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Existence of Solutions to a Class of Nonlinear Convergent Chattering-free Sliding Mode Control Systems

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APPENDIX C
PROOF OF LEMMA 3.3

The following equalities hold:

$$\begin{aligned} V(k + \omega) &= \sum_{h=k+\omega-n\omega}^{k+\omega-1} \Phi^H(k + \omega, h + 1) \text{Im } M(h) \\ &= \sum_{h=k-n\omega}^{k-1} \Phi^H(k, h + 1) \text{Im } M(h) = V(k) \\ &\quad (\text{by (10) and } \omega\text{-periodicity of } M(\cdot)) \end{aligned}$$

for all $k \in \mathbf{Z}$. Thus, $V(\cdot)$ is ω -periodic. Further, these relations follow from Lemma A.1:

$$\begin{aligned} A^H(k)V(k) &= A^H(k) \sum_{h=k-n\omega}^{k-1} \Phi^H(k, h + 1) \text{Im } M(h) \\ &\subset \sum_{h=k-n\omega}^{k-1} \Phi^H(k + 1, h + 1) \text{Im } M(h) \\ &\quad + \Phi^H(k + 1, k + 1) \text{Im } M(k) \\ &= \Phi^H(k + 1, (k + 1) - n\omega) \text{Im } M((k + 1) - n\omega - 1) \\ &\quad + \sum_{h=k+1-n\omega}^k \Phi^H(k + 1, h + 1) \text{Im } M(h) \\ &\subset \sum_{i=0}^{n-1} \{ \Phi^H(k + 1, k + 1 - i\omega) \text{Im } M(k - i\omega) \} \\ &\quad + \sum_{h=(k+1)-n\omega}^{(k+1)-1} \Phi^H(k + 1, h + 1) \text{Im } M(h) \\ &\quad (\text{by Lemma A.1}) \\ &= \sum_{h=(k+1)-n\omega}^{(k+1)-1} \Phi^H(k + 1, h + 1) \text{Im } M(h) = V(k + 1) \end{aligned}$$

for all $k \in \mathbf{Z}$. This completes the proof. $\square$

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Existence of Solutions to a Class of Nonlinear Convergent Chattering-Free Sliding Mode Control Systems

Pushkin Kachroo

Abstract—Sliding mode control is a nonlinear control technique, which is robust against some classes of uncertainties and disturbances. However, this control produces chattering which can cause instability due to unmodeled dynamics and can also cause damage to actuators or the plant. There are essentially two ways to counter the chattering phenomenon. One way is to use higher order sliding mode, and the other way is to add a boundary layer around the switching surface and use continuous control inside the boundary. The problem with the first method is that the derivative of a certain state variable is not available for measurement, and therefore methods have to be used to observe that variable. In the second method, it is important that the trajectories inside the boundary layer do not try to come outside the boundary after entering the boundary layer. Control laws producing chattering-free sliding mode using a boundary layer have been proposed and the existence of solutions to the system using these control laws are presented in this paper.

Index Terms—Differential inclusions, Fillipov's solution, upper semi-continuous.

I. INTRODUCTION

Sliding mode control is a robust nonlinear feedback control technique [1]–[4] which introduces discontinuities in the system differential equations. Due to these discontinuities, sliding mode control systems encounter the drawback of chattering. There are essentially two ways of producing chattering-free performance in sliding mode. One technique for producing chattering-free sliding mode [5]–[7] utilizes higher order sliding mode. In that technique the state equation is differentiated to produce a differential equation which consists of the derivative of the control input, which then is utilized as a new control variable. Hence, this new control variable can be discontinuous while still producing a continuous control input. The difficulty with this technique is that the derivative of the state variable (which is differentiated to produce the derivative of the control in–out in the dynamic equation) is not available for measurement, and hence observers have to be designed to estimate that variable.

One approach for chattering reduction involves introducing a boundary layer around the switching surface and using a continuous control within the boundary layer [2], [3], [8]. The application of simply using a continuous control law inside the boundary layer is not sufficient for producing a chattering-free control. It is essential that once the trajectories enter the boundary layer, they stay inside the boundary layer. To produce a system with this property, a sliding mode control law has been designed which produces a chattering-free system using a boundary layer [8]. This method can still produce a discontinuous control across the boundary. In that case the system is still represented by a discontinuous right-hand side, but the solution does not have chattering. The differential equations representing the control systems with discontinuous right-hand sides are reformulated in terms of differential inclusions [9], and the conditions for solutions for those inclusions are applied to these systems.

