4-19-2017

Diffusive Logistic Equations with Harvesting and Heterogeneity Under Strong Growth Rate

Saeed Shabani Rokn-e-vafa

Sharif University of Technology, saeed.shabani@gmail.com

Hossein T. Tehrani

University of Nevada, Las Vegas, hossein.tehrani@unlv.edu

Follow this and additional works at: https://digitalscholarship.unlv.edu/math_fac_articles

Part of the [Applied Mathematics Commons](#)

Repository Citation

Rokn-e-vafa, S. S., Tehrani, H. T. (2017). Diffusive Logistic Equations with Harvesting and Heterogeneity Under Strong Growth Rate. *Advances in Nonlinear Analysis*, 8(1), 455-467. De Gruyter.

<http://dx.doi.org/10.1515/anona-2016-0208>

This Article is protected by copyright and/or related rights. It has been brought to you by Digital Scholarship@UNLV with permission from the rights-holder(s). You are free to use this Article in any way that is permitted by the copyright and related rights legislation that applies to your use. For other uses you need to obtain permission from the rights-holder(s) directly, unless additional rights are indicated by a Creative Commons license in the record and/or on the work itself.

This Article has been accepted for inclusion in Mathematical Sciences Faculty Publications by an authorized administrator of Digital Scholarship@UNLV. For more information, please contact digitalscholarship@unlv.edu.

Research Article

Saeed Shabani Rokn-e-vafa and Hossein T. Tehrani*

Diffusive logistic equations with harvesting and heterogeneity under strong growth rate

<https://doi.org/10.1515/anona-2016-0208>

Received September 22, 2016; accepted March 1, 2017

Abstract: We consider the equation

$$-\Delta u = au - b(x)u^2 - ch(x) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $b(x)$ and $h(x)$ are nonnegative functions, and there exists $\Omega_0 \subset\subset \Omega$ such that $\{x : b(x) = 0\} = \overline{\Omega_0}$. We investigate the existence of positive solutions of this equation for c large under the strong growth rate assumption $a \geq \lambda_1(\Omega_0)$, where $\lambda_1(\Omega_0)$ is the first eigenvalue of the $-\Delta$ in Ω_0 with Dirichlet boundary condition. We show that if $h \equiv 0$ in $\Omega \setminus \overline{\Omega_0}$, then our equation has a unique positive solution for all c large, provided that a is in a right neighborhood of $\lambda_1(\Omega_0)$. For this purpose, we prove and utilize some new results on the positive solution set of this equation in the weak growth rate case.

Keywords: Logistic equation, harvesting, heterogeneity, strong growth rate, comparison principles, stable solutions

MSC 2010: 35J25, 35J61, 92D25

1 Introduction

In this paper we are interested in the study of existence of positive solutions for a semilinear elliptic equation with logistic type nonlinearity and harvesting under the so-called strong growth assumption. More precisely, we consider the following equation arising in modeling population biology of one species:

$$-\Delta u = au - b(x)u^2 - ch(x) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \quad (1.1)$$

Here Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 3$, and u represents the population density of the species whose growth follows a logistics model. The positive constant a is the so-called linear growth rate and the nonnegative crowding coefficient $b(x) \in C(\overline{\Omega})$ is assumed to be spatially dependent due to the heterogeneity of the environment. Finally, the last term on the right-hand side models the presence of a constant yield harvesting pattern (see [15] for more details). The existence and the structure of the positive solution set of (1.1) has been extensively studied under various assumptions on a , b and h (see [9, 12–15]). Here we are mainly interested in the so-called degenerate logistic case, where $b(x) \geq 0$, $b \neq 0$ and the zero set of b is the closure of some suitably regular sub-domain Ω_0 , that is,

$$\overline{\Omega_0} := \{x : b(x) = 0\} \neq \emptyset, \quad \Omega_0 \subset\subset \Omega,$$

so that our model is a mix of logistic and Malthusian models.

To properly set up our problem and give a review of the state of affairs regarding this equation we start with a few words about the notation. For a bounded smooth domain O in $\mathbb{R}^N$ we let $\lambda_i(\phi, O)$ denote the i -th

Saeed Shabani Rokn-e-vafa, Department of Mathematical Sciences, Sharif University of Technology, Tehran, Iran,

e-mail: saeed.shabani@gmail.com

***Corresponding author: Hossein T. Tehrani**, Department of Mathematical Sciences, University of Nevada, Las Vegas, USA,

e-mail: tehranih@unlv.nevada.edu

Open Access. © 2019 Walter de Gruyter GmbH, Berlin/Boston.

This work is licensed under the Creative Commons Attribution 4.0 Public License.

Brought to you by | University of Nevada, Las Vegas

Authenticated

Download Date | 3/25/19 5:38 PM

eigenvalue of $-\Delta + \phi$ over the region O with Dirichlet boundary condition. We omit the potential ϕ and write $\lambda_i(O)$ if $\phi = 0$. Furthermore, for a solution u of equation (1.1), we let $\mu_i(u)$ denote the i -th eigenvalue of the linearization of (1.1) at u , that is, $\mu_i(u) = \lambda_i(-a + 2b(x)u, \Omega)$. Following the classical terminology, u will be called stable if $\mu_1(u) > 0$, and unstable if $\mu_1(u) < 0$. We also recall the well-known fact that

$$\lambda_i(\phi_1, O) < \lambda_i(\phi_2, O) \quad \text{if } \phi_1 \geq \phi_2 \text{ and } \phi_1 \neq \phi_2.$$

For the case $c = 0$, that is, in the absence of harvesting, the equation

$$-\Delta u = au - b(x)u^2 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad (1.2)$$

has been investigated by a number of authors (cf, [1, 2, 6, 7, 10, 16]) and a complete picture of the structure of the positive solution set is available. Indeed, we have the following theorem.

Theorem 1.1. Assume $b(x) \neq 0$ in Ω .

- (i) If $\Omega_0 = \emptyset$, then for every $a > \lambda_1(\Omega)$, equation (1.2) has a unique positive solution u_a .
- (ii) If $\Omega_0 \neq \emptyset$, then for any $a \in (\lambda_1(\Omega), \lambda_1(\Omega_0))$, equation (1.2) has a unique positive solution u_a . In addition, if $a \geq \lambda_1(\Omega_0)$, then (1.2) has no nonnegative solution except zero.

Furthermore, in either case the curve $a \rightarrow u_a$ is continuous and increasing and the positive solution u_a is stable, i.e., $\mu_1(u_a) > 0$.

This result indicates that equation (1.2) behaves similar to the logistic model for $a \in (\lambda_1(\Omega), \lambda_1(\Omega_0))$, but a dramatic change occurs as the linear growth rate a crosses the threshold value $\lambda_1(\Omega_0)$. In fact, as a approaches the critical value $\lambda_1(\Omega_0)$, the degeneracy of the crowding coefficient $b(x)$ in Ω_0 causes the solution u_a to blow up in Ω_0 . More precisely, we have the following result.

Theorem 1.2 ([8, Theorem 3.6]). Let $a_0 = \lambda_1(\Omega_0)$. Then the following hold:

- (i) $u_a \rightarrow +\infty$ uniformly on $\overline{\Omega}_0$ as $a \nearrow a_0$.
- (ii) $u_a \rightarrow \underline{u}_{a_0}$ uniformly as $a \nearrow a_0$ on any compact subset of $\overline{\Omega} \setminus \overline{\Omega}_0$, where $\underline{u}_{a_0}$ is the minimal positive solution of the boundary blow-up equation

$$-\Delta u = au - b(x)u^2 \quad \text{in } \Omega \setminus \overline{\Omega}_0, \quad u = 0 \quad \text{on } \partial\Omega, \quad u = +\infty \quad \text{on } \partial\Omega_0.$$

Following the terminology of [9], we call $a < \lambda_1(\Omega_0)$ and $a \geq \lambda_1(\Omega_0)$, the weak and strong growth rate case, respectively. As for the existence of positive solutions in the presence of harvesting, Oruganti et al. in [15] considered the case of $b(x) = b > 0$. Their results were then extended to the degenerate logistic case considered here in the weak growth rate regime in [18]. The following two theorems summarize the main results in this case.

Theorem 1.3 ([18, Theorem 2.6]). Suppose that $\lambda_1(\Omega) < a < \lambda_1(\Omega_0)$, $b(x) \neq 0$ and $h(x) \neq 0$. Then there exists $\hat{c}_a > 0$ such that the following hold:

- (i) If $0 \leq c < \hat{c}_a$, then equation (1.1) has a maximal positive solution $\bar{u}_{a,c}$. If $c > \hat{c}_a$, then no solution of (1.1) stays positive in Ω .
- (ii) The curve $c \rightarrow \bar{u}_{a,c}$ is decreasing with respect to the parameter c for $c \in [0, \hat{c}_a)$ and $\bar{u}_{a,c}$ is stable, that is, $\mu_1(\bar{u}_{a,c}) > 0$. Furthermore, $\bar{u}_{a,c}$ is the unique positive stable solution of (1.1).

