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# Feedback Control of Macroscopic Crowd Dynamic Models

Sabiha A. Wadoo, Sadeq Al-nasur, Pushkin Kachroo

**Abstract**—This paper presents design of nonlinear feedback controllers for two different macroscopic models for two-dimensional pedestrian dynamics. The models presented here are based on the laws of conservation of mass and momentum. These models have been developed by extending one-dimension macroscopic vehicle traffic flow models that use two-coupled partial differential equations (PDEs). These models modify the vehicle traffic models so that bi-directional controlled flow is possible. Both models satisfy the conservation principle and are classified as nonlinear, time-dependent, hyperbolic PDE systems. The equations of motion in both cases are described by nonlinear partial differential equations. We address the feedback control problem for both models in the framework of partial differential equations. The objective is to synthesize nonlinear distributed feedback controllers that guarantee stability of a closed loop system.

## I. INTRODUCTION

THE objective of this paper is to design feedback controllers and study their stability properties for an evacuation control system in two dimensions. The evacuation system we are proposing is for evacuating people from large halls and buildings. The hardware for the implementation is assumed to include sensors such as cameras that can calculate in real-time the traffic density as a function of space variables. It is also assumed that a method of indicating the desired speed and direction is available so that people can follow those for evacuation. One method that is implementable with the current available technologies is that of using light matrix on the ceiling. These can be turned on and off in a sequence to indicate how fast and in which direction people should move at different locations. This actuation could also be achieved by providing speakers that are local, i.e. they should not be too loud for all people to hear. They should only let people know close to the speakers where to move and how fast. This way, different commands can be given at different locations. In either case, we can control the vector field of people flow in a continuous manner in space and time.

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The dynamics that are used to model an evacuation system in this paper are based on traffic flow theory [1]. The evacuation system can be modeled like a traffic flow, which can be thought of as the flow of people on a building floor or a corridor. These models are based on the basic law of conservation. In case of an evacuation, the conservation can be stated as “total number of people is conserved in the system”. This kind of systems is distributed, that is both the state and control variables are distributed in time and space. The control objective is to design feedback controllers to remove people from the evacuation area by generating distributed control commands.

There are two main approaches to modeling. One approach is microscopic [2] where each individual is taken into consideration and his behavior is expressed by a set of rules or an equation involving adjacent individuals. The other approach is macroscopic [3]. Here the overall behavior of the flow of people is considered. The area is treated as a series of sections within each of which the density and average velocity of people can be measured for a given time. The changes in these variables may then be described using partial differential equations. The models presented here are macroscopic where the dynamics are represented in terms of density, flow and speed.

This paper uses two different macroscopic models for two-dimensional pedestrian movements. These models have been developed by extending one-dimension macroscopic vehicle traffic flow models that use two-coupled partial differential equations. The coupled PDE equations provide conservation of continuity and movement (momentum). These models modify the vehicle traffic models so that bi-directional controlled flow is possible. One model is taken from [4], which is a macroscopic system model that is derived directly from a microscopic car-following model. This model was selected based on its microscopic-to-macroscopic derivation property, which adds to the macroscopic model an important validation point. The other model, which is also a two-equation PDE system model is proposed by Al-nasur in [5]. This model is an expansion of the one-dimension vehicle traffic model by Aw and Rascle [6], and improved by Rascle [7]. The model does not have a direct micro-to-macro link, but it carries the desired anisotropic nature of traffic flow that the first model

also carries. This property is described as being the natural observed traffic flow behavior in which traffic flow movement is influenced by conditions from current and ahead locations only. This property is different from the isotropic property of fluids, where a fluid particle is influenced from all directions.

There are two approaches to the design of feedback controllers for distributed systems. In the conventional approach, a lumped parameter model having finite dimensions approximates the distributed mathematical model. The controllers are designed based on the resulting linear or nonlinear ordinary differential equation model using known techniques available for such systems [8], [9]. This approach however has certain disadvantages. By neglecting infinite dimensional nature of original system, the design of controllers may result in instability even though the resulting finite dimensional system is stable using same controllers. Moreover, properties like controllability and observability depend on the method of discretization used [10]. Thus in order to avoid errors introduced by spatial discretization it is desirable to formulate the control and stability problem directly in the framework of distributed model in form of partial differential equations. In this paper latter approach is used. Here we extend the feedback control design done in [11]. The evacuation model is presented in a partial differential equation framework. Design of controllers and stability analysis is performed using distributed setting. Sufficient conditions for Lyapunov stability for distributed control are also derived.

