end{array}\right], \quad r=\left[\begin{array}{l}
0.05 \\
0.05
\end{array}\right], \quad d=0.05
$$

It means that the stock price volatility changes as the economy switches from one state to the other while the interest rate keeps constant.

We display the computed option values and its maximum absolute error (MAE)
at 9 stock prices in Tables $5.1-5.8$. We observe that the FDLM converges linearly and the MCLM and MCM converge with the speed of $1 / \sqrt{N}$, which is as expected.

Table 5.1. The FD lattice method for American call option: State 1

| $S$ | $C_{1}$ | $M=500$ | $M=1000$ | $M=2000$ | $M=4000$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 80 | 2.6783 | 2.6793 | 2.6788 | 2.6785 | 2.6783 |
| 85 | 4.0308 | 4.0305 | 4.0312 | 4.0310 | 4.0309 |
| 90 | 5.7589 | 5.7592 | 5.7594 | 5.7593 | 5.7590 |
| 95 | 7.8734 | 7.8742 | 7.8747 | 7.8739 | 7.8737 |
| 100 | 10.3692 | 10.3681 | 10.3687 | 10.3689 | 10.3690 |
| 105 | 13.2281 | 13.2302 | 13.2289 | 13.2287 | 13.2283 |
| 110 | 16.4229 | 16.4241 | 16.4227 | 16.4232 | 16.4229 |
| 115 | 19.9212 | 19.9231 | 19.9221 | 19.9216 | 19.9215 |
| 120 | 23.6884 | 23.6899 | 23.6891 | 23.6887 | 23.6886 |
| MAE |  | $2.15 e-03$ | $1.23 e-03$ | $5.77 e-04$ | $2.65 e-04$ |

Table 5.2. The FD lattice method for American call option: State 2

| $S$ | $C_{2}$ | $M=500$ | $M=1000$ | $M=2000$ | $M=4000$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 80 | 2.2304 | 2.2312 | 2.2309 | 2.2305 | 2.2303 |
| 85 | 3.4766 | 3.4766 | 3.4766 | 3.4768 | 3.4767 |
| 90 | 5.1142 | 5.1148 | 5.1143 | 5.1146 | 5.1144 |
| 95 | 7.1621 | 7.1631 | 7.1632 | 7.1626 | 7.1622 |
| 100 | 9.6187 | 9.6180 | 9.6184 | 9.6185 | 9.6186 |
| 105 | 12.4660 | 12.4682 | 12.4664 | 12.4665 | 12.4662 |
| 110 | 15.6736 | 15.6740 | 15.6735 | 15.6739 | 15.6736 |
| 115 | 19.2043 | 19.2063 | 19.2050 | 19.2045 | 19.2046 |
| 120 | 23.0185 | 23.0201 | 23.0189 | 23.0187 | 23.0187 |
| MAE |  | $2.26 e-03$ | $1.11 e-03$ | $5.19 e-04$ | $2.54 e-04$ |

Table 5.3. The MC lattice method for American call option: State 1

| $S$ | $C_{1}$ | $M=250$ | $M=500$ | $M=1000$ | $M=2000$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 80 | 2.6783 | 2.6870 | 2.6907 | 2.6792 | 2.6757 |
| 85 | 4.0308 | 4.0407 | 4.0423 | 4.0411 | 4.0337 |
| 90 | 5.7589 | 5.7711 | 5.7884 | 5.7643 | 5.7696 |
| 95 | 7.8734 | 7.8857 | 7.8777 | 7.8733 | 7.8630 |
| 100 | 10.3692 | 10.3815 | 10.3882 | 10.3721 | 10.3806 |
| 105 | 13.2281 | 13.2482 | 13.2427 | 13.2290 | 13.2387 |
| 110 | 16.4229 | 16.4389 | 16.4446 | 16.4395 | 16.4269 |
| 115 | 19.9212 | 19.9466 | 19.9330 | 19.9262 | 19.9446 |
| 120 | 23.6884 | 23.7173 | 23.7100 | 23.7014 | 23.7092 |
| MAE |  | $2.88 e-02$ | $2.95 e-02$ | $1.65 e-02$ | $2.34 e-02$ |

Table 5.4. The MC lattice method for American call option: State 2

| $S$ | $C_{2}$ | $M=250$ | $M=500$ | $M=1000$ | $M=2000$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 80 | 2.2304 | 2.1173 | 2.2352 | 2.2270 | 2.2260 |
| 85 | 3.4766 | 3.3362 | 3.4496 | 3.4718 | 3.4764 |
| 90 | 5.1142 | 4.9475 | 5.1273 | 5.1164 | 5.1177 |
| 95 | 7.1621 | 6.9811 | 7.1697 | 7.1701 | 7.1616 |
| 100 | 9.6187 | 9.4278 | 9.6007 | 9.6189 | 9.6203 |
| 105 | 12.4660 | 12.2782 | 12.4828 | 12.4800 | 12.4767 |
| 110 | 15.6736 | 15.4890 | 15.6917 | 15.6837 | 15.6757 |
| 115 | 19.2043 | 19.0434 | 19.2267 | 19.2241 | 19.2144 |
| 120 | 23.0185 | 22.8789 | 23.0272 | 23.0318 | 23.0308 |
| MAE |  | $1.91 e-01$ | $2.70 e-02$ | $1.98 e-02$ | $1.23 e-02$ |

