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Isolated Ramp Metering Feedback Control utilizing Mixed Sensitivity for Desired Mainline Density and the Ramp Queues

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Abstract

This paper presents a feedback control design for isolated ramp metering control. This feedback control design, unlike the existing isolated feedback ramp controllers, also takes into account the ramp queue length. Using a nonlinear H_∞ control design methodology, we formulate the problem in the desired setting to be able to utilize the results of the methodology.

1. Introduction

Ramp metering is a way to improve traffic flow by regulating the ramp inflow to a freeway. By effectively controlling the ramp flow, the traffic density on the mainline freeway can be kept below critical level to provide high throughput, which is congestion free. For this type of operation, many factors have to be considered such as:

- The inflow at the mainline
- The queue holding capacity of the ramp
- Availability of sensors
- The arterial system connected to the ramp system

In this paper we will use the theory of nonlinear H_∞ control to design a ramp control law, which minimizes a weighted function of the ramp queues and the difference of the mainline density to the desired mainline density. In order to design the controller, we need the system dynamics equations. We present the system dynamics next followed by the theory of nonlinear H_∞ control, which is then applied to the ramp control problem.

2. System Modelling [1]

The first step in the design of feedback controllers for ramp metering is to model the system dynamics appropriately. Macroscopic model of the traffic can effectively be used in this context. From the macroscopic perspective, the traffic flow is considered analogous to a fluid flow, which is a distributed parameter system represented by partial differential equations. Mass conservation model of a highway, characterized by $x \in [0, L]$, which is the position on the highway, is given by

$$\frac{\partial}{\partial t}\rho(x,t)=-\frac{\partial}{\partial x}q(x,t) \tag{1}$$

where $\rho(x,t)$ is the density of the traffic as a function of x, and time t, and $q(x,t)$ is the flow at given x, and t. The flow $q(x,t)$ is a function of $\rho(x,t)$, and the velocity v(x,t), as shown below:

$$q(x,t)=\rho(x,t)v(x,t) \tag{2}$$

This model of a highway section is shown in Figure 3.

Figure 3. Segment of Highway Model

There are various static and dynamic models which have been used to represent the relationship between v(x,t) and $\rho(x,t)$ One of the simplest models is the one proposed by Greenshield [2], which hypothesizes a linear relationship between the two variables.

$$v=v_f(1-\frac{\rho}{\rho_{max}}) \tag{3}$$

where v_f is the free flow speed, and ρ_{max} is the jam density.

2.1. Discretized System Dynamics

Many researchers have studied and designed optimal open loop controllers utilizing space and time discretized models of traffic flow. Some researchers have also designed feedback control laws using similar models. The reason for the popularity of these models is that there are many techniques available to deal with discretized systems. The same is also true for feedback control, and hence, in order to utilize the various linear and nonlinear control techniques available for lumped parameter systems, the distributed parameter model is space discretized. For this the highway is subdivided into several sections as shown in Figure 4.

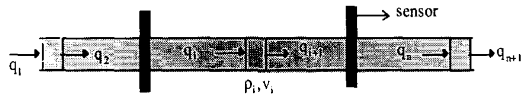

Figure 4. Highway Divided into Sections

Space discretization is performed by dividing the considered highway links into segments. In general the length of each segment is taken to be between .5 mile and 1 mile. This is an approximation that is quite realistic since the traditional sensors like loop detectors along a freeway are generally installed at least 1 mile apart. Although a smaller step size for space dicretization will undoubtedly improve the accuracy of the simulation, in reality it is not possible to measure speed and flow variables at smaller intervals due to limited availability of sensors along freeways. Thus, 1-mile segment length for space discretization appears to be a realistic assumption. On the other hand, the time discretization can be done using very small time steps since traffic data can be downloaded from sensors practically at every second. The space discretized form of (1) produces the following n continuous ODEs for the n sections of the highway.

$$\frac{d}{dt}\rho_i = \frac{1}{\delta_i}[q_i(t) - q_{i+1}(t) + r_i(t) - s_i(t)], i = 1,2,\ldots n \cdot \tag{4}$$