## II. MATHEMATICAL MODELS

In this section mathematical models of the crowd dynamics problem are presented. Both models are similar to the compressible fluid flow and are based on the principle of conservation. The dynamics are described by nonlinear hyperbolic partial differential equations. The models are macroscopic with the dynamics being represented in terms of density, flow and speed. As a result the system is distributed with all the parameters as functions of space and time. There are two PDEs that we use to model the control problem. The first is the equation of conservation of mass and the second is conservation of momentum.

We will discuss the models of a two-dimensional area of dimensions  $L \times L$ . Let  $\Omega = (0, L) \times (0, L)$  be a bounded, open subset of a two-dimensional Euclidean space  $R^2$  and  $\partial\Omega$  be its boundary. Let  $\rho(x, t)$  denote the density of people as a function of position vector  $x$  and time  $t$ . The vector  $x \in \Omega \subset \mathfrak{R}^2$  is expressed in terms of its coordinates as  $x = [x_1, x_2]^T$ . Let  $q_1(x, t)$  and  $q_2(x, t)$  be the flows in  $x_1$

and  $x_2$  directions respectively. The velocity vector fields associated with the flows are given as  $v_1(x, t)$  and  $v_2(x, t)$ . The flux rates  $q_i(x, t)$  in both directions are given as  $q_i(x, t) = \rho(x, t)v_i(x, t)$  with  $i = 1, 2$ . For simplicity the arguments  $x$  and  $t$  are dropped from all dependent functions.

### A. Model 1

The first model is taken from [8]. The model there is classified as a nonlinear, time-varying hyperbolic system of two coupled partial differential equations that describe a macroscopic crowd flow in two-dimension space. The first equation is simply the conservation of continuity that keeps the mass (pedestrians) in conservation. This equation is coupled with a second equation that serves similar objective as the momentum equation in Navier-Stokes for compressible flows. Here it describes the motion in the  $x_1$  and  $x_2$  for crowd flow. The model is derived from a microscopic *car-following* model. Therefore, it establishes a link between micro-to-macro models. The nature of this model is anisotropic, i.e. pedestrians respond to current and front conditions only. This way the model moves away from fluid flow models and gives a similar behavior to the one observed by crowds.

In the first equation, the conservation of continuity which conserves density (pedestrians) will change according to the change in flow at the boundary endpoints only. The conservation equation is given by

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_1)}{\partial x_1} + \frac{\partial(\rho v_2)}{\partial x_2} = 0 \quad (1)$$

with the following initial and boundary conditions

$$\rho(x, t_0) = \rho_0(x), \forall x \in \Omega \quad (2)$$

$$\rho(x, t) = 0 \quad \forall t \in [0, \infty), x \in \partial\Omega \quad (3)$$

The second set of equations represent the pedestrian motion dynamics and they are derived for  $x_1$  and  $x_2$  directions from the microscopic model. They are given by

$$\begin{aligned} \frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_1}{\partial x_2} + \rho \frac{\partial V_1(\rho)}{\partial \rho} \left( \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} \right) &= S_1 \\ \frac{\partial v_2}{\partial t} + v_1 \frac{\partial v_2}{\partial x_1} + v_2 \frac{\partial v_2}{\partial x_2} + \rho \frac{\partial V_2(\rho)}{\partial \rho} \left( \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} \right) &= S_2 \end{aligned} \quad (4)$$

Here  $\rho(x, t) \in H^2[\Omega, \mathfrak{R}]$  with  $H^2[\Omega, \mathfrak{R}]$  being the infinite dimensional Hilbert space of two dimensional like vector function defined on domain  $\Omega$ , whose spatial derivatives upto second order are square integrable with a specified  $L_2$  norm.

