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# Analytical Expressions for Tunneling Time Through Single and Double Barrier Structures

Prabharan Thanikasalarn, R. Venkatasubramanian, *Member, IEEE*, and Marc Cahay, *Member, IEEE*

**Abstract**—In the past, the quantum mechanical tunneling time through simple rectangular barrier has been obtained by various theoretical approaches including the dwell time, the phase delay time, the Larmor clock time and also using the numerical analysis of wave packets. The agreement among these approaches over a range of incident electron energy is far from satisfactory. In this manuscript, analytical expressions for the tunneling time are derived based on the group velocity approach (referred hereafter as the Average Particle Time,  $\tau_{APT}$ ) for single and double rectangular potential barriers under zero bias. The results of the single barrier case, including the limiting value of the tunneling time for various energy limits, are compared with these previous tunneling time calculations. The  $\tau_{APT}$  results provide physically meaningful tunneling times for zero and infinite incident energy limits of the electron. The  $\tau_{APT}$  for the double barrier structure is computed from the analytical solution as a function of the incident energy of the electron for two experimentally studied resonant tunneling structures. For both the single and double barrier cases, the effect of the structure parameters such as barrier width, height, and well width on the  $\tau_{APT}$  are obtained and reported.

## I. INTRODUCTION

RESONANT tunneling through double barrier structures has been the subject of experimental and theoretical study for the past few years due to its potential application in high speed electronic devices within the terahertz regime. One important aspect of the resonant tunneling structures is the traversal time of the electron from one end of the device to the other by the tunneling process. The traversal time for electrons through a rectangular barrier has been studied by various theoretical approaches: the phase-delay method first introduced by Bohm [1] and Wigner [2], the dwell time approach of Smith [3], the Larmor Clock time [4]–[6] and its later generalizations [7], [8] and the numerical studies of wave packets [9]–[12]. Agreement among the results of these various approaches even for the simple case of a single rectangular barrier is poor. We use the group velocity approach developed in [13]–[17] to calculate analytically the

Average Particle Time ( $\tau_{APT}$ ) for the cases of single and double rectangular barrier structures under zero bias.

In Section II, the derivation of the analytical expressions for the two cases is presented. In Section III, the results of  $\tau_{APT}$  are compared with that of the other approaches for the single barrier structures. A detailed comparison of the tunneling time at various energy limits are also made. The results of  $\tau_{APT}$  as a function of incident energy of electron are presented for two experimentally studied double barrier structures [18]–[20]. In the same section, the effect of structure parameters on the tunneling time for both single and double barrier is also presented. Conclusions are presented in Section IV.

## II. DERIVATION OF ANALYTICAL EXPRESSIONS

An integral expression for the  $\tau_{APT}$  for a barrier of width,  $L$ , is given by [14]–[17]:

$$\tau_{APT} = \int_0^L d\tau = 2 \int_0^L \frac{dx}{R(x)} \quad (1)$$

where  $R(x)$  is the real part of the quantum mechanical wave impedance,  $Z(x)$ . The quantum mechanical wave impedance, (QMWI), at any plane  $x$ ,  $Z(x)$ , is defined as [13]:

$$Z(x) = \frac{\Phi(x)}{\Psi(x)} \quad (2)$$

where  $\Psi(x)$  and  $\Phi(x)$  are the electron wave function and its spatial derivative, respectively, for the potential problem of interest. Eqs. (1) and (2) show that knowing the wave function solution to the Schrodinger equation for a typical potential energy profile, the  $Z(x)$  and  $\tau_{APT}$  can be obtained either analytically or numerically. In this manuscript, it is shown that analytical solutions are possible for single and double barrier structures under zero bias.

### A. Single Barrier

The solution to the Schrodinger's equation for a single rectangular barrier structure shown in Fig. 1 is given by [7]:

$$\Psi_1(x) = e^{ikx} + Ae^{-ikx} \quad \dots \quad x < \frac{-d}{2}, \quad (3)$$

$$\Psi_2(x) = Be^{\alpha x} + Ce^{-\alpha x} \quad \dots \quad \frac{-d}{2} < x < \frac{d}{2}, \quad (4)$$

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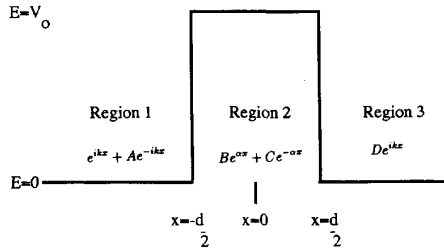


Fig. 1. A single rectangular potential barrier with the corresponding wave function solution for different regions.

$$\Psi_3(x) = De^{ikx} \quad \dots \quad x > \frac{d}{2}. \quad (5)$$

where  $A$ ,  $B$ ,  $C$ , and  $D$  are the complex coefficients,  $\alpha$  is the attenuation constant given by  $\sqrt{2m^*(V_0 - E)/\hbar^2}$ ,  $k$  is the propagation constant given by  $\sqrt{2m^*E/\hbar^2}$ ,  $V_0$  and  $d$  are the height and width of the potential barrier, respectively, and  $E$  is the incident energy of the electron. Using the typical boundary conditions, i.e., the wave function and its derivative are continuous at the potential discontinuities,  $A$ ,  $B$ , and  $C$  can be obtained in terms of  $D$ . Expressions for  $A$ ,  $B$ , and  $C$  in terms of  $D$  are given in Appendix A.