Here, $r_i(t)$ and $s_i(t)$ terms indicate the on-ramp and off-ramp flows. The mathematical model for the highway can be represented in a standard nonlinear state space form for control design purposes.

$$y_j = g_j(\rho_1, \rho_2, \ldots, \rho_n), j = 1,2,\ldots, p, \tag{5}$$

The standard state space form is

$$\begin{aligned} \frac{d}{dt}\mathbf{x}(t) &= \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t)], \\ \mathbf{y}(t) &= \mathbf{g}[\mathbf{x}(t), \mathbf{u}(t)], \\ \mathbf{x}(0) &= \mathbf{x}_0, \end{aligned} \tag{6}$$

where $\mathbf{x} = [\rho_1, \rho_2, \ldots, \rho_n]^T$ and $\mathbf{u}(t)=\mathbf{r}(t)$.

There are various other proposed models, which are more detailed in the description of the system dynamics. The phenomenon of shock waves, which is very well represented in the PDE representation of the system, is modelled by expressing the traffic flow between two contiguous sections of the highway, as the weighted sum of the traffic flows in those two sections, which correspond to the densities in those two sections [3, 4]. A dynamic relationship instead of a static one like (3) has also been proposed by [5] and used successfully.

The model thus obtained can also be time discretized to transform the continuous time model into a discrete time mode. A comprehensive model, which incorporates shock waves, as well as represents the dynamic nature of mean speed propagation, is shown in [6] and is reproduced here for completion. The difference equations

$$\begin{aligned} &\rho_j(k+1) = \rho_j(k) + \frac{T}{\delta_j}[q_{j-1}(k) - q_j(k) + r_j(k) - s_j(k)], \\ &v_j(k+1) = v_j(k) + \frac{T}{\tau}[v_e(\rho_j(k)) - v_j(k)] + \frac{T}{\delta_j} v_j(k)[v_{j-1}(k) - v \\ &-\frac{vT}{\tau\delta_j}[\frac{\rho_j(k+1) - \rho_j(k)}{\rho_j(k) + \vartheta}] \end{aligned} \tag{7}$$

with the relationships

$$\begin{aligned} &q_j(k) = \alpha\rho_j(k)v_j(k) + (1-\alpha)\rho_{j+1}(k)v_{j+1}(k), 0 \le \alpha \le 1, \\ &v_e(\rho) = v_f[1 - (\frac{\rho}{\rho_{\max}})^l]^m, \end{aligned} \tag{8}$$

output measurements of traffic flows q and time mean speeds y, shown as

$$y_j(k) = \gamma v_j(k) + (1-\gamma)v_{j+1}(k), 0 \le \gamma \le 1, \tag{9}$$

and the boundary conditions

$$\begin{aligned} &v_0(k) = y_0(k), \\ &\rho_{n+1}(k) = q_n(k)/y_n(k), \end{aligned} \tag{10}$$

gives the discrete system dynamics, which can be represented in the standard nonlinear discrete time form

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{f}(\mathbf{x}(k), \mathbf{u}(k)), \\ \mathbf{y}(k) &= \mathbf{g}(\mathbf{x}(k), \mathbf{u}(k)), \\ \mathbf{x}(0) &= \mathbf{x}_0, \end{aligned} \tag{11}$$

where control $\mathbf{u}$(k) is the vector of ramp input flows.

If the control actuation is discrete, such as the ones implemented by microprocessors and computers, feedback control laws can be designed based on the discrete model (11), or can be designed using (6) after which the controller can be discretized.

The dynamics of the ramp queue are represented by the conservation equation where the rate of change of the number of vehicles in the queue is equal to the input flow to the queue subtracted from the outflow as seen in (12).