$q(x, t) \in H^2[\Omega, \mathfrak{R}]$ ,  $\rho_0(x) \in H[\Omega, \mathfrak{R}]$ ,  $x \in \Omega \subset \mathfrak{R}^2$  and  $t \in [0, \infty)$ . For the rest of the paper it will be assumed that

the vector spaces are Banach spaces [12]. The velocities  $V_i$  are the desired velocity functions meant to mimic pedestrian behavior given by the velocity-density relation (6), and  $\rho \frac{\partial V_i(\rho)}{\partial \rho}$  is the traffic sound speed at which small

traffic disturbances are propagated relative to the moving traffic stream. The relaxation term  $S_i$  is added to the model to keep speed concentration in equilibrium and is given as

$$S_i = \frac{V_i(\rho) - v_i}{\tau} \quad (5)$$

where  $\tau$  is this process relaxation time. In the above model  $V_i$  are the velocity-density functions that relate the desired pedestrian velocity to its density profile. Many relationships that describe this function can be found in [13]. In this model, we make use of Greenshield's model [14] that assumes the velocity is a linearly decreasing function of density and it is given by

$$V_i(\rho) = v_{fi}(x, t) \left(1 - \frac{\rho}{\rho_{\max}}\right) \quad (6)$$

where  $v_{fi}(x, t)$  is the free flow speed in  $x_i$  direction and  $\rho_{\max}$  is the jam density which is the maximum number of people that could possibly fit a single cell. We have

$$\frac{\partial V_i(\rho)}{\partial \rho} = -\frac{v_{fi}(x, t)}{\rho_m} \quad (7)$$

Again, for the simplicity sake we drop all arguments from the functions. Multiply (4) by  $\rho$  and substitute for the product rules for terms  $\frac{\partial(\rho v_1)}{\partial t}$ ,  $\frac{\partial(\rho v_1^2)}{\partial x_1}$  and  $\frac{\partial(\rho v_1 v_2)}{\partial x_2}$ .

Finally, the substitution of (1) results in the following form of the model

$$\begin{aligned} \frac{\partial(\rho v_1)}{\partial t} + \frac{\partial(\rho v_1^2)}{\partial x_1} + \frac{\partial(\rho v_1 v_2)}{\partial x_2} + \frac{\rho v_1}{\tau} &= \hat{S}_1 \\ \frac{\partial(\rho v_2)}{\partial t} + \frac{\partial(\rho v_2^2)}{\partial x_2} + \frac{\partial(\rho v_1 v_2)}{\partial x_1} + \frac{\rho v_2}{\tau} &= \hat{S}_2 \end{aligned} \quad (8)$$

where  $\hat{S}_i$  is given as

$$\hat{S}_i = F_i v_{fi} \quad (9)$$

$$F_i = \left[ \frac{\rho^2}{\rho_{\max}} \left( \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} \right) + \frac{\rho}{\tau} \left( 1 - \frac{\rho}{\rho_{\max}} \right) \right] \quad (10)$$

Thus, the form for our model written in terms of density  $\rho$  and flow vectors  $q_i$  is as

$$\frac{\partial \rho}{\partial t} + \frac{\partial(q_1)}{\partial x_1} + \frac{\partial(q_2)}{\partial x_2} = 0 \quad (11)$$

and

$$\begin{aligned} \frac{\partial q_1}{\partial t} + \frac{\partial}{\partial x_1} \left( \frac{q_1^2}{\rho} \right) + \frac{\partial}{\partial x_2} \left( \frac{q_1 q_2}{\rho} \right) + \frac{q_1}{\tau} &= \hat{S}_1 \\ \frac{\partial q_2}{\partial t} + \frac{\partial}{\partial x_2} \left( \frac{q_2^2}{\rho} \right) + \frac{\partial}{\partial x_1} \left( \frac{q_1 q_2}{\rho} \right) + \frac{q_2}{\tau} &= \hat{S}_2 \end{aligned} \quad (12)$$

## B. Model2

The second model has been proposed in [5]. Here we present a crowd dynamic model in 2-D space based on a 1-D vehicle traffic flow model. The model does not closely follow fluids and gas-like models. In addition, it allows observed traffic conditions such as traffic flow with the flow in front, rather than with the flow that is upstream. Hence, the model carries the desired anisotropic nature of traffic flow.

