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Feedback Control Design and Stability Analysis of One Dimensional Evacuation System

Sabiha Amin Wadoo, Pushkin Kachroo

Abstract—This paper presents design of nonlinear feedback controllers for two different models representing evacuation dynamics in one dimension. The models presented here are based on the laws of conservation of mass and momentum. The first model is the classical one equation model for a traffic flow based on conservation of mass with a prescribed relationship between density and velocity. The other model is a two equation model in which the velocity is independent of the density. This model is based on conservation of mass and momentum. The equations of motion in both cases are described by nonlinear partial differential equations. We address the feedback control problem for both models. The objective is to synthesize a nonlinear distributed feedback controller that guarantees stability of a closed loop system. The problem of control and stability is formulated directly in the framework of partial differential equations. Sufficient conditions for Lyapunov stability for distributed control are derived.

I. INTRODUCTION

THE objective of this paper is to design feedback controllers and study their stability properties for an evacuation control system in one dimension. The evacuation system we are proposing is for evacuating people from large halls and buildings. The hardware for the implementation is assumed to include sensors such as cameras that can calculate in real-time the traffic density as a function of space variables. It is also assumed that a method of indicating the desired speed and direction is available so that people can follow those for evacuation. One method that is implementable with the current available technologies is that of using light matrix on the ceiling. These can be turned on and off in a sequence to indicate how fast and in which direction people should move at different locations. This actuation could also be achieved by providing speakers that are local, i.e. they should not be too loud for all people to hear. They should only let people know close to the speakers where to move and how fast. This way, different commands can be given at different locations. In either case, we can control the vector field of people flow in a continuous manner in space and time.

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The dynamics that are used to model an evacuation system in this paper are based on traffic flow theory [1]. The evacuation system can be modeled like a traffic flow or a fluid flow, which can be thought of as the flow of people on a building floor or a corridor. These models are based on the basic law of conservation. In case of evacuation the conservation law can be stated as “total number of people is conserved in the system”. The resulting dynamics are given by nonlinear partial differential equations [1]. The system is distributed, that is both the state and control variables are distributed in time and space. The control objective is to design feedback controllers to remove people from the evacuation area by generating distributed control commands.

There are two approaches to the design of feedback controllers for distributed systems. In the conventional approach the distributed mathematical model is approximated by a lumped parameter model having finite dimensions. The spatial discretization of the system is performed using either finite difference or finite element methods. The controllers are then designed on the basis of resulting linear or nonlinear ordinary differential equation model using known techniques available for such systems [2], [3]. This approach however has certain disadvantages. By neglecting infinite dimensional nature of original system, the design of controllers may result in instability even though the resulting finite dimensional system is stable using same controllers. Moreover, properties like controllability and observability depend on the method of discretization used [4]. Thus in order to avoid errors introduced by spatial discretization it is desirable to formulate the control and stability problem directly in the framework of distributed model in form of partial differential equations. In this paper latter approach is used. The evacuation model is presented in a partial differential equation framework. Design of controllers and stability analysis is performed using distributed setting. Sufficient conditions for Lyapunov stability for distributed control are also derived.

There is a wide variety of models available for traffic flow [5]. As such there are number of equations that can model the evacuation system. In this paper we will use two one-dimensional models which describe the behavior of people using a single partial differential equation in the first

case and two partial differential equations in the other. The first one is a basic one-equation model based on the equation of continuity or conservation of mass. This model also has a fundamental relationship between density of people and speed of flow. According to the law of conservation of mass, total flow of people exiting from any section cannot be higher than the total flow of the people that are entering. The number of people moving in and out accounts for the change in density in that area. To represent the flows, Greenshield's model [6] is used to show the dependence of speed on the density of people. The second model is a two-equation model based on conservation of mass and momentum where the velocity is independent of density.

First we discuss the design of nonlinear feedback control for the model based on continuity equation alone. To increase resolution and accuracy of the model we add the equation of conservation of momentum. We discuss the feedback control design for the system described by both equations. Backstepping approach will be used for the control design in this case. In both cases the objective of control design is to synthesize a nonlinear distributed feedback controller that stabilizes the system and guarantees stability in closed loop system.