Table 5.5. The FD lattice method for European call option: State 1

| $S$ | $C_{1}$ | $M=500$ | $M=1000$ | $M=2000$ | $M=4000$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 80 | 2.6604 | 2.6613 | 2.6609 | 2.6606 | 2.6604 |
| 85 | 3.9997 | 3.9993 | 4.0000 | 3.9998 | 3.9998 |
| 90 | 5.7078 | 5.7079 | 5.7082 | 5.7081 | 5.7078 |
| 95 | 7.7932 | 7.7938 | 7.7944 | 7.7937 | 7.7934 |
| 100 | 10.2485 | 10.2471 | 10.2478 | 10.2482 | 10.2484 |
| 105 | 13.0531 | 13.0551 | 13.0538 | 13.0536 | 13.0533 |
| 110 | 16.1771 | 16.1780 | 16.1766 | 16.1773 | 16.1770 |
| 115 | 19.5852 | 19.5870 | 19.5861 | 19.5855 | 19.5855 |
| 120 | 23.2404 | 23.2419 | 23.2410 | 23.2407 | 23.2407 |
| MAE |  | $2.02 e-03$ | $1.17 e-03$ | $5.40 e-04$ | $2.60 e-04$ |

Table 5.6. The FD lattice method for European call option: State 2

| $S$ | $C_{2}$ | $M=500$ | $M=1000$ | $M=2000$ | $M=4000$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 80 | 2.2180 | 2.2187 | 2.2184 | 2.2181 | 2.2179 |
| 85 | 3.4541 | 3.4540 | 3.4541 | 3.4542 | 3.4542 |
| 90 | 5.0759 | 5.0763 | 5.0759 | 5.0762 | 5.0760 |
| 95 | 7.1000 | 7.1008 | 7.1010 | 7.1005 | 7.1001 |
| 100 | 9.5226 | 9.5216 | 9.5221 | 9.5224 | 9.5225 |
| 105 | 12.3229 | 12.3250 | 12.3232 | 12.3233 | 12.3231 |
| 110 | 15.4675 | 15.4676 | 15.4672 | 15.4677 | 15.4675 |
| 115 | 18.9160 | 18.9178 | 18.9166 | 18.9161 | 18.9162 |
| 120 | 22.6253 | 22.6267 | 22.6256 | 22.6254 | 22.6255 |
| MAE |  | $2.10 e-03$ | $1.02 e-03$ | $4.59 e-04$ | $2.39 e-04$ |

Table 5.7. The MC method for European call option: State 1

| $S$ | $C_{1}$ | $M=250000$ | $M=500000$ | $M=1000000$ | $M=2000000$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 80 | 2.6604 | 2.6050 | 2.6593 | 2.6604 | 2.6605 |
| 85 | 3.9997 | 3.9297 | 3.9980 | 3.9996 | 3.9988 |
| 90 | 5.7078 | 5.6255 | 5.7076 | 5.7082 | 5.7071 |
| 95 | 7.7932 | 7.7022 | 7.7940 | 7.7943 | 7.7934 |
| 100 | 10.2485 | 10.1530 | 10.2461 | 10.2483 | 10.2478 |
| 105 | 13.0531 | 12.9571 | 13.0541 | 13.0543 | 13.0527 |
| 110 | 16.1771 | 16.0845 | 16.1772 | 16.1768 | 16.1765 |
| 115 | 19.5852 | 19.4988 | 19.5843 | 19.5864 | 19.5839 |
| 120 | 23.2404 | 23.1622 | 23.2407 | 23.2409 | 23.2409 |
| MAE |  | $9.60 e-02$ | $2.43 e-03$ | $1.22 e-03$ | $1.29 e-03$ |

Table 5.8. The MC method for European call option: State 2

| $S$ | $C_{2}$ | $M=250000$ | $M=500000$ | $M=1000000$ | $M=2000000$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 80 | 2.2180 | 2.1777 | 2.2186 | 2.2180 | 2.2187 |
| 85 | 3.4541 | 3.4002 | 3.4531 | 3.4534 | 3.4540 |
| 90 | 5.0759 | 5.0103 | 5.0737 | 5.0759 | 5.0747 |
| 95 | 7.1000 | 7.0261 | 7.0999 | 7.1009 | 7.0997 |
| 100 | 9.5226 | 9.4445 | 9.5204 | 9.5237 | 9.5219 |
| 105 | 12.3229 | 12.2449 | 12.3218 | 12.3206 | 12.3241 |
| 110 | 15.4675 | 15.3934 | 15.4675 | 15.4687 | 15.4684 |
| 115 | 18.9160 | 18.8485 | 18.9161 | 18.9159 | 18.9161 |
| 120 | 22.6253 | 22.5660 | 22.6247 | 22.6250 | 22.6254 |
| MAE |  | $7.81 e-02$ | $2.20 e-03$ | $2.22 e-03$ | $1.23 e-03$ |

Example 5.2. In this example, we assume that there are two states of economy. The rate matrix for the Markov chain is the same as in Example 5.1. The other parameters are as follows:

$$
\sigma=\left[\begin{array}{l}
0.2 \\
0.2
\end{array}\right], \quad r=\left[\begin{array}{c}
0.1 \\
0.05
\end{array}\right], \quad d=0.08
$$

It means that the interest rate changes as the economy switches from one state to the other while the the stock price volatility keeps constant.