The  $Z(x)$  in the barrier region can be obtained using (2)–(4) as:

$$\frac{\Psi_2'(x)}{\Psi_2(x)} = \frac{\alpha [Be^{\alpha x} - Ce^{-\alpha x}]}{[Be^{\alpha x} + Ce^{-\alpha x}]}. \quad (6)$$

Substituting for  $B$  and  $C$  in terms of  $D$  from (34)–(35) presented in Appendix A, the following expression for  $Z(x)$  in the barrier in terms of the attenuation constant  $\alpha$  and the propagation constant  $k$  is derived:

$$Z(x) = \frac{2\hbar}{m^*} \frac{\alpha [\alpha \sinh(\alpha x') + ik \cosh(\alpha x')]}{[\alpha \cosh(\alpha x') + ik \sinh(\alpha x')]} \quad (7)$$

where  $x' = x - (d/2)$ . Eq. (7) can also be written as

$$Z(x') = \frac{2\hbar}{m^*} \frac{\alpha [(\alpha^2 + k^2) \sinh(\alpha x') \cosh(\alpha x') + ik\alpha]}{[\alpha^2 \cosh^2(\alpha x') + k^2 \sinh^2(\alpha x')]} \quad (8)$$

The real part of  $Z(x')$ ,  $R(x')$ , can be derived from (8) as:

$$Re[Z(x')] = \frac{2\hbar}{m^*} \left[ \frac{\alpha^2 k}{\alpha^2 \cosh^2(\alpha x') + k^2 \sinh^2(\alpha x')} \right]. \quad (9)$$

The  $\tau_{APT}$  through the barrier is then obtained by substituting (9) into (1). Thus, the expression for  $\tau_{APT}$  becomes:

$$\tau_{APT} = 2 \int_{-d}^0 \left[ \frac{m^*}{2\hbar \alpha^2 k} \right] \cdot [\alpha^2 \cosh^2(\alpha x') + k^2 \sinh^2(\alpha x')] dx'. \quad (10)$$

The above expression for  $\tau_{APT}$  is analytically integrable for all values of incident energy of the electron.

When the energy of the incident electron,  $E$ , is less than the barrier height,  $V_0$ , the attenuation constant,  $\alpha$ , is a real quantity, and the electron wave function is decaying in nature. The expression given by (10) integrated to obtain  $\tau_{APT}$  given by:

$$\tau_{APT}^{below} = \left( \frac{m^*}{4\hbar \alpha^3 k} \right) [(k^2 + \alpha^2) \sinh(2\alpha d) + 2\alpha d(\alpha^2 - k^2)]. \quad (11)$$

When the energy of the incident electron,  $E$ , is more than the barrier height,  $V_0$ ,  $\alpha$  is an imaginary quantity, and the electron wave function is propagating in nature. In that case, the expression given by (10) can be analytically integrated to obtain the following expression for  $\tau_{APT}$ :

$$\tau_{APT}^{above} = \left( \frac{m^*}{4\hbar k_B^3 k} \right) [2k_B d(k^2 + k_B^2) - (k^2 - k_B^2) \cdot \sin(2k_B d)] \quad (12)$$

where  $k_B$  and  $k$  are the propagation constants in regions  $x < 0$  and  $x > 0$ , respectively. These propagation constants are given by  $\sqrt{2m^*(E - V_0)/\hbar^2}$  and  $\sqrt{2m^*E/\hbar^2}$ , respectively. The analytical expressions for tunneling time given by (11) and (12) were derived recently by Spiller *et al.* using Bohm's quantum potential approach [21].

1) *Various Energy Limits of  $\tau_{APT}$* : The values for  $\tau_{APT}$  for three limiting cases of the incident energy of the electron, viz.  $E \rightarrow 0$ ,  $E \rightarrow V_0$ , and  $E \rightarrow \infty$  can be derived analytically. The derivation of these limits is discussed in this section.

When the incident energy of the electron approaches zero, the propagation constant  $k$  tends to zero and  $\tau_{APT}$  given by (11) tends to the following limit:

$$\tau_{APT} \rightarrow \infty. \quad (13)$$

When the incident energy of the electron approaches infinity, the propagation constant in the barrier,  $k_B \rightarrow \infty$ . The corresponding limit for  $\tau_{APT}$  from (12) is then:

$$\tau_{APT} \rightarrow \frac{m^*}{4\hbar k_B^4} [2k_B d(2k_B^2)], \quad (14)$$

or equivalently,

$$\tau_{APT} \rightarrow \frac{m^* d}{\hbar k_B} = \tau_{classical}. \quad (15)$$

The  $\tau_{APT}$  tends to the classical time,  $\tau_{classical}$ , which is defined as the time it takes for an electron of same energy and effective mass to traverse a distance equal to the barrier width in the absence of the barrier.