The organization of this paper is as follows. Section II presents the two mathematical models. In section III we formulate the control model and present feedback control design for the first model. This section also studies Lyapunov stability for this model and finally presents some simulation results. Section IV presents the feedback control design and stability analysis for second model. Simulation results for closed loop are presented.

II. MODELING

In this section mathematical models of the evacuation problem are presented. We will discuss the evacuation model of a one dimensional single exit corridor of length L . The model is similar to the one dimensional traffic flow model. Both these models are similar to fluid flow and are based on the principle of conservation. The model is described by a nonlinear hyperbolic partial differential equation.

There are two main approaches to modeling. One approach is microscopic [7] where each individual is taken into consideration and his behavior is expressed by a set of rules or an equation involving adjacent individuals. The other approach is macroscopic [8]. Here the overall behavior of the flow of people is considered. The area is treated as a series of sections within each of which the density and average velocity of people can be measured for a given time. The changes in these variables may then be described using partial differential equations. The models presented here are macroscopic with the dynamics being

represented in terms of density, flow and speed. As a result the system is distributed with all the parameters as functions of space and time. There are two partial differential equations that we use to model the control problem. The first is the equation of conservation of mass and the second is the conservation of momentum.

A. Continuity Equation Based Model

This model is based on equation of conservation of mass. The conservation law of mass in case of an evacuation system means that the number of people is conserved in the system. Let us consider the case of a single exit one dimensional corridor of length L . Let $\rho(x,t)$ denote the density of people as a function of position vector x and time t , $q(x,t)$ the flow at a given x and t , and $v(x,t)$ the velocity vector field associated with the flow. The conservation of mass equation holds and is given by

$$\frac{\partial \rho(x,t)}{\partial t} + \text{div}(q(x,t)) = 0 \quad (1)$$

with initial condition

$$\rho(x, t_0) = \rho_0(x) \quad (2)$$

and a boundary condition

$$\rho(0,t) = 0 \text{ and } \rho(L,t) = 0 \quad \forall t \in [0, \infty) \quad (3)$$

Here $\rho(x,t) \in H^2[(0,L), \mathfrak{X}]$ with $H^2[(0,L), \mathfrak{X}]$ being the infinite dimensional Hilbert space of one dimensional like vector function defined on the interval $[0,L]$ whose spatial derivatives upto second order are square integrable with a specified L_2 norm. $q(x,t) \in H^2[(0,L), \mathfrak{X}]$ and $\rho_0(x) \in H[(0,L), \mathfrak{X}]$. The vectors $x \in [0,L] \subset \mathfrak{X}$ and $t \in [0, \infty)$ denote position and time respectively. For the rest of the paper it will be assumed that the vector spaces are Sobolev spaces (Banach spaces) [9]. The flow $q(x,t)$ is obtained as a product of density and velocity as

$$q(x,t) = \rho(x,t)v(x,t) \quad (4)$$

The dynamics for one dimension are therefore given by

$$\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial (\rho(x,t)v(x,t))}{\partial x} = 0 \quad (5)$$

subject to the initial conditions and boundary conditions given by (2) and (3) respectively. To describe the relationship between velocity vector field $v(x,t)$ and density $\rho(x,t)$ we need one more equation. Here we make use of Greenshields model

$$v(x,t) = v_f(x,t) \left(1 - \frac{\rho(x,t)}{\rho_{\max}} \right) \quad (6)$$

where $v_f(x,t)$ is the free flow speed and ρ_{\max} is the jam density that is the maximum number of people that could possibly fit a single cell. Using (6) in (5) we get modified one-equation model given by

$$\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial}{\partial x} \left[v_f(x,t) \left(1 - \frac{\rho(x,t)}{\rho_{\max}}\right) \rho(x,t) \right] = 0 \quad (7)$$

subject to conditions (1) and (2).