We display the computed option values and its maximum absolute error at 9 stock prices in Tables 5.9-5.16. Again, we observe that the FDLM converges linearly and the MCLM and MCM converge with the speed of $1 / \sqrt{N}$.

Table 5.9. The FD lattice method for American call option: State 1

| $S$ | $C_{1}$ | $M=500$ | $M=1000$ | $M=2000$ | $M=4000$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 80 | 1.1570 | 1.1580 | 1.1574 | 1.1566 | 1.1571 |
| 85 | 2.1011 | 2.0997 | 2.1010 | 2.1011 | 2.1013 |
| 90 | 3.4807 | 3.4800 | 3.4816 | 3.4809 | 3.4809 |
| 95 | 5.3455 | 5.3454 | 5.3446 | 5.3460 | 5.3454 |
| 100 | 7.7118 | 7.7089 | 7.7104 | 7.7111 | 7.7115 |
| 105 | 10.5650 | 10.5668 | 10.5650 | 10.5658 | 10.5652 |
| 110 | 13.8670 | 13.8691 | 13.8683 | 13.8675 | 13.8669 |
| 115 | 17.5663 | 17.5664 | 17.5655 | 17.5667 | 17.5662 |
| 120 | 21.6055 | 21.6055 | 21.6061 | 21.6058 | 21.6055 |
| MAE |  |  | $2.90 e-03$ | $1.45 e-03$ | $8.02 e-04$ |
| $3.60 e-04$ |  |  |  |  |  |

Table 5.10. The FD lattice method for American call option: State 2

| $S$ | $C_{2}$ | $M=500$ | $M=1000$ | $M=2000$ | $M=4000$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 80 | 1.0544 | 1.0553 | 1.0548 | 1.0540 | 1.0544 |
| 85 | 1.9351 | 1.9338 | 1.9350 | 1.9351 | 1.9353 |
| 90 | 3.2364 | 3.2358 | 3.2374 | 3.2366 | 3.2367 |
| 95 | 5.0132 | 5.0133 | 5.0124 | 5.0137 | 5.0132 |
| 100 | 7.2889 | 7.2860 | 7.2874 | 7.2882 | 7.2885 |
| 105 | 10.0565 | 10.0585 | 10.0566 | 10.0574 | 10.0568 |
| 110 | 13.2871 | 13.2894 | 13.2885 | 13.2877 | 13.2870 |
| 115 | 16.9392 | 16.9396 | 16.9385 | 16.9397 | 16.9391 |
| 120 | 20.9695 | 20.9697 | 20.9702 | 20.9698 | 20.9695 |
| MAE |  | $2.82 e-03$ | $1.44 e-03$ | $8.39 e-04$ | $3.50 e-04$ |

Table 5.11. The MC lattice method for Americanu call option: State 1

| $S$ | $C_{1}$ | $M=250$ | $M=500$ | $M=1000$ | $M=2000$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 80 | 1.1570 | 1.1744 | 1.1619 | 1.1716 | 1.1749 |
| 85 | 2.1011 | 2.1307 | 2.1115 | 2.1251 | 2.1296 |
| 90 | 3.4807 | 3.5255 | 3.4958 | 3.5151 | 3.5218 |
| 95 | 5.3455 | 5.4041 | 5.3661 | 5.3902 | 5.4001 |
| 100 | 7.7118 | 7.7840 | 7.7345 | 7.7657 | 7.7790 |
| 105 | 10.5650 | 10.6499 | 10.5923 | 10.6263 | 10.6438 |
| 110 | 13.8670 | 13.9613 | 13.8949 | 13.9308 | 13.9546 |
| 115 | 17.5663 | 17.6609 | 17.5960 | 17.6285 | 17.6593 |
| 120 | 21.6055 | 21.6983 | 21.6377 | 21.6624 | 21.6998 |
| MAE |  | $9.47 e-02$ | $3.22 e-02$ | $6.38 e-02$ | $9.43 e-02$ |

Table 5.12. The MC lattice method for American call option: State 2

| $S$ | $C_{2}$ | $M=250$ | $M=500$ | $M=1000$ | $M=2000$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 80 | 1.0544 | 1.0341 | 1.0561 | 1.0524 | 1.0301 |
| 85 | 1.9351 | 1.9029 | 1.9396 | 1.9327 | 1.8957 |
| 90 | 3.2364 | 3.1921 | 3.2444 | 3.2339 | 3.1781 |
| 95 | 5.0132 | 4.9540 | 5.0252 | 5.0114 | 4.9341 |
| 100 | 7.2889 | 7.2184 | 7.3047 | 7.2899 | 7.1898 |
| 105 | 10.0565 | 9.9826 | 10.0795 | 10.0641 | 9.9427 |
| 110 | 13.2871 | 13.2188 | 13.3160 | 13.3039 | 13.1685 |
| 115 | 16.9392 | 16.8862 | 16.9765 | 16.9701 | 16.8311 |
| 120 | 20.9695 | 20.9485 | 21.0199 | 21.0203 | 20.8924 |
| MAE |  | $7.39 e-02$ | $5.04 e-02$ | $5.08 e-02$ | $1.19 e-01$ |