Finally, when the incident energy of the electron tends to the barrier height, the limiting values for the  $\tau_{APT}$  is obtained from either (11) or (12) as follows:

$$\tau_{APT} \rightarrow \frac{m^* d}{\hbar k} \left[ 1 + \frac{(kd)^2}{3} \right]. \quad (16)$$

From (15),  $\tau_{APT}$  is finite when  $E \rightarrow V_0$  as  $k$  is finite. The limiting values of  $\tau_{dwell}$ ,  $\tau_{phase-delay}$ ,  $\tau_{L-C}$  and  $\tau_{classical}$  were

TABLE I  
THE LIMITS FOR THE TRAVERSAL TIMES; THE DWELL TIME, ( $\tau_{\text{dwell}}$ ) [3], THE PHASE-DELAY TIME, ( $\tau_{\text{phase-delay}}$ ) [1], [2], THE LARMOR CLOCK TIME, ( $\tau_{L-C}$ ) [4]–[7], THE CLASSICAL TRAVERSAL TIME, ( $\tau_{\text{classical}}$ ), AND THE APT TIME, ( $\tau_{\text{APT}}$ ), FOR VARIOUS INCIDENT ENERGY LIMITS

	$E \rightarrow 0$	$E \rightarrow \infty$	$E \rightarrow V_0$
$\tau_{\text{dwell}}$	0	$\frac{m^* d}{\hbar k}$	$\frac{m^* k_0}{\hbar} \left( \frac{4d^2 + 6d/k_0^2}{12 + 3k_0^2 d^2} \right)$
$\tau_{\text{phase-delay}}$	$\infty$	$\frac{m^* d}{\hbar k}$	$\frac{m^* k_0}{\hbar} \left( \frac{4d^2 + 6d/k_0^2}{12 + 3k_0^2 d^2} \right)$
$\tau_{L-C}$	$\left( \frac{m^* k_0^2}{\hbar \alpha^2} \right) \frac{(\alpha^2 - k^2) \sinh^2(\alpha d) + (\alpha d k_0^2 / 2) \sinh(2\alpha d)}{4k^2 \alpha^2 + k_0^4 \sinh^2(\alpha d)}$	$\frac{m^* d}{\hbar k}$	$\sqrt{\tau_x^2 + \tau_y^2}$
$\tau_{\text{classical}}$	$\infty$	$\frac{m^* d}{\hbar k}$	$\frac{m^* d}{\hbar k}$
$\tau_{\text{APT}}$	$\infty$	$\frac{m^* d}{\hbar k}$	$\frac{m^* d}{\hbar k} \left[ 1 + \frac{(kd)^2}{3} \right]$

$$\dagger \quad \tau_x = \left( \frac{m^*}{\hbar} \right) \left( \frac{3d^2 + 2d^4 k_0^2}{12 + 3k_0^2 d^2} \right) \text{ and } \tau_y = \left( \frac{m^* k_0}{\hbar} \right)$$

obtained for three cases of limiting energies,  $E \rightarrow 0$ ,  $E \rightarrow \infty$ , and  $E \rightarrow V_0$  and are listed in Table I for comparison. A discussion of this comparison is presented in Section III.

2) **Double Barrier Structure:** An analytical expression for the  $\tau_{\text{APT}}$  through a symmetrical double rectangular potential barrier structure shown in Fig. 2 is obtained by an approach similar to that used for the single barrier case. The following is the details of the derivation. Using the plane wave solutions, the analytical solution to the Schrodinger equation in the five regions shown in Fig. 2, is given by:

$$\Psi_1(x) = e^{ikx} + Ae^{-ikx} \dots x < 0, \quad (17)$$

$$\Psi_2(x) = Be^{\alpha x} + Ce^{-\alpha x} \dots 0 < x < d, \quad (18)$$

$$\Psi_3(x) = De^{ikx} + Ee^{-ikx} \dots d < x < (d + d_1), \quad (19)$$

$$\Psi_4(x) = Fe^{\alpha x} + Ge^{-\alpha x} \dots (d + d_1) < x < (d_1 + 2d), \quad (20)$$

$$\Psi_5(x) = He^{ikx} \dots x > (d_1 + 2d) \quad (21)$$

where  $d$  and  $d_1$  are the barrier and well widths, respectively, and  $V_0$  is the height of the barrier. Assuming that

$$\text{Re}[Z(x)] = \left( \frac{2\hbar\alpha}{m^*} \right) \left[ \frac{2(B_2 C_1 - B_1 C_2)}{(B_1^2 + B_2^2)e^{2\alpha x} + (C_1^2 + C_2^2)e^{-2\alpha x} + 2(B_1 C_1 + B_2 C_2)} \right]. \quad (23)$$

Then, the  $\tau_{\text{APT}}^{\text{lb}}$  can be obtained from (1) and (22) as:

$$\tau_{\text{APT}}^{\text{lb}} = 2 \int_0^d \left( \frac{m^* dx}{2\hbar\alpha} \right) \left[ \frac{(B_1^2 + B_2^2)e^{2\alpha x} + (C_1^2 + C_2^2)e^{-2\alpha x} + 2(B_1 C_1 + B_2 C_2)}{2(B_2 C_1 - B_1 C_2)} \right]. \quad (24)$$

the wave function and its derivatives are continuous at the interfaces, analytical expressions are obtained for the complex constants  $A$ – $H$  and are reported in Appendix B. The analytical expressions for the real and imaginary parts of the complex coefficients appear in the final expression for  $\tau_{\text{APT}}$  which will be discussed later. They are also given in Appendix B. The real and imaginary parts of the complex constants are subscripted 1 and 2, respectively.