A. Two Equation Model

The model presented earlier is one of the original models representing the traffic flow dynamics. Since its appearance a number of other models have appeared in literature where velocity is independent of density. One such model is being presented here. In this section we consider a higher order model or more precisely a system of two partial differential equations for a one dimensional corridor. This model consists of conservation of mass equation coupled with a second equation based on the principle of conservation of momentum. The first equation is the conservation of mass equation (5). The second equation is derived from conservation of momentum for one dimensional flow [10] given by

$$\frac{\partial(\rho(x,t)v(x,t))}{\partial t} + \frac{\partial(\rho(x,t)v(x,t)^2)}{\partial x} = -\frac{\partial p(x,t)}{\partial x}$$

where $p(x,t) \in H^1[(0,L), \mathfrak{R}]$ is pressure. The relationship between density and flow is given by $q(x,t) = \rho(x,t)v(x,t)$. Thus we have the dynamics of an evacuation system given by following two-equation model

$$\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial q(x,t)}{\partial x} = 0 \quad (8)$$

and

$$\frac{\partial q(x,t)}{\partial t} + \frac{\partial}{\partial x} \left(\frac{q(x,t)^2}{\rho(x,t)} \right) = -\frac{\partial p}{\partial x} \quad (9)$$

with initial conditions

$$\rho(x,t_0) = \rho_0(x), \quad q(x,t_0) = q_0(x) \quad (10)$$

and subject to boundary conditions

$$\begin{aligned} \rho(0,t) = \rho(L,t) = 0 \\ q(0,t) = q(L,t) = 0 \quad \forall t \in [0, \infty) \end{aligned} \quad (11)$$

II. FEEDBACK CONTROL FOR CONTINUITY EQUATION MODEL

A. Continuity Equation Control Model

To formulate the control problem based on the continuity equation model we need to choose a control variable. To do so we use Greenshields model to represent the relationship between traffic density and the velocity field which is given by (6) as

$$v(x,t) = v_f(x,t) \left(1 - \frac{\rho(x,t)}{\rho_{\max}}\right)$$

with $v_f(x,t)$ being free flow speed and ρ_{\max} the jam density. In this model we take free flow velocity vector field $v_f(x,t)$ as the distributed control variable denoted by $u(x,t)$. If the density at a location is zero then the speed

at that location will be the free flow speed. However, with the actuation system implemented, we can tell people to change the speed. Also the traffic density affects the achievable speeds, therefore we choose $v_f(x,t)$ as the control variable, giving us the following representation

$$\frac{\partial \rho(x,t)}{\partial t} = -\frac{\partial}{\partial x} \left[[u(x,t) \left(1 - \frac{\rho(x,t)}{\rho_{\max}}\right) \rho(x,t)] \right] \quad (12)$$

where $u(x,t) \in H[(0,L), \mathfrak{R}]$ is the control variable.

B. State Feedback Control

Here we address the problem of synthesizing a distributed state feedback controller $u(x,t)$ that stabilizes origin ($\rho(x,t) = 0$) of system (12). More specifically we consider control law of the form

$$u(x,t) = F(\rho(x,t))$$

which makes origin of the closed loop dynamics exponentially stable. Here $F(\rho)$ is a nonlinear operator mapping $H^2[(0,L), \mathfrak{R}]$ into $H^1[(0,L), \mathfrak{R}]$. Designing the operator $F(\rho)$ as

$$F(\rho(x,t)) = -\left[\left(1 - \frac{\rho(x,t)}{\rho_{\max}}\right) \rho(x,t) \right]^{-1} k \frac{\partial \rho(x,t)}{\partial x} \quad (13)$$

we get the following closed loop dynamics

$$\frac{\partial \rho(x,t)}{\partial t} - k \frac{\partial^2 \rho(x,t)}{\partial x^2} = 0 \quad (14)$$

with boundary conditions given by (1) and (2). Introducing a differential operator A on $H^2[(0,L), \mathfrak{R}]$ defined by

$$A\phi = k \frac{\partial^2 \phi}{\partial x^2}, \quad \forall \phi \in D(A) \quad (15)$$