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II. BACKGROUND

Let a single input nonlinear system be defined as

$$\dot{x}^{(n)} = f(\mathbf{x}, t) + b(\mathbf{x}, t)u(t). \quad (1)$$

Here, $\mathbf{x}(t) = [x(t) \dot{x}(t) \cdots x^{(n-1)}(t)]^T$ is the state vector, u is the control input, and x is the output state. The superscript n on $x(t)$ signifies the order of differentiation.

A time-varying surface $S(t)$ is defined by equating variable $s(t)$ to zero, where

$$s(t) = \left(\frac{d}{dt} + \gamma \right)^{n-1} \tilde{x}(t). \quad (2)$$

Here, γ is a design constant and $\tilde{x}(t) = x(t) - x_d(t)$ is the error in the output state where $x_d(t)$ is the desired output state. The switching condition

$$\frac{1}{2} \frac{d}{dt} (s(t)^2) \leq -\eta |s(t)|, \quad \eta > 0 \quad (3)$$

makes the surface $S(t)$ an invariant set. All trajectories outside $S(t)$ point toward the surface, and trajectories on the surface remain there. It takes finite time to reach the surface $S(t)$ from outside. Moreover, the definition (2) implies that once the surface is reached, the convergence to zero error is exponential. Chattering is caused by nonideal switching around the switching surface. Delay in digital implementation causes $s(t)$ to pass to the other side of the surface, which in turn produces chattering.

Consider a second-order system

$$\ddot{x}(t) = f(\mathbf{x}, t) + u(t) \quad (4)$$

where $f(\mathbf{x}, t)$ is generally nonlinear and/or time-varying and is estimated as $\hat{f}(\mathbf{x}, t)$, $u(t)$ is the control input, and $x(t)$ is the output, desired to follow trajectory $x_d(t)$. The estimation error on $f(\mathbf{x}, t)$ is assumed to be bounded by some known function $F = F(\mathbf{x}, t)$, so that

$$|\hat{f}(\mathbf{x}, t) - f(\mathbf{x}, t)| \leq F(\mathbf{x}, t). \quad (5)$$

We define a sliding variable according to (4)

$$s(t) = \left(\frac{d}{dt} + \gamma \right) \tilde{x}(t) = \dot{\tilde{x}}(t) + \gamma \tilde{x}(t). \quad (6)$$

The next two theorems give controls that guarantee the satisfaction of the switching condition (3).

Theorem 1 [2]: For a single-input second-order nonlinear lumped parameter system, affine in control, given by (4), where $\mathbf{x} \in R^2$, $u \in R$, $x \in R$, and $f: R^2 \times R^+ \rightarrow R$, choosing the control law as

$$u(t) = \hat{u}(t) - k(\mathbf{x}, t) \operatorname{sgn}(s(t))$$

with

$$k(\mathbf{x}, t) = F(\mathbf{x}, t) + \eta, \quad \text{and} \quad \hat{u}(t) = -f + \ddot{x}_d - \gamma \ddot{\tilde{x}} \quad (7)$$

satisfies the invariant condition of (3).

Results for a second-order system with uncertain control gain are given by the following theorem [2].

Theorem 2: For a single-input second-order nonlinear lumped parameter system, affine in control, given by

$$\ddot{x}(t) = f(\mathbf{x}, t) + b(\mathbf{x}, t)u(t)$$

where

$$0 \leq b_{\min}(\mathbf{x}, t) \leq b(\mathbf{x}, t) \leq b_{\max}(\mathbf{x}, t) \quad (8)$$

where $\mathbf{x} \in R^2$, $u \in R$, $x \in R$, $b: R^2 \times R^+ \rightarrow R$, and $f: R^2 \times R^+ \rightarrow R$, control law

$$u(t) = \hat{b}(\mathbf{x}, t)^{-1} [\hat{u}(t) - k(\mathbf{x}, t) \operatorname{sgn}(s(t))] \quad (9)$$

where

$$k(\mathbf{x}, t) \geq \alpha(\mathbf{x}, t) (F(\mathbf{x}, t) + \eta) + (\alpha(\mathbf{x}, t) - 1) |\hat{u}(t)| \quad (10)$$

$$\hat{b}(\mathbf{x}, t) = \sqrt{b_{\min}(\mathbf{x}, t) b_{\max}(\mathbf{x}, t)} \quad (11)$$

and

$$\alpha(\mathbf{x}, t) = \sqrt{b_{\max}(\mathbf{x}, t) / b_{\min}(\mathbf{x}, t)} \quad (12)$$

ensures the invariant condition of (3).