Table 5.13. The FD lattice method for European call option: State 1

| $S$ | $C_{1}$ | $M=500$ | $M=1000$ | $M=2000$ | $M=4000$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 80 | 1.1427 | 1.1438 | 1.1431 | 1.1423 | 1.1428 |
| 85 | 2.0708 | 2.0692 | 2.0705 | 2.0707 | 2.0709 |
| 90 | 3.4218 | 3.4208 | 3.4227 | 3.4220 | 3.4220 |
| 95 | 5.2398 | 5.2395 | 5.2388 | 5.2403 | 5.2397 |
| 100 | 7.5342 | 7.5307 | 7.5325 | 7.5334 | 7.5339 |
| 105 | 10.2830 | 10.2851 | 10.2829 | 10.2839 | 10.2833 |
| 110 | 13.4402 | 13.4426 | 13.4418 | 13.4408 | 13.4400 |
| 115 | 16.9469 | 16.9472 | 16.9457 | 16.9475 | 16.9467 |
| 120 | 20.7400 | 20.7399 | 20.7409 | 20.7404 | 20.7399 |
| MAE |  | $3.55 e-03$ | $1.73 e-03$ | $8.36 e-04$ | $3.64 e-04$ |

Table 5.14. The FD lattice method for European call option: State 2

| $S$ | $C_{2}$ | $M=500$ | $M=1000$ | $M=2000$ | $M=4000$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 80 | 1.0412 | 1.0422 | 1.0416 | 1.0408 | 1.0413 |
| 85 | 1.9068 | 1.9054 | 1.9066 | 1.9067 | 1.9069 |
| 90 | 3.1807 | 3.1798 | 3.1817 | 3.1809 | 3.1809 |
| 95 | 4.9117 | 4.9114 | 4.9107 | 4.9122 | 4.9116 |
| 100 | 7.1152 | 7.1116 | 7.1134 | 7.1143 | 7.1148 |
| 105 | 9.7750 | 9.7770 | 9.7748 | 9.7758 | 9.7752 |
| 110 | 12.8498 | 12.8521 | 12.8514 | 12.8504 | 12.8495 |
| 115 | 16.2841 | 16.2842 | 16.2828 | 16.2846 | 16.2838 |
| 120 | 20.0160 | 20.0158 | 20.0168 | 20.0164 | 20.0159 |
| MAE |  | $3.57 e-03$ | $1.74 e-03$ | $8.39 e-04$ | $3.67 e-04$ |

Table 5.15. The MC method for European call option: State 1

| $S$ | $C_{1}$ | $M=250000$ | $M=500000$ | $M=1000000$ | $M=2000000$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 80 | 1.1427 | 1.1602 | 1.1592 | 1.1437 | 1.1380 |
| 85 | 2.0708 | 2.0989 | 2.0973 | 2.0724 | 2.0629 |
| 90 | 3.4218 | 3.4631 | 3.4607 | 3.4241 | 3.4101 |
| 95 | 5.2398 | 5.2960 | 5.2928 | 5.2431 | 5.2239 |
| 100 | 7.5342 | 7.6061 | 7.6020 | 7.5386 | 7.5138 |
| 105 | 10.2830 | 10.3699 | 10.3650 | 10.2883 | 10.2580 |
| 110 | 13.4402 | 13.5412 | 13.5355 | 13.4464 | 13.4110 |
| 115 | 16.9469 | 17.0602 | 17.0539 | 16.9540 | 16.9139 |
| 120 | 20.7400 | 20.8637 | 20.8568 | 20.7477 | 20.7038 |
| MAE |  | $1.24 e-01$ | $1.17 e-01$ | $7.73 e-03$ | $3.62 e-02$ |

Table 5.16. The MC method for European call option: State 2

| $S$ | $C_{2}$ | $M=250000$ | $M=500000$ | $M=1000000$ | $M=2000000$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 80 | 1.0412 | 1.0333 | 1.0357 | 1.0385 | 1.0570 |
| 85 | 1.9068 | 1.8939 | 1.8979 | 1.9022 | 1.9323 |
| 90 | 3.1807 | 3.1618 | 3.1676 | 3.1738 | 3.2184 |
| 95 | 4.9117 | 4.8860 | 4.8940 | 4.9021 | 4.9633 |
| 100 | 7.1152 | 7.0824 | 7.0926 | 7.1028 | 7.1814 |
| 105 | 9.7750 | 9.7351 | 9.7475 | 9.7595 | 9.8554 |
| 110 | 12.8498 | 12.8035 | 12.8180 | 12.8317 | 12.9437 |
| 115 | 16.2841 | 16.2320 | 16.2483 | 16.2635 | 16.3897 |
| 120 | 20.0160 | 19.9592 | 19.9770 | 19.9933 | 20.1317 |
| MAE |  | $5.68 e-02$ | $3.90 e-02$ | $2.27 e-02$ | $1.16 e-01$ |

Example 5.3. In this example, we assume that there are four states of economy.
The rate matrix for the Markov chain is assumed to be

$$
Q=\left[\begin{array}{cccc}
-1.8 & 0.80 & 0.40 & 0.60 \\
0.70 & -1.50 & 0.30 & 0.50 \\
0.24 & 0.45 & -1.24 & 0.55 \\
0.25 & 0.70 & 0.40 & -1.35
\end{array}\right]
$$

The other parameters are as follows:

$$
\sigma=\left[\begin{array}{c}
0.3 \\
0.2 \\
0.4 \\
0.18
\end{array}\right], \quad r=\left[\begin{array}{l}
0.05 \\
0.05 \\
0.05 \\
0.05
\end{array}\right], \quad d=0.05
$$

As in Example 5.1, the stock price volatility changes as the economy switches from one state to the other while the interest rate keeps constant.

