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A Quasi-Classical Logic for Classical Mathematics

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A QUASI-CLASSICAL LOGIC FOR CLASSICAL MATHEMATICS

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ABSTRACT

Classical mathematics is a form of mathematics that has a large range of application; however, its application has boundaries. In this paper, I show that Sperber and Wilson's concept of relevance can demarcate classical mathematics' range of applicability by demarcating classical logic's range of applicability. Furthermore, I introduce how to systematize Sperber and Wilson's concept of relevance into a quasi-classical logic that can explain classical logic's and classical mathematics' range of applicability.

INTRODUCTION

Classical mathematics is the most robust and commonly used form of mathematics. It is the kind of mathematics that economist use to calculate a country's GDP. It is the kind of mathematics manufacturers use to produce cars. It is the kind of mathematics that engineers use to build spaceships. Not only can experts rely on classical mathematics as a deductive tool, but they can also apply it to a greater range of mathematical problems than any competing form of mathematics. As a system, classical mathematics has axioms. Axioms are basic rules that, when combined with rules of inference, govern the way a system operates and reaches conclusions. Classical mathematicians attempt to discover and prove the conclusions that follow from the axioms of classical mathematics.

Each system of mathematics uses a system of logic. Classical mathematics uses classical logic. The role of the logic that a form of mathematics uses is to provide some of the basic principles that govern the operation of the mathematics. The logic is also able to explain mathematics by allowing the construction of proofs about objects and properties of mathematics. This makes the relation between a mathematics system and its logic intimate – the logic allows us to see into the mathematics.

However, despite the seemingly unshakable nature of classical mathematics its acceptability is questionable. Alternative programs, such as the one developed by the intuitionists, compete with classical mathematics. Some of these competing schools arise from disagreement over which axioms are acceptable. The law of excluded middle (LEM) is one of those controversial axioms. Below is the schema of the law, followed by its traditional interpretation.

$$p \vee \neg p$$

This schema expresses that the proposition, *p*, is true or its negation is true. For instance, either Mercury is geologically active or it is not the case that Mercury is geologically active. The structure of LEM applies, or at least is supposed to apply, to any meaningful sentence. Here is a non-traditional interpretation of LEM that includes the schema's universal application to sentences: for all *x* (where *x* can be any sentence), either *x* is true or it is not the case that *x* is true.

There are differing reasons for rejecting LEM. The constructivists, who do not consider LEM to be an axiom, require a proof of LEM in order for it to be true. However, they cannot derive LEM for all possible cases. The finitude of human experience does not allow the construction of the infinite number of derivations necessary to prove that LEM holds. Others simply do not find the axiom to be self-evident. The restriction to the two options that LEM expresses does not seem to be obvious. The possibility of a third option or infinitely many options remains. Kurt Gödel famously claimed, “the axioms force themselves upon us as being true” (Gödel, 484). Those who question the self-evidence of LEM do not find the dilemma that it expresses to force itself upon us. What I consider the best reasons for rejecting LEM are counterexamples of the following kind.

(1) Either an A musical note is green or it is not the case that an A musical note is green.

(2) Either the purpose of human life is to be happy or it is not the case that the purpose of human life is to be happy.

(3) Either the largest natural number is even or it is not the case that the largest natural number is even.

(4) Either there will be a sea battle tomorrow or it is not the case that there will be a sea battle tomorrow.

The counterexamples are cases of LEM. If LEM is correct, then the disjunction must be true in all cases. Any case of LEM that is false or appears to be confused qualifies as a counterexample to LEM.

Sentences (1), (2), (3), and (4) present cases of LEM where the predicates employed in the embedded sentences have an unusual relation to the subjects of those sentences. In sentence (1), it is not clear how a sound could satisfy (bear the feature of) having or not having a color. The concept of sound and the concept of color have no overlap; the concept of greenness cannot combine with the concept of sound into a complex concept. This makes the attribution of color to sound a category mistake. A category mistake is attributing a property to a thing that cannot possibly have that property. For instance, to claim that this paper is anxious would be a category mistake. Talking about the anxiousness of this paper does not make sense because this paper is inanimate. In a similar manner, a sound cannot possibly have a color. It is nonsensical to associate color and sound together, at least in any way that is stronger than correlation. In other words, color is irrelevant to sound. The result is that the sentence 'an A musical note is green' is nonsense or meaningless.

To be fair, the defenders of LEM would not accept sentence (1) to be a genuine case of LEM. In order for a sentence to be an instance of LEM, the sentences must be meaningful. Since sentence (1) is not meaningful (by category mistake), sentence (1) is not a real case of LEM. Consequently, this example is not a genuine counterexample. The defenders of LEM are able to cherry pick away these kinds of cases by setting the standard that sentences must be meaningful. This is a fair move. There are good reasons to reject the use of meaningless sentences into the LEM schema.

For sentence (2), assume that natural processes are entirely responsible for the rise of humans, and that entities have purpose only if an intelligent being gives purpose to those entities when creating them. The purpose of a kitchen knife, for instance, is to cut.

However, a rock does not have purpose because its coming to existence does not include an intelligent being giving it purpose. Humans, like rocks, are a product of an unguided natural process. Humans came to exist without an intelligent being giving them purpose. On this view, human life does not have purpose. However, the disjunction in sentence (2) presupposes that human life has purpose. The problem is that, according to our background assumption, this presupposition is false. Human life does not have purpose (according to the earlier stipulations). The presupposition being false creates a problem for this case of LEM because both disjuncts rely on the presupposition. Each disjunct is attributing features to the purpose of human life, but that attribution would only make sense if human life really had purpose. Since the presupposition is false, the disjunction appears mistaken. The assumption that human life has purpose acts as a foundation for the disjuncts that allows the disjunction to make sense. Without the foundation (because it is a false assumption), there is no warrant to assert the disjunction.