To remove the chattering, a thin boundary of thickness ϕ around the switching surface is defined as

$$B(t) = \{\mathbf{x}(t), |s(t)| < \phi\}. \quad (13)$$

We can guarantee that all the trajectories outside the boundary layer are attracted toward the boundary by imposing the following condition [2]:

$$|s(t)| \geq \phi(t) \Rightarrow \frac{1}{2} \frac{d}{dt} s(t)^2 \leq (\dot{\phi}(t) - \eta) |s(t)|. \quad (14)$$

The following theorem gives the result for chattering reduction for system (4) while satisfying (14).

Theorem 3: For a single-input second-order nonlinear lumped parameter system, affine in control, given by (6), where $\mathbf{x} \in R^2$, $u \in R$, $x \in R$, and $f: R^2 \times R^+ \rightarrow R$, control law

$$u(t) = \hat{u}(t) - k(\mathbf{x}, t) \operatorname{msat}(a(\mathbf{x}, t), s(t), \phi) \quad (15)$$

with $k(\mathbf{x}, t) = F(\mathbf{x}, t) + \eta$ ensures the invariant condition (14). Moreover, when $|s(t)| \leq \phi$, the variable $s(t)$ passes through the first-order low pass filter given by

$$\dot{s}(t) = -\gamma s(t) + (-\Delta f(\mathbf{x}_d, t) + o(\xi)) \quad (16)$$

where $\Delta f(\mathbf{x}, t) = \hat{f}(\mathbf{x}, t) - f(\mathbf{x}, t)$, $o(\xi)$ represents the term of relatively small magnitude caused by using an desired state instead of actual state vector in (16). The function $\operatorname{msat}(a(\mathbf{x}, t), s(t), \phi)$ is defined as

$$\operatorname{msat}(a(\mathbf{x}, t), s(t), \phi) = a(\mathbf{x}, t) s(t) / \phi, \quad \text{for } |s(t)| < \phi \quad (17)$$

$$\operatorname{msat}(a(\mathbf{x}, t), s(t), \phi) = \operatorname{sgn}(s(t)), \quad \text{otherwise.} \quad (18)$$

Notice that the msat function is discontinuous at $s(t) = \phi$. If the trajectories on both sides of the boundary face inwards, i.e., toward $S(t)$, the discontinuity does not produce any problems. This is the case when the input $-\Delta f(\mathbf{x}_d) + o(\xi)$ to the first order filter is an impulse input.

Now if the input $-\Delta f(\mathbf{x}_d) + o(\xi)$ to the first order filter is a step input, then the variable $s(t)$ has a steady state value. This problem is solved by forcing the trajectories on both sides of the boundary to face inwards, for which an integral action is needed as explained next by Theorem 4.

Theorem 4: For a single-input second-order nonlinear lumped parameter system, affine in control, given by (4), where $\mathbf{x} \in R^2$, $u \in R$, $x \in R$, and $f: R^2 \times R^+ \rightarrow R$, and when $-\Delta f(\mathbf{x}_d, t) + o(\xi)$ is time invariant, control law

$$u(t) = \hat{u}(t) - k(\mathbf{x}, t) \operatorname{int}(a(\mathbf{x}, t), j(\mathbf{x}, t), s(t), \phi) \quad (19)$$

where

$$\begin{aligned} \operatorname{int}(a(\mathbf{x}, t), j(\mathbf{x}, t), s(t), \phi) &= a(\mathbf{x}, t) s(t) / \phi + \frac{j(\mathbf{x}, t)}{\phi} \int_0^t s(t) dt \quad \text{for } |s(t)| \leq \phi \\ \operatorname{int}(a(\mathbf{x}, t), j(\mathbf{x}, t), s(t), \phi) &= \operatorname{sgn}(s(t)), \quad \text{otherwise.} \end{aligned} \quad (20)$$

with $k(\mathbf{x}, t) = F(\mathbf{x}, t) + \eta$ ensures the invariant condition (14). Moreover, if $\exists t' \text{ s.t. } |s(t')| \leq \phi$ then we can choose $a(\mathbf{x}, t)$ and $j(\mathbf{x}, t)$ such that $s(t) \rightarrow 0$.

Extending the argument in the same vein, if the filter input has a term with Laplace transform m/p^n , we introduce n integral terms.