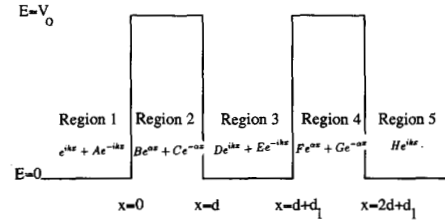


Fig. 2. A symmetrical double rectangular potential barrier with the wave function solutions for different regions.

The  $Z(x)$  can be obtained from the wave function solution involving the complex constants  $B$ – $G$  given by (35)–(53) reported in Appendix B. Then, the total tunneling traversal time through the structure, the  $\tau_{\text{APT}}^{\text{total}}$ , can be obtained from the individual tunneling traversal times through the two barriers and the well region:

$$\tau_{\text{APT}}^{\text{total}} = \tau_{\text{APT}}^{\text{lb}} + \tau_{\text{APT}}^{\text{well}} + \tau_{\text{APT}}^{\text{rb}} \quad (22)$$

where  $\tau_{\text{APT}}^{\text{lb}}$ ,  $\tau_{\text{APT}}^{\text{well}}$ , and  $\tau_{\text{APT}}^{\text{rb}}$  are traversal times in the left barrier, well, and right barrier regions, respectively.

The  $\tau_{\text{APT}}$  for the Left Barrier Region:

The  $\text{Re}[Z(x)]$  can be obtained from (2), (17), and (42)–(45) as:

Performing the integration, the following expression results:

$$\tau_{\text{APT}}^{\text{lb}} = \left[ \frac{m^*}{4\hbar\alpha(B_2 C_1 - B_1 C_2)} \right] \quad (25)$$

$$\left( \frac{B_1^2 + B_2^2}{2\alpha} \right) [e^{2\alpha d} - 1] - \left[ \frac{C_1^2 + C_2^2}{2\alpha} \right] \cdot [e^{-2\alpha d} - 1] + 2d(B_1 C_1 + B_2 C_2). \quad (26)$$

### The $\tau_{APT}$ for the Potential Well Region:

By an approach similar to that employed for the left barrier, an analytical expression for the  $\tau_{APT}^{well}$  can be obtained from the wave function solution in the well given by (19) as:

$$\tau_{APT}^{well} = \int_d^{d+d_1} \left[ p + \frac{q}{2k} \cos(2kx) + \frac{r}{2k} \sin(2kx) \right] dx \quad (27)$$

where  $p$ ,  $q$ , and  $r$  are given by

$$p = \frac{D_1^2 + D_2^2 + E_1^2 + E_2^2}{D_1^2 + D_2^2 - E_1^2 - E_2^2} \quad (28)$$

$$q = \frac{D_1 E_1 + D_2 E_2}{D_1^2 + D_2^2 - E_1^2 - E_2^2} \quad (29)$$

and

$$r = \frac{D_1 E_2 - D_2 E_1}{D_1^2 + D_2^2 - E_1^2 - E_2^2} \quad (30)$$

where  $D_1$ ,  $D_2$ ,  $E_1$ , and  $E_2$  are given by (46)–(49) given in Appendix B. The expression given by (26) can be integrated to obtain the following analytical expression for the  $\tau_{APT}$  in the well region,  $\tau_{APT}^{well}$ :

$$\tau_{APT}^{well} = \left( \frac{m^*}{2\hbar k} \right) \left[ p d_1 + \left( \frac{q}{2k} \right) \sin 2k(d + d_1) - \sin(2kd) - \left( \frac{r}{2k} \right) \cos 2k(d + d_1) + \cos(2kd) \right] \quad (31)$$

### The $\tau_{APT}$ for the Right Barrier Region:

By an approach similar to that employed for the left barrier and the well, the  $\tau_{APT}^{rb}$  can be obtained as follows:

$$\tau_{APT}^{rb} = 2 \int_{d+d_1}^{2d+d_1} \left( \frac{m^* dx}{2\hbar \alpha} \right) \left[ \frac{(F_1^2 + F_2^2) e^{2\alpha x} + (G_1^2 + G_2^2) e^{-2\alpha x} + 2(F_1 G_1 + F_2 G_2)}{2(F_2 G_1 - F_1 G_2)} \right] \quad (32)$$

where  $F_1$ ,  $F_2$ ,  $G_1$ , and  $G_2$  are given by (50)–(53) given in Appendix B. Upon integration, the following analytical expression for  $\tau_{APT}^{rb}$ , results:

$$\tau_{APT}^{rb} = \left[ \frac{m^*}{4\hbar \alpha (F_2 G_1 - F_1 G_2)} \right] \cdot \left[ \frac{F_1^2 + F_2^2}{2\alpha} \right] [e^{2\alpha(d_1 + 2d)} - e^{2\alpha(d_1 + d)}] - \left[ \frac{G_1^2 + G_2^2}{2\alpha} \right] \cdot [e^{-2\alpha(d_1 + 2d)} - e^{-2\alpha(d_1 + d)}] + 2d(F_1 G_1 + F_2 G_2) \quad (33)$$

## III. RESULTS AND DISCUSSIONS

### A. Single Barrier

A plot of  $\tau_{APT}$  and the transmission coefficient versus the normalized incident energy ( $E < V_o$ ) for a rectangular potential barrier with a barrier of height 0.3 eV and width of 200 Å is shown in Fig. 3. The transmission coefficient