Sentence (3) is another instance of LEM that, like sentence (2), has presupposition failure. In this case, the sentence presupposes that there is a largest natural number. Since the natural numbers go on infinitely, there cannot be a largest natural number. Thus, sentence (3) is asserting a predicate about an object that does not exist. The presupposition is false and each disjunct relies on the presupposition. Consequently, the assertion of the disjunction is mistaken.

The defenders of LEM may argue that this is a mistaken interpretation of the sentence. In fact, sentence (3) says that either there exists a largest natural number and that number is even or it is not the case that there exists a largest natural number and that number is even. On this interpretation, there is no presupposition. This may be a result of a divergence in intuitions of what the sentence means, but apply this interpretation to the following sentence: 'Zeus is a god or it is not the case that Zeus is a god'. The allies of LEM can interpret this sentence in two ways. (1) Either Zeus exists and he is a god or it is not the case that Zeus exists and he is a god. (2) Either Zeus exists in Greek mythology and he is a god or it is not the case that Zeus exists in Greek mythology and he is a god. It is not clear which interpretation is correct. The former follows the structure that the allies of LEM use for sentence (3), but discussion about Zeus is typically in the context of Greek mythology. The allies of LEM cannot definitively select one of the interpretations because the metaphysical status of Zeus is not in the content of the sentence 'Zeus is a god or it is not the case that Zeus is a god'. The status of Zeus is a presupposition and the presupposition may vary from context to context. Without any context, the presupposition is an abstraction and not in the content of the sentence. Similarly, whether a largest natural number exists is not in the content of sentence (3) – the sentence is not considering whether a largest natural number exists. The sentence is simply considering whether the largest natural number is even or not. The existence of a largest natural number is the concern of some investigation before the assertion of sentence (3).

For sentence (4), assume that the future is undecided. Consequently, all claims about the future lack truth-value. The disjuncts in sentence (4) are claims about the future and, therefore, lack truth-value. With these metaphysical obligations about fate and time, sentence (4) no longer appears to be an appropriate case of LEM. In order for sentence (4) to hold at least one of the disjuncts must be true, however, neither disjunct is true. This means, not surprisingly, that sentence (4) also lacks a truth-value and fails to be true. Therefore, on this assumption, sentence (4) is also a counterexample to LEM.

The rules of deductive logic have a high standard to meet – they should not have any counterexamples. This means that when there is a counterexample, then there are grounds to reject the rule at hand.

It is, therefore, quite serious to reject LEM for any reason. LEM is central to classical mathematics. Without it, classical mathematics is lost. That is, many of the proofs in classical mathematics, even at early stages, rely on LEM. Reconciling the rejection of LEM with the acceptance of classical mathematics would require replacing classical logic with some non-classical logic that does not posit LEM as an axiom. However, a mathematics that uses that non-classical logic could no longer apply LEM in proofs. Therefore, doing so constructs a new mathematics. However, mathematicians have thus far not generated a system worthy of replacing classical mathematics. The competing mathematics systems are not able to maintain the full range of applicability that classical mathematics has, not to mention their lesser practicality.

Thus, there appears to be an impasse. Accepting classical mathematics appears to require accepting LEM. Rejecting LEM appears to require rejecting classical mathematics. Neither option is appealing. Classical mathematics is our most powerful and useful system of mathematics. However, there are counterexamples to LEM. The problem is that losing classical mathematics comes at too great of a cost.

There is a way to resolve this impasse – to accept classical mathematics and reject LEM. The solution requires addressing a fundamental problem in classical logic that is the source of counterexamples to LEM and other concerns with the axioms.

There are two categories of axioms in classical logic. First, there are axioms that have limited application. False conditions and other axioms like introduction and elimination rules set a restriction – this is an implicit application of relevance. For instance, the conjunction introduction rule is the inference that when any two sentences are true a conjunction of those sentences is also true. Therefore, in cases where there are two true sentences, the construction of the conjunction of those sentences follows. These rules allow a filtration of when it is acceptable to introduce conjunctions. Second, there are axioms that have no restriction and are intended to apply in all scenarios. There are certain implicit restrictions, like the requirement of meaningful sentences, but they are not enough – as shown by sentences (2), (3), and (4). The law of bivalence, the law of non-contradiction, and LEM are all examples of axioms from the second-category. Classical logic allows the implementation of these second-category axioms at almost any time within its formal language. It is precisely for this reason that there are counterexamples for LEM.

The second-category axioms need restrictions that can determine when the second-category axioms are irrelevant and prevent their application in those cases. I propose integrating the needed relevance restrictions into a new quasi-classical logic. A quasi-classical logic is a logic that contains classical logic within it but also has additional logical parts. The application of this quasi-classical logic to classical mathematics will allow us to partition out cases where the second-category axioms do not apply. The result is a demarcation of when the practice of classical mathematics can use second-category axioms of classical logic. With this demarcation of when the classical mathematician can help herself to classical logic, there is also a limit set on the application of classical mathematics. In this paper, I introduce this new logic and begin the first step of creating it. I will refer to this new logic as ‘axiomatic relevance logic’ (ARL).

AXIOMATIC RELEVANCE LOGIC

ARL replaces second-category axioms of classical logic with new axioms. These new axioms simply have the second-category axioms of classical logic as a disjunct in a disjunction where the other disjunct is the irrelevance of the second-category axiom of classical logic. Consequently, falsifying the irrelevance disjunct would yield the second-category axioms of classical logic as a formula or theorem. A case-by-case elimination of irrelevance would yield the respective axiom as a formula for a particular domain. An execution of these derivations will demarcate the cases and domains in which the classical mathematician can use classical logic. ARL will be able to partition philosophically problematic cases as irrelevant, and demarcate appropriate cases that constitute the range of application for classical mathematics. Conveniently, ARL would also leave the practice of classical mathematics mostly, if not completely, unaltered.

