Optimal Control of Pedestrian Evacuation in a Corridor

A. Shende
Virginia Polytechnic Institute and State University

Pushkin Kachroo
University of Nevada, Las Vegas, pushkin@unlv.edu

C. Konda Reddy

M. Singh

Proceedings of the 2007 IEEE
Intelligent Transportation Systems Conference
Seattle, WA, USA, Sept. 30 - Oct. 3, 2007

Optimal Control of Pedestrian Evacuation in a Corridor

Apoorva Shende, Pushkin Kachroo, C. Konda Reddy and Mahendra Singh

***Abstract*— The problem of optimal control of pedestrian evacuation from a corridor has been addressed. The corridor has been treated as a one dimensional link in the building network from which the pedestrians have to be evacuated. The governing flow equations are derived from the discretized continuity equation and a flow density relation for the pedestrian flow. Necessary conditions for the optimal control of these differential equations are developed using the calculus of variations method. The necessary conditions constitute a 2 point boundary value problem that has to be solved for the state, the co-state and the optimal control. Method of steepest descent is used to solve this problem. Numerical results are presented in the end for a test case.**

I. INTRODUCTION

Evacuation Dynamics and Control is a field of critical importance given the threats of modern urban life. Evacuation can be classified as broadly belonging to 2 types: 1) Evacuation from a built up facility 2) Evacuation from a locality. While the 1st kind involves moving pedestrians out of the buildings quickly the 2nd type mainly involves appropriate routing of vehicular traffic. This paper involves optimal evacuation of pedestrians from the building facilities. Pedestrian dynamics has been studied using 2 types of models 1) macroscopic and 2) microscopic. The macroscopic models typically deal with the pedestrian flow as a continuum where as the microscopic models deal with the movement and interactions of the individual pedestrians. Various cellular automata and simulation based models are available to deal with the pedestrian flow at the microscopic level ([1],[2], [3], [4]). At the macroscopic level the pedestrian flow is analyzed in terms of the variables like pedestrian flow density and flow velocity. Various models developed for macroscopic traffic flow (*ch. 2* of [5])can be readily extended to capture the dynamics of pedestrian flow. The dynamics of pedestrian flow described at the macroscopic level can be utilized to develop control for quick evacuation. Feedback controls for evacuation have been developed based on macroscopic models in [6] for the 1-D case and in [7] for the 2-D case. A wealth of literature exists dealing with dynamic traffic assignment ([5],[8], [9]). A Building structure can be considered to be a network of rooms, corridors, stairways and exits. While the rooms and exits can be thought of as the sources and sinks of the pedestrian flow respectively, the corridors and stairways can be thought of as circulation elements which will be the key elements in routing the pedestrians out of the buildings. A controlled flow of pedestrians in these links can achieve a fast evacuation of the pedestrians from these links and hence from the entire facility. Studies have also been done to design network links of proper dimensions to accommodate a quick evacuation using queuing theory [10].

Apoorva Shende is with the Department of Engineering Science and Mechanics, Virginia Tech, Blacksburg, VA 24061, USA apoorva@vt.edu
Pushkin Kachroo is with the Department of Electrical and Computer Engineering, Virginia Tech, Blacksburg, VA 24061, USA pushkin@vt.edu
C. Konda Reddy is with the Nonlinear Systems Lab (NSL), Aerospace and Ocean Engineering, Virginia Tech, Blacksburg, VA 24061, USA konda@vt.edu
Mahendra Singh is with the Department of Engineering Science and Mechanics, Virginia Tech, Blacksburg, VA 24061, USA mpsingh@vt.edu

This paper is organized as follows: In section II a mathematical model is developed for the pedestrian flow using the continuity equation and the Greensheilds model described in [5]. In section III necessary conditions for the optimal control are stated using the calculus of variations approach. The procedure for this can be found in [11]. Following this the method of steepest descent is described in section IV that solves for the control satisfying these necessary conditions numerically. This method is also described in [11]. The numerical results using this method are stated for a test case in section V. In the last section VI the conclusions are summarized.