For system (8), the system trajectories inside the boundary layer can be expressed as

$$\begin{aligned} \dot{s}(t) = & (f(\mathbf{x}, t) - b(\mathbf{x}, t)\hat{b}(\mathbf{x}, t)^{-1}\hat{f}(\mathbf{x}, t)) \\ & + (1 - b(\mathbf{x}, t)\hat{b}(\mathbf{x}, t)^{-1})(-\ddot{x}_d(t) + \gamma\dot{x}(t)) \\ & - b(\mathbf{x}, t)\hat{b}(\mathbf{x}, t)^{-1}k(\mathbf{x}, t) \operatorname{sgn}(s(t)) \end{aligned} \quad (21)$$

or

$$\dot{s}(t) = -b(\mathbf{x}, t)\hat{b}(\mathbf{x}, t)^{-1}k(\mathbf{x}, t) \operatorname{sgn}(s(t)) - i(\mathbf{x}, t) \quad (22)$$

with

$$\begin{aligned} i(\mathbf{x}, t) = & -(f(\mathbf{x}, t) - b(\mathbf{x}, t)\hat{b}(\mathbf{x}, t)^{-1}\hat{f}(\mathbf{x}, t)) \\ & - (1 - b(\mathbf{x}, t)\hat{b}(\mathbf{x}, t)^{-1})(-\ddot{x}_d(t) + \gamma\dot{x}(t)). \end{aligned} \quad (23)$$

The input to the filter for the $s(t)$ variable is $-i(\mathbf{x}_d, t)$ and therefore this term should be analyzed as explained for system (4) to ascertain which function should replace the sgn function for chattering reduction and error convergence.

These results can be generalized to a class of nonlinear systems by using the internal model principle approach [10]. Consider the nonlinear systems (4) and (8). Denote the input to the filters, for both nonlinear systems, by $d(\mathbf{x}_d, t)$. Note that $\mathbf{x}_d(t)$ is a function of time so that we can write the input as $d(t)$. The disturbance $d(t)$ satisfies

$$A(p)d(t) = 0. \quad (24)$$

Corollary 1: For a single-input second-order nonlinear lumped parameter system, affine in control, given by (4), where $\mathbf{x} \in R^2$, $u \in R$, $x \in R$, and $f: R^2 \times R^+ \rightarrow R$, and when $-\Delta f(\mathbf{x}_d, t) + o(\xi)$ is a disturbance $d(t)$ and the dynamics inside the boundary layer are $\dot{s} = -g(s) + d(t)$, where the Laplace transform of $g(s)$ is $[R(p) + T(p)/A(p)]S(p)$; $R(p)$ and $T(p)$ are polynomials in p ; $S(p)$ is the Laplace transform of $s(t)$, control law

$$u = \hat{u} - k \operatorname{gen}(s) \quad (25)$$

where

$$\operatorname{gen}(s) = \operatorname{sgn}(s) \quad \text{for } |s| \geq \phi$$

and

$$\operatorname{gen}(s) = g(s)/k \quad \text{for } |s| < \phi \quad (26)$$

with $k(\mathbf{x}, t) = F(\mathbf{x}, t) + \eta$ ensures the invariant condition (14). Moreover, if $\exists t' \text{ s.t. } |s(t')| \leq \phi$ then we can choose $a(\mathbf{x}, t)$ and $j(\mathbf{x}, t)$ such that $s(t) \rightarrow 0$.

III. EXISTENCE OF SOLUTIONS

Let us consider the following differential equation:

$$\dot{\mathbf{x}} = f(\mathbf{x}, t) \quad (27)$$

where $f: R^n \times R \rightarrow R^n$ is essentially locally bounded and measurable. The solution of this differential equation is defined by Filippov by the following theorem.

Definition (Filippov's Solution to Differential Equations with Discontinuous Right-Hand Sides [9]): A vector function $\mathbf{x}(\cdot)$ is the solution of (1) on $[t_0, t_1]$ in the sense of Filippov, if $\mathbf{x}(\cdot)$ is absolutely continuous on $[t_0, t_1]$, and for almost all $t \in [t_0, t_1]$ it satisfies the following differential inclusion:

$$\dot{\mathbf{x}} \in K[f](\mathbf{x}, t). \quad (28)$$

There are two equivalent definitions for $K[f](\mathbf{x}, t)$. The two definitions are described in [11]–[13]. We will use one of the definitions here as

$$K[f](\mathbf{x}, t) \equiv \bigcap_{\delta > 0} \bigcap_{\mu N = 0} \overline{\operatorname{co}} f(B(\mathbf{x}, \delta) - N, t) \quad (29)$$

where N denotes all sets of Lebesgue measure zero.