We display the computed option values for the first two states and their maximum absolute error at 9 stock prices in Tables 5.9-5.16. We have the same observation
about the convergence of our methods as in previous examples.
Table 5.17. The FD lattice method for American call option: State 1

| $S$ | $C_{1}$ | $M=500$ | $M=1000$ | $M=2000$ | $M=4000$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 80 | 2.8635 | 2.8647 | 2.8639 | 2.8631 | 2.8635 |
| 85 | 4.2271 | 4.2258 | 4.2269 | 4.2270 | 4.2272 |
| 90 | 5.9623 | 5.9613 | 5.9629 | 5.9623 | 5.9624 |
| 95 | 8.0819 | 8.0812 | 8.0809 | 8.0821 | 8.0817 |
| 100 | 10.5820 | 10.5790 | 10.5806 | 10.5814 | 10.5818 |
| 105 | 13.4453 | 13.4463 | 13.4450 | 13.4458 | 13.4455 |
| 110 | 16.6437 | 16.6450 | 16.6446 | 16.6440 | 16.6434 |
| 115 | 20.1435 | 20.1433 | 20.1425 | 20.1439 | 20.1434 |
| 120 | 23.9094 | 23.9090 | 23.9099 | 23.9096 | 23.9093 |
| MAE |  | $3.01 e-03$ | $1.43 e-03$ | $6.34 e-04$ | $2.38 e-04$ |

Table 5.18. The FD lattice method for American call option: State 2

| $S$ | $C_{2}$ | $M=500$ | $M=1000$ | $M=2000$ | $M=4000$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 80 | 1.9741 | 1.9757 | 1.9748 | 1.9738 | 1.9742 |
| 85 | 3.1164 | 3.1152 | 3.1163 | 3.1164 | 3.1165 |
| 90 | 4.6622 | 4.6612 | 4.6629 | 4.6623 | 4.6623 |
| 95 | 6.6433 | 6.6426 | 6.6422 | 6.6436 | 6.6432 |
| 100 | 9.0659 | 9.0624 | 9.0643 | 9.0652 | 9.0657 |
| 105 | 11.9132 | 11.9144 | 11.9129 | 11.9138 | 11.9134 |
| 110 | 15.1501 | 15.1518 | 15.1513 | 15.1506 | 15.1499 |
| 115 | 18.7324 | 18.7324 | 18.7314 | 18.7329 | 18.7323 |
| 120 | 22.6125 | 22.6124 | 22.6132 | 22.6128 | 22.6124 |
| MAE |  | $3.54 e-03$ | $1.68 e-03$ | $7.46 e-04$ | $2.80 e-04$ |

Table 5.19. The MC lattice method for American call option: State 1

| $S$ | $C_{1}$ | $M=250$ | $M=500$ | $M=1000$ | $M=2000$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 80 | 2.8635 | 2.8189 | 2.8052 | 2.6861 | 3.0678 |
| 85 | 4.2271 | 4.1706 | 4.1464 | 4.0105 | 4.4768 |
| 90 | 5.9623 | 5.8998 | 5.8662 | 5.7183 | 6.2503 |
| 95 | 8.0819 | 8.0104 | 7.9745 | 7.8186 | 8.3985 |
| 100 | 10.5820 | 10.5015 | 10.4714 | 10.3103 | 10.9147 |
| 105 | 13.4453 | 13.3840 | 13.3450 | 13.1811 | 13.7843 |
| 110 | 16.6437 | 16.5869 | 16.5602 | 16.3951 | 16.9777 |
| 115 | 20.1435 | 20.1031 | 20.0757 | 19.9204 | 20.4643 |
| 120 | 23.9094 | 23.8959 | 23.8714 | 23.7190 | 24.2117 |
| MAE |  |  |  | $8.05 e-02$ | $1.11 e-01$ | $2.72 e-01 \quad 3.39 e-01$.

Table 5.20. The MC lattice method for American call option: State 2

| $S$ | $C_{2}$ | $M=250$ | $M=500$ | $M=1000$ | $M=2000$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 80 | 1.9741 | 2.0293 | 2.1358 | 1.9160 | 1.9107 |
| 85 | 3.1164 | 3.1874 | 3.3116 | 3.0459 | 3.0415 |
| 90 | 4.6622 | 4.7408 | 4.8893 | 4.5827 | 4.5790 |
| 95 | 6.6433 | 6.7381 | 6.8963 | 6.5582 | 6.5551 |
| 100 | 9.0659 | 9.1557 | 9.3300 | 8.9804 | 8.9766 |
| 105 | 11.9132 | 12.0143 | 12.1850 | 11.8333 | 11.8298 |
| 110 | 15.1501 | 15.2651 | 15.4223 | 15.0839 | 15.0763 |
| 115 | 18.7324 | 18.8540 | 19.0030 | 18.6823 | 18.6750 |
| 120 | 22.6125 | 22.7354 | 22.8776 | 22.5863 | 22.5760 |
| MAE |  | $1.23 e-01$ | $2.72 e-01$ | $8.55 e-02$ | $8.93 e-02$ |