3. Background (Nonlinear $\mathbf{H}_\infty$ Control)

Consider the system

$$\begin{aligned} &\dot{\mathbf{x}} = \mathbf{a}(\mathbf{x}) + \mathbf{b}(\mathbf{x})\mathbf{u} + \mathbf{g}(\mathbf{x})\mathbf{d}_1, \mathbf{a}(\mathbf{0}) = \mathbf{0} \\ &\mathbf{y} = \mathbf{c}(\mathbf{x}) + \mathbf{d}_2, \mathbf{c}(\mathbf{0}) = \mathbf{0} \\ &\mathbf{z} = \begin{bmatrix} \mathbf{h}(\mathbf{x}) \\ \mathbf{u} \end{bmatrix}, \ \mathbf{h}(0) = \mathbf{0} \end{aligned} \tag{12}$$

where $\mathbf{x} = (\mathbf{x}_1, ..., \mathbf{x}_n)$ are local coordinates for a C^∞ state space manifold M, $\mathbf{u} \in R^m$ are the control inputs, $\mathbf{d}_1 \in R^r$ and $\mathbf{d}_2 \in R^p$ the exogenous inputs consisting of reference and/or disturbance signals, $\mathbf{y} \in R^p$ the measured outputs, and $\mathbf{z} \in R^s$ outputs to be controlled. The system (12) is identified by G. For a full-state measurement case **y**=**x**. The controller is identified by K. The closed loop system in Figure 5 will be denoted by $\Omega(G/K)$

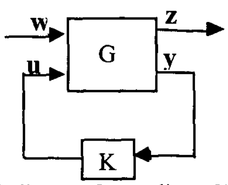

Figure 5 Block diagram for nonlinear H_∞ formulation

Definition 1: The closed loop system $\Omega(G/K)$ is said to have L_2 gain less than or equal to γ for some _ > 0 if

$$\int_0^T \|\mathbf{z}(t)\|^2 dt \le \gamma^2 \int_0^T \|\mathbf{w}(t)\|^2 dt + b(\mathbf{x}_0) \tag{13}$$

$\forall$ T > 0 and **w**(t) $\in$ L_2[0, T], where $b(\mathbf{x}_0)$ is a positive constant depending on initial condition $\mathbf{x}_0$.

State Feedback H_∞ Control Problem: Find a state feedback controller K:u=u(x) if any, such that the closed loop system Ω(G/K) is asymptotically stable and has L_2-gain $\le \gamma$.

Solution: [7-11] If there exists a smooth function V(**x**)$\ge$0 which satisfies the Hamilton-Jacobi (HJ) inequalities

$$V_{\mathbf{x}}(\mathbf{x})\mathbf{a}(\mathbf{x}) + \frac{1}{2}V_{\mathbf{x}}(\mathbf{x})\left[\frac{1}{\gamma^2}\mathbf{g}(\mathbf{x})\mathbf{g}^T(\mathbf{x}) - \mathbf{b}(\mathbf{x})\mathbf{b}^T(\mathbf{x})\right]\mathbf{V}_{\mathbf{x}}^T(\mathbf{x})$$
$$+\frac{1}{2}\mathbf{h}^T(\mathbf{x})\mathbf{h}(\mathbf{x}) \le 0, V(\mathbf{0}) = 0 \tag{14}$$

and we set

$$\mathbf{u}_* = -\mathbf{b}^T(\mathbf{x})V_{\mathbf{x}}(\mathbf{x}) \tag{15}$$

then the closed loop system $\Omega(G/\mathbf{u}_*)$ has gain at most γ. Moreover if V(**x**) has a strict local minimum at **x**=0 and the system

$$\dot{x} = \mathbf{a}(\mathbf{x})$$
$$\mathbf{z} = \begin{bmatrix} \mathbf{h}(\mathbf{x}) \\ -\mathbf{b}^T(\mathbf{x})V_{\mathbf{x}}^T(\mathbf{x}) \end{bmatrix} \tag{16}$$

is zero-state detectable (i.e. $\dot{\mathbf{x}} = \mathbf{a}(\mathbf{x})$ and $\mathbf{z}(\mathbf{x}(t)) \equiv \mathbf{0}$ for t$\ge$0 $\Rightarrow \lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$), then **x**=**0** is a locally asymptotically stable equilibrium of

$$\dot{\mathbf{x}} = \mathbf{a}(\mathbf{x}) - \mathbf{b}(\mathbf{x})\mathbf{b}^T(\mathbf{x})V_{\mathbf{x}}^T(\mathbf{x}) \tag{17}$$

If additionally, V has a global strict minimum at **x**=**0** and V is proper (so the inverse image of a compact set under V is again compact), then **x**=**0** is a globally asymptotically stable equilibrium of (17).