II. PEDESTRIAN FLOW MODEL

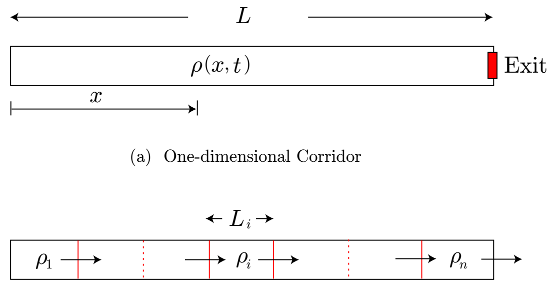
Consider a one-dimensional corridor as shown in Fig. 1 (a). The corridor is assumed to be of length L with an exit at one end of the corridor. The classical *continuity equation* (conservation of mass) of fluid flow is used as a basic equation to model the flow of pedestrians through the corridor. The one-dimensional continuity equation is given by the following partial differential equation:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial q}{\partial x}, \qquad (1)$$

where $\rho(x,t)$ represents the pedestrian density at a distance x along the length of the corridor at time t and $q(x,t)$ is the discharge at point x at time t.

Equation (1) can be approximated by a set of n ordinary differential equations by dividing the corridor into n sections and integrating (1) over a section under the assumption of no spatial variation of ρ within the section. The sections are schematically shown in Fig. 1(b). The set of ordinary differential equations is given by:

$$\dot{\rho}_1 = -\frac{q_1}{L_1} \qquad (2)$$



(a) One-dimensional Corridor

(b) Disretized model of the Corridor

Fig. 1. Schematic of a 1-D corridor

$$\dot{\rho}_i = -\frac{q_i - q_{i-1}}{L_i}, \quad 2 \le i \le n, \tag{3}$$

where $\rho_i(t)$ is the pedestrian density in the i^{th} section and $q_i(t)$ is the discharge at the i^{th} interface. The equation for the 1^{st} section (2) is different from the equations for the other sections (3) because there is no input discharge into section 1. In order to find the discharges at the interfaces $q_i(t)$ in terms of the densities $\rho_i(t)$, we need to assume a functional dependence of flow velocity v with the pedestrian density ρ. In the present paper, we will use the *Greensheilds* model of traffic flow[5]. According to the Greensheilds model

$$v = v_f(1 - \frac{\rho}{\rho_m}), \quad 0 \le \rho \le \rho_m, \tag{4}$$

where v_f represents the free flow velocity when the pedestrian density is zero and ρ_m represents the critical density at which the flow velocity becomes zero. Since

$$q = \rho v,$$

it follows from (4) that

$$q = \rho(1 - \frac{\rho}{\rho_m})v_f. \tag{5}$$

Using (5) in (2) and (3), we get the following equations governing the dynamics of the system.

$$\dot{\rho}_1 = -\frac{\rho_1(1 - \frac{\rho_1}{\rho_m})v_{f_1}}{L_1}, \tag{6}$$

$$\dot{\rho}_i = -\frac{\rho_i(1 - \frac{\rho_i}{\rho_m})v_{f_i} - \rho_{i-1}(1 - \frac{\rho_{i-1}}{\rho_m})v_{f_{i-1}}}{L_i}. \tag{7}$$

The constraint in the Greensheilds model (4) results in the following constraints on the states ρ_i of the system:

$$0 \le \rho_i \le \rho_m \quad \forall \quad 1 \le i \le n. \tag{8}$$

For normalizing these equations we can divide them by ρ_m. We get the following equations in which the quantities with "tilde" are the normalized quantities.