This model is also a nonlinear, time-varying hyperbolic system of two PDE's. The first equation is the same conservation of continuity, but the second equation is different. In this model, a convective derivative is used instead of the space derivative for the pressure term. Due to this, traffic flow changes with respect to space and time, and not space only as in Payne [15] and Whitham [16] model. Based on this modification, the second equation is given by

$$\begin{aligned} \frac{\partial}{\partial t} (v_1 + P_1) + v_1 \frac{\partial}{\partial x_1} (v_1 + P_1) + v_2 \frac{\partial}{\partial x_2} (v_1 + P_1) &= S_1 \\ \frac{\partial}{\partial t} (v_2 + P_2) + v_1 \frac{\partial}{\partial x_1} (v_2 + P_2) + v_2 \frac{\partial}{\partial x_2} (v_2 + P_2) &= S_2 \end{aligned} \quad (13)$$

The pressure functions  $P_1$  and  $P_2$  are functions of density and velocity and are given by

$$\begin{aligned} P_1(\rho, v) &= \frac{v_1 \rho^{\gamma+1}}{\beta - \rho^{\gamma+1}} \\ P_2(\rho, v) &= \frac{v_2 \rho^{\gamma+1}}{\beta - \rho^{\gamma+1}} \end{aligned} \quad (14)$$

Equation (14) is valid for  $\gamma > 0$  and  $\beta > \rho_m^{\gamma+1}$ . We modified these functions to serve two main purposes; first is to maintain the increasing property of these functions with respect to density as in the 1-D model, second, to be able to simulate bi-directional pedestrian motion and in the same time preserve the model anisotropic property. We rewrite the system in (13) in the following conservation form

$$\begin{aligned} \frac{\partial}{\partial t} (\rho(v_1 + P_1)) + \frac{\partial}{\partial x_1} (\rho v_1 (v_1 + P_1)) + \frac{\partial}{\partial x_2} (\rho v_2 (v_1 + P_1)) &= \rho S_1 \\ \frac{\partial}{\partial t} (\rho(v_2 + P_2)) + \frac{\partial}{\partial x_1} (\rho v_1 (v_2 + P_2)) + \frac{\partial}{\partial x_2} (\rho v_2 (v_2 + P_2)) &= \rho S_2 \end{aligned}$$

The functions  $S_1$  and  $S_2$  given as before by (5). Using (6) we can write the above set of equations as

$$\begin{aligned} \frac{\partial(\rho v_1)}{\partial t} + \frac{\partial(\rho v_1^2)}{\partial x_1} + \frac{\partial(\rho v_1 v_2)}{\partial x_2} + \frac{\rho v_1}{\tau} + \hat{P}_1 &= \hat{s}_1 \\ \frac{\partial(\rho v_2)}{\partial t} + \frac{\partial(\rho v_2^2)}{\partial x_2} + \frac{\partial(\rho v_1 v_2)}{\partial x_1} + \frac{\rho v_2}{\tau} + \hat{P}_2 &= \hat{s}_2 \end{aligned} \quad (15)$$

where  $\hat{P}_i$  and  $\hat{s}_i$  are given as

$$\hat{P}_i = \frac{\partial(\rho P_1)}{\partial t} + \frac{\partial(\rho v_1 P_1)}{\partial x_1} + \frac{\partial(\rho v_2 P_1)}{\partial x_2} \quad (16)$$

$$\hat{s}_i = f_i v_{f_i} \quad (17)$$

$$f_i = \frac{\rho}{\tau} \left(1 - \frac{\rho}{\rho_{\max}}\right) \quad (18)$$

Thus the form for our model written in terms of density  $\rho$  and flow vectors  $q_i$  is as

$$\frac{\partial \rho}{\partial t} + \frac{\partial(q_1)}{\partial x_1} + \frac{\partial(q_2)}{\partial x_2} = 0 \quad (19)$$

and

$$\frac{\partial q_1}{\partial t} + \frac{\partial}{\partial x_1} \left(\frac{q_1^2}{\rho}\right) + \frac{\partial}{\partial x_2} \left(\frac{q_1 q_2}{\rho}\right) + \frac{q_1}{\tau} + \hat{P}_1 = \hat{s}_1 \quad (20)$$

$$\frac{\partial q_2}{\partial t} + \frac{\partial}{\partial x_2} \left(\frac{q_2^2}{\rho}\right) + \frac{\partial}{\partial x_1} \left(\frac{q_1 q_2}{\rho}\right) + \frac{q_2}{\tau} + \hat{P}_2 = \hat{s}_2$$