The early stages of creating ARL concentrate on accepting new axioms for the system and the theoretical notion of relevance ARL uses. In the remainder of this section, I will briefly discuss proposing new axioms, present/explain the concept of relevance used by ARL, and introduce an axiom of ARL that can be used to show that LEM is a theorem in the cases that make up the realm of classical mathematics. The introduction of the new axiom is the first step in creating ARL, and it will include a detailed demonstration of the role of relevance.

Proposing a new axiom is not unheard of. Historically there have been new axioms proposed and accepted. However, there is warrant to accept a new axiom only when the axiom has sufficient justification. Penelope Maddy, in her article "Believing the Axioms", provides a general framework for justifying axioms. Axioms have intrinsic support when they are "obvious, self-evident" (Maddy, "Believing the Axioms. I" 482). Axioms have extrinsic support when the consequences of their application are pragmatically significant (Ibid.). An axiom can be pragmatically significant by allowing mathematicians and logicians to find new ways to derive long-standing theorems, solve new problems, and explain logic or mathematics.

Perhaps the most well-known case of the explicit acceptance of an axiom is the axiom of choice. The axiom of choice states that if there are infinitely many non-empty sets, then an algorithmic selection of one element from each set can create a new set. The intrinsic and extrinsic justifications for this axiom are strong (Maddy, "Believing the Axioms. I" 487). It has intrinsic support due to its self-evidence. The axiom of choice is simply obvious. It makes sense that the collection of one element from each set can constitute a new set. It has extrinsic support due to the consequences of the axiom. The proofs for many important theorems depend on the axiom of choice. In fact, the axiom is indispensable for a number of areas in mathematics, including abstract algebra and the logic of mathematics (Maddy, "Believing the Axioms. I" 488).

The axiom of choice also has bizarre consequences. When one combines it with Zermelo-Fraenkel set theory, one can derive the Banach-Tarksi paradox. The paradox states that when a sphere gets cut into at least five pieces, then there is a way to reassemble the five pieces into two spheres that are each equal in volume to the original sphere. The doubling of volume is an absurd consequence of the axiom of choice. However, support for the axiom of choice was so strong that mathematicians still accepted it.

The term ‘relevance’ is multi-faceted, in that it has many different meanings. Consequently, there are potentially many concepts of relevance. ARL is a form of relevance logic. Therefore, it seems natural to consider the concepts of relevance that other relevance logics have. However, other relevance logics are concerned with maintaining relevance at each step of a proof. It is within the deductive process (derivations) that these relevance logics apply the concept of relevance. ARL, on the other hand, is concerned with determining when the second-category axioms of classical logic are relevant. The relevance status of an axiom is an input into the deductive apparatus. That means that the relevance status of an instance of a second-category axiom of classical logic is not the conclusion of a proof; rather, it is a premise. Due to the radically different role of relevance in ARL, the concepts of relevance that relevance logics use are too narrow and irregular. ARL uses ‘relevance’ in its ordinary sense. The ordinary notion of relevance has broad application, precision, flexibility, is able to filter out counterexamples, and the accuracy of its use in individual scenarios is subject to deliberation. Dan Sperber and Deirdre Wilson propose a theoretical concept of relevance in linguistics and cognitive science that has these features. I will use Sperber and Wilson’s concept of relevance for ARL.

Contextual effect is the defining feature – the necessary and sufficient conditions – of Sperber and Wilson's concept of relevance (Sperber & Wilson, *Relevance* 122).[1] Contextual effect is a particular kind of interaction between new and old information. In a given case, old information constitutes the context. Introducing new information into that context has contextual effect if it increases justification for already known information, falsifies information, or implicates new information.

If a given assumption has contextual effect in a given context, then it is relevant in that context. If a given assumption does not have contextual effect in a given context, then it is irrelevant in that context.

Sperber and Wilson, in *Relevance Communication & Cognition*, present three ways the introduction of an assumption into a context can fail to produce contextual effect. In the first, the assumption is new information but unrelated to the information in the context. In the second, the assumption is identical to information already in the context, so the introduction of the assumption into the context does not strengthen any old information. In the third, the assumption is inconsistent with information in the context and is not strong enough to warrant rejection or falsification of the information in the context that brings about the inconsistency.

Sperber and Wilson also propose an additional fourth way that an assumption can fail to obtain contextual effect. There is a distinction between cases in which an assumption seems to produce contextual effect and cases in which an assumption genuinely produces contextual effect. If processing an assumption in a context produces a false or useless conclusion/implication, then it fails to have positive contextual effect.

[1] Dan Sperber and Deirdre Wilson use 'cognitive effect' interchangeably with 'contextual effect'. This paper uses the latter term in order to preserve consistency.

This can occur if one uses a false assumption.[2] If processing an assumption in a context yields a true conclusion or implication, then it succeeds in having positive contextual effect. Only positive contextual effect qualifies as contextual effect (Sperber & Wilson, "Relevance Theory" 251-252).[3] As an example of an assumption that fails to produce positive contextual effect, imagine the following context. In my room, where I am writing this paper, there is a table. On this table, there is a transparent plastic bottle. In this plastic bottle, there are six fluid ounces of a clear, colorless liquid. All of this information constitutes a particular context. The introduction of the assumption that the liquid in the plastic bottle is Iordanov Vodka produces a number of conclusions. Some of these conclusions are: (1) there is alcohol in the plastic bottle, (2) there are six ounces of vodka in the plastic bottle, (3) the liquid in the plastic bottle is very expensive, and (4) the liquid in the plastic bottle has a sweet aftertaste. At first, the assumption appears to have contextual effect, because it is able to produce new conclusions. The problem is that the liquid in the plastic bottle is actually water. Consequently, conclusions (1) through (4) are false. As a result, the contextual effect is not positive. Since only positive contextual effect counts as contextual effect, the assumption has no contextual effect.

Assumptions that produce positive contextual effect in a given context are relevant in that context. Relevance comes in degrees. Some assumptions are very relevant in a given context and some assumptions are hardly relevant in a given context. Sperber

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[2] A conclusion is useless in that it is not worth having. For instance, if an output is not enlightening information by virtue of lacking a truth-value, then it is useless.