Table 5.21. The FD lattice method for European call option: State 1

| $S$ | $C_{1}$ | $M=500$ | $M=1000$ | $M=2000$ | $M=4000$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 80 | 2.8438 | 2.8451 | 2.8443 | 2.8434 | 2.8439 |
| 85 | 4.1940 | 4.1927 | 4.1937 | 4.1939 | 4.1941 |
| 90 | 5.9091 | 5.9080 | 5.9097 | 5.9091 | 5.9092 |
| 95 | 7.9998 | 7.9992 | 7.9988 | 8.0001 | 7.9997 |
| 100 | 10.4601 | 10.4570 | 10.4586 | 10.4595 | 10.4599 |
| 105 | 13.2703 | 13.2715 | 13.2700 | 13.2708 | 13.2704 |
| 110 | 16.3995 | 16.4011 | 16.4007 | 16.4000 | 16.3993 |
| 115 | 19.8120 | 19.8122 | 19.8110 | 19.8125 | 19.8120 |
| 120 | 23.4699 | 23.4699 | 23.4707 | 23.4703 | 23.4698 |
| MAE |  | $3.08 e-03$ | $1.46 e-03$ | $6.49 e-04$ | $2.43 e-04$ |

Table 5.22. The FD lattice method for European call option: State 2

| $S$ | $C_{2}$ | $M=500$ | $M=1000$ | $M=2000$ | $M=4000$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 80 | 1.9638 | 1.9654 | 1.9644 | 1.9634 | 1.9639 |
| 85 | 3.0977 | 3.0965 | 3.0976 | 3.0977 | 3.0978 |
| 90 | 4.6300 | 4.6289 | 4.6308 | 4.6301 | 4.6301 |
| 95 | 6.5903 | 6.5895 | 6.5891 | 6.5906 | 6.5901 |
| 100 | 8.9820 | 8.9783 | 8.9803 | 8.9812 | 8.9817 |
| 105 | 11.7853 | 11.7868 | 11.7850 | 11.7860 | 11.7855 |
| 110 | 14.9617 | 14.9637 | 14.9631 | 14.9622 | 14.9615 |
| 115 | 18.4630 | 18.4635 | 18.4620 | 18.4637 | 18.4630 |
| 120 | 22.2379 | 22.2381 | 22.2388 | 22.2383 | 22.2378 |
| MAE |  | $3.63 e-03$ | $1.72 e-03$ | $7.64 e-04$ | $2.87 e-04$ |

Table 5.23. The MC method for European call option: State 1

| $S$ | $C_{1}$ | $M=250000$ | $M=500000$ | $M=1000000$ | $M=2000000$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 80 | 2.8438 | 2.9349 | 2.8451 | 2.8899 | 2.9006 |
| 85 | 4.1940 | 4.3068 | 4.1991 | 4.2531 | 4.2650 |
| 90 | 5.9091 | 6.0403 | 5.9177 | 5.9795 | 5.9920 |
| 95 | 7.9998 | 8.1443 | 8.0110 | 8.0783 | 8.0913 |
| 100 | 10.4601 | 10.6118 | 10.4726 | 10.5429 | 10.5562 |
| 105 | 13.2703 | 13.4225 | 13.2821 | 13.3530 | 13.3666 |
| 110 | 16.3995 | 16.5471 | 16.4097 | 16.4790 | 16.4928 |
| 115 | 19.8120 | 19.9508 | 19.8197 | 19.8856 | 19.8996 |
| 120 | 23.4699 | 23.5969 | 23.4745 | 23.5358 | 23.5496 |
| MAE |  | $1.52 e-01$ | $1.25 e-02$ | $8.28 e-02$ | $9.63 e-02$ |

Table 5.24. The MC method for European call option: State 2

| $S$ | $C_{2}$ | $M=250000$ | $M=500000$ | $M=1000000$ | $M=2000000$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 80 | 1.9638 | 1.9626 | 1.9294 | 2.0065 | 2.0451 |
| 85 | 3.0977 | 3.0983 | 3.0550 | 3.1568 | 3.1974 |
| 90 | 4.6300 | 4.6323 | 4.5804 | 4.7033 | 4.7453 |
| 95 | 6.5903 | 6.5940 | 6.5360 | 6.6739 | 6.7168 |
| 100 | 8.9820 | 8.9866 | 8.9253 | 9.0707 | 9.1147 |
| 105 | 11.7853 | 11.7893 | 11.7281 | 11.8735 | 11.9186 |
| 110 | 14.9617 | 14.9647 | 14.9061 | 15.0449 | 15.0913 |
| 115 | 18.4630 | 18.4647 | 18.4108 | 18.5380 | 18.5854 |
| 120 | 22.2379 | 22.2377 | 22.1898 | 22.3023 | 22.3504 |
| MAE |  | $4.59 e-03$ | $5.72 e-02$ | $8.88 e-02$ | $1.33 e-01$ |

Example 5.4. In this example, we assume that there are four states of economy. The rate matrix for the Markov chain is the same as in Example 5.3. The other parameters are as follows:

$$
\sigma=\left[\begin{array}{l}
0.2 \\
0.2 \\
0.2 \\
0.2
\end{array}\right], \quad r=\left[\begin{array}{l}
0.05 \\
0.10 \\
0.08 \\
0.05
\end{array}\right], \quad d=0.08
$$

As in Example 5.2, the interest rate changes as the economy switches from one state to the other while the the stock price volatility keeps constant.

As in Example 3, we display the computed option values for the first two states and their maximum absolute error at 9 stock prices in Tables $5.25-5.32$. Also, we observe that the FDLM converges linearly and the MCLM and MCM converge with the speed of $1 / \sqrt{N}$, which is as expected.