### III. FEEDBACK CONTROL DESIGN

In this section we design feed back controllers for the two-equation models of the evacuation system using backstepping approach adopted in [11]. The approach is similar in principle to feedback control by backstepping for ordinary differential equations [17].

#### A. Control Model 1

To formulate the control problem we need to choose a control variable. To do so we use Greenshield's model to represent the relationship between traffic density and the velocity field given by (6). In this model we take free flow velocity vector fields  $v_{f_i}$  as the distributed control variables denoted by  $u_i \in H[\Omega, \mathfrak{R}]$ . If the density at a location is zero then the speed at that location will be the free flow speed. However, with the actuation system implemented, we can tell people to change the speed. Also the traffic density affects the achievable speeds, therefore we choose  $v_{f_i}$  as the control variable, thus giving us the following control model for system (11) and (12)

$$\frac{\partial \rho}{\partial t} + \frac{\partial(q_1)}{\partial x_1} + \frac{\partial(q_2)}{\partial x_2} = 0 \quad (21)$$

and

$$\frac{\partial q_1}{\partial t} + \frac{\partial}{\partial x_1} \left(\frac{q_1^2}{\rho}\right) + \frac{\partial}{\partial x_2} \left(\frac{q_1 q_2}{\rho}\right) + \frac{q_1}{\tau} = F_1 u_1 \quad (22)$$

$$\frac{\partial q_2}{\partial t} + \frac{\partial}{\partial x_2} \left(\frac{q_2^2}{\rho}\right) + \frac{\partial}{\partial x_1} \left(\frac{q_1 q_2}{\rho}\right) + \frac{q_2}{\tau} = F_2 u_2$$

where the functions  $F_i$  are given by (10). To use the control design scheme in [11] we need to put the above control model in a standard form given by following

$$\frac{\partial \rho}{\partial t} = -\sum_{i=1}^2 \frac{\partial q_i}{\partial x_i} \quad (23)$$

$$\frac{\partial q_i}{\partial t} = \bar{u}_i, \quad i=1,2 \quad (24)$$

The modified control variables are given as

$$\bar{u}_1 = -\frac{\partial}{\partial x_1} \left(\frac{q_1^2}{\rho}\right) - \frac{\partial}{\partial x_2} \left(\frac{q_1 q_2}{\rho}\right) + \frac{q_1}{\tau} + F_1 u_1 \quad (25)$$

$$\bar{u}_2 = -\frac{\partial}{\partial x_2} \left(\frac{q_2^2}{\rho}\right) - \frac{\partial}{\partial x_1} \left(\frac{q_1 q_2}{\rho}\right) + \frac{q_2}{\tau} + F_2 u_2$$

#### B. Control Model 2

For this model (19)-(20) also use of free flow velocity as the control variable gives us the following control model

$$\frac{\partial \rho}{\partial t} + \frac{\partial(q_1)}{\partial x_1} + \frac{\partial(q_2)}{\partial x_2} = 0 \quad (26)$$

and

$$\frac{\partial q_1}{\partial t} + \frac{\partial}{\partial x_1} \left(\frac{q_1^2}{\rho}\right) + \frac{\partial}{\partial x_2} \left(\frac{q_1 q_2}{\rho}\right) + \frac{q_1}{\tau} + \hat{P}_1 = f_1 u_1 \quad (27)$$

$$\frac{\partial q_2}{\partial t} + \frac{\partial}{\partial x_2} \left(\frac{q_2^2}{\rho}\right) + \frac{\partial}{\partial x_1} \left(\frac{q_1 q_2}{\rho}\right) + \frac{q_2}{\tau} + \hat{P}_2 = f_2 u_2$$

where the functions  $f_i$  are given by (18). The above system of equations can be written as