[3] This paper categorizes the lack of positive contextual effect as a fourth way in which contextual effect can fail to obtain, since Sperber and Wilson only implicitly discuss it when listing the categories in *Relevance Communication & Cognition*. There is, however, an explicit discussion of positive contextual effect in "Relevance Theory"; however, the article does not include a categorization of the different ways that contextual effect can fail to obtain.

and Wilson use the productivity and processing effort of an assumption to determine its degree of relevance. An assumption operates as an input into a context. The contextual effect operates as the output from introducing the assumption into the context. The amount of contextual effect/output produced is the productivity of the assumption/input. The number of assumptions/inputs needed to produce the contextual effect is the processing effort. Therefore, output per input or contextual effect per assumption amounts to the degree of relevance (Sperber & Wilson, *Relevance Communication & Cognition* 125). The value of the quotient relates directly to the degree of relevance.

When determining an assumption's degree of relevance, Sperber and Wilson propose comparative and quantitative assessments. A comparative analysis of relevance compares the contextual effect and processing effort of two or more assumptions. Assumptions with more contextual effect and less processing effort overall are more relevant than assumptions with less contextual effect and more processing effort overall. A quantitative analysis of relevance would measure contextual effect and processing effort. The measurements would generate a value. An analysis of the value's place on a standard spectrum of potential values would determine the degree of relevance.[4]

Relevance in ARL filters out cases where axioms of classical logic are inapplicable. The counterexamples for LEM listed earlier are examples of cases where

[4] Sperber and Wilson have their reservations about a quantitative analysis of relevance. Quantifying processing effort requires a reduction of mental processing to elementary cognitive operations or some alternative way to track mental effort. These elementary operations and other ways to track mental effort are, thus far, unknown to cognitive scientists. In addition, understanding of contextual effect, Sperber & Wilson believe, requires non-quantitative confirmation values. Counting or tracking contextual implication may be alternatives, but either way, cognitive science is currently unable to give an absolute account of contextual effect. Simply put, a quantitative analysis of relevance hits a pragmatic wall.

LEM is inapplicable. The goal of the concept of relevance is to evaluate problematic cases as involving irrelevance. We can use the counterexamples for LEM as a test for the appropriateness of Sperber and Wilson's concept of relevance for ARL.

(1) Either an A musical note is green or it is not the case that an A musical note is green.

In sentence (1), the assumption/input is that an A musical note is green or it is not the case than an A musical note is green. The context is the information about an A musical note, like its pitch, ability to have volume, particular location, and so on. The information that makes up the context is a little vague, for instance, the note having a particular location (but not specifying it). This is a result of sentence (1) being somewhat abstract. The input is unrelated to the information in the context in virtue of the attribution of a color to a sound being a category mistake. The input simply cannot interact with the context. Consequently, the feature of being green or not being green cannot produce contextual effect in the context. The input not being able to produce contextual effect means that the input is irrelevant in the given context.

The result is that Sperber and Wilson's concept of relevance provides an alternative way to dismiss sentences that the allies of LEM dismiss. However, Sperber and Wilson's concept of relevance can also dismiss other cases that are problematic for LEM, like sentences (2), (3) and (4). In fact, Sperber and Wilson's concept of relevance is the only tool/standard that ARL needs in order to filter out problematic cases.

Consider

(2) Either the purpose of human life is to be happy or it is not the case that the purpose of human life is to be happy.

In sentence (2), the assumption/input is the purpose of human life is to be happy or it is not the case that the purpose of human life is to be happy. The assumption, however, relies on human life having a purpose. According to the stipulations mentioned with the initial analysis of sentence (2), the sentence relies on a false presupposition. However, the presupposition being true is a requirement for an acceptable assertion of the assumption. Since the assertion of the assumption is not acceptable, it cannot act as an input into any context. Without any input, there cannot be an output, no matter what the context is (in this case, the features of human life comprise the context). The result is that the assumption fails to produce contextual effect and, accordingly, the assumption is irrelevant.[5]

Now consider

(3) Either the largest natural number is even or it is not the case that the largest natural number is even.

In sentence (3), the assumption/input is that the largest natural number is even or it is not the case that the largest natural number is even. The assumption, like the assumption for

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[5] The context that sentence (2) selects does not matter because without an input there cannot be contextual effect in any context. Nevertheless, for clarity, the context that sentence (2) selects is human life and all of the related information.

sentence (2), relies on a false presupposition. By relying on a false presupposition, the assumption is not able to interact with the information in the context (the nature of the natural numbers) and thus fails to produce any output. Consequently, the assumption cannot produce contextual effect and is irrelevant.

Finally, consider

(4) Either there will be a sea battle tomorrow or it is not the case that there will be a sea battle tomorrow.

In sentence (4), the assumption is that there will be a sea battle tomorrow or it is not the case that there will be a sea battle tomorrow. This assumption lacks a truth-value, however, it can still be an input into the context. The context is the time-period of tomorrow and other related information. The following output follows: there will be violence tomorrow. This output, however, also lacks a truth-value. By lacking a truth-value, the output is not worthwhile or a significant conclusion. Therefore, sentence (4) fails to produce positive contextual effect and involves irrelevance.

The ability of Sperber and Wilson's concept of relevance to explain and rule out the counterexamples is evidence of its appropriateness for this project. In ARL, relevance is all or nothing. Assumptions with low degrees of relevance (where there is little contextual effect) are relevant. Assumptions with no contextual effect are irrelevant. It is possible for ARL to classify relevant assumption with low yields as irrelevant. However, it is not clear how this would be useful, and it would certainly introduce complications.

