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Abstract

This paper presents the estimation of travel time on highways based on macroscopic modelling. The focus is on real-time values as compared to average or static values. The macroscopic models are used for distributed and time/space lumped settings and corresponding travel time estimation functions and algorithms are developed. The implications of these algorithms for the implementation of various incident management and traffic control strategies are also discussed.

1. Introduction

The accurate and practical estimation of travel times using on-line traffic flow information obtained from traffic sensors continue to be an active research area. Estimated travel times can be used in many ITS applications raging from the dissemination of real-time route guidance information to the development of pro-active traffic control measures. In this paper we introduce a travel time estimation procedure that employs well-know macroscopic traffic flow characteristics.

2. System Modelling

The first step in the design of estimators for travel time is to model the system dynamics appropriately. Macroscopic model of the traffic can effectively be used in this context. From the macroscopic perspective, the traffic flow is considered analogous to a fluid flow, which is a distributed parameter system represented by partial differential equations. Mass conservation model of a highway, characterized by $x \in [0, L]$, which is the position on the highway, is given by

$$\frac{\partial}{\partial t}\rho(x,t) = \frac{\partial}{\partial x} q(x,t) \qquad (1)$$

where $\rho(x,t)$ is the density of the traffic as a function of x, and time t, and $q(x,t)$ is the flow at given x, and t. The flow $q(x,t)$ is a function of $\rho(x,t)$, and the velocity $v(x,t)$, as shown below:

$$q(x,t) = \rho(x,t)v(x,t) \qquad (2)$$

This model of a highway section is shown in Figure 1.

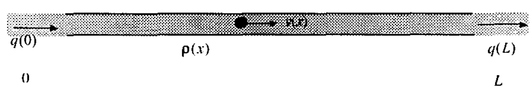

Figure 1: Segment of Highway Model

There are various static and dynamic models which have been used to represent the relationship between $v(x,t)$ and $\rho(x,t)$. One of the most widely used models is the one proposed by Greenshield [1], which hypothesizes a linear relationship between the two variables.

$$v = v_f\left(1 - \frac{\rho}{\rho_{\max}}\right) \qquad (3)$$

where v_f is the free flow speed, and $\rho_{\max}$ is the jam density.

3. Burgers' Equation for Macroscopic Modeling

The system represented by equations (1-3) is an infinite dimensional representation of the traffic, since it has infinite state variables. There are two ways to travel time estimators for such a system. One way is to work in the infinite dimensional domain, and design an estimator, which then can be discretized. Another way is to space discretize (Partial Differential Equation) PDE

(1) to obtain an (Ordinary Differential Equation) ODE representation of the system.

The modeling of traffic in the PDE domain provides a reasonably accurate model of the traffic system, especially since phenomena such as shock waves are effectively represented. Hence, it would be highly desirable to design an estimator directly utilizing this model. However, the model shown previously shows a static velocity relationship with density. A more appropriate model that uses a dynamic relationship can be used. Burgers' equation is just a way of representing the same hydrodynamic traffic flow model in a different form utilizing the diffusion behavior of the traffic.

In the PDE context, researchers have used Burgers' equation to model the traffic flow [2,3]. Burger's equation was introduced by Burgers [4,5,6] as a simple model for turbulence, where ε is a viscosity coefficient. The following is borrowed from [2] to show how the traffic problem can be modelled as Burgers' equation.

$$\frac{\partial}{\partial t}\rho(x,t)+\rho(x,t)\frac{\partial}{\partial x}\rho(x,t)=\varepsilon\frac{\partial^2}{\partial x^2}\rho(x,t) \tag{4}$$

In order to account for the fact that drivers look ahead and modify their speeds accordingly, (3) can be replaced by

$$v_e=v_f(1-\frac{\rho}{\rho_{\max}})-D(\frac{\partial\rho}{\partial x})/\rho \tag{5}$$

Using (5) and the fact $q=\rho v_e$, relationship (2) now can be replaced by

$$q(x,t)=\rho(x,t)v(x,t)-D\frac{\partial}{\partial x}\rho(x,t) \tag{6}$$

where D is a diffusion coefficient [19] given by

$$D=\tau v_r^2 \tag{7}$$

where v_r is a random velocity, and τ is a parameter. Diffusion is a useful concept mentioned by many researchers as an extension to the existing traffic flow models to improve their realism [2,3,7]. Diffusion term represents "the diffusion effect" due to the fact that each driver's gaze is concentrated on the road in front of him/her, so that he/she adjusts his/her speed according to the concentration ahead. This adjustment creates a dependence of flow on concentration gradient that leads to an effective diffusion. This models the gradual rather than instantaneous reduction of speed by the drivers in response to the shock waves. The diffusion term D has the units of velocity multiplied with those of length, such as $mile^2/hour$. Combining equations (1) and (6) gives

$$[\frac{\partial}{\partial t}\rho(x,t)+v_f\frac{\partial}{\partial x}\rho(x,t)]-2\frac{\rho}{\rho_{\max}}v_f\frac{\partial}{\partial x}\rho(x,t)-D\frac{\partial^2}{\partial x^2}\rho(x,t)=0 \tag{8}$$