$$\frac{\partial \rho}{\partial t} = -\sum_{i=1}^2 \frac{\partial q_i}{\partial x_i} \quad (28)$$

$$\frac{\partial q_i}{\partial t} = \bar{u}_i, \quad i=1,2 \quad (29)$$

with the modified control variables being

$$\bar{u}_1 = -\frac{\partial}{\partial x_1} \left(\frac{q_1^2}{\rho}\right) - \frac{\partial}{\partial x_2} \left(\frac{q_1 q_2}{\rho}\right) + \frac{q_1}{\tau} + \hat{P}_1 + f_1 u_1 \quad (30)$$

$$\bar{u}_2 = -\frac{\partial}{\partial x_2} \left(\frac{q_2^2}{\rho}\right) - \frac{\partial}{\partial x_1} \left(\frac{q_1 q_2}{\rho}\right) + \frac{q_2}{\tau} + \hat{P}_2 + f_2 u_2$$

The control model (28)-(29) is similar to (23)-(24). We will use this standard form to design the feedback controllers for both models in next section.

#### C. State Feedback Control Using Backstepping

Here we address the problem of synthesizing a distributed state feedback controller  $\bar{u}_i$  that stabilizes origin ( $\rho=0, q=0$ ) of control systems (23)-(24) and (28)-(29).

We will use the standard form (23)-(24) for both models to design the feedback controllers. More specifically we consider control law

$$\bar{u}_i = F_i(\rho, q)$$

such that origin of closed loop dynamics is asymptotically stable.  $F_i$  is a nonlinear operator mapping  $H^2[\Omega, \mathfrak{R}]$  into  $H^1[\Omega, \mathfrak{R}]$ . First we design control law for equation (23) where  $q_i$  can be viewed as an input. We proceed to design a control law  $q_i = G_i(\rho)$  to stabilize origin  $\rho = 0$ .  $G_i$  is a nonlinear operator mapping  $H^2[\Omega, \mathfrak{R}]$  into  $H^2[\Omega, \mathfrak{R}]$ . By choosing the following control law

$$q_i = G_i(\rho) = \int_0^{x_i} \frac{\partial^2 \rho}{\partial m^2} dm, \quad i = 1, 2 \quad (31)$$

we can rewrite (23) as

$$\frac{\partial \rho}{\partial t} = \nabla^2 \rho \quad (32)$$

which is the heat equation. We want to find the Lyapunov functional for (23) which will ensure that its origin of closed loop dynamics (32) is asymptotically stable.

#### D. Stability analysis using Lyapunov

The stability problem is to establish sufficient conditions for which the origin of the closed loop dynamics (32) is exponentially stable. Within the framework of our system the definition of stability in terms of Lyapunov amounts to establishing conditions for which the null state of the linear system (32) is exponentially asymptotically stable with respect to the specified norm i.e.  $\|\rho(t)\|_2 \rightarrow 0$  as  $t \rightarrow \infty$ . For our system we have chosen the following  $L_2$  norm defined by

$$\|\rho\|_2 \rightarrow \left( \int_{\Omega} |\rho|^2 d\Omega \right)^{1/2} \quad (33)$$

We consider the following Lyapunov functional  $V(t)$

$$V(t) = \frac{1}{2} \|\rho\|_2^2 = \frac{1}{2} \int_{\Omega} |\rho|^2 d\Omega \quad (34)$$

Using the norm properties we can easily see that  $V(t)$  is a positive definite function. The time rate of change of  $V(t)$  is given as

$$\frac{dV(t)}{dt} = \int_{\Omega} \rho \frac{\partial \rho}{\partial t} d\Omega \quad (35)$$

Using (32) and integrating (35) we get

$$\frac{dV(t)}{dt} = \left[ k\rho \sum_{i=1}^2 \frac{\partial \rho}{\partial x_i} \Big|_{x \in \partial\Omega} \right] - k \int_{\Omega} \sum_{i=1}^2 \left( \frac{\partial \rho}{\partial x_i} \right)^2 d\Omega$$