Theorem 1.4 ([18, Theorem 2.8]). Under the assumptions of Theorem 1.3, there exists $\epsilon > 0$ such that for $a \in (\lambda_1(\Omega), \lambda_1(\Omega) + \epsilon)$, the following hold:

- (i) Equation (1.1) has exactly two positive solutions $\bar{u}_{a,c}$ and $\underline{u}_{a,c}$ for $c \in (0, \hat{c}_a)$, exactly one positive solution $\hat{u}_a$ with $c = \hat{c}_a$, and no positive solution for $c > \hat{c}_a$.
- (ii) The Morse index $M(u)$ is 1 for $u = \underline{u}_{a,c}$ when $c \in [0, \hat{c}_a)$, and $\hat{u}_a$ is degenerate with $\mu_1(\hat{u}_a) = 0$.
- (iii) All solutions lie on a smooth curve Σ that, on (c, u) space, starts from $(0, 0)$, continues to the right, reaches the unique turning point at $c = \hat{c}_a$ where it turns back, then continues to the left without any turnings until it reaches $(0, u_a)$, where u_a is the unique positive solution of (1.1) with $c = 0$.

At this point it is worth making a few comments. Firstly, we note that under the weak growth rate assumption, any positive solution $u_{a,c}$ of equation (1.1) is a sub-solution of (1.2), and therefore, by a classical comparison

result, is point wise bounded by the unique solution u_a of equation (1.2), that is, $u_{a,c}(x) \leq u_a(x)$ for $x \in \Omega$. This observation plays a crucial role in the proof of both existence and nonexistence results of Theorem 1.3 above. In particular, in the light of this uniform (with respect to c) point wise bound, the nonexistence result for positive solutions and c large, that is, the fact that equation (1.1) does not have a positive solution as c crosses the critical value $\hat{c}_a$ is rather obvious. In fact, as positive solutions remain uniformly bounded, heuristically one does not expect survival of the species (i.e., existence of a positive density distribution u) as the harvesting rate c approaches infinity.

However, we note that if $a \geq \lambda_1(\Omega_0)$, then such a uniform point wise bound is not available. In fact, as mentioned before, as a increases toward $\lambda_1(\Omega_0)$, the solution u_a blows up in $\overline{\Omega}_0$ impeding existence of a positive solution for the pure logistic equation (1.2) for $a \geq \lambda_1(\Omega_0)$. On the other hand, it seems reasonable to inquire whether one may be able to offset the absence of crowding effect in Ω_0 through the presence of a strong harvesting term, and therefore prove the existence of a positive solution in this case. To the best of our knowledge, the question of existence of positive solutions to equation (1.1) in the strong growth rate regime has not been considered before, and the above observations were our initial motivation for taking up this study here.

In this work we provide some results in this direction. Since a simple application of the implicit function theorem (see Proposition 2.8) provides the necessary and sufficient condition for the existence of positive solutions for c small, our concentration here is on the question of existence of positive solutions for c large. In particular, we show that under the strong growth rate assumption, $h(x) \equiv 0$ in $\Omega \setminus \overline{\Omega}_0$ is a necessary condition for existence of positive solutions as $c \nearrow \infty$. Furthermore, under the same condition, we will establish an existence (and somewhat surprisingly) uniqueness result for positive solutions of (1.1) for all c large in a right neighborhood of the threshold growth rate $a = \lambda_1(\Omega_0)$ (see Theorem 3.11).

Our approach is based on variational and topological arguments and makes extensive use of classical elliptic estimates and comparison principles. In Section 2 we provide some background and preliminary results and consider the basic setup of our problem. In Section 3 we consider the case $h(x) \equiv 0$ in $\Omega \setminus \overline{\Omega}_0$ and then finish the proof of our main existence and uniqueness result.

2 Preliminaries

We start with the consideration of equation (1.1), by stating our main hypotheses. Here Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 3$. The constant c is nonnegative and throughout we assume the following:

(A1) $b(x)$ and $h(x)$ ($\neq 0$) are nonnegative $C^\alpha(\overline{\Omega})$ functions.

(A2) There exists a smooth region $\Omega_0 \neq \emptyset$ such that $\overline{\Omega}_0 \subset \Omega$ and $b(x) \equiv 0$ for $x \in \overline{\Omega}_0$, and $b(x) > 0$ on $\Omega \setminus \overline{\Omega}_0$.

In this section we first gather some useful background material and then present some preliminary results. Throughout this paper, we will repeatedly use the saddle-node bifurcation result of Crandall and Rabinowitz [4], which we recall below.

Theorem 2.1 (Saddle-node bifurcation at a turning point [4]). *Let X and Y be Banach spaces and assume that $(\lambda_0, u_0) \in \mathbb{R} \times X$. Let F be a continuously differentiable mapping of an open neighborhood V of (λ_0, u_0) into Y . Suppose that the following hold:*

(i) $\dim N(F_u(\lambda_0, u_0)) = \text{codim } R(F_u(\lambda_0, u_0)) = 1$, $N(F_u(\lambda_0, u_0)) = \text{span}\{w_0\}$,

(ii) $F_\lambda(\lambda_0, u_0) \notin R(F_u(\lambda_0, u_0))$.

If Z is a complement of $\text{span}\{w_0\}$ in X , then the solutions of $F(\lambda, u) = F(\lambda_0, u_0)$ near (λ_0, u_0) form a curve $(\lambda(s), u(s)) = (\lambda_0 + \tau(s), u_0 + s w_0 + z(s))$, where $s \rightarrow (\tau(s), z(s)) \in \mathbb{R} \times Z$ is a continuously differentiable function near $s = 0$ and $\tau(0) = \tau'(0) = 0$, $z(0) = z'(0) = 0$. Moreover, if F is k times continuously differentiable, then so are $\tau(s)$ and $z(s)$.

Furthermore, by [19, Theorem 2.4], we have

$$\tau''(0) = -\frac{\langle l, F_{uu}(\lambda_0, u_0)[w_0, w_0] \rangle}{\langle l, F_\lambda(\lambda_0, u_0) \rangle},$$

where $l \in Y^*$ satisfies $N(l) = R(F_u(\lambda_0, u_0))$.

The following two results provide additional useful information for the pure logistic equation set in a bounded domain $\Omega \subset \mathbb{R}^N$ with $N \geq 3$:

$$-\Delta u = au - b(x)u^2, \quad x \in \Omega. \quad (2.1)$$

Lemma 2.2 ([8]). *Suppose that $u > 0$ and $v > 0$ are, respectively, $C^2(\Omega)$ sub and super solutions of (2.1), that is,*

$$\Delta v + av - b(x)v^2 \leq 0 \leq \Delta u + au - b(x)u^2 \quad \text{in } \Omega.$$

Furthermore, assume that

$$\limsup_{x \rightarrow \partial\Omega} (u - v) \leq 0.$$

Then $u \leq v$ in Ω .

Next we recall a result of Du and Huang (see [8]) on the existence of boundary blow-up solutions for

$$-\Delta u = au - b(x)u^2, \quad \text{in } \Omega \setminus \overline{\Omega}_0, \quad u = 0 \quad \text{on } \partial\Omega, \quad u = +\infty \quad \text{on } \partial\Omega_0. \quad (2.2)$$

Theorem 2.3 ([8, Theorem 2.4]). *For any $a \in (-\infty, +\infty)$, (2.2) has a minimal positive solution $\underline{U}_a$ and a maximal positive solution $\overline{U}_a$, in the sense that any positive solution u of (2.2) satisfies $\underline{U}_a(x) \leq u(x) \leq \overline{U}_a(x)$.*

We are finally ready to state our first result. In the following section we will have occasions where knowing the structure of the set of all (not just positive) solutions of equation (1.1) in the weak growth rate case is of great value. The following provides a picture, similar to one obtained in Theorem 1.3 above, for the set of all solutions.