Here $D(A) \subset H^2[(0,L), \mathfrak{R}]$ is the domain of operator A defined as

$$D(A) = \left\{ \phi \in H^2 : \phi', \phi'' \in H^2[0,L]; \phi(0) = \phi(L) = 0 \right\}$$

The system dynamics can be written using operator (15) which makes the dynamics look like ordinary differential equation in Sobolev (Banach) spaces [11], [12]. Thus the abstract version or state space representation of (14) can be put into the form

$$\frac{d}{dt} \rho(t) = A\rho(t), \quad t > 0; \quad \rho(0) = \rho_0 \quad (16)$$

The operator A is known to generate a strongly continuous semigroup $U(t)$ of bounded linear operators on a normed linear space $H^2[(0,L), \mathfrak{R}]$. The system motion starting from any initial state $\rho(t_0)$ at time t_0 is defined by $U(t)\rho(t_0)$.

Thus a partial differential equation can be regarded as an evolution system where $U(t)$ evolves ρ_0 forward in time.

C. Stability analysis using Lyapunov

The stability problem is to establish sufficient conditions for which the origin of the closed loop dynamics (14) is exponentially stable. Within the framework of our system the definition of stability in terms of Lyapunov can be established as follows.

Definition

An equilibrium state ρ_{eq} of a dynamical system (14) is stable with respect to a specified norm $\|\rho(x, t)\|$ if for every real number $\varepsilon > 0$ there exists a real number $\delta(\varepsilon, t_0) > 0$ such that

$$\|\rho(t_0) - \rho_{eq}\| < \delta \Rightarrow \|U(t)\rho(t_0) - \rho_{eq}\| < \varepsilon \quad \forall t > t_0$$

If in addition $\|U(t)\rho(t_0) - \rho_{eq}\| \rightarrow 0$ as $t \rightarrow \infty$ then the equilibrium is said to be asymptotically stable. Furthermore, if there exist two positive constants a and b such that

$$\|U(t)\rho(t_0) - \rho_{eq}\| \leq a \|\rho(t_0) - \rho_{eq}\| e^{-b(t-t_0)} \quad \forall t > t_0$$

is satisfied, then ρ_{eq} is said to be exponentially asymptotically stable. More precisely the problem is to find a condition under which operator A generates an exponentially stable semigroup $U(t)$ that satisfies the following growth property with respect to a specified norm

$$\|U(t)\|_2 \leq a e^{-bt} \quad t \geq 0 \quad (17)$$

where $b > 0$. In other words we say that A generates an exponentially stable semigroup $U(t)$ [13]. The determination of conditions for which estimate (17) is satisfied amounts to establishing conditions for which the null state of the linear system (14) is exponentially asymptotically stable with respect to the specified norm i.e. $\|\rho(t)\|_2 \rightarrow 0$ as $t \rightarrow \infty$. It should be noted here that for infinite dimensional systems stability with respect to one norm does not necessarily imply stability with respect to others unlike finite dimensional systems where all norms are equivalent. For our system we have chosen the following L_2 norm defined by

$$\|\rho(t, x)\|_2 \rightarrow \left[\int_0^L |\rho(t, x)|^2 dx \right]^{1/2} \quad (18)$$

This norm represents the total energy of the system at any time. Thus the exponential for this norm implies that the systems total energy goes to zero as time goes to infinity.

Let us consider a Lyapunov functional $V(t)$ for the system (14). Here $V : H^2[0, L] \rightarrow \mathfrak{R}_+$ is a smooth functional of the form

$$V(t) = \frac{1}{2} \|\rho(t, x)\|_2^2 = \frac{1}{2} \int_0^L |\rho(t, x)|^2 dx \quad (19)$$

Using the norm properties we can easily see that $V(t)$ is a positive definite function. The time rate of change of $V(t)$ using Leibniz rule in (19) is given as

$$\frac{dV(t)}{dt} = \int_0^L \rho(x, t) \frac{\partial \rho(x, t)}{\partial t} dx \quad (20)$$