The first term vanishes by boundary condition (3). For the second integral we make use of Gagliardo-Nirenberg-Sobolev Inequality [18]. The inequality as applied to our case states

$$\|\rho\|_2 \leq C \|\nabla \rho\|_2 \quad (36)$$

where  $C$  is a positive real number. Using (36) it can be shown that

$$\int_{\Omega} \sum_{i=1}^2 \left( \frac{\partial \rho}{\partial x_i} \right)^2 d\Omega \geq C^{-2} \int_{\Omega} \rho^2 d\Omega$$

The rate of change of  $V(t)$  can be thus be bounded by

$$\frac{dV(t)}{dt} \leq -\beta V(t)$$

It follows that

$$V(t) \leq V(t_0) e^{-\beta(t-t_0)}$$

or

$$\|\rho(t)\|_2 \leq \|\rho(t_0)\|_2 e^{-\beta(t-t_0)}$$

with  $\beta = 2kC^{-2}$ . As long as  $\beta > 0$ , null state of closed loop system (32) is exponentially stable using control law (31).

#### E. Control law

As we have already shown the origin of (32) is asymptotically exponentially stable and there exists a Lyapunov functional which ensures this stability. From the knowledge of this function we want to design a smooth feedback control to stabilize origin of the overall closed system for (23)-(24). Rewriting (23) in closed loop form we have

$$\frac{\partial \rho}{\partial t} = -\sum_{i=1}^2 \frac{\partial G_i(\rho)}{\partial x_i} - \sum_{i=1}^2 \frac{\partial}{\partial x_i} (q_i - G_i(\rho))$$

Defining error variables  $z_i = q_i - G_i(\rho)$ , where the error variable  $z_i \in H^2[(\Omega, \mathfrak{R})]$ , result in the following modified dynamics

$$\frac{\partial \rho}{\partial t} = -\sum_{i=1}^2 \frac{\partial G_i(\rho)}{\partial x_i} - \sum_{i=1}^2 \frac{\partial z_i}{\partial x_i} \quad (37)$$

$$\frac{\partial z_i}{\partial t} = u_{ni} \quad (38)$$

where  $u_{ni} = \bar{u}_i - \frac{\partial G_i(\rho)}{\partial t}$ ,  $i = 1, 2$  are the new control variables. Now let us choose the Lyapunov functional for the overall system as

$$\begin{aligned} V_a(t) &= V(t) + \frac{1}{2} \|z(x, t)\|_2^2 \\ &= \frac{1}{2} \int_{\Omega} |\rho|^2 dx + \frac{1}{2} \int_{\Omega} \sum_{i=1}^2 |z_i|^2 d\Omega \end{aligned} \quad (39)$$

The time rate of change of this functional from [11] can be shown to be bounded by choosing the following control law

$$u_{ni} = -\hat{k} z_i - z_i^{-1} \rho \frac{\partial z_i}{\partial x_i} \quad (40)$$

This control law yields

$$\frac{dV_a(t)}{dt} \leq -\beta V(t) - \hat{k} \|z\|_2^2 = -2\beta V_a(t)$$

with  $\hat{k} = 2\beta > 0$ . This shows that the origin for modified

system (37)-(38) is asymptotically exponentially stable. It can be proved from  $z_i = q_i - G_i(\rho)$  that the origin for actual closed loop system (23) and (24) is also asymptotically exponential stable. Thus the feedback control law for the standard form of both the system is given by the following partial differential-integral equations:

$$\bar{u}_i = u_{ni} + \frac{\partial G_i(\rho)}{\partial t} \quad (41)$$

with  $u_{ni} = -\hat{k}z_i - z_i^{-1} \rho \frac{\partial z_i}{\partial x_i}$ . The actual control law  $u_i$  can

be evaluated from (25) for model 1 and from (30) for model 2. These control laws respectively result in the exponentially stability of closed loop dynamics for both models.

#### IV. SIMULATION RESULTS

In this section we show the simulation results for the closed loop dynamics for model 1 using controller (41). We have only included the plots for model 1 as the plots for model 2 are similar to these. We show the density profiles at different time instants. The solution is shown as contour plot snapshots in figure (1) and a mesh plot snapshots in figure (2). As is seen from plots density at every point in space is decreasing exponentially with time.