Theorem 2.4. *Suppose that $\lambda_1(\Omega) < a < \lambda_1(\Omega_0)$ and (A1) and (A2) hold. Then there exists $\bar{c}_a > 0$ such that the following hold:*

- (i) *If $0 \leq c < \bar{c}_a$, then equation (1.1) has a maximal solution $\bar{u}_{a,c}$, and if $c > \bar{c}_a$, then (1.1) has no solution.*
- (ii) *The curve $c \rightarrow \bar{u}_{a,c}$ is decreasing with respect to the parameter c for $c \in [0, \bar{c}_a]$ and $\bar{u}_{a,c}$ is stable, that is, $\mu_1(\bar{u}_{a,c}) > 0$. Furthermore, $\bar{u}_{a,c}$ is the unique stable solution of (1.1).*
- (iii) *For $c = \bar{c}_a$, there exists a degenerate solution $\bar{u}_a$, i.e., $\mu_1(\bar{u}_a) = 0$, and equation (1.1) has another unstable solution for c near and to the left of $\bar{c}_a$.*

Proof. As arguments similar to the ones used in the proof of this result will be needed repeatedly later on, we will provide all the details here so as to be able to refer to them later on. First note that if equation (1.1) has a solution (c_1, u_1) , then either $u_1 \leq 0$ or the comparison lemma above applied on the set $\{x : u_1(x) > 0\}$ implies that $u_1^+ \leq u_a$, where u_a is the unique positive solution of equation (1.2). Multiplying equation (1.1) with φ_1 , the first eigenfunction of $-\Delta$ in Ω with Dirichlet boundary condition, and then integrating over Ω , we obtain

$$c_1 \leq \frac{(a - \lambda_1(\Omega)) \int_{\Omega} u_a \varphi_1}{\int_{\Omega} h(x) \varphi_1},$$

implying that $\bar{c}_a$ is well defined. Next for $p > n$ let $X = \{u \in W^{2,p}(\Omega) : u = 0 \text{ on } \partial\Omega\}$ and $Y = L^p(\Omega)$. We define $F: \mathbb{R} \times X \rightarrow Y$ by $F(c, u) = \Delta u + au - b(x)u^2 - ch(x)$. Note that if $F(c_1, u_1) = 0$, that is, if u_1 is a solution of (1.1) with harvesting rate $c = c_1$, and if $\mu_1(u_1) > 0$, then by applying the implicit function theorem, we can continue the curve of solutions forward (i.e., for $c \geq c_1$) with respect to c . Moreover, denoting the curve of solutions by $(c, u) = (c, u(c))$, it will be decreasing as $v := \frac{\partial u}{\partial c}(c_1)$ solves the equation

$$-\Delta v + (-a + 2b(x)u_1)v = -h(x).$$

Hence, $\mu_1(u(c)) = \lambda_1(-a + 2b(x)u(c))$ is decreasing with respect to c as well. Therefore, the curve of solutions starting at $(0, u_a)$, can be continued to the right until a point (c_0, u_0) with $\mu_1(u_0) = 0$. Next applying the saddle-node bifurcation (Theorem 2.1), one easily sees that the curve turns back at the degenerate solution u_0 , therefore generating a second unstable solution in a left neighborhood of c_0 . Furthermore, the curve of solutions obtained above is indeed the curve of maximal solutions. To see this, first note that if equation (1.1)

has a solution at (c_1, u_1) , then (1.1) will have a maximal solution $(c_1, \bar{u}_1)$ with $\mu_1(\bar{u}_1) \geq 0$. Indeed, u_1, u_a is a pair of sub-super solutions of (2.1), therefore the comparison lemma above applied on the set $\{x : u_1(x) > 0\}$ implies that $u_1^+ \leq u_a$. Next, considering u_1, u_a as an ordered pair of sub-super solutions of (1.1) for $c = c_1$, the standard iteration process starting at the super solution u_a will provide the maximal solution $\bar{u}_1 = \bar{u}_1(c_1)$ of (1.1) for $c = c_1$, which, by construction, will have $\mu_1(\bar{u}_1) \geq 0$ (see [17]). Also, as $\bar{u}_1(c_1)$ is a sub-solution of (1.1) with $c = c_2$, if $c_1 > c_2$, it is clear that $\bar{u}(c)$ is decreasing in c and therefore left continuous. In addition, for $0 \leq c < c_0$, since $u(c) \leq \bar{u}(c)$, we have $\mu_1(\bar{u}(c)) \geq \mu_1(u(c)) > 0$. This implies that the implicit function theorem applies at every point $(c, \bar{u}(c))$, from which one easily concludes that $\bar{u}(c)$ is right continuous as well. Hence, the curve of maximal solutions $c \rightarrow \bar{u}(c)$ is continuous and decreasing for $0 \leq c < c_0$. Finally, as u_a is the unique positive (and therefore maximal) solution of equation (1.1) for $c = 0$, we have $u(0) = \bar{u}(0) = u_a$, which together with the fact that both curves $(c, u(c))$ and $(c, \bar{u}(c))$ are continuous, decreasing and constitute of stable solutions yield $u(c) = \bar{u}(c)$ for $0 \leq c < c_0$.

Next we will show that equation (1.1) does not have a solution for $c > c_0$. Otherwise, if u_1 is a solution of (1.1) for some $c_1 > c_0$, then as was shown above, (1.1) has a maximal solution $\bar{u}_1$ at c_1 with $\mu_1(\bar{u}_1) \geq 0$. Now depending on whether $\mu_1(\bar{u}_1) > 0$ or $\mu_1(\bar{u}_1) = 0$, we can continue the curve of solutions from $(c_1, \bar{u}_1)$ backwards (i.e., for $c < c_1$) in an increasing fashion, initially using either the implicit function theorem or the saddle-node bifurcation theorem, and then (as $\mu_1(u(c))$ becomes positive) through the implicit function theorem all the way to $c = c_0$. Thus, equation (1.1) will have a solution u at $c = c_0$ with $\mu_1(u) > 0$. Since $u \leq \bar{u}(c_0) = u(c_0)$, we get $\mu_1(u(c_0)) > 0$, which is a contradiction. Finally, a similar argument yields uniqueness of stable solutions. The proof of the theorem is now complete. $\square$

Remark 2.5. It is obvious that $\hat{c}_a \leq \bar{c}_a$. In the remarks on page 3614 of [15], Oruganti, Shi and Shivaji claim, in passing, that $\bar{c}_a = \hat{c}_a$. Although the validity of this claim for arbitrary $\lambda_1(\Omega) < a < \lambda_1(\Omega_0)$ is not clear to us, we are able to show that if $h \equiv 0$ in $\bar{\Omega} \setminus \Omega_0$, then $\hat{c}_a = \bar{c}_a$ for a sufficiently close to $\lambda_1(\Omega_0)$ (see Lemma 3.5).

The next result, covering the strong growth rate case, is crucial in the proof of our main result in Section 3.

Theorem 2.6. Assume $a \geq \lambda_1(\Omega_0)$. Then any solution u (not necessary positive) of (1.1) is unstable, that is, $\mu_1(u) < 0$.

Proof. Let u_0 be a solution of equation (1.1) with $c = c_0 > 0$. Let $\varphi > 0$ denote the first eigenfunction of the linearization of (1.1) at u_0 , that is,

$$-\Delta\varphi = a\varphi - 2b(x)u_0\varphi + \mu_1(u_0)\varphi.$$

Multiplying the above equation by φ_1^* , the first eigenfunction of $-\Delta$ in Ω_0 with Dirichlet boundary condition, and integrating over Ω_0 , we obtain

$$\int_{\partial\Omega_0} \varphi \partial_\nu \varphi_1^* + (\lambda_1(\Omega_0) - a) \int_{\Omega_0} \varphi \varphi_1^* = \mu_1(u_0) \int_{\Omega_0} \varphi \varphi_1^*,$$

which readily yields $\mu_1(u_0) < 0$. $\square$

Remark 2.7. Recalling the classical result (see [17]) that a solution u obtained between an ordered pair of sub and super solutions through the classical iteration procedure (starting at the sub or the super solution) will automatically satisfy $\mu_1(u) \geq 0$, the above result, in particular, indicates that in our search for positive solutions of (1.1) in the strong growth rate case, we cannot utilize the technique of sub and super solutions.

We end this section by providing a necessary and sufficient condition for existence of positive solutions of (1.1) for c small. In fact, since for $a \geq \lambda_1(\Omega_0)$, $u = 0$ is the unique nonnegative solution of (1.1) with $c = 0$, a simple application of implicit function theorem yields the following result.

Proposition 2.8. Assume (A1) and (A2), and suppose that $a > \lambda_1(\Omega_0)$ and $a \neq \lambda_i(\Omega)$ for all $i > 1$. Then equation (1.1) has a positive solution for $0 < c < \sigma$ (with some $\sigma > 0$) if and only if the linear equation

$$-\Delta v = av - h(x) \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega \quad (2.3)$$

has a positive solution.