$$\dot{\tilde{\rho}}_1 = -\tilde{\rho}_1(1 - \tilde{\rho}_1)b_1\tilde{v}_{f_1} \tag{9}$$

$$\dot{\tilde{\rho}}_i = \tilde{\rho}_{i-1}(1 - \tilde{\rho}_{i-1})b_i\tilde{v}_{f_{i-1}} - \tilde{\rho}_i(1 - \tilde{\rho}_i)b_i\tilde{v}_{f_i} \tag{10}$$

$$\tilde{\rho}_i = \frac{\rho_i}{\rho_m}$$
$$\tilde{v}_{f_i} = \frac{v_{f_i}}{L}$$
$$b_i = \frac{L}{L_i}$$

$\tilde{\rho}_i$ is the normalized non-dimensional density in the i^{th} section. $\tilde{v}_{f_i}$ is the normalized free flow velocity however its not non-dimensional but can be made non-dimensional if we multiply it with a time constant. The parameter b_i is the ratio of the total length to the length of the section. If $b_i = n \; \forall i$ then we have a case of n equal length sections.

III. Optimal Control Formulation for Pedestrian Flow

In this section we state the necessary conditions for the optimal control of pedestrian flow described by the above equations (9) and (10). In III-A we give the state equations and there by define the control variables. In III-B we define the cost function that we intend to minimize. Then using the state equations and the cost function we develop the necessary conditions for the optimal control in III-C by the calculus of variations method described in *ch. 5* of [11]. The problem considered here is that of free final state and fixed final time.

A. State Equations

For the purpose of developing an optimal control for the system of equations (9) and (10) we write them in the following form by dropping the tildes. Henceforth the normalized quantities will be represented without the tildes.

$$\dot{\rho}_1 = -\rho_1(1 - \rho_1)b_1 v_{f_1} \tag{11}$$

$$\dot{\rho}_i = \rho_{i-1}(1 - \rho_{i-1})b_i v_{f_{i-1}} - \rho_i(1 - \rho_i)b_i v_{f_i} \tag{12}$$

$$\dot{v_{f_i}} = u_i \tag{13}$$

The control variable $\mathbf{u}$ is introduced in the last equation (13). As is apparent from the equation the control input represents the time rate of change of free flow velocities in the various sections. State vector in this formulation is

$$\mathbf{x} = \begin{bmatrix} \rho \\ \mathbf{v_f} \end{bmatrix} \tag{14}$$

We collect the right hand sides of the equations (11), (12) and (13) in the vector $\mathbf{a}$ given as follows.

$$a_1(\mathbf{x}(t), \mathbf{u}(t), t) = -\rho_1(1 - \rho_1)b_1 v_{f_1} \tag{15}$$

$$a_i(\mathbf{x}(t), \mathbf{u}(t), t) = \rho_{i-1}(1-\rho_{i-1})b_i v_{f_{i-1}} - \rho_i(1-\rho_i)b_i v_{f_i}$$
$$2 \leq i \leq n \tag{16}$$

$$a_{v_{f_i}}(\mathbf{x}(t),\mathbf{u}(t), t) = u_i \qquad 1 \leq i \leq n \tag{17}$$

B. Cost Function

For the development of the optimal control we use the following as the cost function:

$$J(\mathbf{u}(t)) = \int_{t_0}^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t)d\tau + h(\mathbf{x}(t_f), t_f), \tag{18}$$

where

$$g(\mathbf{x}(t), \mathbf{u}(t), t) = \sum_{i=1}^{n} {\rho_i}^2 + {u_i}^2 \tag{19}$$

and

$$h(\mathbf{x}(t_f), t_f) = \sum_{i=1}^{n} \rho_i(t_f)^2 . \tag{20}$$

Here, $h(\mathbf{x}(t_f), t_f)$ represents the terminal cost. By defining the cost function by equation (18) we are assured that its minimization will ensure low pedestrian density as well as low rate of change of free flow velocity at every instant of time. Since both the attributes are highly desirable from the point of view of evacuation, the particular choice of cost function given by (18) is justified.