For the finite-time horizon problem, where final time T is finite, the solution is given by $\mathbf{u} = -\mathbf{b}^T(\mathbf{x})V_{\mathbf{x}}(t, \mathbf{x})$, where $V(t, \mathbf{x}) \ge 0$ satisfies the following HJ equation.

$$V_t(t,\mathbf{x}) + V_{\mathbf{x}}(\mathbf{x})\mathbf{a}(\mathbf{x}) + \frac{1}{2}V_{\mathbf{x}}(\mathbf{x})\left[\frac{1}{\gamma^2}\mathbf{g}(\mathbf{x})\mathbf{g}^T(\mathbf{x}) - \mathbf{b}(\mathbf{x})\mathbf{b}^T(\mathbf{x})\right]\mathbf{V}_{\mathbf{x}}^T$$
$$+\frac{1}{2}\mathbf{h}^T(\mathbf{x})\mathbf{h}(\mathbf{x}) = 0, V(T,\mathbf{x}) = V_f(\mathbf{x}) \tag{18}$$

The solution for the finite-time can be derived from a min-max differential game perspective [12].

Measurement Feedback H_∞ Control Problem: Find a dynamic feedback controller

$$K: \begin{cases} \dot{\eta} = \kappa(\eta) + \ell(\eta) y \\ \mathbf{u} = \mathbf{m}(\eta) \end{cases} \tag{19}$$

so that the closed loop system Ω(G/K) is asymptotically stable and has L_2-gain $\le \gamma$.

Solution: [28,29,30, 32,34] A necessary condition for the existence of solutions for which the closed loop system has a smooth storage function is that there exists a solution V(**x**)$\ge$0 of (14) as well as a solution R(**x**)$\ge$0 of

$$R_{\mathbf{x}}(\mathbf{x})\mathbf{a}(\mathbf{x}) + \frac{1}{2\gamma^2}R_{\mathbf{x}}(\mathbf{x})\mathbf{g}(\mathbf{x})\mathbf{g}^T(\mathbf{x})R_{\mathbf{x}}^T(\mathbf{x})$$
$$+\frac{1}{2}\mathbf{h}^T(\mathbf{x})\mathbf{h}(\mathbf{x}) - \frac{1}{2}\gamma^2\mathbf{c}^T(\mathbf{x})\mathbf{c}(\mathbf{x}) \le 0, R(\mathbf{0}) = 0 \tag{20}$$

such that V(**x**)$\le$R(**x**) for all **x**.

Conversely, conditions (14) and (20) are sufficient to solve, at least locally, the measurement feedback-problem. A more complicated version of equation (20), involving an "information-state" in combination with (14), leads to compensators that solve the problem. However, these compensators are in general infinite-dimensional. This is an ongoing area of research, which is beyond the scope of this paper.

4 Ramp Control Design

We present two control laws for isolated ramp metering control: one using space discretized dynamics and the other one using space and time discretized dynamics.

4.1 Continuous-time Case

In order to illustrate the ideas discussed above, we have designed a feedback control law for a space discretized system. The isolated ramp metering problem area is shown in Figure 5 below.

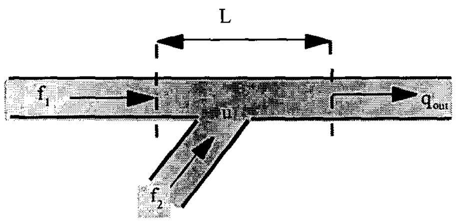

Figure 5. Traffic Flow for an Isolated Ramp Metering

The dynamic equation for this problem is given by

$$\begin{aligned}&\frac{d}{dt}\rho=\frac{1}{L}[q_1(t)-q_{out}(t)+r(t)]\\&\frac{d}{dt}l=q_2(t)-r(t)\\&\frac{d}{dt}r=u(t)\end{aligned}\tag{21}$$

The symbols used in (21) are defined below:

- ρ: Traffic density on the mainline
- l: Number of vehicles (queue length) on the ramp
- L: Length of the mainline section
- r: Rate of flow of traffic into the mainline from the ramp
- q_1: Traffic flow entering the mainline section from the highway
- q_2: Traffic flow entering the ramp
- q_{out}: Traffic flow leaving the mainline section

According to Greenshield formula we have

$$q_{out}(t)=v_f\rho(1-\frac{\rho}{\rho_m})\tag{22}$$

where

- v_f: Freeflow speed on mainline
- ρ_m: Jam density on the mainline

Therefore, we can replace 21 by

$$\begin{aligned}&\frac{d}{dt}\rho=\frac{1}{L}[q_1-v_f\rho(1-\frac{\rho}{\rho_m})+r]\\&\frac{d}{dt}l=q_2-r\\&\frac{d}{dt}r=u\end{aligned}\tag{23}$$

Then defining the error e as $\rho-\rho_c$, making the substitution of $\rho=e+\rho_c$ into (23), and assuming L=1 the following equations are obtained

$$\begin{aligned}&\frac{d}{dt}\rho=q_1-v_f(e+\rho_c)\left(1-\frac{e+\rho_c}{\rho_m}\right)+r\\&\frac{d}{dt}l=q_2-r\\&\frac{d}{dt}r=u\end{aligned}\tag{23b}$$

We can now define the state vector $\mathbf{x}$ as $\mathbf{x}=[e, l, r]'$. We can now present the ramp metering system in the form given by (12).

$$f=\frac{d}{dt}\begin{bmatrix}e\\l\\r\end{bmatrix}=\begin{bmatrix}-v_f(e+\rho_c)\left(1-\frac{e+\rho_c}{\rho_m}\right)+r\\-r\\0\end{bmatrix}+\begin{bmatrix}q_1\\q_2\\0\end{bmatrix}+\begin{bmatrix}0\\0\\1\end{bmatrix}u\tag{24}$$

The objective of the control can be taken as

$$\mathbf{z}=\begin{bmatrix}e\\l\end{bmatrix}\tag{25}$$

Using this formulation, we can obtain the controller by solving the Hamilton Jacobi equation like (18) associated with this system.

4.1.1 Nonlinear H_∞ Solution for Two Cost Functions

The two cost functions to be extremized are (26) and (27). These equations are posed such that the disturbance $\mathbf{q}$ will be maximized and the arguments of $\mathbf{z}$ and the control u will be minimized.

$$J_a=\int_0^\infty\tfrac{1}{2}\left(\mathbf{z}^T\mathbf{z}+u^2-\gamma^2\mathbf{q}^T\mathbf{q}\right)dt\tag{26}$$

$$J_b=\int_0^\infty\tfrac{1}{2}\left(\mathbf{z}^T\mathbf{z}+u^2-\gamma^2\mathbf{q}^T\mathbf{q}-k^2r^2\right)dt\tag{27}$$

The pre-Hamiltonian of (26) and (27) are (28) and (29), respectively

$$H_{pa}=\tfrac{1}{2}\left(\mathbf{z}^T\mathbf{z}+u^2-\gamma^2\mathbf{q}^T\mathbf{q}\right)+\ddot{\mathbf{e}}^Tf\tag{28}$$

$$H_{pb}=\tfrac{1}{2}\left(\mathbf{z}^T\mathbf{z}+u^2-\gamma^2\mathbf{q}^T\mathbf{q}-k^2r^2\right)+\ddot{\mathbf{e}}^Tf\tag{29}$$

Where $\ddot{\mathbf{e}}=\begin{bmatrix}\lambda_1\\\lambda_2\\\lambda_3\end{bmatrix}$ and are the Lagrange multipliers that provide the constraints along with (24).

The stationarity conditions for the problem are

$$\frac{\partial H_p}{\partial u}=0\tag{30}$$

and

$$\frac{\partial H_p}{\partial \mathbf{q}}=0\tag{31}$$

These conditions ensure that the control is minimized and the disturbance is maximized.