Remark 2.9. In regards to the existence of a positive solution to (2.3), we may write $h(x) = h_1\varphi_1 + \tilde{h}(x)$, where φ_1 is the first eigenfunction of $-\Delta$ in Ω with Dirichlet boundary condition and $\int_{\Omega} \tilde{h}(x)\varphi_1 = 0$. Then clearly v , the unique solution of equation (2.3), is given by

$$v = \frac{h_1}{a - \lambda_1(\Omega)}\varphi_1 + (\Delta + a)^{-1}(\tilde{h}(x)) > 0.$$

Therefore, there exists $h_1^* > 0$, depending on a and $\tilde{h}$, such that if $h_1 > h_1^*$, then $v > 0$.

3 Existence and nonexistence results in the strong growth rate case

In this section we will consider the existence of positive solutions of equation (1.1) for $a \geq \lambda_1(\Omega_0)$ and c large. Our first result provides a necessary condition for existence of positive solutions for c large.

Theorem 3.1. Assume that $a \geq \lambda_1(\Omega)$ and $h \not\equiv 0$ in $\Omega \setminus \overline{\Omega}_0$. Then there exists $0 < \bar{c}_a < +\infty$ such that equation (1.1) does not have a nonnegative solution for $c > \bar{c}_a$.

Proof. Let u be a nonnegative solution of (1.1). Using the comparison Lemma 2.2, we have $u \leq \underline{U}_a$ in $\Omega \setminus \overline{\Omega}_0$, where $\underline{U}_a$ is the minimal boundary blow-up solution given in Theorem 2.3. Let $\Omega_n = \{x \in \Omega : d(x, \Omega_0) > \frac{1}{n}\}$. Clearly, $h \not\equiv 0$ in Ω_n for all n large. Fix such an n and let $\varphi_{n,1} > 0$ be the first eigenfunction of $-\Delta$ in Ω_n with Dirichlet boundary condition. Multiplying equation (1.1) with $\varphi_{n,1}$ and then integrating over Ω_n , we obtain

$$\int_{\Omega_n} -\Delta u \varphi_{n,1} = a \int_{\Omega_n} u \varphi_{n,1} - \int_{\Omega_n} b(x) u^2 \varphi_{n,1} - c \int_{\Omega_n} h(x) \varphi_{n,1},$$

which implies

$$c \int_{\Omega_n} h(x) \varphi_{n,1} \leq (a - \lambda_1(\Omega_n)) \int_{\Omega_n} u \varphi_{n,1} - \int_{\partial\Omega_n} u \partial_\nu \varphi_{n,1},$$

where ν is the outer normal vector of $\partial\Omega_n$. Therefore, we have

$$c \int_{\Omega_n} h(x) \varphi_{n,1} \leq (a - \lambda_1(\Omega_n)) \int_{\Omega_n} \underline{U}_a \varphi_{n,1} - \int_{\partial\Omega_n} \underline{U}_a \partial_\nu \varphi_{n,1}$$

if $a > \lambda_1(\Omega_n)$, and

$$c \int_{\Omega_n} h(x) \varphi_{n,1} \leq - \int_{\partial\Omega_n} \underline{U}_a \partial_\nu \varphi_{n,1}$$

if $a \leq \lambda_1(\Omega_n)$. Therefore, c has a bound independent of u . $\square$

In the rest of this section we will assume that $h \equiv 0$ in $\overline{\Omega} \setminus \Omega_0$. In order to prove the existence of positive solutions of (1.1) for $a \geq \lambda_1(\Omega_0)$ and c large, we first show that (1.1) has an unstable solution for large c as a approaches $\lambda_1(\Omega_0)$ from below. Using this and the fact that unstable solutions can not blow up as $a \nearrow \lambda_1(\Omega_0)$, we prove the existence of a positive solution for $a = \lambda_1(\Omega_0)$ by considering the limit of these unstable solutions. Next, using the degree theory on cones we are able to establish the same result for a in a right neighborhood of $\lambda_1(\Omega_0)$. We start this procedure by first considering the behavior of $\hat{c}_a$ as $a \nearrow \lambda_1(\Omega_0)$.

Theorem 3.2. Assume that $h \equiv 0$ in $\Omega \setminus \overline{\Omega}_0$. Then $\hat{c}_a \rightarrow +\infty$ as $a \nearrow \lambda_1(\Omega_0)$.

Proof. Fix $c > 0$ arbitrary large. We will show that (1.1) has a positive solution for a sufficiently close $\lambda_1(\Omega_0)$ by constructing an ordered pair of sub and super solutions. Clearly, u_a is a super solution of (1.1). To construct a sub-solution, we consider

$$\underline{u} = \begin{cases} w_n(x), & x \in \Omega \setminus \overline{\Omega}_0, \\ n + m\varphi_1^*(x), & x \in \overline{\Omega}_0, \end{cases}$$

where m and n are large positive constants (given below), $\varphi_1^*(x)$ denotes the first eigenfunction of $-\Delta$ in Ω_0 with Dirichlet boundary condition, and w_n is the positive solution of the equation

$$-\Delta w = aw - b(x)w^2 \quad \text{in } \Omega \setminus \overline{\Omega}_0, \quad w = 0 \quad \text{on } \partial\Omega, \quad w = n \quad \text{on } \partial\Omega_0,$$

whose existence is shown in [8, Lemma 2.3]. We claim that $\underline{u}$ is a weak sub-solution of (1.1). First note that $\underline{u}$ is continuous. Next let $0 \leq \phi \in C_c^\infty(\Omega)$. We have

$$\begin{aligned} \int_{\Omega} \nabla \underline{u} \nabla \phi &= \int_{\Omega \setminus \overline{\Omega}_0} \nabla w_n \nabla \phi + m \int_{\Omega_0} \nabla \varphi_1^* \nabla \phi \\ &= \int_{\Omega \setminus \overline{\Omega}_0} (-\Delta w_n) \phi + m \int_{\Omega_0} (-\Delta \varphi_1^*) \phi + \int_{\partial\Omega_0} (m \partial_\nu \varphi_1^* - \partial_\nu w_n) \phi \\ &= a \int_{\Omega_0 \setminus \overline{\Omega}_0} (w_n - b(x)w_n^2) \phi + \lambda_1(\Omega_0)m \int_{\Omega_0} \varphi_1^* \phi + \int_{\partial\Omega_0} (m \partial_\nu \varphi_1^* - \partial_\nu w_n) \phi. \end{aligned}$$

Therefore,

$$\int_{\Omega} \nabla \underline{u} \nabla \phi - \int_{\Omega} (a\underline{u} + b(x)\underline{u}^2 - ch(x))\phi \leq \int_{\Omega_0} (m(\lambda_1(\Omega_0) - a)\|\varphi_1^*\|_\infty + c\|h(x)\|_\infty - an)\phi + \int_{\partial\Omega_0} (m \partial_\nu \varphi_1^* - \partial_\nu w_n) \phi.$$

Now we assume that $a \in [\lambda_1(\Omega_0) - \epsilon_0, \lambda_1(\Omega_0))$ for some ϵ_0 small. First we take n large enough such that $(\lambda_1(\Omega_0) - \epsilon_0)n - c\|h\|_\infty \geq 1$. Next, using the fact that $\max_{\partial\Omega_0} \partial_\nu \varphi_1^* < 0$, we choose m sufficiently large so that $m \partial_\nu \varphi_1^* - \partial_\nu w_n < 0$ on $\partial\Omega_0$. Finally, we choose a close enough to $\lambda_1(\Omega_0)$ such that $u_a > n + m\varphi_1^*$ in $\overline{\Omega}_0$ and $m(\lambda_1(\Omega_0) - a)\|\varphi_1^*\|_\infty < 1$. Then $\underline{u}$ is a weak sub-solution and, in addition, $\underline{u} \leq u_a$ (note that $h \equiv 0$ on $\Omega \setminus \overline{\Omega}_0$). So, equation (1.1) has a positive solution between $\underline{u}$ and u_a . This completes the proof. $\square$

Before stating our next result, we recall that by the anti-maximum principle (see [3]), there exists $\delta_h > 0$ such that for $\lambda_1(\Omega_0) < a < \lambda_1(\Omega_0) + \delta_h$, the following equation has a unique positive solution ψ_a :

$$-\Delta \psi = a\psi - h(x), \quad \text{in } \Omega_0, \quad \psi = 0 \quad \text{on } \partial\Omega_0. \quad (3.1)$$

Lemma 3.3. *There exists $0 < \delta < \delta_h$ such that if u_n is a sequence of nonnegative solutions of equation (1.1), with $a = a_n$, $c = c_n$ such that $\lambda_1(\Omega) < a_n < \lambda_1(\Omega_0) + \delta$, $c_n \geq 0$ and $\|u_n\|_\infty \rightarrow +\infty$. Then $u_n \rightarrow +\infty$ uniformly on $\overline{\Omega}_0$.*

We note that this lemma is in fact valid for arbitrary nonnegative $h(x)$.