Using (12) and integrating (18) we get

$$\frac{dV(t)}{dt} = \left[k \rho(x, t) \frac{\partial \rho(x, t)}{\partial x} \Big|_0^L \right] - k \int_0^L \left(\frac{\partial \rho(x, t)}{\partial x} \right)^2 dx$$

The first term vanishes by boundary condition (3). For the second integral we make use of Gagliardo-Nirenberg-Sobolev Inequality [14]. The inequality as applied to our case states

$$\|\rho(x, t)\|_2 \leq C \|\nabla \rho(x, t)\|_2 \quad (21)$$

where C is a positive real number. Using (21) we have

$$\int_0^L \left(\frac{\partial \rho(x, t)}{\partial x} \right)^2 dx \geq C^{-2} \int_0^L \rho^2(t, x) dx$$

The rate of change of $V(t)$ can be thus be bounded by

$$\begin{aligned} \frac{dV(t)}{dt} &\leq -kC^{-2} \int_0^L \rho^2(t, x) dx \\ &\leq -2kC^{-2} V(t) = -\beta V(t) \end{aligned}$$

It follows that

$$V(t) \leq V(t_0) e^{-\beta(t-t_0)}$$

or

$$\|\rho(t, x)\|_2 \leq \|\rho(t_0, x)\|_2 e^{-\beta(t-t_0)}$$

with $\beta = 2kC^{-2}$. As long as $\beta > 0$, null state of (14) is exponentially stable and condition (15), $\|U(t)\|_2 \leq e^{-\beta(t-t_0)}$ is satisfied with $b = \beta$. Thus equilibrium of closed loop system (14) using feedback control (13) is exponentially stable.

D. Simulation Results

This section shows simulation results for closed loop system (14) using the control law (13). The numerical method used is the Explicit Finite-Difference method [15].

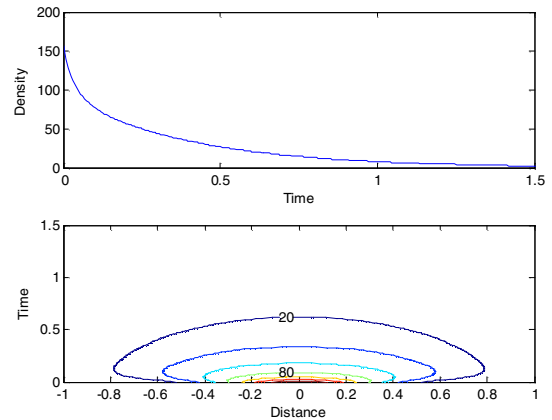


Fig. 1. Density contours for closed loop dynamics for one equation model. Contour lines vary from 150 to 80

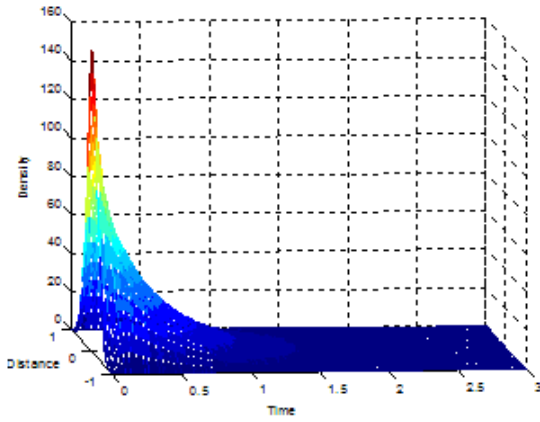


Fig. 2. Density plot for closed loop dynamics for one equation model.

The solution is shown as a contour plot in fig. 1 and a mesh plot in fig. 2. As is seen from plots density at every point in space is decreasing exponentially with time.

IV. FEEDBACK CONTROL FOR TWO- EQUATION MODEL

In this section we design a feedback control for the two-equation model of the evacuation system given by (8) and (9) subject to boundary conditions (10)-(11) using backstepping approach. The Lyapunov functional (19) which was used as a stability analysis tool for one-equation model will be used as a feedback control design tool for this system. The design of feedback control is done in such a way that Lyapunov functional or its derivative has certain properties that guarantee boundedness or convergence to an equilibrium point.