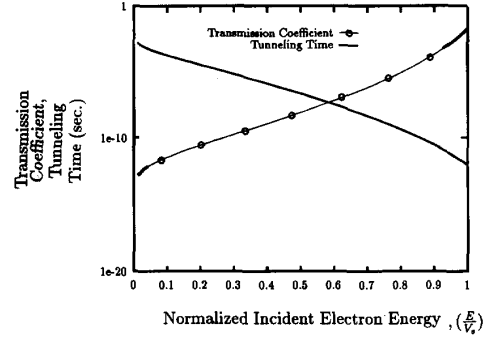


Fig. 3. Plot of the transmission coefficient and the APT time, for a single rectangular potential barrier for  $E < V_o$ , with barrier width 200 Å and barrier height 0.3 eV.

increases with incident energy,  $E$ , as expected. The  $\tau_{APT}$  decreases with increasing energy. It is noted that the  $\tau_{APT}$  approaches infinity in the limit of zero energy like in the case of a classical electron. This can be readily seen from (13). A plot of  $\tau_{APT}$  and the transmission coefficient versus the normalized incident energy ( $E > V_o$ ) for the same structure is shown in Fig. 4. It is noted that the  $\tau_{APT}$  oscillates with a small amplitude. When the incident energy of the electron close to the barrier height, i.e.,  $E \rightarrow V_o$ , the  $\tau_{APT}$  is large, but finite as reported in Table I. In the limit of  $E \rightarrow \infty$ ,  $\tau_{APT}$  reaches the classical limit as reported in Table I.

A plot of the dwell time, the phase-delay time, the Larmor clock time, the  $\tau_{APT}$  and the classical traversal time versus normalized incident energy ( $E < V_o$ ) is shown in Fig. 5, for a single potential barrier structure with a barrier height of 0.3 eV and a width of 200 Å. It is observed that the  $\tau_{APT}$  is greater than the classical time for all in-

cident energies. The  $\tau_{APT}$  tends to infinity in the limit of no incident energy, implying that the electron takes infinite time to traverse the distance when it possess no energy. For  $E < V_o$ , the dwell time and the phase-delay time are less than the classical traversal time. The Larmor Clock time is below the classical time for a range of incident energy, and above the classical time for the rest of the incident energy interval below the barrier.

A plot of the transmission coefficient, the dwell time, the phase-delay time, the Larmor Clock time, the  $\tau_{APT}$  and the classical traversal time vs. normalized incident energy ( $E > V_o$ ) is shown in Fig. 6 for the same structure. In this case, all the traversal times are above the classical traversal time. This is also supported by the limit values reported in Table I. It is observed that the dwell time, the phase-delay time, and the Larmor Clock time attain a maximum value when the transmission coefficient is maximum, and reaches a minimum when the transmission is

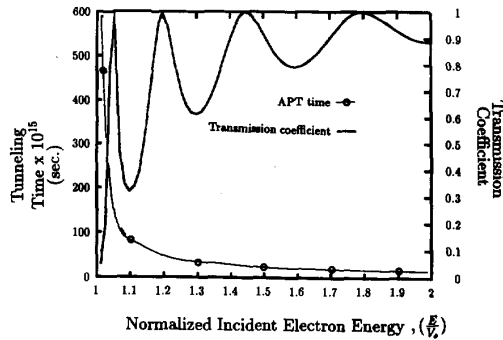


Fig. 4. Plot of the transmission coefficient, and the APT time, for a single rectangular potential barrier for  $E > V_0$ , with barrier width 200 Å and barrier height 0.3 eV.

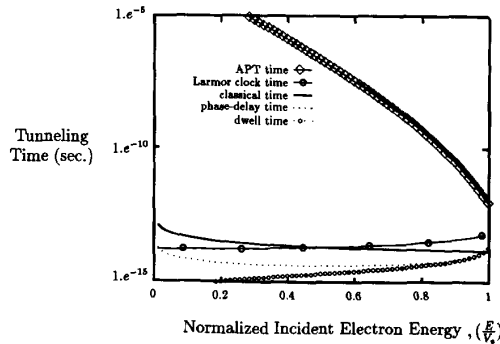


Fig. 5. Plot of the traversal times: dwell time, phase-delay time, Larmor Clock time, APT time, and classical Time for  $E < V_0$ , with barrier width 200 Å and barrier height 0.3 eV.

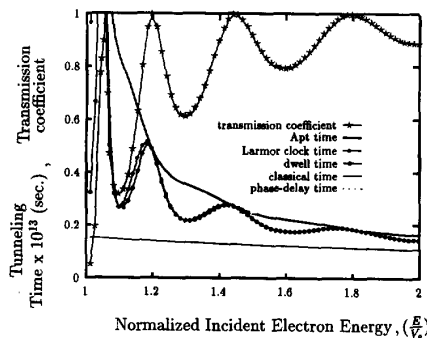


Fig. 6. Plot of the traversal times: dwell time, phase-delay time, Larmor Clock time, APT time, classical Time, and the transmission coefficient for  $E > V_0$ , with barrier width 200 Å and barrier height 0.3 eV.

minimum. Whereas, the  $\tau_{APT}$  exhibits a kink when the transmission coefficient is maximum, and reaches a maximum when transmission is minimum. In other words, according to  $\tau_{APT}$ , the electron travels fastest at resonant energies, whereas according to the other approaches, the electron travels fastest at nonresonant energies. All the traversal times approach the classical time limit at very high incident energies.