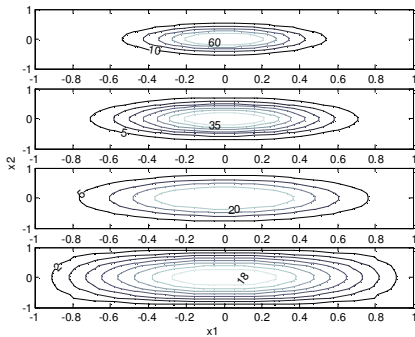


Fig.1: Snapshots of density contours at different time intervals for closed loop dynamics for model 1.

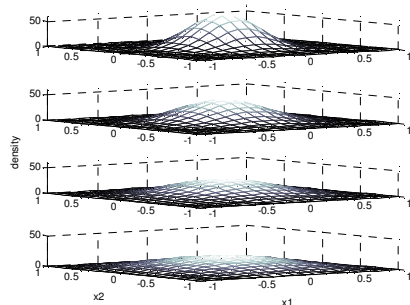


Fig.2: Density plot snapshots at different time instants for closed loop dynamics for model 1

#### REFERENCES

- [1] P. Kachroo and K. Ozbay, *Feedback Control Theory for Dynamic Traffic Assignment*. Springer-Verlag, 1999, ch. 2.
- [2] E. N. Holland, "A generalised stability criterion for Motorway traffic," *Tranpn. Res.*, vol. B-32, pp. 141-154, 1998
- [3] M.J. Lighthall and G.B. Whitham, "On kinematic waves I: A theory of traffic flow on long crowded roads," *Proc. of the Royal Society of London*, A-229, pp. 281-345, 1955.
- [4] S. Al-nasur and P. Kachroo, "A Microscopic-to-Macroscopic Crowd Dynamic model," *Proceedings of the IEEE Intelligent Transportation Systems Conference*, Toronto, Canada, 2006.
- [5] S. Al-nasur and P. Kachroo, "Derivations of Two Macroscopic Crowd Dynamic Models," *Transportation Research Board*, submitted for publication.
- [6] Aw, A., and M. Rascle. Reconstruction of 'second order' models of traffic flow, *SIMA J. Appl. Math.*, 60, 2000, pp. 916-938.
- [7] Rascle, M. An Improved Macroscopic Model of Traffic Flow: Derivation and Links with the Lighthill-Whitham Model, *Mathematical and Computer Modeling*, 35, 2002 pp. 581-590.
- [8] D. Dochain, J. P. Babary, and M. N. Tali-Manaarn, "Modeling and adaptive control of nonlinear distributed parameter bioreactors via orthogonal collocation," *Automatica.*, vol. 68, pp. 873-883, 1992.
- [9] P. K. Gundepudi and J. C. Friedly, "Velocity control of hyperbolic partial differential equation systems with single characteristic variable," *Chem. Sci.*, vol. 53, pp. 4055-4072, 1998.
- [10] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer-Verlag, New York, 1983.
- [11] S.A. Wadoo and P. Kachroo, "Feedback Control Design and Stability Analysis of Two Dimensional Evacuation System," *Proceedings of the IEEE Intelligent Transportation Systems Conference*, Toronto, Canada, 2006.
- [12] P. D. Christofides, *Nonlinear and Robust Control of PDE Systems*. Birkhauser, 2000.
- [13] J. V. Morgan, *Numerical Methods for Macroscopic Traffic Models*. PhD Thesis, University of Reading, 2002.
- [14] B. D. Greenshields, "A study in highway capacity, highway research board," *Proceedings*, vol. 14, pp. 458, 1935.
- [15] Payne, H.J. Models of freeway traffic and control, *Math. Models Publ. Sys. Simul. Council Proc.*, 28, 1971, pp. 51-61.
- [16] Whitham, G.B. *Linear and Nonlinear Waves*, John Wiley, NY. 1974.
- [17] H. K. Khalil, *Nonlinear Systems*. Prentice Hall, 1996.
- [18] R. Howard, "The sharp Sobolev inequality and the Banchoff-Pohl inequality on surfaces," *Proceedings of the American Mathematical Society.*, vol. 126, issue 9, pp. 2779-2787, 1998.