A. Two Equation Control Model

In the two equation model (8) and (9) we choose divergence of pressure $\frac{\partial p(x,t)}{\partial x}$ as distributed control variable $u(x,t)$ which gives us the following control model

$$\begin{aligned} \frac{\partial \rho(x,t)}{\partial t} &= -\frac{\partial q(x,t)}{\partial x} \\ \frac{\partial q(x,t)}{\partial t} &= -\frac{\partial}{\partial x} \left(\frac{q(x,t)^2}{\rho(x,t)} \right) + u(x,t) \end{aligned}$$

This system can be rewritten in the form

$$\frac{\partial \rho(x,t)}{\partial t} = -\frac{\partial q(x,t)}{\partial x} \quad (22)$$

$$\frac{\partial q(x,t)}{\partial t} = \bar{u}(x,t) \quad (23)$$

where $\bar{u}(x,t) = -\frac{\partial}{\partial x} \left(\frac{q(x,t)^2}{\rho(x,t)} \right) + u(x,t)$

B. State Feedback Control Using Backstepping

Here we address the problem of synthesizing a distributed state feedback controller $\bar{u}(x,t)$ that stabilizes origin

($\rho(x,t) = 0, q(x,t) = 0$) of control system (22)-(23). More specifically we consider control law

$$\bar{u}(x,t) = \bar{F}(\rho(x,t), q(x,t))$$

such that origin of closed loop dynamics are exponentially stable. \bar{F} is a nonlinear operator mapping $H^2[(0,L), \mathfrak{R}]$ into $H^1[(0,L), \mathfrak{R}]$. The control strategy adopted here is similar in principle to feedback control by backstepping for ordinary differential equations [16].

First we design control law for equation (22) where $q(x,t)$ can be viewed as an input. We proceed to design a control law $q(x,t) = G(\rho(x,t))$ to stabilize origin $\rho(x,t) = 0$. G is a nonlinear operator mapping $H^2[(0,L), \mathfrak{R}]$ into $H^2[(0,L), \mathfrak{R}]$. With control law

$$q(x,t) = G(\rho(x,t)) = -\int_0^x \frac{\partial^2 \rho(x,t)}{\partial m^2} dm \quad (24)$$

we can rewrite (22) as

$$\frac{\partial \rho(x,t)}{\partial t} = \frac{\partial^2 \rho(x,t)}{\partial x^2} \quad (25)$$

which is similar to (14). As we have already shown the origin of this equation is asymptotically exponentially stable. In addition there exists a Lyapunov functional

$$V(t) = \frac{1}{2} \|\rho(t, x)\|_2^2 = \frac{1}{2} \int_0^L |\rho(t, x)|^2 dx$$

for this system which satisfies $\frac{dV(t)}{dt} \leq -\beta V(t)$ and

ensures the stability of above equation. From the knowledge of this function we want to design a smooth feedback control to stabilize origin of the overall system. Rewriting (22)

$$\frac{\partial \rho(x,t)}{\partial t} = -\frac{\partial G(\rho(x,t))}{\partial x} - \frac{\partial}{\partial x} (q(x,t) - G(\rho(x,t)))$$

Defining a new variable $z(x,t) = q(x,t) - G(\rho(x,t))$, with $z(x,t) \in H^2[(0,L), \mathfrak{R}]$, results in the following dynamics.

$$\frac{\partial \rho(x,t)}{\partial t} = -\frac{\partial G(\rho(x,t))}{\partial x} - \frac{\partial z(x,t)}{\partial x} \quad (26)$$

$$\frac{\partial z(x,t)}{\partial t} = u_n(x,t) \quad (27)$$

where $u_n(x,t) = \bar{u}(x,t) - \frac{\partial G(\rho(x,t))}{\partial t}$ is the new control variable. Now let us choose the Lyapunov functional for the overall system as

$$\begin{aligned} V_a(t) &= V(t) + \frac{1}{2} \|z(x,t)\|_2^2 \\ &= \frac{1}{2} \int_0^L |\rho(x,t)|^2 dx + \frac{1}{2} \int_0^L |z(x,t)|^2 dx \end{aligned} \quad (28)$$