The dependence of  $\tau_{APT}$  on the barrier width with  $E <$

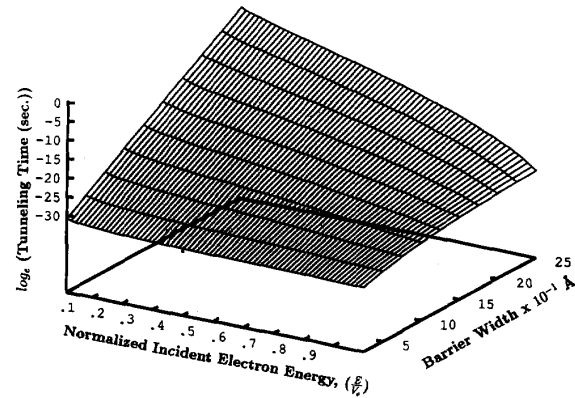


Fig. 7. Three-dimensional surface plot of the APT time, for the case of  $E < V_0$  with the barrier height 1.0 eV and barrier width in the range 25 to 250 Å.

$V_0$  is shown in Fig. 7, for a barrier height of 1.0 eV and for a barrier width in the range of 25 Å to 250 Å. It is observed that the  $\tau_{APT}$  increases with the barrier width for the same incident energy of the electron. The dependence of  $\tau_{APT}$  on barrier width, with incident energy of the electron more than the barrier height is shown in Fig. 8 for the same structure. The oscillations in the  $\tau_{APT}$  with energy are more pronounced for thicker barriers.

## B. Double Barrier

The transmission coefficient and  $\tau_{APT}$  are obtained as a function of incident energy of the electron for two experimentally studied symmetric double barrier structures [18]–[20].

**Case 1:**  $d = 50$  Å,  $d_1 = 50$  Å, and  $V_0 = 0.23$  eV. [18]:

The plot of  $\tau_{APT}$  and the transmission coefficient vs. the electron incident energy is shown in Fig. 9. There is one resonant energy state at 0.0791 eV which is less than the barrier height. This value of 0.0791 eV agrees with that obtained from experiments [18]. At this resonant energy value, the  $\tau_{APT}$  exhibits a kink (local minimum).

**Case 2:**  $d = 25$  Å,  $d_1 = 45$  Å, and  $V_0 = 1.0$  eV. [20]:

The plot of  $\tau_{APT}$  and the transmission coefficient vs. the incident energy of the electron is shown in Fig. 10. It is observed that there are two resonant energy states, one at 0.154 eV and the other at 0.581 eV below the barrier height. These values agree well with the values obtained from numerical solution obtained using SEQUAL [22]. At these resonant energy levels,  $\tau_{APT}$  exhibits a kink (local minimum).

**1) Effect of Barrier Width on the APT Time:** A three-dimensional surface plot of  $\tau_{APT}$  is shown in Fig. 11, for a range of the barrier widths from 30 Å to 100 Å with the well width fixed at 30 Å and the barrier height fixed at 0.3 eV. The  $\tau_{APT}$  approaches infinity when  $E \rightarrow 0$ . The formation of troughs in the  $\tau_{APT}$  at resonances indicate that at these resonant energy levels, the  $\tau_{APT}$  is minimum. As the barrier thickness is increased, the formation of the resonant energy levels is more pronounced and  $\tau_{APT}$  for a

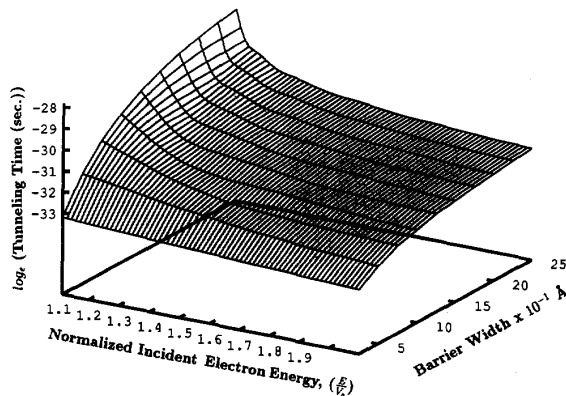


Fig. 8. Three-dimensional surface plot of the APT time, for the case of  $E > V_0$  with the barrier height 1.0 eV and barrier width in the range 25 to 250 Å.

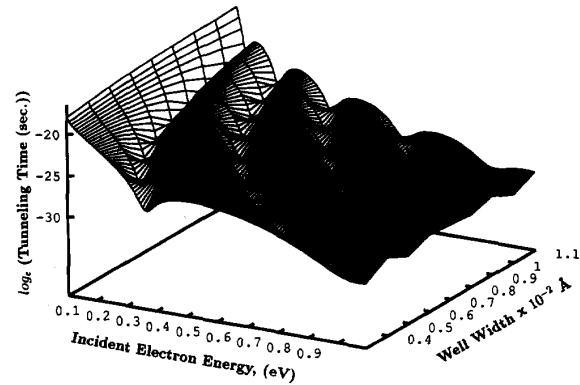


Fig. 11. Three-dimensional surface plot of the APT time for  $E < V_0$ . The barrier height is 0.3 eV and the barrier width in the range from 30 to 100 Å and the well width is 30 Å.