4.1.1a Derivation of the Optimal Control for J_a

For the cost function defined in (26) the following demonstrates the method used to find the optimal control.

$$\frac{\partial H_p}{\partial u}=0=u+\lambda_3\Rightarrow u=-\lambda_3\tag{32}$$

$$\frac{\partial H_p}{\partial \mathbf{q}}=0=\gamma^2\mathbf{q}+\begin{bmatrix}\lambda_1\\\lambda_2\end{bmatrix}\Rightarrow q_1=\frac{\lambda_1}{\gamma^2},q_2=\frac{\lambda_2}{\gamma^2}\tag{33}$$

Substituting (32) and (33) into (28) the following results

$$H_a = \tfrac{1}{2}(e^2 + I^2 + \lambda_3^2 - \frac{\lambda_1^2}{\gamma^2} - \frac{\lambda_2^2}{\gamma^2})$$
$$+\lambda_1\left(-v_f(e+\rho_c)\left(1-\frac{e+\rho_c}{\rho_m}\right)+\frac{\lambda_1}{\gamma^2}+r\right) \tag{34}$$
$$+\lambda_2\left(\frac{\lambda_2}{\gamma}-r\right)-\lambda_3^2$$

Another necessary condition that needs to be satisfied is the costate equations

$$\frac{\partial H_a}{\partial \mathbf{x}} = -\begin{bmatrix}\dot{\lambda}_1\\ \dot{\lambda}_2\\ \dot{\lambda}_3\end{bmatrix} = \begin{bmatrix} e+v_f\left(2\frac{\rho_c+e}{\rho_m}-1\right)\lambda_1 \\ I \\ \lambda_1-\lambda_2 \end{bmatrix} \tag{35}$$

From the above costate equation it can be seen that

$$\lambda_2 = -\int I\,\mathrm{d}t \tag{36}$$

This result can then be substituted into the $\dot{\lambda}_3$ equation that results in

$$-\dot{\lambda}_3 = \lambda_1 + \int I\,\mathrm{d}t \tag{37}$$

Now recognizing the relationship between u and λ_3 the following can be stated.

$$u = -\lambda_3 \Rightarrow \dot{u} = -\dot{\lambda}_3 = \dot{r} \tag{38}$$

And λ_1 can be solved for and $\dot{\lambda}_1$ can be found.

$$\lambda_1 = \dot{u} - \int I\,\mathrm{d}t \Rightarrow \dot{\lambda}_1 = \ddot{u} - I \tag{39}$$

Then substituting these results into the $\dot{\lambda}_1$ equation the following equation, independent of λ_i, results

$$\ddot{u} = I - e - v_f\left(\frac{\rho_c+e}{\rho_m} - \left(1-\frac{\rho_c+e}{\rho_m}\right)\right)\left(\dot{u} - \int I\,\mathrm{d}t\right) \tag{40}$$

From this a state equation for the feedback system can be written, the states for the system are as follows:

$$\begin{aligned} I &= \int I \Rightarrow \dot{I} = I \\ u_1 &= u \\ u_2 &= \dot{u}_1 \\ \dot{u}_2 &= \ddot{u} \end{aligned} \tag{41}$$

Adding these states to the states already defined creates the following state equations

$$\begin{bmatrix}\dot{e}\\ \dot{I}\\ \dot{r}\\ \dot{I}\\ \dot{u}_1\\ \dot{u}_2\end{bmatrix} = \begin{bmatrix} -v_f(e+\rho_c)\left(1-\frac{(e+\rho_c)}{\rho_m}\right)+q_1+r \\ q_2-r \\ u_1 \\ I \\ u_2 \\ I-e-v_f\left(2\frac{(\rho_c+e)}{\rho_m}-1\right)(u_2-I) \end{bmatrix} \tag{42}$$

4.1.1b Derivation of the Optimal Control for J_b

Following is the analysis used to find the optimal control for (27)

The stationarity conditions for the problem are the same as before, $\frac{\partial H_p}{\partial u} = 0$ and $\frac{\partial H_p}{\partial \mathbf{q}} = 0$.