The time rate of change of this functional using (26) and (27) is given as

$$\frac{dV_a(t)}{dt} = \int_0^L \rho(x,t) \frac{\partial^2 \rho(x,t)}{\partial x^2} dx$$

$$+ \int_0^L \rho(x,t) \frac{\partial z(x,t)}{\partial x} dx + \int_0^L z(x,t) u_n(x,t) dx$$

Using (21) we know that the first term is bounded by $-\beta V(t)$. Therefore

$$\frac{dV_a(t)}{dt} \leq -\beta V(t) + \int_0^L \rho(x,t) \frac{\partial z(x,t)}{\partial x} dx + \int_0^L z(x,t) u_n(x,t) dx$$

We have to choose a new control law $u_n(x,t)$ in such a manner that the time derivative of new functional or the sum of second and third terms is also bounded. Choosing

$$u_n(x,t) = -\hat{k}z(x,t) - z(x,t)^{-1} \rho(x,t) \frac{\partial z(x,t)}{\partial x} \quad (29)$$

yields

$$\frac{dV_a(t)}{dt} \leq -\beta V(t) - \hat{k} \|z(x,t)\|_2^2 = -2\beta V_a(t)$$

with $\hat{k} = 2\beta > 0$. This shows that the origin is asymptotically exponentially stable. Since $G(0) = 0$, we can conclude that the origin for actual closed loop system (22) and (23) is asymptotically exponential stable. The control law is given by the partial differential-integral equation

$$u(x,t) = u_n(x,t) + \frac{\partial G(\rho(x,t))}{\partial t} + \frac{\partial}{\partial x} \left(\frac{q(x,t)^2}{\rho(x,t)} \right) \quad (30)$$

with $u_n(x,t)$ given by (29) as

$$u_n(x,t) = -\hat{k}z(x,t) - z(x,t)^{-1} \rho(x,t) \frac{\partial z(x,t)}{\partial x}$$

where $z(x,t) = q(x,t) - G(\rho(x,t))$. Hence the closed loop dynamics for two equation model are exponentially stable.

C. Simulation

Here we show simulation results for closed loop system (22) and (23) using controller (30). The numerical technique used to simulate the system is Lax-Friedrichs Method [17]. The density plots are same as in fig. 1 and 2.

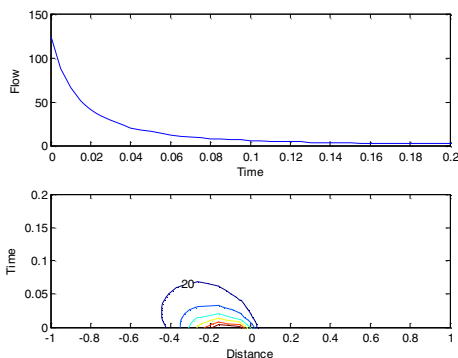


Fig. 3. Flow contours for closed loop dynamics for two equation model. Contour lines vary from 150 to 20

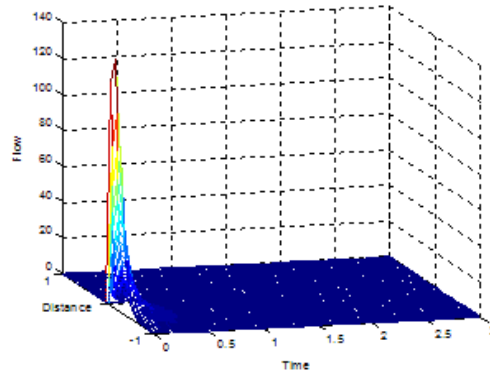


Fig. 4. Flow plot for closed loop dynamics for two equation model.

The flow plots are shown in fig. 3 and 4. The flow at every point in space is decreasing exponentially with time

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