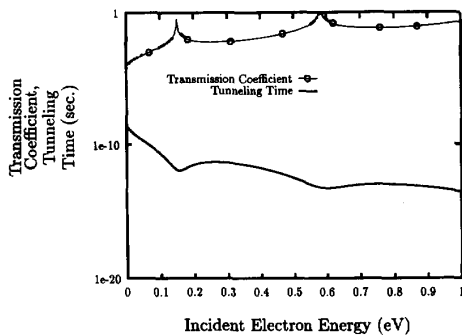


Fig. 9. Plot of the transmission coefficient and the APT time for a symmetrical double rectangular potential barrier structure with a barrier height 0.23 eV, barrier width 50 Å and a well width 50 Å, for  $E < V_0$ .

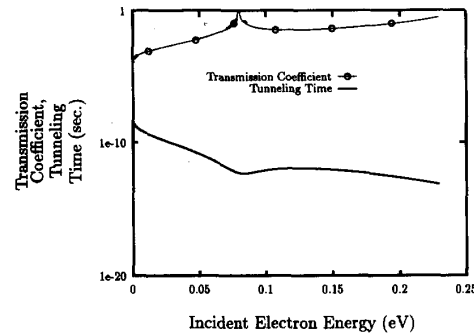


Fig. 12. Three-dimensional surface plot of the APT time for  $E < V_0$ . The barrier height is 1.0 eV and barrier width 30 Å and the well width in the range from 30 to 110 Å.

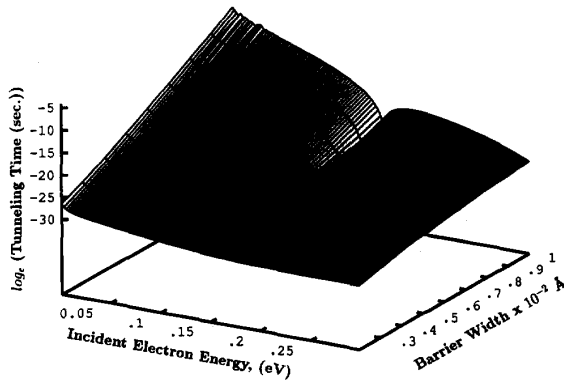


Fig. 10. Plot of the transmission coefficient and the APT time for  $E < V_0$  for a symmetrical double rectangular potential barrier structure with a barrier height 1.0 eV, barrier width 25 Å and a well width 45 Å.

very thick barrier, at resonance, is larger than that for a thin barrier.

2) *Effect of Well Width on the APT Time:* A three-dimensional surface plot of the  $\tau_{APT}$  is shown in Fig. 12, for a range of well widths from 30 Å to 110 Å with the barrier width fixed at 30 Å, and the barrier height fixed

at 1.0 eV. The formation of troughs in the  $\tau_{APT}$  at resonances indicate that at these resonant energy levels, the  $\tau_{APT}$  is minimum. More troughs appear in the tunneling time as the well width increases indicating more resonant levels appear within the barrier height.

#### IV. CONCLUSION

Analytical expressions for the quantum mechanical tunneling time,  $\tau_{APT}$ , for rectangular single and double potential barriers has been derived based on the group velocity concept [13]–[17]. The results of  $\tau_{APT}$  for the single barrier case is compared with that of various other approaches (the dwell time, the phase-delay time and the Larmor clock time). It is shown that  $\tau_{APT}$  gives physically meaningful results in the limits of zero and infinite incident energy of the electron. The  $\tau_{APT}$  results for the double barrier case are obtained for two experimentally studied structures and are reported as a function of incident energy of the electron. In a typical resonant tunneling device,  $\tau_{APT}$  is found to be minimum at resonance (energies with unit transmission coefficient). Depending on the structure parameters,  $\tau_{APT}$  can vary from a few nanoseconds to a few picoseconds.

## APPENDIX A

The analytical expressions for  $A$ ,  $B$ , and  $C$  in terms of  $D$  are:

$$A = \frac{D(\alpha^2 + k^2) \sinh(\alpha d)}{i2k\alpha}, \quad (34)$$

$$B = \frac{De^{ikd/2}(\alpha + ik)}{2\alpha e^{\alpha d/2}}, \quad (35)$$

and

$$C = \frac{De^{ikd/2}(\alpha - ik)}{2\alpha e^{-\alpha d/2}}. \quad (36)$$

## APPENDIX B

The analytical expressions for the complex coefficients  $A$ ,  $B$ ,  $C$ ,  $D$ ,  $E$ ,  $F$ , and  $G$ , used in (17)–(21) can be obtained in terms of the complex amplitude  $H$ . More explicitly, they can be written as:

$$A = \frac{e^{i2kd}}{i2k\alpha e^{\alpha d}} \text{left} \left( \frac{\alpha + ik}{\alpha - ik} \right) [(k^2 - \alpha^2) + (\alpha - ik)^2 e^{i2kd_1}] \sinh(\alpha d) + i2k\alpha \cosh(\alpha d) - \left( \frac{\alpha + ik}{\alpha - ik} \right), \quad (37)$$