Irrelevance is a disjunct in some of the axioms of ARL. Each second-category axiom of classical logic potentially has cases that are counterexamples. In ARL, the predicates employed in particular instances of the second-category axioms of classical logic either can apply to the objects the instance picks out and their context, or the predicates are irrelevant to the objects and their context.[6] In order to account for the relation between the second-category axioms of classical logic and relevance, ARL replaces each second category axiom of classical logic with a new axiom.

Below is the schema for the first of these new axioms, which I refer to as the 'law of excluded irrelevance middle' (LEIM).

$$((p \vee \neg p) \vee I\ulcorner p \urcorner)$$

The schema expresses that the sentence, *p,* is true or its negation is true, or the sentence involves irrelevance. '$I\ulcorner p \urcorner$' is a meta-linguistic predicate. Its interpretation is that *p* involves irrelevance. More specifically, its interpretation is that *p* has an assumption and selects a context such that the assumption is not able to produce contextual effect in the context.[7] ARL's domain of discourse, *U*, also holds the following relation: $I\ulcorner \phi \urcorner \subseteq U$. '$I\ulcorner \phi \urcorner$' is the collection of all sentences that involve irrelevance. '$I\ulcorner \phi \urcorner$' is a subset if *U*.

[6] Pre-theoretic notions of the new axioms of ARL are simply a disjunction where one disjunct is a second category axiom of classical logic and the other disjunct is irrelevance.

[7] LEIM can also have the schema '$((p \vee \neg p) \vee \div p)$', where '÷' is a single place sentential operator for irrelevance. The following relation would hold of ARL's domain of discourse: $\div\phi \subseteq U$. That is, the set of all sentences that involve irrelevance is a subset of all the sentences in the domain of discourse. This approach to LEIM, however, encroaches on the law bivalence.

The following is a simple mathematical example of a derivation of LEM from LEIM in the realm of baby addition. After presenting baby addition, I will show how ARL, with LEIM, can demarcate baby addition as within classical logic's and classical mathematics' range of applicability.

The language of the system that will allow us to model baby addition consists of a vocabulary and formation rules.

The vocabulary consists of '0', 'S', '+', and '='.

'0' is a constant.
'S' is a successor function, where 'Sα' means the successor of α.
'+' is an addition function, where '$\alpha + \beta$' means the combination of α and β.
'=' is an identity predicate, where '$\alpha = \beta$' means that α is identical to β.

The system has the following formation rules.

'0' is a term.
If α is a term, then Sα is a term.
If α is a term and β is a term, then Sα+ β is a term.
Nothing else is a term.
Closed or ground terms do not have variables.
If α and β are terms, then $\alpha = \beta$ is a well-formed formula (wff).
If σ is a wff, then $\neg\sigma$ is a wff.
If σ and θ are wffs, then (σ & θ) is a wff.
If σ and θ are wffs, then (σ v θ) is a wff.
If σ and θ are wffs, then ($\sigma \rightarrow \theta$) is a wff.
If σ is a wff, then $\forall x(\sigma)$ is a wff.
If σ is a wff, then $\exists y(\sigma)$ is a wff.

The following are the axioms of baby addition

Axioms		Explanation
1.	$0 \neq S\alpha$	0 is not identical to the successor of any term.
2.	$(S\alpha = S\beta) \rightarrow (\alpha = \beta)$	If the successor of α is identical to the successor of β, then α is identical to β.
3.	$0 + \alpha = \alpha$	0 plus α is identical to α.
4.	$\alpha + S\beta = S(\alpha + \beta)$	α plus the successor of β is identical to the successor of α plus β.

The logic of baby addition is propositional logic with Leibniz's law and parenthesis reduction. Parenthesis reduction allows us to remove extraneous parenthesis, i.e., we can reformulate 'S(S0)' as 'SS0' with a single application of parenthesis reduction. Leibniz's law allows us to replace equivalences, i.e. if $\alpha = \beta$ and α is the successor of 0, then β is the successor of 0 by Leibniz's law.

This system is sufficient to do baby addition. That is, this system is capable of proving all equations of baby addition. I will use '1 + 2 = 3' as an example. The equation '1 + 2 = 3' translates to 'S0 + SS0 = SSS0'. Below is the proof of 'S0 + SS0 = SSS0'.

1.	S0 + SS0 = S(S0 + S0)	premise (Axiom 4)
2.	S0 + S0 = S(S0 + 0)	premise (Axiom 4)
3.	S0 + 0 = S0	premise (Axiom 3)
4.	S0 + S0 = S(S0)	2,3, Leibniz's law
5.	S0 + S0 = SS0	4, parenthesis reduction
6.	S0 + SS0 = S(SS0)	1,5, Leibniz's law
7.	S0 + SS0 = SSS0	6, parenthesis reduction

ARL can demarcate baby addition as within the range of application of classical logic, and consequently as a part of classical mathematics, by demarcating each element of the system that models baby addition as within the range of application of classical logic. A necessary condition for an element to be in the range of application of classical logic is for LEM to hold for that element. That means that ARL must be able to get an instance LEM from an instance of LEIM when the content of the instances are about an element of the system of baby addition.

A proof of a case of LEM would require setting up the entire system of ARL. This is not accomplishable within the limits of this paper, because creating ARL in its entirety is too large a project. However, a sample derivation can show how these proofs would work. Below is a sample derivation.

The following are the interpretations of the sentences.

Symbol	Interpretation
E	R selects an assumption, α, and a context, β, such that α produces contextual effect in β.
¬E	It is not the case that R selects an assumption, α, and a context, β, such that α produces contextual effect in β.
R	Axiom I is a rule that governs the system of baby addition.
¬R	It is not the case that axiom I is a rule that governs the system of baby addition.
I⌜R⌝	'Axiom I is a rule that governs the system of baby addition' involves irrelevance (has an assumption and selects a context such that the assumption fails to produce contextual effect in the context).
¬I⌜R⌝	It is not the case that 'Axiom I is a rule that governs the system of baby addition' involves irrelevance (has an assumption and selects a context such that the assumption fails to produce contextual effect in the context).