An alternate definition is based on a control representation of the system [9] as

$$\dot{\mathbf{x}} = f(\mathbf{x}, t, u_1(\mathbf{x}, t), u_2(\mathbf{x}, t), \dots, u_p(\mathbf{x}, t)) \quad (30)$$

which at the discontinuities of $u_i(\mathbf{x}, t)$ $i = 1, 2, \dots, p$ (all $u_i(\mathbf{x}, t)$ being independent of each other), can be represented by the following differential inclusion:

$$\dot{\mathbf{x}} \in F(\mathbf{x}, t, U_1(\mathbf{x}, t), U_2(\mathbf{x}, t), \dots, U_p(\mathbf{x}, t)) \quad (31)$$

$U_i(\mathbf{x}, t)$ $i = 1, 2, \dots, p$, being closed convex sets containing all the limit points of $u_i(\mathbf{x}, t)$.

For analyzing set valued maps, we will use the following definitions [14].

- 1) The distance between two points $\mathbf{x}, \mathbf{y} \in R^n$ is $\|\mathbf{x} - \mathbf{y}\|$.
- 2) The distance between a point $\mathbf{x} \in R^n$ and a set $A \subset R^n$ is $d(\mathbf{x}, A) \triangleq \inf\{\|\mathbf{x} - \mathbf{a}\|: \mathbf{a} \in A\}$.
- 3) The separation of a set $A \subset R^n$ from $B \subset R^n$ is $\rho(A, B) \triangleq \sup\{d(\mathbf{a}, B): \mathbf{a} \in A\}$.

A set $A(p)$ is upper semicontinuous at p_0 , if given any $\varepsilon, \exists \delta > 0$ s.t. $\|p - p_0\| < \delta \Rightarrow \rho(A(p), A(p_0)) < \varepsilon$.

For the existence of solutions to the differential inclusions (28) and (31), the following two conditions should be satisfied: 1) the set $K(\mathbf{x}, t)$ is compact and convex for $(\mathbf{x}, t) \in R^n \times R^+$ and 2) $K(\mathbf{x}, t)$ is upper semicontinuous on $R^n \times R^+$.

Lemma 1: System (8) with control (9) is upper semicontinuous in $\mathbf{x}$, when definitions (28) or (31) are used, and is upper semicontinuous in t and $\mathbf{x}$ for definition (31) and for (28) if an additional condition γ is assumed [9, p. 68].

Hence, a solution exists to the differential inclusions (28) and (31) generated by (8) and (9), since the right-hand side sets are compact and convex, and additionally they are upper semicontinuous. These results are easily generalized for n th-order system (1) with control law of type (9).

As an example, let us show the upper semicontinuity of system (8) with (9). If we make the transformation $y_1 = \hat{x}$, $y_2 = \dot{\hat{x}}$, then the system can be written as

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= f - \ddot{x}_d + b\hat{b}^{-1}[-f + \ddot{x}_d - \gamma\dot{\hat{x}}] - b\hat{b}^{-1}k \operatorname{sgn}(s(t)) \end{aligned} \quad (32)$$

which in the vector form can be represented as

$$\dot{\mathbf{y}} = \psi + Bu^* = F(\mathbf{y}, t) \quad (33)$$

where, $\mathbf{y} = [y_1, y_2]^T$, $\psi = [y_2, f - \ddot{x}_d + b\hat{b}^{-1}[-f + \ddot{x}_d - \gamma\dot{\hat{x}}]]^T$, $B = [0, -b\hat{b}^{-1}k]^T$, and $u^* = \operatorname{sgn}(s(t))$. For $(\bar{\mathbf{y}}, \bar{t}) \notin SF(\bullet, \bullet)$ is upper semicontinuous, because $F(\bullet, \bar{t})$ is continuous and the γ

condition is assumed. For $(\bar{\mathbf{y}}, \bar{t}) \in S$, we will show that the separation between $F(\mathbf{y}, \bar{t})$ and $F(\bar{\mathbf{y}}, \bar{t})$ is continuous.