For (27) this turns out to be the same also

$$\frac{\partial H_p}{\partial u} = 0 = u + \lambda_3 \Rightarrow u = -\lambda_3 \quad \text{and}$$

$$\frac{\partial H_p}{\partial \mathbf{q}} = 0 = \gamma^2\mathbf{q} + \begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix} \Rightarrow q_1 = \frac{\lambda_1}{\gamma^2}, q_2 = \frac{\lambda_2}{\gamma^2}$$

Substituting (32) and (33) into (29) the following results

$$H_a = \tfrac{1}{2}(e^2 + I^2 + \lambda_3^2 - \frac{\lambda_1^2}{\gamma^2} - \frac{\lambda_2^2}{\gamma^2} - k^2 r)$$
$$+\lambda_1\left(-v_f(\rho_c+e)\left(1-\frac{\rho_c+e}{\rho_m}\right)+\frac{\lambda_1}{\gamma^2}+r\right)+\lambda_2\left(\frac{\lambda_2}{\gamma}-r\right)-\lambda_3^2 \tag{43}$$

Another necessary condition that needs to be satisfied is the resulting costate equations

$$\frac{\partial H_a}{\partial \mathbf{x}} = -\begin{bmatrix}\dot{\lambda}_1\\ \dot{\lambda}_2\\ \dot{\lambda}_3\end{bmatrix} = \begin{bmatrix} e+v_f\left(2\frac{\rho_c+e}{\rho_m}-1\right)\lambda_1 \\ I \\ \lambda_1-\lambda_2-k^2 r \end{bmatrix} \tag{44}$$

From the above costate equation it can be seen that $\lambda_2 = -\int I\,\mathrm{d}t$

This result is the same as above also, and is substituted into the $\dot{\lambda}_3$ equation yields a different result than above

$$-\dot{\lambda}_3 = \lambda_1 + \int I\,\mathrm{d}t - k^2 r \tag{45}$$

Then, as above we use the relationship of (38) to solve for λ_1 which in this case is

$$\lambda_1 = \dot{u} - \int I\,\mathrm{d}t + k^2 r \tag{46}$$

And using the same argument as in (39)

$$\dot{\lambda}_1 = \ddot{u} - I + k^2\dot{r} = \ddot{u} - I + k^2 u \tag{47}$$

Now substituting (46) and (47) into the $\dot{\lambda}_1$ equation of (44) the following results

$$\ddot{u} = I - k^2 u - e\left(\frac{2(e+\rho_c)}{\rho_m}-1\right)(\dot{u} - I + k^2 r) \tag{48}$$

Using the definitions in (41) the state equations utilizing the control can be written as follows

$$\begin{bmatrix}\dot{e}\\ \dot{I}\\ \dot{r}\\ \dot{I}\\ \dot{u}_1\\ \dot{u}_2\end{bmatrix} = \begin{bmatrix} -v_f(e+\rho_c)\left(1-\frac{(e+\rho_c)}{\rho_m}\right)+q_1+r \\ q_2-r \\ u_1 \\ I \\ u_2 \\ I-k^2u_1-e-\left(2\frac{e+\rho_c}{\rho_m}-1\right)(u_2-I+k^2 r) \end{bmatrix} \tag{49}$$

Where $q_1 = (u - I + k^2 r)/\gamma^2$ and $q_2 = -I/\gamma^2$.

4.1.2 Discretization of the Resulting System

In discretizing the above equations it is important to realize there are certain limits physically placed on the system. The limits on this system are as follows

$$\begin{gathered} 0 \le \rho \le \rho_m \Rightarrow -\rho_c \le e \le \rho_m - \rho_c \\ 0 \le q_1 \le \rho_m \\ 0 \le q_2 \\ 0 \le l \\ 0 \le r \\ u_{1,\min} \le u_1 \le u_{1,\max} \end{gathered} \tag{50}$$

Since there are constraints on the controls and the states the method chose for descretization of the system is the Euler method which is demonstrated by the following

$$\begin{aligned} &\frac{dx}{dt} = f(x) \approx \frac{x(t+h) - x(t)}{(t+h) - t} = \frac{x(t+h) - x(t)}{h} \\ &\Rightarrow \frac{dx(i)}{dt} = f(x,i) \approx \frac{x(i+1) - x(i)}{h} \\ &x(i+1) = x(i) + h \cdot f(x,i) \end{aligned} \tag{51}$$