If we introduce a moving reference frame

$$\xi(x,t)=-x+v_f t \tag{9}$$

and non-dimensionalize $\rho(x,t)$ by $\rho_{\max}/2$, and t by t_0, equation (7) gets transformed to

$$\frac{\partial}{\partial t}\rho(\xi,t)+\rho\frac{\partial}{\partial\xi}\rho(\xi,t)-\frac{1}{R_e}\frac{\partial^2}{\partial\xi^2}\rho(\xi,t)=0 \tag{10}$$

Here, R_e is a dimensionless constant, and is analogous to Reynolds number in fluid dynamics. R_e is given by

$$R_e=(\frac{v_f}{v_r})^2\frac{t_0}{\tau} \tag{11}$$

Equation (10) shows the Burgers' equation formulation of the traffic flow problem. Some researchers have also worked on the conservation law

$$\frac{\partial}{\partial t}\rho(x,t)+\rho(x,t)\frac{\partial}{\partial x}\rho(x,t)=\varepsilon\frac{\partial^2}{\partial x^2}\rho(x,t) \tag{12}$$

with a solution obtained by taking the following limit. where $\rho^\varepsilon(x,t)$ satisfies (4) [8-13]. Using this form reduces the Burgers' equation formulation into the

$$\rho(x,t)=\lim_{\varepsilon\to 0}\rho^\varepsilon(x,t) \tag{13}$$

classical traffic model with no diffusion. So, even in the PDE domain, we can try to work on the same model as the classical traffic models.

4. Discretized System Dynamics

Many researchers have studied traffic utilizing space and time discretized models of traffic flow. The reason for the popularity of these models is that there are many techniques available to deal with discretized systems. For this the highway is subdivided into several sections as shown in Figure 2.

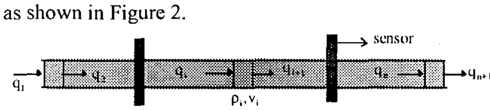

Figure 2. Highway Divided into Sections

Dividing the considered highway links into multiple segments performs space-discretization. In general the length of each segment is taken to be between .5 mile and 1 mile. This is an approximation that is quite realistic since the traditional sensors like loop detectors along a freeway are generally installed at least 1 mile apart. Although a smaller step size for space discretization will undoubtedly improve the accuracy of the simulation, in reality it is not possible to measure speed and flow variables at smaller intervals due to limited availability of sensors along freeways. Thus, 1-mile segment lengths for space discretization appear to be a realistic assumption. On the other hand, the time discretization can be done using very small time steps since traffic data can be downloaded from sensors practically at every second. With the use of more sensors such as CCTVs we can obtain information at higher resolution to have smaller space discretiztion steps.

The space discretized form of (1) produces the following n continuous ODEs for the n sections of the highway.

$$\frac{d}{dt}\rho_i = \frac{1}{\delta_i}[q_i(t) - q_{i+1}(t)], i = 1,2,...n \cdot \tag{14}$$

Equation (14) combined with (2), (3) (or (5), (6) depending on the decision to include the diffusion term) gives the mathematical model for a highway, which can be represented in a standard nonlinear state space form. The standard state space form is

$$\frac{d}{dt}y(t) = f[y(t), u(t)],$$
$$y(0) = y_0, \tag{15}$$

where $y = [\rho_1, \rho_2, ..., \rho_n]^T$ and $u(t) = q_1(t)$.

There are various other proposed models, which are more detailed in the description of the system dynamics. The phenomenon of shock waves, which is very well represented in the PDE representation of the system, is modeled by expressing the traffic flow between two contiguous sections of the highway, as the weighted sum of the traffic flows in those two sections that correspond to the densities in those two sections [14, 15]. A dynamic relationship instead of a static one like (3) has also been proposed by [16] and used successfully.