C. Calculus of Variation

We now use the calculus of variations approach to develop the Euler Lagrange equations that need to be satisfied by the optimal control and the corresponding states. The Hamiltonian for the equations

$$\dot{\mathbf{x}} = \mathbf{a}(\mathbf{x}, \mathbf{u}, t)$$

is given by

$$H(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}(t), t) = g(\mathbf{x}(t), \mathbf{u}(t), t) + \mathbf{p}^T \mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t). \tag{21}$$

In the above equation the vector $\mathbf{p}$ is the *co-state* vector. For equations (11), (12) and (13) the Hamiltonian is given by

$$\begin{aligned} H(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}(t), t) \quad &= \quad \sum_{i=1}^{n}({\rho_i}^2 + u_i^2) \\ &+ \quad p_1(-\rho_1(1-\rho_1)b_1 v_{f_1}) \\ &+ \quad \sum_{i=2}^{n} p_i(\rho_{i-1}(1-\rho_{i-1})b_i v_{f_{i-1}} \\ &- \quad \rho_i(1-\rho_i)b_i v_{f_i}) + \sum_{i=1}^{n} p_{v_{f_i}} u_i \end{aligned} \tag{22}$$

1) Necessary Conditions: We now state Euler Lagrange equations which specify the necessary conditions for the control to be a optimal

$$\dot{\mathbf{x}}^* = \frac{\partial H}{\partial \mathbf{p}}(\mathbf{x}*, \mathbf{u}*, \mathbf{p}*, t) \tag{23}$$

These equations (23) are the state equations given by (11), (12) and (13).

$$\dot{\mathbf{p}}^* = -\frac{\partial H}{\partial x}(\mathbf{x}*, \mathbf{u}*, \mathbf{p}*, t) \tag{24}$$

(24) gives the co-state equations that are specified below.

$$\dot{p}_i = \begin{pmatrix} -(2\rho_i + p_i(-1+2\rho_i)b_i v_{f_i} + \\ p_{i+1}(1-2\rho_i)b_{i+1}v_{f_i}) \end{pmatrix} \tag{25}$$

$$1 \leq i \leq n-1$$

$$\dot{p}_n = -(2\rho_n + p_n(-1+2\rho_n)b_n v_{f_n}) \tag{26}$$

$$\dot{p}_{v_{f_i}} = -(p_i(-\rho_i(1-\rho_i)b_i) + p_{i+1}(\rho_i(1-\rho_i)b_{i+1})) \tag{27}$$

$$\dot{p}_{v_{f_n}} = -(p_n(-\rho_n(1-\rho_n)b_n)) \tag{28}$$

Another necessary condition on the Hamiltonian is that its partial derivative with respect to the control must vanish for all times $t_0 \leq t \leq t_f$.

$$\frac{\partial H}{\partial \mathbf{u}}(\mathbf{x}*, \mathbf{u}*, \mathbf{p}*, t) = 2u_i^* + p_{v_{f_i}} = 0 \tag{29}$$

We choose the case in which final time t_f is specified and the final state $\mathbf{x}(t_f)$ is free. Hence we get the following boundary conditions.

At the initial time t_0 we have,

$$\rho^*(t_0) = \rho_0 \tag{30}$$

$$\mathbf{v}^*_{\mathbf{f_i}}(t_0) = 0 \tag{31}$$

At the final time t_f we have,

$$\mathbf{p}^*(t_f) = \frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) \tag{32}$$

$$p_i^*(t_f) = 2\rho_i(t_f) \tag{33}$$

$$p^*_{v_{f_i}} = 0 \tag{34}$$

This is a typical 2 point boundary value problem that occurs in optimal control. The boundary conditions are a part of the necessary conditions.