- 1) Take $(\mathbf{y}, \bar{t}) \notin S$, then, since $(\bar{\mathbf{y}}, \bar{t})$ is a point in R^n

$$\begin{aligned} \rho(F(\mathbf{y}, \bar{t}), F(\bar{\mathbf{y}}, \bar{t})) &= \inf_{\bar{u}^*} \|\bar{\psi} - \psi + \bar{B}\bar{u}^* - Bu^*\| \\ &\leq \|\bar{\psi} - \psi\| + \|\bar{B}\bar{u}^* - Bu^*\|. \end{aligned} \quad (34)$$

- 2) Take $(\mathbf{y}, \bar{t}) \in S$, then

$$\begin{aligned} \rho(F(\mathbf{y}, \bar{t}), F(\bar{\mathbf{y}}, \bar{t})) &= \sup_{u^*} \inf_{\bar{u}^*} \|\bar{\psi} - \psi + \bar{B}\bar{u}^* - Bu^*\| \\ &\leq \|\bar{\psi} - \psi\| + \sup_{u^*} \inf_{\bar{u}^*} \|\bar{B}\bar{u}^* - Bu^*\| \\ &\leq \|\bar{\psi} - \psi\| + \sup_{u^*} \inf_{\bar{u}^*} \|\bar{B}(\bar{u}^* - u^*) + (\bar{B} - B)u^*\| \\ &\leq \|\bar{\psi} - \psi\| + \sup_{u^*} [\|(\bar{B} - B)u^*\| \\ &\quad + \|\bar{B}\| \inf_{\bar{u}^*} \|(\bar{u}^* - u^*)\|]. \end{aligned} \quad (35)$$

Now applying Utkin's equivalent control [4] at $(\mathbf{y}, \bar{t}) \in S$ and $(\bar{\mathbf{y}}, \bar{t}) \in S$, we get

$$\begin{aligned} \rho(F(\mathbf{y}, \bar{t}), F(\bar{\mathbf{y}}, \bar{t})) &\leq \|\bar{\psi} - \psi\| + \|(\bar{B} - B)u_{eq}^*\| \\ &\quad + \|\bar{B}\| \|(\bar{u}_{eq}^* - u_{eq}^*)\|. \end{aligned} \quad (36)$$

Note that we can also take the control value at the discontinuity to be bounded by a continuous function, as was used in [14]. Since $\psi(\Sigma)$, $B(\bullet)$, and $u_{eq}^*(\Sigma)$ are continuous, then given any $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$, $\exists \delta_1, \delta_2, \delta_3$, such that

$$\begin{aligned} \|(\bar{\mathbf{y}}, \bar{t}) - (\mathbf{y}, t)\| < \delta_1 &\Rightarrow \|\bar{\psi} - \psi\| < \varepsilon_1 \\ \|(\bar{\mathbf{y}}, \bar{t}) - (\mathbf{y}, t)\| < \delta_2 &\Rightarrow \|(\bar{B} - B)u_{eq}^*\| < \varepsilon_2 \\ \|(\bar{\mathbf{y}}, \bar{t}) - (\mathbf{y}, t)\| < \delta_3 &\Rightarrow \|\bar{B}\| \|(\bar{u}_{eq}^* - u_{eq}^*)\| < \varepsilon_3. \end{aligned} \quad (37)$$

Hence, given any ε , we take $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ such that $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon$, then $\exists \delta = \min(\delta_1, \delta_2, \delta_3)$ such that

$$\|(\bar{\mathbf{y}}, \bar{t}) - (\mathbf{y}, t)\| < \delta \Rightarrow \rho(F(\mathbf{y}, t), F(\bar{\mathbf{y}}, \bar{t})) < \varepsilon. \quad (38)$$

The same can be proven for 1) using (34) as is shown here for 2) using (35).

Now, we will study the discontinuous chattering-free control laws (15), (19), and (25). The main principle in these laws is that the velocity fields on both sides of the boundary of the boundary layer should face inwards. For instance [8], the filter inside the boundary layer, generated by system (4) is

$$\dot{s} = -g(s) + d(t). \quad (39)$$

The function $g(s)$ is designed based on the knowledge of the form of disturbance $d(t)$. For instance, if $d(t)$ was just an impulse, then $g(s)$ would be γs , on the other hand if it was a step input, then the Laplace transform of $g(s(t))$ would also have poles to compensate for that [8].

A sufficient condition which forces the vector fields at the inside boundary to face inwards is that $s(t_i)\dot{s}(t_i) < 0$ where t_i is the time such that $|s(t_i)| = \phi$. This can be achieved by choosing the appropriate gains on the functions used inside the boundary layer. As an example, the dynamics for system (4) inside the boundary layer are

$$\dot{s}(t) = -\gamma s(t) + d(t). \quad (40)$$

Hence, the condition for this takes the form $\gamma\phi > d(t_i)$. This design is used when impulse disturbances are expected. For step input the dynamics would be

$$\dot{s}(t) = k_1 s(t) + k_2 \int_0^t s(\tau) d\tau + d(t). \quad (41)$$

Since trajectories entering from the outside would not have the integral term, the integration is initiated when the boundary is hit. Therefore, the condition for (41) becomes $k_1\phi > d(t_i)$. Another condition needed is that once a trajectory enters the boundary layer, it should not come out of the boundary. This condition can be stated as $\forall t > t_i \|\dot{s}\|_\infty < \phi$. These conditions can be designed for by making restrictions on the percent overshoot of the system inside the boundary.