The model thus obtained can also be time discretized to transform the continuous time model into a discrete time mode. A comprehensive model, which incorporates shock waves, as well as represents the dynamic nature of mean speed propagation is derived from [17] and shown below. The difference equations

$$\rho_j(k+1) = \rho_j(k) + \frac{T}{\delta_j}[q_{j-1}(k) - q_j(k)],$$
$$v_j(k+1) = v_j(k) + \frac{T}{\tau}[v_e(\rho_j(k)) - v_j(k)] + \frac{T}{\delta_j}v_j(k)[v_{j-1}(k) - v_j(k)] \tag{16}$$
$$-\frac{vT}{\tau\delta_j}[\frac{\rho_j(k+1) - \rho_j(k)}{\rho_j(k) + \vartheta}]$$

with the relationships

$q_j(k) = \alpha\rho_j(k)v_j(k) + (1-\alpha)\rho_{j+1}(k)v_{j+1}(k), 0 \leq \alpha \leq 1$

$$v_e(\rho) = v_f[1 - (\frac{\rho}{\rho_{max}})^l]^m, \tag{17}$$

output measurements of traffic flows q and time mean speeds m, shown as

$$m_j(k) = \gamma v_j(k) + (1-\gamma)v_{j+1}(k), 0 \leq \gamma \leq 1, \tag{18}$$

and the boundary conditions

$$v_0(k) = y_0(k), \tag{19}$$
$$\rho_{n+1}(k) = q_n(k)/y_n(k),$$

gives the discrete system dynamics, which can be represented in the standard nonlinear discrete time form

$$y(k+1) = f(x(k), u(k)), \tag{20}$$
$$m(k) = g(x(k), u(k)),$$
$$x(0) = x_0,$$

where control $u(k)$ is the input flow at the first upstream section.

5. Travel Time Estimation Based on only Current State

Travel time estimation algorithms and functions can be designed using the distributed or lumped parameter models of the traffic. We will develop appropriate models using the different modeling paradigms. This section will present models without forecasting that will approximate the travel time based on only the current state values. The next section will present more accurate estimators that will use forecasting.

5.1 Distributed Static Velocity Based Estimation

The system model for travel time estimation for this is given by equations (1-3). The derivation utilizes the infinitesimal section with length dx and then integrating it over the entire length to get the travel time, as shown in Figure 3.

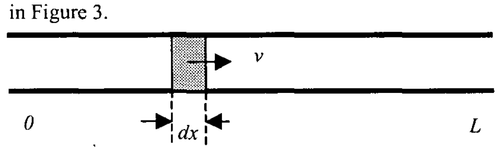

Figure 3 Infinitesimal Section

The travel time in the section dx is given by dividing the section length by the section velocity

$$dt_r = \frac{dx}{v(\rho(t,x))} \tag{21}$$

We use the variable T for travel time and it is a function of the actual time variable t, indicating that the estimated travel time changes with time as the density profile changes with time. Integrating (21) gives us the travel time estimate for the entire section.

$$\int_0^T dt_r = \int_0^L \frac{dx}{v(\rho(t,x))} = \int_0^L \frac{dx}{v_f(1 - \frac{\rho}{\rho_{max}})} \tag{22}$$

This integral can be calculated if the density profile of the section is known. This can be known exactly if distributed sensors are used or can approximated by aggregated values obtained by spot sensors. We can also use simulation methods that would require solving the hydro-dynamic equation with appropriate boundary conditions.

5.2 Distributed Dynamic Velocity Based Estimation

The system model for travel time estimation for this is given by equations (4,5). The derivation, as before utilizes the infinitesimal section with length dx and then integrating it over the entire length to get the travel time. The travel time in the section dx is given by dividing the section length by the section velocity as in (21). The difference in this model is that when we do the integration, the relationship (5) is used instead of (3).

$$\int_0^T dt_r = \int_0^L \frac{dx}{v(\rho(t,x))} = \int_0^L \frac{dx}{v_f(1-\frac{\rho}{\rho_{max}}) - D(\frac{\partial \rho}{\partial x})/\rho} \quad (23)$$

This integral can be calculated if the density profile of the section is known. This can be known exactly if distributed sensors are used or can approximated by aggregated values obtained by spot sensors. The approximation of the density gradient can also be done using aggregate values. If we use simulation methods, then we would have to obtain the solution of Burgers' equation.

5.3 Discrete Space Based Estimation

The system model for travel time estimation for this is given by equations (14, 2, 3). The derivation utilizes a discrete section and then summing over all the sections on the entire length to get the travel time, as shown in Figure 4.