Proof. Up to a subsequence, $a_n \rightarrow a^*$ with $\lambda_1(\Omega_0) \leq a^* \leq \lambda_1(\Omega_0) + \delta$. We divide the proof into several steps.

Step 1: $u_n \rightarrow +\infty$ uniformly in every compact subsets of Ω_0 . Note that $c_n \leq \|u_n\|_\infty$. Thus, up to a subsequence, $c_n/\|u_n\|_\infty \rightarrow \alpha < +\infty$. Let $v_n = u_n/\|u_n\|_\infty$. From equation (1.1) we have

$$-\Delta v_n = a_n v_n - b(x)\|u_n\|_\infty v_n^2 - \frac{c_n}{\|u_n\|_\infty} h(x). \quad (3.2)$$

Since $-\Delta v_n \leq a_n v_n$ and $\|v_n\|_\infty = 1$, there exists $v_0 \in H_0^1(\Omega)$ such that $v_n \rightarrow v_0$ weakly in $H_0^1(\Omega)$ and strongly in $L^p(\Omega)$ for $p > 1$. Multiplying (3.2) by $v_n/\|u_n\|_\infty$ and then integrating over Ω , we get

$$\frac{1}{\|u_n\|_\infty} \int_{\Omega} |\nabla v_n|^2 = \frac{a_n}{\|u_n\|_\infty} \int_{\Omega} v_n^2 - \int_{\Omega} b(x)v_n^3 - \frac{c_n}{\|u_n\|_\infty} \int_{\Omega} h(x)v_n.$$

Thus, $\int_{\Omega} b(x)v_0^3 = 0$, and so $v_0 \equiv 0$ in $\overline{\Omega} \setminus \Omega_0$.

Next, multiplying (3.2) by $\phi \in C_c^\infty(\Omega_0)$ and integrating over Ω_0 , we have

$$\int_{\Omega_0} \nabla v_n \nabla \phi = a_n \int_{\Omega_0} v_n \phi - \frac{c_n}{\|u_n\|_\infty} \int_{\Omega_0} h(x)\phi.$$

Passing n to infinity, we obtain

$$\int_{\Omega_0} \nabla v_0 \nabla \phi = a^* \int_{\Omega_0} v_0 \phi - \alpha \int_{\Omega_0} h(x) \phi.$$

Therefore, v_0 is a nonnegative weak solution of the equation

$$-\Delta v_0 = a^* v_0 - \alpha h(x), \quad \text{in } \Omega_0, \quad v_0 = 0 \quad \text{on } \partial\Omega_0.$$

If $a^* = \lambda_1(\Omega_0)$, then, thanks to the Fredholm alternative, we have $\alpha = 0$, since $h(x) \geq 0$ and $h(x) \not\equiv 0$. Therefore, $v_0 = 0$ or $v_0 = \varphi_1^* > 0$ in Ω_0 (recall that φ_1^* is the first eigenfunction of $-\Delta$ with Dirichlet boundary condition in Ω_0).

We claim that $v_0 \neq 0$. In fact, if $v_0 = 0$, then, by equation (3.2), $-\Delta v_n \leq a_n v_n$, so that

$$0 \leq v_n \leq (a_n + 1)(-\Delta + 1)^{-1} v_n \rightarrow 0$$

uniformly in Ω , as $v_n \rightarrow 0$ in $L^p(\Omega)$ for $p > 1$. This contradicts $\|v_n\|_\infty = 1$.

Next if $\lambda_1(\Omega_0) < a^* \leq \lambda_1(\Omega_0) + \delta$, then assuming that $\alpha = 0$, we have $v_0 = 0$, as $a^* \neq \lambda_1(\Omega_0)$. But this will again yield a contradiction as above. Thus, $\alpha > 0$ and $v_0 = \alpha \psi_{a^*} > 0$, the unique positive solution of equation (3.1).

Now standard elliptic estimates (see [11]) imply that $u_n \rightarrow +\infty$ uniformly in every compact subset of Ω_0 , completing the proof of step 1.

Next we let $u_n(z_n) = \min_{x \in \overline{\Omega_0}} u_n(x)$ and show that $u_n(z_n) \rightarrow \infty$. This will be done in the next two steps. First note that since by assumption $\partial\Omega_0$ is $C^{2,\gamma}$, it satisfies a uniform interior ball assumption, i.e., there exists $R > 0$ such that for every $x \in \partial\Omega_0$, there exists a ball $B_x = B_R(y)$ with radius R and center y such that $B_x \subset \overline{\Omega_0}$ and $B_x \cap \partial\Omega_0 = \{x\}$. Let K be a compact subset of Ω_0 such that $B_{R/2}(y) \subset K$ for all $x \in \partial\Omega_0$.

Step 2: Let $u_n(z_n) = \min_{x \in \overline{\Omega_0}} u_n(x)$. If $\{u_n(z_n)\}$ is bounded, then $z_n \in \partial\Omega_0$ for all n large. Using some of the ideas of the proof of [8, Lemma 3.3], we argue by contradiction by assuming that for a subsequence, still denoted by (z_n) , we have $z_n \in \Omega_0$. Then step 1 implies $z_n \rightarrow \partial\Omega_0$. Hence, there exists $x_n \in \partial\Omega_0$ such that $z_n \in B_{x_n} \setminus B_{R/2}(y_n)$. Next we consider the auxiliary functions

$$\eta_n(x) = e^{-\sigma|x-y_n|^2} - e^{-\sigma R^2},$$

and show that there exists a sequence $\beta_n \rightarrow \infty$ such that

$$u_n(x) \geq u_n(z_n) + \beta_n \eta_n(x) \quad \text{in } B_{x_n} \setminus B_{R/2}(y_n). \quad (3.3)$$

To start with, we fix a suitably chosen large $\sigma > 0$, so that η_n satisfy the following properties on B_{x_n} :

$$\begin{aligned} \eta_n(x) &= 0, & x &\in \partial B_{x_n}, \\ \eta_n(x) &> 0, & x &\in B_{x_n} \setminus B_{R/2}(y_n), \end{aligned} \quad (3.4)$$

$$\begin{aligned} \eta_n(x) &= e^{-\sigma R^2/4} - e^{-\sigma R^2} < e^{-\sigma R^2/4}, & x &\in \partial B_{\frac{R}{2}}(y_n), \\ \partial_{v_n} \eta_n(x_n) &= 2\sigma R^2 e^{-\sigma R^2} > 0, & \text{where } v_n &= \frac{y_n - x_n}{|y_n - x_n|}, \end{aligned} \quad (3.5)$$

and

$$\begin{aligned} \Delta \eta_n + a_n \eta_n - e^{-\sigma R^2} h(x) &= (4\sigma^2 |x - y_n|^2 - 2\sigma N + a_n) e^{-\sigma |x - y_n|^2} - (a_n + h(x)) e^{-\sigma R^2} \\ &> (4\sigma^2 |x - y_n|^2 - 2\sigma N - h(x)) e^{-\sigma R^2} > 0, & x &\in B_{x_n} \setminus B_{R/2}(y_n). \end{aligned}$$

Note that the choice of σ depends only on R and the function $h(x)$. Let $\alpha_n = \min_K u_n(x)$. The rest of the proof proceeds by considering the two cases $a^* = \lambda_1(\Omega_0)$ and $a^* > \lambda_1(\Omega_0)$ separately.

If $a^* = \lambda_1(\Omega_0)$, since $u_n/\|u_n\|_\infty \rightarrow \varphi_1^* > 0$ in Ω_0 and $c_n/\|u_n\|_\infty \rightarrow 0$, then $\alpha_n/c_n \rightarrow +\infty$. Hence, we can choose $\beta_n \rightarrow \infty$ so that $c_n e^{\sigma R^2} \leq \beta_n \leq \frac{1}{2} \alpha_n e^{\sigma R^2/4}$. Therefore, for $x \in B_{x_n} \setminus B_{R/2}(y_n)$, we have

$$(-\Delta - a_n)(u_n(x) - (u_n(z_n) + \beta_n \eta_n(x))) \geq (\beta_n e^{-\sigma R^2} - c_n) h(x) \geq 0$$

and, in addition,

$$u_n(x) \geq u_n(z_n) + \beta_n \eta_n(x), \quad x \in \partial(B_{x_n} \setminus B_{R/2}(y_n)).$$

Since $a_n \rightarrow \lambda_1(\Omega_0) < \lambda_1(B_{x_n})$, the maximum principle finally yields (3.3). Now taking $x = z_n$ in (3.3), we have $\beta_n \eta_n(z_n) \leq 0$, which, since $z_n \in B_{x_n} \setminus B_{R/2}(y_n)$, contradicts (3.4).