IV. Numerical Solution: The Method of Steepest Descent

The Method of Steepest Descent was used to compute the optimal control in a piecewise constant manner. This is described in *ch. 6* of [11]. This is an iterative method in which the control profile is updated in every iteration at every discretized time instant at which it was initially assumed. An initial piecewise constant control profile is assumed and the states are computed by forward integration. The the boundary conditions for the co-state $p_i^*(t_f)$ are computed using equations (32) and (33). Then equations (25), (26), (27) and (28) are integrated backwards in time from $t = t_f$ to $t = t_0$. Using the costate values $\frac{\partial H(t)}{\partial \mathbf{u}}$ is calculated using the first equality in (29) at every discrete time instant at which the control value $\mathbf{u}$ is assumed. Since the control profile was arbitrarily chosen this in general will not be zero. As the necessary condition (29) requires this quantity to be zero, the control will have to be updated in the direction of the steepest descent of the hamiltonian H at every discrete time instant. This corresponds to the negative gradient of H with respect to $\mathbf{u}$, $-\frac{\partial H}{\partial \mathbf{u}}$. Hence we have the following control update law at every iteration

$$\mathbf{u}^{(i+1)} = \mathbf{u}^{(i)} - \tau \frac{\partial H^{(i)}}{\partial \mathbf{u}} \tag{35}$$

τ in (35) has to be chosen such that the cost function in equation (18) has to continually decrease in every iteration. This usually is done by an ad hoc strategy. The iterations stop when

$$\|\frac{\partial H^{(i)}}{\partial \mathbf{u}}\| \leq tol_1 \tag{36}$$

or

$$|(J^{(i+1)} - J^i)| \leq tol_2, \tag{37}$$

where tol_1 and tol_2 are pre-specified tolerances.

V. Numerical Results

A test case was considered for the above optimal control formulation. A corridor was divided into 2 sections and each was assigned a initial normalized pedestrian density of $\rho_{i0} = 0.6$. The initial free flow velocities were set to $v_{f_{i0}} = 0$ as the crowd is stationary initially. The sections are taken to be of equal length, so $b_i = 2$. The initial time and final times are $t_0 = 0$ and $t_f = 10s$ respectively. The interval $[t_0, t_f]$ was divided into subintervals of $0.1s$. The method of steepest descent described above was used to obtain the optimal control. A matlab code was written to solve the 2-point boundary value problem above and the plots were obtained for ρ_i, vf_i, u_i, p_i, p_{vf_i} versus time. A simple euler solver was used to solve the ode's involved.

It can be seen in Fig. 3 that almost all of the population is evacuated in a period of 10s. From Fig. 7 it is apparent that the value of $\frac{\partial H}{\partial u_i}$ is very close to zero. This was forced using a low value of tolerance, tol_1 in equation (36). Thus the requirement of optimality in equation 29 is closely satisfied. In reality when implementing this control strategy we can give instructions based on the profile of $\mathbf{v_f}$ in Fig. 4 rather than $\mathbf{u}$ in Fig. 2.