The following is a proof of $(R \vee \neg R)$

1.	$((R \vee \neg R) \vee I_{\ulcorner R \urcorner})$	premise (instance of LEIM)
2.	$(\neg I_{\ulcorner R \urcorner} \leftrightarrow E)$	premise (definitional truth)
3.	E	premise (positive fact)
4.	$(E \rightarrow \neg I_{\ulcorner R \urcorner})$	2, biconditional elimination
5.	$\neg I_{\ulcorner R \urcorner}$	3, 4, modus ponens
6.	$(R \vee \neg R)$	1, 5, disjunctive syllogism

The first line of the proof is an instance of LEIM. The second line is a definitional truth about Sperber and Wilson's concept of relevance. It is not the case that the sentence, R, has irrelevance if and only if that sentence selects an assumption, α, and a context, β, such that α produces contextual effect in β. The truth expressed by line two simply states that contextual effect and sentences involving relevance always occur together or always fail to occur together.[8] The third line claims that the sentence, R, has contextual effect. The assumption that R selects is that axiom I is a rule that governs the system of baby addition. The context that R selects is baby addition and other related information. This assumption produces the output that any successor of a term is not identical to 0. This is a lot of information. In fact, an infinite amount of outputs can result from this assumption. For instance, S0 is not identical to 0, SS0 is not identical to 0, and so on ad infinitum. Furthermore, these outputs are true. S0 translates to 1 and SS0 translates to 2; certainly 1 is not identical to 0 and 2 is not identical to 0. Since the assumption that R selects is able to yield true outputs, the assumption produces contextual effect. The fourth line uses the biconditional elimination rule to derive that if

[8] Failing to have irrelevance is equivalent to succeeding to have relevance.

R selects an assumption, α, and a context, β, such that α produces contextual effect in β, then it is not the case that R involves irrelevance. The fifth line uses modus ponens to detach the consequent of line four, and the sixth line uses the fourth line to eliminate the '$I\ulcorner_{R}\urcorner$' disjunct from line one by applying disjunctive syllogism. The result is that LEM holds for R ('Axiom 1 is a rule that governs the system of baby addition).

A similar derivation for each element of the system of baby addition would show that LEM holds for the system of baby addition. Executing these derivations for each relevance axiom of ARL would show that baby addition is within the range of application of classical logic and thus a part of classical mathematics. Cases where the irrelevance predicate from an instance of LEIM remains are cases in which LEM does not hold, and those cases are, therefore, not within the range of application of classical logic and not a part of classical mathematics.

JUSTIFYING THE LAW OF EXCLUDED IRRELEVANCE MIDDLE

The adoption of LEIM as an axiom of ARL requires justification. It is obvious that properties and negated properties do not exhaust logic. There are various cases in which properties cannot attach to subjects, sometimes even in imagination. Irrelevance is a means to account for those remaining cases. More importantly, raw intuition and language already suggest that these problematic cases involve irrelevance. It is only natural to integrate the instrument of relevance to the second-category axioms of classical logic. Unlike LEM, LEIM does not account for those abnormal cases by forcing negation on them. Rather, it admits our fundamental intuitions about them – that they are misguided, categorically awry, involve inapplicability of the predicate employed, and yielding irrelevance. The logical structure of LEIM accurately models the apparent organization of relevance in the universe, establishing its self-evidence.

Extrinsic justification, on the other hand, is difficult to provide this early in the project. The fruitfulness of the axiom, and even the system as a whole, has yet to be determined. If ARL yields LEM and other essential rules in enough sub-domains of classical mathematics to generate classical mathematics, then there would be strong extrinsic justification. There is, however, considerably weaker extrinsic justification for LEIM. The axiom's ability to explain counterexamples to LEM demonstrates explanatory power. The axiom's ability to explain cases that the defenders of LEM already reject also demonstrates explanatory power. The explanations are also remarkably intuitive. The explanatory power of ARL confirms to some degree the appropriateness of integrating Sperber and Wilson's concept of relevance into a logic.

There is also the potential for LEIM to permit a more detailed analysis of logic and mathematics. By partitioning out irrelevant cases, further logical processing of irrelevant cases becomes possible. Relevant cases, on the other hand, are reducible from a LEIM schema to a LEM schema, where a more traditional analysis is employable. Whether this is substantial extrinsic justification is unclear, but even allowing the possibility of processing a new category of logical sentences – those involving irrelevance – is an advantage.

DISCUSSION

The success or failure of this project depends on the acceptability of ARL. Without showing that ARL is an acceptable system, there can be no genuine confidence in the correctness of this project. Thus far, there is only a framework of the system and one step in its construction. The implementation of the framework will determine whether it is any good. Successful implementation indicates good theorizing. Unsuccessful implementation indicates that the framework is flawed. After all, even a framework that appears workable may have fatal problems. In the case of ARL, complete implementation would be a full attempt at constructing the logic. In the case of this program for vindicating classical mathematics within a non-classical logic, implementation would be a full attempt at constructing ARL and the derivations to demarcate certain elements within the range of classical logic and classical mathematics. This is the true test for the endeavor at hand.

There is no denying it; this program is in its infancy. Thus far, I have essentially laid down a general framework so there may be something to implement in the first place. An inspection of this framework, I admit, is not enough to determine conclusively that the program is feasible. Certainly, there may be lethal concerns that are currently unforeseeable. Nevertheless, laying down the right foundation is essential for success. ARL is a logic that captures more of the overall structure of logic, in which an integration of relevance creates new rules and new consequences. There is an admission of complexity in order to resolve philosophical problems primarily in classical mathematics, but also in classical logic. This early in the project the concern is whether this maneuver makes sense and is feasible. I argue in its favor – that we need to introduce the complexity of relevance into logic in order to resolve certain issues, and doing so is acceptable. This is the topic of concern in this stage, and the construction of the entire system should commence after its analysis.