The existence of solution of this chatter-free discontinuous system is based on using definition (28) or (31) at the discontinuity of the system, which in this case is present at the boundary of the boundary layer. The technique is similar to the one used for (32). As an example, consider system (8) with the control law (15) assuming that the control design to achieve the same inward direction of the vector fields on both sides of the boundary edge has been used. Instead of (32), we obtain

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= f - \ddot{x}_d + b\hat{b}^{-1}[-f + \ddot{x}_d - \gamma\hat{x}] \\ &\quad - b\hat{b}^{-1}k \text{msat}(a, s, \phi) \end{aligned} \quad (42)$$

which in the vector form can be represented as

$$\dot{\mathbf{y}} = \psi + Bu^* = F(\mathbf{y}, t) \quad (43)$$

where, $\mathbf{y} = [y_1, y_2]^T$, $\psi = [y_2, f - \ddot{x}_d + b\hat{b}^{-1}[-f + \ddot{x}_d - \gamma\hat{x}]]^T$, $B = [0, -b\hat{b}^{-1}, k]^T$, and $u^* = \text{msat}(a, s, \phi)$. Let us define the set S as the set of discontinuities, which essentially is $S = \{\mathbf{y}: s = \phi\}$. For $(\bar{\mathbf{y}}, \bar{t}) \notin SF(\bullet, \bullet)$ is upper semicontinuous, because $F(\bullet, \bar{t})$ is continuous and the γ condition is assumed. For $(\bar{\mathbf{y}}, \bar{t}) \in S$, we will show that the separation between $F(\mathbf{y}, \bar{t})$ and $F(\bar{\mathbf{y}}, \bar{t})$ is continuous. We also define that at the discontinuity, the control is bounded by a continuous function as $|u| < \sigma(\mathbf{y})$, for $\mathbf{y} \in S$.

- 1) Take $(\mathbf{y}, \bar{t}) \notin S$, then, since $(\bar{\mathbf{y}}, \bar{t})$ is a point in R^n

$$\begin{aligned} \rho(F(\mathbf{y}, \bar{t}), F(\bar{\mathbf{y}}, \bar{t})) &= \inf_{\bar{u}^*} \|\bar{\psi} - \psi + \bar{B}\bar{u}^* - Bu^*\| \\ &\leq \|\bar{\psi} - \psi\| + \|\bar{B}\bar{u}^* - Bu^*\|. \end{aligned} \quad (44)$$

- 2) Take $(\mathbf{y}, \bar{t}) \in S$, then

$$\begin{aligned} \rho(F(\mathbf{y}, \bar{t}), F(\bar{\mathbf{y}}, \bar{t})) &= \sup_{u^*} \inf_{\bar{u}^*} \|\bar{\psi} - \psi + \bar{B}\bar{u}^* - Bu^*\| \\ &\leq \|\bar{\psi} - \psi\| + \sup_{u^*} \inf_{\bar{u}^*} \|\bar{B}\bar{u}^* - Bu^*\| \\ &\leq \|\bar{\psi} - \psi\| + \sup_{u^*} \inf_{\bar{u}^*} \|\bar{B}(\bar{u}^* - u^*) + (\bar{B} - B)u^*\| \\ &\leq \|\bar{\psi} - \psi\| + \sup_{u^*} [\|(\bar{B} - B)u^*\| \\ &\quad + \|\bar{B}\| \inf_{\bar{u}^*} \|(\bar{u}^* - u^*)\|]. \end{aligned} \quad (45)$$