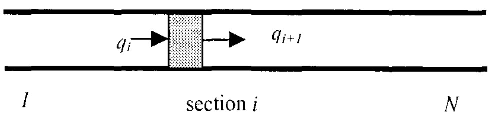

Figure 4. Discrete Time Modelling Section

The travel time in the given discrete section is given by dividing the section length by the section velocity

$$T_i = \frac{\Delta_i}{v(\rho_i(t))} \quad (24)$$

We use the variable T_i for travel time in a section i and Δ_i as the length of section i. Summing (24) gives us the travel time estimate for the entire section.

This sum can be calculated if the density of each section is known. We can also use simulation methods that

$$T = \sum_{i=1}^{N} T_i = \sum_{i=1}^{N} \frac{\Delta_i}{v(\rho_i(t))} = \sum_{i=1}^{N} \frac{\Delta_i}{v_{fi}(1-\frac{\rho_i}{\rho_{max\,i}})} \quad (25)$$

would require solving the set of ordinary differential equations with appropriate input flows.

5.4 Discrete Time and Space Based Estimation

For this case, the travel time functions for sections and the total length remain the same as (24) and (25). The only difference in this case is that the dynamic equations are in terms of difference equations instead of differential equations.

6. Corrected Travel Time Estimation

Travel time estimation functions used in section 5 above give a travel time function that is the sum of (or integral of) travel times experienced in all the sections at the same time. However, if we take the travel time to be the travel time expected to be experienced by a vehicle currently at the upstream boundary of the first section, then we use only the current expected travel time for that section, and then sequentially calculate travel times expected at the expected arrival times of the vehicle at other sections. This idea is used to develop the equations for the various models below.

6.1 Distributed Static Velocity Based Estimation

Distributed static velocity based estimation of travel time function can be performed by defining a travel time function on a highway of length L as a function of x and t as $T(t,x)$. This function gives the travel time at time t from the start of the highway section to the point given by x. Hence, function $T(t,L)$ gives the travel time function for the entire section. The derivation of this function is given in terms of solution of a set of two partial differential equations that are derived from the conditions depicted in Figure 5.

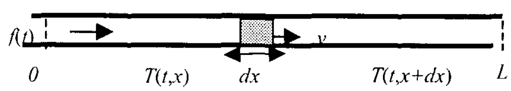

Figure 5 Partial Differential Equation for Travel Time Function Derivation

From Figure 5, we can see that the time to travel the distance dx is given by

$$dt_r = \frac{dx}{v_f(1-\frac{\rho(t+T(t,x),x)}{\rho_m})} \quad (26)$$

Now, using the definition of the travel time function, we get

$$T(t,x+dx)=T(t,x)+\frac{dx}{v_f(1-\frac{\rho(t+T(t,x),x)}{\rho_m})} \quad (27)$$

Taking the limit $dx \rightarrow 0$, we get the following partial differential equation

$$v_f(1-\frac{\rho(t+T(t,x),x)}{\rho_m})\frac{\partial T(t,x)}{\partial x}-1=0 \quad (28)$$

The travel time at time t for the highway section is given by $T(t,L)$ that is obtained by solving (1) and (28), with (2), (3), and the following boundary conditions.

$$T(t,0)=0 \quad (29)$$

$$\rho v_f(1-\frac{\rho(t,0)}{\rho_m})=f(t) \quad (30)$$

6.2 Distributed Dynamic Velocity Based Estimation

The difference between the static and the dynamic velocity case comes from the velocity expression. From Figure 5, we can see that the time to travel the distance dx is given by

$$dt_r=\frac{dx}{v_f(1-\frac{\rho(t+T(t,x),x)}{\rho_m})-D\frac{\partial\rho(t+T(t,x),x)}{\partial x}/\rho(t+T(t,x),x)} \quad (31)$$

Now, using the definition of the travel time function, we get

$$T(t,x+dx)=T(t,x)+\frac{dx}{v_f(1-\frac{\rho(t+T(t,x),x)}{\rho_m})-D\frac{\partial\rho(t+T(t,x),x)}{\partial x}/\rho(t+T(t,x),x)} \quad (32)$$

Taking the limit $dx \rightarrow 0$, we get the following partial differential equation

$$[v_f(1-\frac{\rho(t+T(t,x),x)}{\rho_m})-D\frac{\partial\rho(t+T(t,x),x)}{\partial x}/\rho(t+T(t,x),x)]\frac{\partial T(t,x)}{\partial x}-1=0 \quad (33)$$

The travel time at time t for the highway section is given by $T(t,L)$ that is obtained by solving (1) and (33), with (3), (5), (6), and the following boundary conditions.

$$T(t,0)=0 \quad (34)$$

$$\rho v_f(1-\frac{\rho(t,0)}{\rho_m})-D\frac{\partial}{\partial x}\rho(t,0)=f(t) \quad (35)$$

6.3 Discrete Space Based Estimation

The discrete time based travel time estimation is done by using travel time estimation applied in a sequential fashion from upstream to downstream. To show this analysis, let us consider Figure 6.