$$B = \frac{He^{i2kd}(\alpha + ik)}{i4k\alpha^2 e^{\alpha d}} [(k^2 - \alpha^2) + e^{i2kd_1}(\alpha - ik)^2 \sinh(\alpha d) + i2k\alpha \cosh(\alpha d)], \quad (38)$$

$$C = \frac{He^{i2kd}(\alpha + ik)}{i4k\alpha^2 e^{-\alpha d}} [(k^2 - \alpha^2) + e^{i2kd_1}(\alpha + ik)^2 \sinh(\alpha d) + i2k\alpha \cosh(\alpha d)], \quad (39)$$

$$D_1 = \frac{2k\alpha \cos(kd) \cosh(\alpha d) + (k^2 - \alpha^2) \sin(kd) \sinh(\alpha d)}{2k\alpha}, \quad (48)$$

$$D_2 = \frac{2k\alpha \cosh(\alpha d) \sin(kd) - (k^2 - \alpha^2) \cos(kd) \sinh(\alpha d)}{2k\alpha}, \quad (49)$$

$$D = \frac{He^{ikd}}{i2k\alpha} [(k^2 - \alpha^2) \sinh(\alpha d) + i2k\alpha \cosh(\alpha d)], \quad (40)$$

$$E = \frac{He^{ik(3d+2d_1)}}{i2k\alpha} (\alpha^2 + k^2) \sinh(\alpha d), \quad (41)$$

$$F = \frac{He^{ik(2d+d_1)}(\alpha + ik)}{2\alpha e^{\alpha(2d+d_1)}}, \quad (42)$$

and

$$G = \frac{He^{ik(2d+d_1)}(\alpha - ik)}{2\alpha e^{-\alpha(2d+d_1)}} \quad (43)$$

Resolving the complex constants into real and imaginary parts, the following expressions result:

$$B_1 = \alpha(k^2 - \alpha^2)(1 - e^{-2\alpha d}) \sin^2(kd_1) + 2\alpha^2 k(1 - e^{-2\alpha d}) \sin(kd_1) \cos(kd_1) + k(k^2 - \alpha^2)(1 - e^{-2\alpha d}) \sin(kd_1) \cos(kd_1) + \alpha k^2(1 - e^{-2\alpha d})(1 - 2 \sin^2(kd_1)) - \alpha k^2(1 + e^{-2\alpha d}), \quad (44)$$

$$B_2 = k(k^2 - \alpha^2)(1 - e^{-2\alpha d}) \sin^2(kd_1) + 2\alpha k^2(1 - e^{-2\alpha d}) \sin(kd_1) \cos(kd_1) - \alpha(k^2 - \alpha^2)(1 - e^{-2\alpha d}) \sin(kd_1) \cos(kd_1) - \alpha^2 k(1 - e^{-2\alpha d})(1 - 2 \sin^2(kd_1)) + \alpha^2 k(1 + e^{-2\alpha d}), \quad (45)$$

$$C_1 = \alpha(k^2 - \alpha^2)(e^{2\alpha d} - 1) \cos^2(kd_1) - 2\alpha^2 k(e^{2\alpha d} - 1) \sin(kd_1) \cos(kd_1) + k(k^2 - \alpha^2)(e^{2\alpha d} - 1) \sin(kd_1) \cos(kd_1) + \alpha k^2(e^{2\alpha d} - 1)(-1 + 2 \cos^2(kd_1)) + \alpha k^2(e^{2\alpha d} + 1), \quad (46)$$

$$C_2 = \alpha(k^2 - \alpha^2)(e^{2\alpha d} - 1) \cos^2(kd_1) \sin(kd_1) + 2\alpha k^2(e^{2\alpha d} - 1) \sin(kd_1) \cos(kd_1) - k(k^2 - \alpha^2)(e^{2\alpha d} - 1) \cos^2(kd_1) + \alpha^2 k(e^{2\alpha d} - 1)(-1 + 2 \cos^2(kd_1)) + \alpha^2 k(e^{2\alpha d} + 1), \quad (47)$$

$$E_1 = \frac{(\alpha^2 + k^2) \sin[k(3d + 2d_1)] \sinh(\alpha d)}{2k\alpha}, \quad (50)$$

$$E_2 = \frac{-(\alpha^2 + k^2) \cos[k(3d + 2d_1)] \sinh(\alpha d)}{2k\alpha}, \quad (51)$$

$$F_1 = \frac{\alpha \cos[k(2d + d_1)] - k \sin[k(2d + d_1)]}{2\alpha e^{\alpha(2d + d_1)}}, \quad (52)$$

$$F_2 = \frac{\alpha \cos[k(2d + d_1)] + k \sin[k(2d + d_1)]}{2\alpha e^{\alpha(2d + d_1)}}, \quad (53)$$



$$G_1 = \frac{\alpha \cos [k(2d + d_1)] + k \sin [k(2d + d_1)]}{2\alpha e^{-\alpha(2d + d_1)}}, \quad (54)$$

$$G_2 = \frac{\alpha \sin [k(2d + d_1)] - k \cos [k(2d + d_1)]}{2\alpha e^{-\alpha(2d + d_1)}}. \quad (55)$$

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