In the second case, that is, $a^* > \lambda_1(\Omega_0)$, by step 1 we have $0 < \alpha = \lim_{n \rightarrow +\infty} c_n / \|u_n\|_\infty$ and $v_n \rightarrow v_0 = \alpha \psi_{a^*}$. Thus,

$$\lim_{n \rightarrow +\infty} \frac{\alpha_n}{c_n} = \lim_{n \rightarrow +\infty} \frac{\frac{\alpha_n}{\|u_n\|_\infty}}{\frac{c_n}{\|u_n\|_\infty}} \geq \frac{\alpha \min_{x \in K} \psi_{a^*}(x)}{\alpha} \geq \min_{x \in K} \psi_{a^*}(x).$$

From (3.1) and Remark 2.9, we have

$$\psi_{a^*} = \frac{h_1}{a^* - \lambda_1(\Omega_0)} \varphi_1^* + (\Delta + a^*)^{-1}(\tilde{h}(x)).$$

Hence, by further decreasing δ , we can guarantee that

$$\min_{x \in K} \psi_{a^*}(x) = \min_{x \in K} \left(\frac{h_1}{a^* - \lambda_1(\Omega_0)} \varphi_1^* + (\Delta + a^*)^{-1}(\tilde{h}(x)) \right) > e^{3\sigma R^2/4}.$$

Now we may proceed as in the previous case, obtaining (3.3) and a contradiction as before. The proof of step 2 is now complete.

Step 3: Let $u_n(z_n) = \min_{x \in \overline{\Omega_0}} u_n(x)$. Then $u_n(z_n) \rightarrow \infty$. To argue by contradiction, we assume that $u_n(z_n)$ is bounded, i.e., $u_n(z_n) \leq M$ for some $M > 0$ independent of n . Then, by step 2, we have $z_n \in \partial\Omega_0$. We follow the argument in step 2 where now $z_n = x_n$, and conclude

$$u_n(x) \geq u_n(x_n) + \beta_n \eta_n(x) \quad \text{in } B_{x_n} \setminus B_{R/2}(y_n) \quad (3.6)$$

for some sequence $\beta_n \rightarrow \infty$. Lemma 2.3 in [8] implies that the equation

$$-\Delta w_n = a_n w_n - b(x) w_n^2, \quad \text{in } \Omega \setminus \overline{\Omega_0}, \quad w_n = 0 \quad \text{on } \partial\Omega, \quad w_n = u_n(x_n) \quad \text{on } \partial\Omega_0,$$

has a unique positive solution and by the comparison lemma, $u_n(x) \geq w_n(x)$ in $\overline{\Omega} \setminus \Omega_0$. Similarly, w , the unique positive solution of

$$-\Delta w = a^* w - b(x) w^2, \quad \text{in } \Omega \setminus \overline{\Omega_0}, \quad w = 0 \quad \text{on } \partial\Omega, \quad w = M \quad \text{on } \partial\Omega_0,$$

satisfies $w_n(x) \leq w(x)$ in $\overline{\Omega} \setminus \Omega_0$. Thus, $\|w_n\|_{L^\infty(\Omega \setminus \overline{\Omega_0})}$ is bounded, and therefore standard elliptic estimates imply that $\{w_n\}$ is bounded in $C^1(\overline{\Omega} \setminus \Omega_0)$, and so, in particular, $|\nabla w_n(x_n)|$ remains uniformly bounded. Since $u_n(x_n) = w_n(x_n)$ and $u_n(x) \geq w_n(x)$ in $\overline{\Omega} \setminus \Omega_0$, we have

$$\partial_{v_n}(x_n) \leq \partial_{v_n} w_n(x_n) \leq M_0 \quad (3.7)$$

for some $M_0 > 0$ independent of n . On the other hand, using (3.6) and taking into account (3.5), we obtain

$$\partial_{v_n} u_n(x_n) \geq \beta_n \partial_{v_n} \eta_n(x_n) \rightarrow +\infty,$$

contradicting (3.7). This completes the proof of step 3, and therefore the proof of the lemma. $\square$

At this point for $\lambda_1(\Omega) < a < \lambda_1(\Omega_0)$, we define

$$\underline{c}_a = \sup\{c \geq 0 : \text{equation (1.1) has a nonnegative solution on } \partial P\},$$

where ∂P is the boundary of the cone

$$P = \left\{ u \in C_0^1(\overline{\Omega}) : u(x) > 0 \text{ for } x \in \Omega \text{ and } \frac{\partial u}{\partial \nu}(x) < 0 \text{ for } x \in \partial\Omega \right\}$$

and $C_0^1(\overline{\Omega})$ denotes the subspace of functions in $C^1(\overline{\Omega})$ which are zero on the boundary of Ω . Note that $u = 0$ is a nonnegative solution of (1.1) with $c = 0$, so $\underline{c}_a$ is well defined and $\underline{c}_a \leq \hat{c}_a$. In the following lemma we show that $\underline{c}_a$ does not go to infinity as $a \nearrow \lambda_1(\Omega_0)$.

Lemma 3.4. $\limsup_{a \nearrow \lambda_1(\Omega_0)} \underline{c}_a$ is bounded.

Proof. Assume on the contrary that there exists a sequence a_n such that $a_n \nearrow \lambda_1(\Omega_0)$ and $\underline{c}_{a_n} \rightarrow +\infty$. Let $u_n \in \partial P$ be a sequence of nonnegative solutions of equation (1.1) with $a = a_n$ and $\underline{c}_{a_n} - \epsilon \leq c_n \leq \underline{c}_{a_n}$ for some $\epsilon > 0$. Since $c_n \leq \|u_n\|_\infty$, by Lemma 3.3, we have $u_n \rightarrow +\infty$ uniformly in $\overline{\Omega}_0$. Thus, for large n , we have $u_n > 0$ on $\partial\Omega_0$. Now clearly we can choose $M_n > 0$ large so that

$$(-\Delta + M_n)u_n = (a_n + M_n)u_n - b(x)u_n^2 \geq 0 \quad \text{in } \Omega \setminus \overline{\Omega}_0.$$

Hence, by the maximum principle and the Hopf boundary lemma $u_n > 0$ in Ω and $\partial u/\partial \nu < 0$ on $\partial\Omega$, contradicting $u_n \in \partial P$. $\square$

Our next result shows that for $\underline{c}_a < c < \hat{c}_a$ equation (1.1) has an unstable positive solution.

Lemma 3.5. *There exists $\delta > 0$ such that if $a \in (\lambda_1(\Omega_0) - \delta, \lambda_1(\Omega_0))$ and $\underline{c}_a < c < \hat{c}_a$, then (1.1) has an unstable positive solution.*

Proof. Since $\hat{c}_a \rightarrow +\infty$ as a approaches $\lambda_1(\Omega_0)$, Lemma 3.4 implies the existence of $\delta > 0$ such that $\underline{c}_a < \hat{c}_a$ if $a \in (\lambda_1(\Omega_0) - \delta, \lambda_1(\Omega_0))$. Next, given such an a , we fix $\underline{c}_a < c_0 < \hat{c}_a$ and $M > \hat{c}_a$. For the positive constants σ , K_σ and $c \in [c_0, M]$, we define

$$T_\sigma = \left\{ u \in C_0^1(\overline{\Omega}) : \sigma\varphi_1 \leq u \leq u_a \text{ in } \Omega \text{ and } \frac{\partial u_a}{\partial \nu} \leq \frac{\partial u}{\partial \nu} \leq \sigma \frac{\partial \varphi_1}{\partial \nu} \text{ on } \partial\Omega \right\},$$

where φ_1 is the first eigenfunction of $-\Delta$ in Ω with Dirichlet boundary condition and $A_{\sigma,c} : T_\sigma \rightarrow C_0^1(\overline{\Omega})$ is such that

$$A_{\sigma,c} = (-\Delta + K_\sigma)^{-1}((a + K_\sigma)u - b(x)u^2 - ch(x)).$$

Firstly, taking into account the definition of $\underline{c}_a$, a simple limiting argument implies that for σ sufficiently small, equation (1.1) has no solution on ∂T_σ for $c \in [c_0, M]$ (where ∂T_σ denotes the relative boundary of T_σ in P). Fixing such a σ , we then take K_σ sufficiently large so that $A_{\sigma,c}$ maps $T_{\sigma,c}$ into P . Now for $c \in [c_0, M]$, we have

$$\deg(I - A_{\sigma,c}, T_\sigma, 0) = \deg(I - A_{\sigma,M}, T_\sigma, 0) = 0. \quad (3.8)$$