Using the bound on the control input at S , we get

$$\begin{aligned} \rho(F(\mathbf{y}, \bar{t}), F(\bar{\mathbf{y}}, \bar{t})) &\leq \|\bar{\psi} - \psi\| + \|(\bar{B} - B)\| \sigma((\mathbf{y}, \bar{t})) \\ &\quad + \|\bar{B}\| |\sigma((\mathbf{y}, \bar{t})) - \sigma((\bar{\mathbf{y}}, \bar{t}))|, \\ &\quad \text{for } \sigma((\bar{\mathbf{y}}, \bar{t})) < \sigma((\mathbf{y}, \bar{t})) \end{aligned} \quad (46a)$$

$$\begin{aligned} \rho(F(\mathbf{y}, \bar{t}), F(\bar{\mathbf{y}}, \bar{t})) &\leq \|\bar{\psi} - \psi\| + \|(\bar{B} - B)\| \sigma((\mathbf{y}, \bar{t})), \\ &\quad \text{for } \sigma((\bar{\mathbf{y}}, \bar{t})) \geq \sigma((\mathbf{y}, \bar{t})). \end{aligned} \quad (46b)$$

Since $\psi(\bullet)$, $B(\bullet)$, and $u_{eq}^*(\bullet)$ are continuous, then given any $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ and the conditions of (46a), $\exists \delta_1, \delta_2, \delta_3$, such that

$$\begin{aligned} \|(\bar{\mathbf{y}}, \bar{t}) - (\mathbf{y}, t)\| < \delta_1 &\Rightarrow \|\bar{\psi} - \psi\| < \varepsilon_1 \\ \|(\bar{\mathbf{y}}, \bar{t}) - (\mathbf{y}, t)\| < \delta_2 &\Rightarrow \|(\bar{B} - B)\sigma\| < \varepsilon_2 \\ \|(\bar{\mathbf{y}}, \bar{t}) - (\mathbf{y}, t)\| < \delta_3 &\Rightarrow \|\bar{B}\| |\sigma(\bar{\mathbf{y}}, \bar{t}) - \sigma(\mathbf{y}, t)| < \varepsilon_3. \end{aligned} \quad (47)$$

Hence, given any ε , we take $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ such that $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon$, then $\exists \delta = \min(\delta_1, \delta_2, \delta_3)$ such that

$$\|(\bar{\mathbf{y}}, \bar{t}) - (\mathbf{y}, t)\| < \delta \Rightarrow \rho(F(\mathbf{y}, t), F(\bar{\mathbf{y}}, \bar{t})) < \varepsilon. \quad (48)$$

The same can be proven for (46b) and also for (46a) using (44) as is shown here for (46b) using (45).

IV. CONCLUSIONS

We presented a sliding mode control which uses a boundary layer for chattering reduction. Inside the boundary layer control laws are designed using the internal model principle and are implemented using functions which replace the signum function in sliding mode. Although, the new control laws produce chattering-free systems, they can still have discontinuities. Therefore, the existence of solutions to these systems was studied in this paper. The existence of solutions to these discontinuous systems was based on the Filippov's solutions to differential equations, which have discontinuous right-hand sides. The discontinuous systems were reformulated in the form of differential inclusions for analysis.

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On Feedback Invariance Properties for Systems over a Principal Ideal Domain

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Abstract—Simple necessary and sufficient conditions for the solvability of many control problems for linear systems over a field are based on the equivalence between the (A, B) -invariance property and the $(A + BF)$ -invariance, or feedback invariance, property. For systems over a ring, this equivalence is no longer true and many results of the geometric control theory cannot be extended. In this paper we will present new, algorithmically checkable characterizations of the $(A + BF)$ -invariance property for systems defined over a principal ideal domain and a new solvability condition for the disturbance rejection problem.

Index Terms—Geometric approach, linear system, ring models.

I. INTRODUCTION

In the theory of linear systems over a field, the geometric approach [13], [3] has allowed a better vision and has provided elegant solutions to many control problems such as the disturbance decoupling problem, the block decoupling problem, and the model matching problem. At the same time, models with coefficients over a ring turned out to be very useful in describing several interesting classes of systems such as delay-differential systems and systems depending on a vector of parameters [5], [6], [12]. Therefore, many authors worked at an extension of the geometric theory to systems over a ring [4]–[6], [9], [10].

The main difficulty one faces when dealing with systems over a ring is that an (A, B) -invariant subspace is not necessarily $(A + BF)$ -invariant and the necessary and sufficient solvability conditions of many control problems, requiring the existence of a feedback law, become very difficult to check. Only in introducing restrictive assumptions can feasible conditions be obtained.

In this paper, extending the results of [1] and [2], we present algorithmic procedures to check the $(A + BF)$ -invariance property of a submodule and to establish the existence of $(A + BF)$ -invariant submodules included in, or containing, a given submodule. So, under mild hypotheses on the system, the solvability conditions of several control problems can be checked. An application of these results to the disturbance rejection problem and an illustrative example are also presented.

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