Table 5.25. The FD lattice method for American call option: State 1

| $S$ | $C_{1}$ | $M=500$ | $M=1000$ | $M=2000$ | $M=4000$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 80 | 0.9592 | 0.9602 | 0.9596 | 0.9588 | 0.9592 |
| 85 | 1.7828 | 1.7816 | 1.7827 | 1.7827 | 1.7829 |
| 90 | 3.0165 | 3.0161 | 3.0174 | 3.0167 | 3.0167 |
| 95 | 4.7232 | 4.7234 | 4.7224 | 4.7237 | 4.7231 |
| 100 | 6.9361 | 6.9338 | 6.9350 | 6.9356 | 6.9359 |
| 105 | 9.6600 | 9.6620 | 9.6601 | 9.6608 | 9.6602 |
| 110 | 12.8759 | 12.8782 | 12.8772 | 12.8764 | 12.8757 |
| 115 | 16.5516 | 16.5521 | 16.5512 | 16.5521 | 16.5516 |
| 120 | 20.6513 | 20.6514 | 20.6517 | 20.6515 | 20.6512 |
| MAE |  | $2.32 e-03$ | $1.39 e-03$ | $7.79 e-04$ | $2.20 e-04$ |

Table 5.26. The FD lattice method for American call option: State 2

| $S$ | $C_{2}$ | $M=500$ | $M=1000$ | $M=2000$ | $M=4000$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 80 | 1.1488 | 1.1498 | 1.1492 | 1.1484 | 1.1489 |
| 85 | 2.0891 | 2.0875 | 2.0889 | 2.0890 | 2.0892 |
| 90 | 3.4653 | 3.4645 | 3.4661 | 3.4654 | 3.4654 |
| 95 | 5.3286 | 5.3285 | 5.3277 | 5.3290 | 5.3285 |
| 100 | 7.6966 | 7.6938 | 7.6953 | 7.6960 | 7.6964 |
| 105 | 10.5559 | 10.5575 | 10.5558 | 10.5566 | 10.5561 |
| 110 | 13.8679 | 13.8698 | 13.8691 | 13.8684 | 13.8678 |
| 115 | 17.5799 | 17.5799 | 17.5791 | 17.5803 | 17.5798 |
| 120 | 21.6320 | 21.6318 | 21.6325 | 21.6322 | 21.6319 |
| MAE |  | $2.81 e-03$ | $1.33 e-03$ | $6.94 e-04$ | $2.21 e-04$ |

Table 5.27. The MC lattice method for American call option: State 1

| $S$ | $C_{1}$ | $M=250$ | $M=500$ | $M=1000$ | $M=2000$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 80 | 0.9592 | 0.9684 | 0.9434 | 0.9586 | 0.9563 |
| 85 | 1.7828 | 1.7981 | 1.7591 | 1.7821 | 1.7785 |
| 90 | 3.0165 | 3.0406 | 2.9836 | 3.0156 | 3.0558 |
| 95 | 4.7232 | 4.7527 | 4.6823 | 4.7215 | 4.7749 |
| 100 | 6.9361 | 6.9735 | 6.8857 | 6.9342 | 6.9984 |
| 105 | 9.6600 | 9.7048 | 9.6095 | 9.6592 | 9.7059 |
| 110 | 12.8759 | 12.9258 | 12.8298 | 12.8772 | 12.9308 |
| 115 | 16.5516 | 16.6039 | 16.5158 | 16.5586 | 16.6140 |
| 120 | 20.6513 | 20.7082 | 20.6382 | 20.6693 | 20.7201 |
| MAE |  | $5.70 e-02$ | $5.05 e-02$ | $1.81 e-02$ | $6.88 e-02$ |

Table 5.28. The MC lattice method for American call option: State 2

| $S$ | $C_{2}$ | $M=250$ | $M=500$ | $M=1000$ | $M=2000$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 80 | 1.1488 | 1.1719 | 1.1450 | 1.1728 | 1.1559 |
| 85 | 2.0891 | 2.1245 | 2.0843 | 2.1266 | 2.1004 |
| 90 | 3.4653 | 3.5202 | 3.4609 | 3.5177 | 3.4699 |
| 95 | 5.3286 | 5.3978 | 5.3262 | 5.3945 | 5.3358 |
| 100 | 7.6966 | 7.7850 | 7.6964 | 7.7728 | 7.7069 |
| 105 | 10.5559 | 10.6593 | 10.5615 | 10.6373 | 10.5049 |
| 110 | 13.8679 | 13.9827 | 13.8802 | 13.9470 | 13.8171 |
| 115 | 17.5799 | 17.6947 | 17.6009 | 17.6516 | 17.5340 |
| 120 | 21.6320 | 21.7370 | 21.6587 | 21.6923 | 21.5958 |
| MAE |  | $1.15 e-01$ | $2.67 e-02$ | $8.14 e-02$ | $5.10 e-02$ |