A practical concern for ARL is the requirement for multiple derivations, like the sample derivation, to demarcate elements of mathematics as within the range of application of classical mathematics. The sample derivation shows that LEM holds for a particular element, which is a necessary condition for that element to be within the range of application of classical mathematics, but it is not sufficient. The sufficient conditions for demarcating an element as within the range of application of classical mathematics is to show that all second category axioms of classical logic hold for that element. There must be a multitude of derivations, like the sample derivation, for each element. Demarcating baby addition within the range of application of classical mathematics would require an even greater number of derivations. Doing so is a long and tedious process. However, the demarcation process is not as bad as it may seem. Demarcating an element or system requires executing the derivations once. Demarcating a particular element is not an ongoing process unless the relevance analysis of that element is unclear.[9] Writing the derivations may require a lot of paper, but they are short, simple, and a one-time task.

There is an infinite regress concern for ARL. LEIM is acting as mechanism to determine when LEM applies. However, there also needs to be an evaluation of the applicability of LEIM. In order for LEM to be derivable for a given case, LEIM must be relevant. The introduction of a new axiom, call it LEIM+, should be able to determine the applicability of LEIM. LEIM+ is a disjunction in which one disjunct is LEIM and the other disjunct is a meta-meta-linguistic sentence for irrelevance (where the meta-

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[9] The relevance analysis of an element being unclear is particularly likely if threshold for relevance is at some degree of relevance rather than the mere production of any contextual effect. Certainly, ARL can handle any adjustment in relevance thresholds.

linguistic sentence involves irrelevance – *P* involving irrelevance involves irrelevance). The problem is that LEIM+ also needs to be applicable. Determining the applicability of LEIM+ requires another axiom, call it LEIM++. LEIM++ is a disjunction in which one disjunct is LEIM+ and the other disjunct is a meta-meta-meta-linguistic sentence for irrelevance. However, there needs to be another axiom to determine the applicability of LEIM++, call it LEIM+++. LEIM+++ is a disjunction in which one disjunct is LEIM++ and the other disjunct is a meta-meta-meta-meta-linguistic sentence for irrelevance. The regression of relevance axioms continues infinitely. Furthermore, the infinite regress is vicious because each relevance axiom relies on the succeeding relevance axiom to determine whether it is applicable. Each relevance axiom relying on the succeeding relevance axiom means that there is no way to determine the relevance of any of the relevance axioms, because any selection of an axiom as a starting point would require a succeeding axiom to determine its applicability.

If ARL is a system committed to a vicious infinite regress of relevance axioms, then it is unintelligible – this would undermine justification for the adoption of the original relevance axiom. Therefore, the original relevance axiom of ARL (LEIM) would be unacceptable. However, the original relevance axioms of ARL are essential to ARL. Those are the axioms that allow ARL to be a demarcation tool for classical mathematics. Without these axioms, the ARL project fails.

However, the line of reasoning that leads to an infinite regress concern is problematic. LEIM is the application of Sperber & Wilson's concept of relevance for a particular case. The notion of another relevance axiom, LEIM+, is questioning whether the use of Sperber and Wilson's concept of relevance is applicable for a particular case.

LEIM+ determines the applicability of Sperber and Wilson's concept of relevance to a particular case by applying Sperber and Wilson's concept of relevance to Sperber and Wilson's concept of relevance. The proponent of the infinite regress is trying to discover the applicability of relevance – this involves looking at relevance the wrong way. Consider the following scenario: A blue pen glued to a red pen. A regress concern may follow. In order for the glue to connect the blue pen to the red pen something must connect the blue pen to the glue and another thing must connect the glue to the red pen. However, something would also have to connect the blue pen to the thing that connects the blue pen to the glue and so on. An explanation of a blue pen glued to a red pen involves an infinite regress. Therefore, the notion of two things connecting with glue is unintelligible. The problem with this analysis is that it involves confusion about how glue works. There is no worry about how the glue connects to the pens; the glue is the connecting mechanism itself. In other words, glue is in the business of connecting, and that is it. An infinite regress does not follow. Similarly, an infinite regress does not follow from LEIM. LEIM tests the applicability of LEM on a case-by-case basis. There is no need to test the applicability of an applicability-testing device with LEIM+, which uses the same applicability-testing device as LEIM.[10] The applicability of LEM is already in the business application (questioning the applicability-testing device for its acceptability is a different objection). Therefore, there is not a regress of relevance axioms.

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[10] There is a distinction between testing the applicability of LEIM and questioning whether or no LEIM is correct. Rejecting the application of Sperber & Wilson's concept of relevance does not generate an infinite regress, but simply denies the legitimacy of LEIM.

The sample derivation may appear to use a question begging proof strategy for deriving a second-category axiom of classical logic. The third line of the sample derivation is a premise that states that the assumption selected by R produces contextual effect in the context selected by R. Introducing this information as a premise is assuming the conclusion. The question at hand is whether R involves relevance. The production or lack of production, of contextual effect determines the relevance of a sentence. The third line assumes that R can produce contextual effect and, thereby, assumes that R involves relevance (or that it is not the case that R involves irrelevance).

In response, I claim that introducing the third line into the proof as a premise is not assuming the conclusion. The explanation of line three shows how the assumption selected by R is able to produce contextual effect in the context selected by R – the assumption generates new information. This is a positive fact about the relation that the components selected by R have. The third line is expressing this fact with the symbol 'E'. If a contemplation of the assumption and the context selected by R were to indicate that the assumption does not produce contextual effect, then line three would express that fact with '¬E'. Accordingly, the proof would show that LEM does not hold for R (or that LEM is irrelevant in the case of R).

There can also be a proof of E. However, those who view the sample derivation as assuming the conclusion would probably not find the proof of E to be any different. However, the premise from the proof of E that would be under question is difficult to deny as a premise. Here is a proof of E.

The following are the interpretations of the propositions.