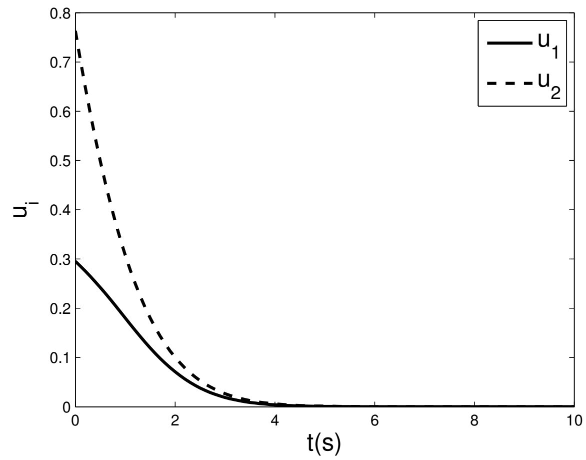


Fig. 2. Optimal control u_1 and u_2 as a function of time,t

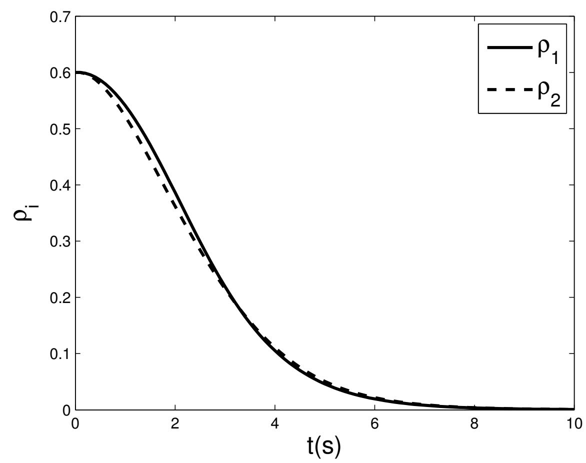


Fig. 3. Evolution of pedestrian densities ρ_1 and ρ_2 as a function of time,t

VI. Conclusion

An optimal control strategy based on a appropriate objective function has been developed for the pedestrian evacuation from a corridor. Equation for 1-D pedestrian flow have been developed in section II. The state and the control parameters are defined and the optimal control formulation for these equations is done in section III using the calculus of variations approach. The time interval for the evacuation process has been discretized and the method of steepest descent is used to get an open loop optimal control for the problem. The numerical results show that a smooth control profile is obtained as well as a smooth free flow velocity

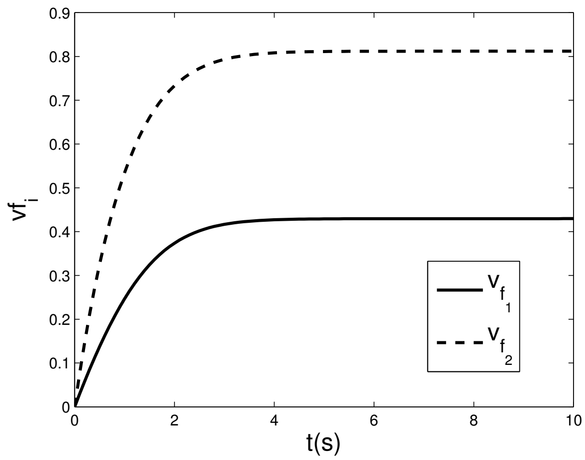


Fig. 4. Evolution of free flow velocities v_{f_1} and v_{f_2} as a function of time,t

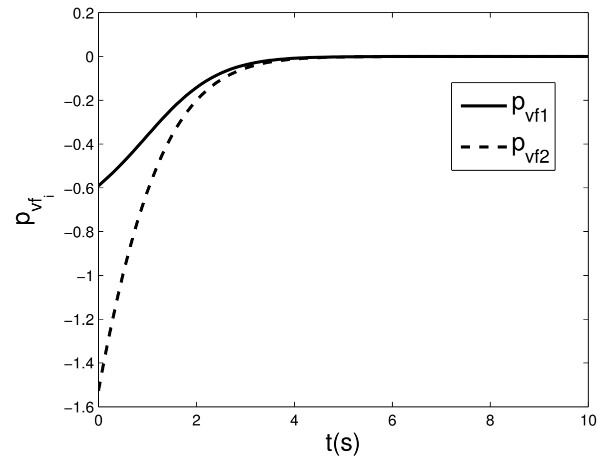


Fig. 6. Evolution of co-state, $p_{v_{f_1}}$ and $p_{v_{f_2}}$ as a function of time,t