Table 5.29. The FD lattice method for European call option: State 1

| $S$ | $C_{1}$ | $M=500$ | $M=1000$ | $M=2000$ | $M=4000$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 80 | 0.9412 | 0.9423 | 0.9416 | 0.9408 | 0.9413 |
| 85 | 1.7434 | 1.7420 | 1.7432 | 1.7433 | 1.7435 |
| 90 | 2.9378 | 2.9370 | 2.9387 | 2.9380 | 2.9380 |
| 95 | 4.5778 | 4.5776 | 4.5768 | 4.5783 | 4.5777 |
| 100 | 6.6849 | 6.6815 | 6.6833 | 6.6842 | 6.6846 |
| 105 | 9.2494 | 9.2516 | 9.2494 | 9.2503 | 9.2497 |
| 110 | 12.2351 | 12.2376 | 12.2368 | 12.2358 | 12.2349 |
| 115 | 15.5898 | 15.5903 | 15.5887 | 15.5906 | 15.5898 |
| 120 | 19.2543 | 19.2542 | 19.2552 | 19.2547 | 19.2542 |
| MAE |  | $3.40 e-03$ | $1.66 e-03$ | $8.68 e-04$ | $2.68 e-04$ |

Table 5.30. The FD lattice method for European call option: State 2

| $S$ | $C_{2}$ | $M=500$ | $M=1000$ | $M=2000$ | $M=4000$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 80 | 1.1315 | 1.1325 | 1.1319 | 1.1310 | 1.1316 |
| 85 | 2.0522 | 2.0505 | 2.0518 | 2.0520 | 2.0522 |
| 90 | 3.3937 | 3.3926 | 3.3946 | 3.3938 | 3.3939 |
| 95 | 5.2008 | 5.2003 | 5.1996 | 5.2012 | 5.2006 |
| 100 | 7.4834 | 7.4797 | 7.4816 | 7.4826 | 7.4831 |
| 105 | 10.2205 | 10.2223 | 10.2203 | 10.2213 | 10.2207 |
| 110 | 13.3666 | 13.3686 | 13.3680 | 13.3671 | 13.3663 |
| 115 | 16.8632 | 16.8633 | 16.8620 | 16.8638 | 16.8631 |
| 120 | 20.6479 | 20.6474 | 20.6485 | 20.6482 | 20.6477 |
| MAE |  | $3.67 e-03$ | $1.74 e-03$ | $7.80 e-04$ | $2.89 e-04$ |

Table 5.31. The MC method for European call option: State 1

| $S$ | $C_{1}$ | $M=250000$ | $M=500000$ | $M=1000000$ | $M=2000000$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 80 | 0.9412 | 0.9535 | 0.9507 | 0.9332 | 0.9419 |
| 85 | 1.7434 | 1.7636 | 1.7587 | 1.7299 | 1.7441 |
| 90 | 2.9378 | 2.9680 | 2.9604 | 2.9175 | 2.9385 |
| 95 | 4.5778 | 4.6197 | 4.6088 | 4.5497 | 4.5785 |
| 100 | 6.6849 | 6.7393 | 6.7248 | 6.6486 | 6.6854 |
| 105 | 9.2494 | 9.3160 | 9.2977 | 9.2044 | 9.2493 |
| 110 | 12.2351 | 12.3133 | 12.2913 | 12.1820 | 12.2344 |
| 115 | 15.5898 | 15.6786 | 15.6533 | 15.5297 | 15.5886 |
| 120 | 19.2543 | 19.3518 | 19.3234 | 19.1877 | 19.2522 |
| MAE |  | $9.75 e-02$ | $6.91 e-02$ | $6.66 e-02$ | $2.14 e-03$ |

Table 5.32. The MC method for European call option: State 2

| $S$ | $C_{1}$ | $M=250000$ | $M=500000$ | $M=1000000$ | $M=2000000$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 80 | 1.1315 | 1.1044 | 1.1313 | 1.1403 | 1.1304 |
| 85 | 2.0522 | 2.0086 | 2.0515 | 2.0660 | 2.0500 |
| 90 | 3.3937 | 3.3302 | 3.3926 | 3.4137 | 3.3903 |
| 95 | 5.2008 | 5.1150 | 5.1991 | 5.2276 | 5.1960 |
| 100 | 7.4834 | 7.3746 | 7.4811 | 7.5173 | 7.4770 |
| 105 | 10.2205 | 10.0890 | 10.2172 | 10.2609 | 10.2121 |
| 110 | 13.3666 | 13.2145 | 13.3624 | 13.4130 | 13.3564 |
| 115 | 16.8632 | 16.6932 | 16.8583 | 16.9149 | 16.8515 |
| 120 | 20.6479 | 20.4627 | 20.6421 | 20.7036 | 20.6346 |
| MAE |  | $1.85 e-01$ | $5.79 e-03$ | $5.57 e-02$ | $1.33 e-02$ |

## CHAPTER 6

## CONCLUSIONS

We have considered the problems of pricing options under regime switching model of stock price processes. Since the option prices can be computed by either solving the variational inequality problem or evaluating the expectation by Monte Carlo simulation, we have proposed and implemented two numerical methods correspondingly. The advantage of these methods is their flexibility to compute the option prices for any given stock price at any given time.

The first method is based on discretizing the partial differential inequalities by the explicit finite difference scheme. The method is called the finite difference lattice method, which is studied in Chapter 3. In order to solve the resulting linear complimentary problems, we have given a detailed account for the Chandrasekaran and Lemke Methods in Chapter 2. The second method is based on the Monte Carlo simulation of the Markov chain. It is named as the Monte Carlo lattice method and studied in Chapter 4. Numerical examples are given to examine these methods in Chapter 5. It has been shown that the proposed methods provides fast and accurate approximations of option prices. Hence they should be helpful for practitioners working in this field.

The future work will be extended our methods for pricing of options under the regime switching model with jumps ([7]).

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