Symbol	Interpretation
E	R selects an assumption, α, and a context, β, such that α produces contextual effect in β.
O	An analysis of the assumption, α, in context, β, is able to generate positive output(s).
A	Axiom I is a rule that governs the system of baby addition (the assumption that R selects).
C	A conjunction of the claims of baby addition and certain related information (the context that R selects).
R	Axiom I is a rule that governs the system of baby addition.

The following is a proof of E.

1.	$(O \rightarrow E)$	premise
2.	A	premise
3.	C	premise
4.	$((A \& C) \rightarrow O)$	premise
5.	$(A \& C)$	2, 3, conjunction introduction
6.	O	4, 5, modus ponens
7.	E	1, 6, modus ponens

The first line claims that if the analysis of the assumption, α, in context, β, is able to generate positive output(s), then R selects an assumption, α, and a context, β, such that α produces contextual effect in β. This is a definitional truth about contextual effect. The second line is the assumption that R selects, namely that axiom I is a rule governing the system of baby addition. The third line is the context that R selects, namely, the identification of the vocabulary of baby addition and the recursive rules of the system. The fourth line claims that if we take the given assumption and the given context, then the analysis of that assumption in that context generates positive output(s). The fifth line is the conjunction of line two and three. The sixth line detaches the consequent of line four with line five by using modus ponens. The seventh line detaches the consequent of line one with line six by using modus ponens.

Again, there can be a criticism for the proof of E as question begging. The ability of an assumption to produce positive output(s) in a context determines whether the sentence selecting that assumption and context has contextual effect. The fourth line assumes that the given assumption produce positive output(s) in the given context. However, line four is under question. There must be a demonstration of the production of output(s).

The criticism that the proof of E involves question begging breaks down at this point. When considering the given assumption in the given context, positive outputs arise.[11] An *apriori* analysis of the given assumption in the given context shows that there is a generation of outputs. There is no simpler or more fundamental justification for the fourth line that does not revert to proofs in syllogistic logic. The claim in line four is true and breaking down 'E' into other sentences in propositional logic is not possible. The claim being true warrants its use as a premise. Therefore, having line four as a premise in the proof does not make the proof assume its conclusion.

There can also be a dismissal of LEIM by criticizing the dismissal of sentences (2) and (3) by appealing to presupposition failure in sentences. Even if the defenders of

[11] See page 22 for examples of outputs that the given assumption generates in the given context.

LEM are interpreting sentences (2) and (3) in an unintuitive way, their interpretation is still viable. Logic is a blunt tool. It cannot capture all of the complexities of language and reasoning. The duty of the philosopher is to select and integrate the most essential parts of logic during logical investigation. The remaining features, though useful in certain scenarios, are disposable. Certain unintuitive interpretations are permissible in order to squeeze certain complexities into a limited logic system. Let there be an unintuitive interpretation of sentences (2) and (3). A consequence of this view is partitioning irrelevance cases or counterexamples as false rather than involving irrelevance. This may not model the full complexity of the nature of logic, but it is enough to allow classical logic to operate. In other words, classical logic is filtering out problematic cases by declaring them false. They may not be immediately distinguishable from non-problematic cases that have the truth-value 'false', but classical logic is still rejecting problematic cases in some way. The result is that there is no demarcation problem. Whenever the classical mathematician goes beyond classical mathematics' range of applicability classical logic will allow her to arbitrarily apply the truth-value 'false' to those problematic claims.

It is true that the practice of classical mathematics can handle problematic cases by categorizing them as having the truth-value 'false'. It is also agreeable that the philosopher of logic must attempt to integrate into a system only what is essential. The disagreement arises over the essentialness of relevance. A consequence of accepting unintuitive interpretations of sentences (2) and (3) is the grouping of irrelevant cases together with false sentences. This is not acceptable. The philosopher who does not find a claim that involves irrelevance to be essentially distinct from a false claim is confused.

It is obvious that involving irrelevance is a different logical status than being false. If the difference between a claim being false and a claim involving irrelevance is minor feature of logic, then the disposal of the distinction is tolerable. However, the distinction appears to be a central issue. Therefore, there should be an integration of relevance into logic.

An unintuitive interpretation of counterexamples is also not a genuine resolution to the question of demarcation. Categorizing irrelevant cases as having the truth-value 'false' fails to draw a demarcation line. ARL, on the other hand, is able to take all of the objects that classical logic categorizes as having the truth-value 'false' and separate them into two categories: those that have false propositional content, and those that involve irrelevance. The latter cases constitute the collection of cases in which classical mathematics is not applicable and, therefore, a clear demarcation line arises. As a result, the classical mathematician can use the full force of classical logic in all scenarios within the demarcation line.

CONTINUATION OF PROJECT

The next phase of this project is to formulate and adopt all of the other new axioms of ARL that replace the second-category axioms of classical logic. The procedure involves integrating a relevance component into each of the second-category axioms of classical logic. After presenting and adopting all of the new axioms, ARL must undergo examination to see if it is actually a properly functioning system. If ARL survives our most thorough inspections for inconsistency, invalidity, unsoundness, and incompleteness, then it is acceptable as a system of logic.

Considering the differences between ARL and other traditional logics, I suspect that ARL is a logic that can actually work. However, it is far too early to know this with any notable degree of confidence. Establishing ARL as a logic system would permit its use in deriving second-category axioms of classical logic in particular cases. The best strategy for deriving the second-category axioms of classical logic would be first to target sub-domains that include the most basic proofs of classical mathematics. The next target would be the sub-domain for the succeeding level of proofs. This pattern would continue until enough sub-domains have the necessary second-category axioms of classical logic to allow the practice of classical mathematics without classical logic. Building up classical mathematics in this manner would complete the project. However, ARL also permits the pursuit of further refinements within classical mathematics. Additional investigation would identify realms or cases that ARL could explain away or reorganize. The extent of these refinements is also unclear, but it is an interesting potential contribution to contemplate.

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