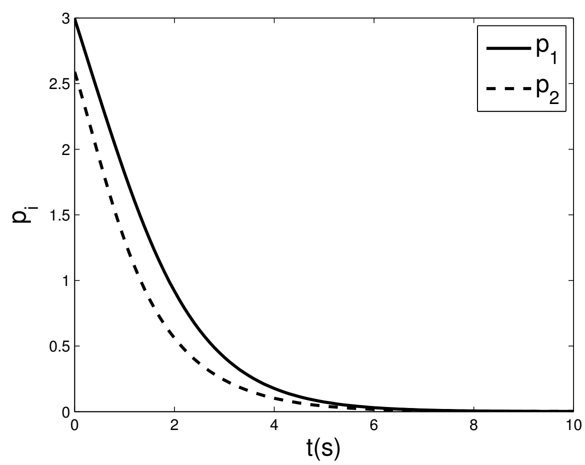


Fig. 5. Evolution of co-state, p_1 and p_2 as a function of time,t

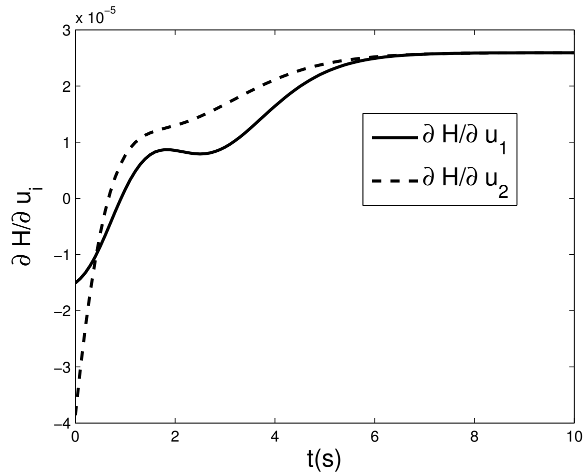


Fig. 7. $\frac{\partial H}{\partial u_i}$ as a function of time, t

profile is obtained. It is also seen that most of the population gets evacuated in the 10 s time period considered here.

ACKNOWLEDGMENT

This research is supported by National Science Foundation through grant no. CMS-0428196 with Dr. S. C. Liu as the Program Director. This support is gratefully acknowledged. Any opinion, findings, and conclusions or recommendations expressed in this study are those of the writers and do not necessarily reflect the views of the National Science Foundation.

REFERENCES

[1] S. P. Hoogendoorn and P. H. L. Bovy, "Pedestrian route-choice and activity scheduling theory and models," *Transpn. Res. B.*, vol. 38, pp. 169–190, 2004.

[2] S. P. Hoogendoorn, "Pedestrian flow behavior: Theory and applications." Research Report Vk2001.002, Transportation and Traffic Engineering Section, Delft University of Technology, Tech. Rep., 2001.

[3] M. R. Hill, "Spatial structure and decision-making of pedestrian route selection through an urban environment," Ph.D. dissertation, University Microfilms International, 1982.

[4] D. Helbing, I. Farkas, and T. Viscek, "Simulating dynamic features of escape panic," *Nature*, vol. 407, pp. 487–490, 2000.

[5] P. Kachroo and K. Ozaby, *Feedback Control Theory for Dynamic Traffic Assignment*. Springer-Verlag, 1999.

[6] S. Wadoo and P. Kachroo, "Feedback control design and stability analysis of one dimensional evacuation system," in *Proc. IEEE ITSC Conference*, 2006, accepted for publication.

[7] ——, "Feedback control design and stability analysis of two dimensional evacuation system," in *Proc. IEEE ITSC Conference*, 2006, accepted for publication.

[8] M. Papageorgiou, *Applications of Automatic Control Concepts to Traffic Flow Modelling and Control*. Springer-Verlag, 1983.

[9] R. T. T.L. Friesz and B. Wie, "Dynamic network traffic assignment considered as a continuous time optimal control problem," *Operations Research*, vol. 37, pp. 893–901, 1989.

[10] D. L. Bakuli and J. M. Smith, "Resource allocation in state-dependent

emergency evacuation networks," *European Journal of Operations Research*, vol. 89, pp. 543–555, 1996.
[11] D. Kirk, *Optimal Control Theory: An Introduction*. New York: Dover Publications, Inc, 2004.