On the other hand, by Theorem 1.3, for any $c < \hat{c}_a$, (1.1) has a unique positive stable solution $\bar{u}_{a,c}$. Moreover, using the fixed point index calculation of Dancer in [5], it is easily seen that

$$\text{ind}(A_{\sigma,c}, \bar{u}_{a,c}) = 1. \quad (3.9)$$

Thus, (3.8) and (3.9) imply that for $c \in [c_0, \hat{c}_a)$, equation (1.1) has another solution $\underline{u}_{a,c}$ in T_σ which, by the uniqueness of stable solutions, satisfies $\mu_1(\underline{u}_{a,c}) \leq 0$. We claim that in fact $\mu_1(\underline{u}_{a,c}) < 0$. Indeed, if $\mu_1(\underline{u}_{a,c}) = 0$, then applying the saddle-node bifurcation theorem, there exist $\epsilon > 0$ and an open neighborhood $O \subset C(\overline{\Omega})$ of $\underline{u}_{a,c}$, such that the solution set of (1.1) in $I = (c - \epsilon, c + \epsilon) \times O$ is an $\supset$ -shaped curve. Moreover, the upper part of the curve consists of stable solutions. However, by the uniqueness of the curve of stable solutions, this can only happen if $c = \hat{c}_a$. This completes the proof. $\square$

Theorem 3.6. *Assume that $a = \lambda_1(\Omega_0)$. There exists $c_0 > \limsup_{a \nearrow \lambda_1(\Omega_0)} \underline{c}_a$ such that for all $c > c_0$, equation (1.1) has a positive solution.*

Proof. First fix $c > \limsup_{a \nearrow \lambda_1(\Omega_0)} \underline{c}_a$. Lemma 3.5 implies the existence of a sequence of unstable positive solutions u_n of equation (1.1) with $a = a_n < \lambda_1(\Omega_0)$ and $a_n \nearrow \lambda_1(\Omega_0)$. We claim that $\|u_n\|_\infty$ is bounded. Otherwise, $\|u_n\|_\infty \rightarrow +\infty$, and therefore by Lemma 3.3, $u_n \rightarrow +\infty$ uniformly in $\overline{\Omega}_0$. Hence, there exist n_1 and n_2 such that $u_{n_2} > u_{n_1}$ in $\overline{\Omega}_0$ and $a_{n_2} > a_{n_1}$. Now an application of the comparison lemma in $\Omega \setminus \overline{\Omega}_0$ easily implies $u_{n_2} \geq u_{n_1}$ in all of Ω (note that u_{n_2} is a super solution and u_{n_1} a solution of the logistic equation $-\Delta u = a_{n_1}u - b(x)u^2$ on $\Omega \setminus \overline{\Omega}_0$). Also, as $a_{n_1} < a_{n_2}$, we have that u_{n_1} is a sub-solution of (1.1) with $a = a_{n_2}$. Therefore, equation (1.1) for $a = a_{n_2}$ has a minimal solution u_0 between u_{n_1} and u_{n_2} with $\mu_1(u_0) \geq 0$ (see [17, Theorem 4.1]). Hence, $\mu_1(u_{n_2}) \geq \mu_1(u_0) \geq 0$, contradicting the fact that u_{n_2} is unstable.

Therefore, $\|u_n\|_\infty$ is bounded and $u_n \rightarrow u_c^*$ weakly in $H_0^1(\Omega)$ and strongly in $L^p(\Omega)$ for some $u_c^* \in H_0^1(\Omega)$. Obviously, u_c^* is a nonnegative solution of (1.1) with $a = \lambda_1(\Omega_0)$.

Therefore, for all $c > \limsup_{a \nearrow \lambda_1(\Omega_0)} c_a$, equation (1.1) has a nonnegative solution u_c^* . Since $c \leq \|u_c^*\|_\infty$, by Lemma 3.3, we get $u_c^* \rightarrow +\infty$ uniformly in $\overline{\Omega_0}$ as $c \rightarrow +\infty$. Therefore, there exists c_0 such that $u_c^* > 0$ in $\overline{\Omega_0}$ for $c > c_0$, and a further application of the maximum principle in $\Omega \setminus \overline{\Omega_0}$ (see the proof of Lemma 3.4) yields $u_c^* > 0$ in Ω . $\square$

Lemma 3.7. *Let $\lambda_1(\Omega_0) \leq a \leq \lambda_1(\Omega_0) + \delta$ with δ as in Lemma 3.3. There exists $c_1 = c_1(a)$ such that every positive solution of (1.1) with $c \geq c_1$ is nondegenerate.*

Proof. Fix an a . Assume on the contrary that u_n is a sequence of solutions of (1.1) corresponding to $c_n \rightarrow +\infty$ and $\mu_{i_n}(u_n) = 0$ for some $i_n \in \mathbb{N}$. From Lemma 3.3, we have $u_n \rightarrow +\infty$ uniformly in $\overline{\Omega_0}$. Thus, for a subsequence $u_{n_1} < u_{n_2} < u_{n_3} < \dots$ in $\overline{\Omega_0}$ and therefore (by an application of the comparison lemma, as in the proof of Theorem 3.6) in all of Ω . Since $\mu_i(u) = \lambda_i(-a + 2b(x)u)$, we have $\mu_i(u_{n_j}) > 0$ for all $j > 1$ and $i \geq i_{n_1}$. On the other hand, since $\mu_{i_{n_j}}(u_{n_j}) = 0$, we have $i_{n_j} < i_{n_1}$. Therefore, there exists a fixed $k \leq i_{n_1}$ such that $\mu_k(u_{n_j}) = 0$ for all $j \in \mathbb{N}$. This contradicts the fact that the sequence u_{n_j} is strictly increasing, and therefore so is $\mu_k(u_{n_j})$. $\square$

Lemma 3.8. *Let $\lambda_1(\Omega_0) \leq a \leq \lambda_1(\Omega_0) + \delta$ with δ as in Lemma 3.3. There exists $c_2 = c_2(a)$ such that for $c \geq c_2$, equation (1.1) has at most one positive solution.*

Proof. Fix an a . Assume on the contrary that u_n and $\bar{u}_n$ are two sequences of positive solutions of equation (1.1) corresponding to $c_n \rightarrow +\infty$. If we subtract the equations for u_n and $\bar{u}_n$, then we obtain that $\lambda_{i_n}(-a + b(x)(u_n + \bar{u}_n)) = 0$ for some $i_n \in \mathbb{N}$. By Lemma 3.3, we have $u_n + \bar{u}_n \rightarrow +\infty$ uniformly in $\overline{\Omega_0}$. Thus, there exists a subsequence $u_{n_j} + \bar{u}_{n_j}$ such that $u_{n_1} + \bar{u}_{n_1} < u_{n_2} + \bar{u}_{n_2} < u_{n_3} + \bar{u}_{n_3} < \dots$ in $\overline{\Omega_0}$, and therefore by the comparison lemma in all of Ω . Therefore, $\lambda_i(-a + b(x)(u_{n_j} + \bar{u}_{n_j})) > 0$ for all $j > 1$ and $i \geq i_{n_1}$. On the other hand, since $\lambda_{i_{n_j}}(-a + b(x)(u_{n_j} + \bar{u}_{n_j})) = 0$, we have $i_{n_j} < i_{n_1}$. We can now continue as in the proof of Lemma 3.7 and reach a contradiction as before. $\square$

The next two results prepare the ground for the application of degree theory arguments in order to prove our main existence result on positive solutions of (1.1) for all $\lambda_1(\Omega_0) \leq a \leq \lambda_1(\Omega_0) + \delta$ and c large. In what follows, we let

$$c^* = \max\{c_0, c_1(\lambda_1(\Omega_0)), c_2(\lambda_1(\Omega_0))\}.$$

Note that the above results imply that equation (1.1) for $a = \lambda_1(\Omega_0)$ and $c \geq c^*$ has a unique positive solution which, in addition, is nondegenerate.