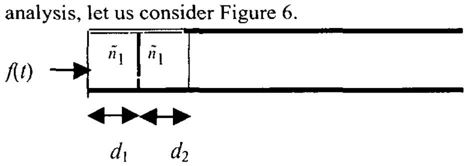

Figure 6 Travel Time Function Derivation for Discrete Space Systems

The travel time in section 1 is given by

$$T_1=\frac{d_1}{v_{f_1}(1-\frac{\rho_1}{\rho_{m_1}})} \quad (36)$$

Now, we need to calculate the travel time in section 2 at a time that occurs after the travel time in section 1, i.e.

$$T_2=\frac{d_2}{v_{f_2}(1-\frac{\rho_2(t+T_1)}{\rho m_2})} \quad (37)$$

In order to calculate $\rho_2(t+T_1)$, we will solve the ODEs for section 1 and 2, which are given by

$$\dot{\rho}_1=f(t)-v_{f_1}(1-\frac{\rho_1}{\rho m_1}) \quad (38)$$

$$\dot{\rho}_2=v_{f_1}\rho_1(1-\frac{\rho_1}{\rho m_1})-\rho_2 v_{f_2}(1-\frac{\rho_2}{\rho m_2}) \quad (39)$$

We can similarly calculate the travel time in all the rest of the links in a sequential fashion by proceeding from upstream to downstream. We see that for a general link,

$$T_i=\frac{d_i}{v_{f_i}(1-\frac{\rho_i(t+T_{i-1})}{\rho_{m_i}})} \quad (40)$$

and

$$\dot{\rho}_i=v_{f_{i-1}}\rho_{i-1}(1-\frac{\rho_{i-1}}{\rho m_{i-1}})-\rho_i v_{f_i}(1-\frac{\rho_i}{\rho m_i}) \quad (41)$$

6.4 Discrete Time and Space Based Estimation

For the discrete time and space case, the development is very similar to the case above except that we have difference equations instead of ODEs. The travel time in section 1 is given by (36) and in section 2 by (37). However, we need to solve

$$\rho_1(k+1)=\rho_1(k)+T[f(t)-v_{f_1}(1-\frac{\rho_1}{\rho_{m_1}})] \quad (42)$$

$$\rho_2(k+1)=\rho_2(k)+T[v_{f_1}\rho_1(1-\frac{\rho_1}{\rho_{m_1}})-\rho_2 v_{f_2}(1-\frac{\rho_2}{\rho_{m_2}})] \quad (43)$$

We can similarly calculate the travel time in all the rest of the links in a sequential fashion by proceeding from upstream to downstream. The difference equation for a general link will be

$$\rho_i(k+1)=\rho_i(k)+T[v_{f_{i-1}}\rho_{i-1}(1-\frac{\rho_{i-1}}{\rho_{m_{i-1}}})-\rho_i v_{f_i}(1-\frac{\rho_i}{\rho_{m_i}})] \quad (44)$$

7. Implications to Prediction, Incident Management and Traffic Control

One point that we notice in the analysis in section 6 is the need to perform prediction of the function f(t) for some time in the future. For the distributed system modelling, the prediction time is same as the travel time on the entire highway section, and for the discretized models, we have to predict for the time period that is equal to the sum of travel times on all but the last (downstream) section. We can perform the prediction on a time period longer than the expected travel time and use those values to drive the travel time estimation.

During incidents the model parameters will change, such as jam density. Using these values, we can calculate the travel times. We will also need to know how long these values will be used. This information can be fed from incident management systems that perform incident duration estimations [18]. Some traffic control problems use travel time as the driver for the control design, such as dynamics traffic routing [19, 20]. Previously static travel time functions have been used for the feedback control design of these controllers. It is in general difficult to use the corrected travel time functions for those control designs, but attempts can be made in order to produce better controllers.

8. Conclusions

In this paper we developed travel time functions based on macroscopic models of highways. The models chosen were of distributed type, as well as space, and space-time lumped types. Travel time functions were derived based on only current state values, but these functions were not accurate representations of the expected travel times. Hence, prediction based travel time functions which, represent the dynamic nature of processing needed for the calculations of the travel times were derived.

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