For (42) the discrete-time system appears as follows

$$\begin{aligned} &e(i+1) = e(i) + h \cdot \left(v_f (e(i) + \rho_c) \left(1 - \frac{e(i) + \rho_c}{\rho_m} \right) + q_1(i) + r(i) \right) \\ &l(i+1) = l(i) + h \cdot (q_2(i) - r(i)) \\ &r(i+1) = r(i) + h \cdot u_1(i) \\ &I(i+1) = I(i) + h \cdot l(i) \\ &u_1(i+1) = u_1(i) + h \cdot u_2(i) \\ &u_2(i+1) = u_2(i) + h \cdot \left(l(i) - e(i) - v_f \left(2\frac{e(i) + \rho_c}{\rho_m} - 1 \right) (u_2(i) - I(i)) \right) \end{aligned} \tag{52}$$

Where h is the size of the time step. The discrete time system for (49) is very similar to (52) and will not be shown here.

4.2 Discrete-time Case

The dynamic equation for the space and time discretized form is

$$\begin{aligned} &\rho(k+1) = \rho(k) + \frac{T}{L}[f_1 - v_f \rho (1 - \frac{\rho}{\rho_m}) + u] \\ &q(k+1) = q(k) + T(f_2 - u) \end{aligned} \tag{?}$$

We can transform these equations also into a state space form like (24) and use a similar technique for the discrete time case as we used in the continuous time case. However, this will not be shown in this paper mainly due to space limitations.

5 Conclusions

A new design technique for isolated ramp metering control was presented. This technique minimizes an objective function which consists of weighted average of terms containing ramp queue length and the difference between the mainline traffic density and the desired mainline traffic density. The design is based on nonlinear H_∞_control and requires solving a Hamilton Jacobi inequality.

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References

1. Kachroo, P., Ozbay, K., Kang, S., and Burns, J. A., "System Dynamics and Feedback Control Problem Formulations for Real Time Dynamic Traffic Routing," Mathl. Comput. Modelling, Vol. 27, No. 9-11, pp. 27-49, 1998.
2. Gazis, Denos C., "Traffic Science", "Edie, Leslie C., 'Flow Theories'", pp. 43, John Wiley and Sons, 1974.
3. Richards, P. I., Shock waves on the highway. Oper. Res., 1956, 42-51.
4. Berger, C.R., and Shaw, L., Discrete event simulation of freeway traffic. Simulation council Proc., Series 7, 1977, 85-93.
5. Payne, H. J., Models of freeway traffic and control. Simulation council proc., 1971, 51-61.
6. Papageorgiou, *Applications of Automatic Control Concepts to Traffic Flow Modelling and Control*, Springer-Verlag, 1983.
7. J. A. Ball, J. W. Helton, and M. L. Walker, H_∞ Control for Nonlinear Systems with Output Feedback, IEEE Trans. on Automatic Control, Vol. 00, No. 00, April 1993.
8. W. M. Lu, and J. C. Doyle, H_∞ Control of Nonlinear Systems via Output Feedback: Controller Parametrization, IEEE Trans. on Automatic Control, Vol. 39, No. 12, December 1994.
9. Isidori, A., and Kang, W., "H_∞ Control via Measurement Feedback for General Nonlinear Systems", IEEE TRANS. on Aut. Control, vol. 40, No. 3, March 1995.
10. van der Schaft, A. J., "Nonlinear State Space H_∞ Control Theory", Perspectives in Control, Feb. 1993.
11. van der Schaft, A. J., L_2-gain and Passivity Techniques in Nonlinear Control, Springer LNCIS #218, Springer, 1996.
12. Basar, T., and Berhard, P., "H_∞-Optimal Control and Related Minimax Design Problems: A Dynamic Game Approach", Birkhouser, Boston, 1991.
13. Krener, A. J., "Necessary and Sufficient Conditions for Nonlinear Worst Case (H_∞) Control and Estimation, Journal of Mathematical Systems, Estimation and Control, Vol. 4, No. 4, pp. 1-25, 1994.