Lemma 3.9. *Let $\delta > 0$ be defined as in Lemma 3.3 and let $d \geq c^*$ be a given constant. There exists $K > 0$, depending on δ and d , such that if u is a nonnegative solution of (1.1) with $\lambda_1(\Omega_0) \leq a \leq \lambda_1(\Omega_0) + \delta$ and $c^* \leq c \leq d$, then $\|u\|_\infty < K$.*

Proof. Assume on the contrary that there exists a sequence u_n of nonnegative solutions of (1.1) with $a = a_n$ and $c^* \leq c = c_n \leq d$, where $\lambda_1(\Omega_0) \leq a_n \leq \lambda_1(\Omega_0) + \delta$, and $\|u_n\|_\infty \rightarrow +\infty$. Now, by Lemma 3.3, we have $u_n \rightarrow +\infty$ in $\overline{\Omega_0}$. Denoting the unique positive solution of (1.1) for $a = \lambda_1(\Omega_0)$ and $c = d$ by u^* , we have for n_0 large, $u_{n_0} > u^*$ in $\overline{\Omega_0}$, and then by the comparison lemma in all of Ω (note that u_{n_0} is a super solution and u^* a solution of the logistic equation $-\Delta u = \lambda_1(\Omega_0)u - b(x)u^2$ on $\Omega \setminus \overline{\Omega_0}$). Hence, u^*, u_{n_0} is an ordered pair of sub-super solution of (1.1) with $a = \lambda_1(\Omega_0)$ and $c = d$, and therefore (1.1) has a solution u_0 (achieved with the iteration process starting at u_{n_0}) between u_c^* and u_{n_0} with $\mu_1(u_0) \geq 0$. This contradicts Theorem 2.6. $\square$

Lemma 3.10. *There exists $c_3 > 0$ such that (1.1) has no solution on ∂P , provided that $\lambda_1(\Omega_0) \leq a \leq \lambda_1(\Omega_0) + \delta$ with δ as in Lemma 3.3 and $c \geq c_3$.*

Proof. Assume on the contrary that $u_n \in \partial P$ is a sequence of solutions of (1.1) with $a = a_n$ and $c = c_n$ with $\lambda_1(\Omega_0) \leq a \leq \lambda_1(\Omega_0) + \delta$ and $c_n \rightarrow +\infty$. By Lemma 3.3, $u_n \rightarrow +\infty$ uniformly in $\overline{\Omega_0}$, and therefore, by the maximum principle (as in the proof of Lemma 3.4), $u_n > 0$ in Ω for large n , contradicting $u_n \in \partial P$. $\square$

We are now ready to state the main result of this paper.

Theorem 3.11. Let $\lambda_1(\Omega_0) \leq a \leq \lambda_1(\Omega_0) + \delta$ with δ as in Lemma 3.3. Then, for $c \geq \max\{c^*, c_3\}$ equation (1.1) has an unstable positive solution. Furthermore, for each a , there exists $c(a)$, such that for $c \geq c(a)$, the positive solution is unique and nondegenerate.

Proof. The only statement that requires a proof is the existence of a positive solution for $c \geq \max\{c^*, c_3\}$. The proof uses similar arguments as in the proof of Theorem 3.6, so we shall brief them here. Fixing a $c \geq \max\{c^*, c_3\}$, we define

$$T_{e,K} = \left\{ u \in C_0^1(\overline{\Omega}) : e\varphi_1 \leq u \leq K \text{ in } \Omega \text{ and } \frac{\partial}{\partial \nu}(u - e\varphi_1) \leq 0 \text{ on } \partial\Omega \right\},$$

where e and K are positive constants. First notice that by Lemmas 3.9 and 3.10, we can choose e sufficiently small and K sufficiently large so that (1.1) has no solution on $\partial T_{e,K}$ (the relative boundary of $T_{e,K}$ in P) for $\lambda_1(\Omega_0) \leq a \leq \lambda_1(\Omega_0) + \delta$. Next we define $A_a : C_0^1(\overline{\Omega}) \rightarrow C_0^1(\overline{\Omega})$ by

$$A_a(u) = (-\Delta + L)^{-1}((a + L)u - b(x)u^2 - ch(x)).$$

By taking $L > 0$ sufficiently large, we may assume that $(a + L)u - b(x)u^2 - ch(x)$ is increasing in $[e\varphi_1(x), K]$ for all $x \in \Omega$ and that $A_{a,c}$ maps $T_{e,K}$ into P . Indeed, if $u \in T_{e,K}$ and $u_0 = A_a(u)$, then we have

$$\begin{aligned} u_0 &= (-\Delta + L)^{-1}((a + L)u - b(x)u^2 - ch(x)) \\ &\geq (-\Delta + L)^{-1}((a + L)e\varphi_1 - b(x)e^2\varphi_1^2 - ch(x)) \geq 0, \end{aligned}$$

as $(a + L)e\varphi_1 - b(x)e^2\varphi_1^2 - ch(x) \geq 0$ for $L > 0$ large. Hence, for e small and L large, $\deg(I - A_a, T_{e,K}, 0)$ is admissible for $\lambda_1(\Omega_0) \leq a \leq \lambda_1(\Omega_0) + \delta$. Now, by Lemmas 3.7 and 3.8, we have

$$\deg(I - A_{\lambda_1(\Omega_0)}, T_{e,K}, 0) \neq 0.$$

Hence,

$$\deg(I - A_a, T_{e,K}, 0) = \deg(A_{\lambda_1(\Omega_0)}, T_{e,K}, 0) \neq 0$$

for all $a \in [\lambda_1(\Omega_0), \lambda_1(\Omega_0) + \delta]$. The proof is now complete. $\square$

Finally, it is an interesting open problem to study existence of positive solutions of (1.1) for all c large when a is large and away from $\lambda_1(\Omega_0)$. In particular, for any $a > \lambda_1(\Omega_0)$, it is easily seen (through a familiar limiting argument) that the existence of a positive solution to the equation

$$-\Delta u = au - h(x) \quad \text{in } \Omega_0, \quad u = 0 \quad \text{on } \partial\Omega_0,$$

is a necessary condition for existence of a positive solutions to (1.1) as $c \nearrow \infty$. Although the techniques used in this paper seem inadequate to deal with the question of sufficiency of this necessary condition, we do believe that some of the ideas used here should be of value in dealing with this problem.

References

- [1] S. Alama and G. Tarantello, On the solvability of a semilinear elliptic equation via an associated eigenvalue problem, *Math. Z.* **221** (1996), no. 3, 467–493.
- [2] A. Ambrosetti and J. L. Gámez, Branches of positive solutions for some semilinear Schrödinger equations, *Math. Z.* **224** (1997), no. 3, 347–362.
- [3] P. Clément and L. A. Peletier, An anti-maximum principle for second-order elliptic operators, *J. Differential Equations* **34** (1979), no. 2, 218–229.
- [4] M. G. Crandall and P. H. Rabinowitz, Bifurcation, perturbation of simple eigenvalues and linearized stability, *Arch. Ration. Mech. Anal.* **52** (1973), 161–180.
- [5] E. N. Dancer, On the indices of fixed points of mappings in cones and applications, *J. Math. Anal. Appl.* **91** (1983), no. 1, 131–151.
- [6] E. N. Dancer, Some remarks on classical problems and fine properties of Sobolev spaces, *Differential Integral Equations* **9** (1996), no. 3, 437–446.

- [7] M. A. del Pino, Positive solutions of a semilinear elliptic equation on a compact manifold, *Nonlinear Anal.* **22** (1994), no. 11, 1423–1430.
- [8] Y. Du and Q. Huang, Blow-up solutions for a class of semilinear elliptic and parabolic equations, *SIAM J. Math. Anal.* **31** (1999), no. 1, 1–18.
- [9] Y. Du and J. Shi, Allee effect and bistability in a spatially heterogeneous predator-prey model, *Trans. Amer. Math. Soc.* **359** (2007), no. 9, 4557–4593.
- [10] J. M. Fraile, P. Koch-Medina, J. López-Gómez and S. Merino, Elliptic eigenvalue problems and unbounded continua of positive solutions of a semilinear elliptic equation, *J. Differential Equations* **127** (1996), no. 1, 295–319.
- [11] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Classics Math., Springer, Berlin, 2001.
- [12] P. Girão and H. Tehrani, Positive solutions to logistic type equations with harvesting, *J. Differential Equations* **247** (2009), no. 2, 574–595.
- [13] P. M. Girão, Bifurcation curves of a logistic equation when the linear growth rate crosses a second eigenvalue, *Nonlinear Anal.* **74** (2011), no. 1, 94–113.
- [14] P. M. Girão and M. Pérez-Llanos, Bifurcation curves of a diffusive logistic equation with harvesting orthogonal to the first eigenfunction, *J. Math. Anal. Appl.* **403** (2013), no. 2, 376–390.
- [15] S. Oruganti, J. Shi and R. Shivaji, Diffusive logistic equation with constant yield harvesting. I: Steady states, *Trans. Amer. Math. Soc.* **354** (2002), no. 9, 3601–3619.
- [16] T. Ouyang, On the positive solutions of semilinear equations $\Delta u + \lambda u - hu^p = 0$ on the compact manifolds, *Trans. Amer. Math. Soc.* **331** (1992), no. 2, 503–527.
- [17] D. H. Sattinger, Monotone methods in nonlinear elliptic and parabolic boundary value problems, *Indiana Univ. Math. J.* **21** (1971/72), 979–1000.
- [18] S. Shabani, Diffusive Holling type-II predator-prey system with harvesting of prey, *J. Math. Anal. Appl.* **410** (2014), no. 1, 469–482.
- [19] J. Shi, Persistence and bifurcation of degenerate solutions, *J. Funct. Anal.* **169** (1999), no. 2, 494–531.