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# Distributed Lagrange Multiplier/Fictitious Domain Finite Element Method for a Transient Stokes Interface Problem with Jump Coefficients

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# Distributed Lagrange Multiplier/Fictitious Domain Finite Element Method for a Transient Stokes Interface Problem with Jump Coefficients

**Andrew Lundberg[1], Pengtao Sun[1,*], Cheng Wang[2] and Chen-song Zhang[3]**

**Abstract:** The distributed Lagrange multiplier/fictitious domain (DLM/FD)-mixed finite element method is developed and analyzed in this paper for a transient Stokes interface problem with jump coefficients. The semi- and fully discrete DLM/FD-mixed finite element scheme are developed for the first time for this problem with a moving interface, where the arbitrary Lagrangian-Eulerian (ALE) technique is employed to deal with the moving and immersed subdomain. Stability and optimal convergence properties are obtained for both schemes. Numerical experiments are carried out for different scenarios of jump coefficients, and all theoretical results are validated.

**Keywords:** Transient Stokes interface problem, jump coefficients, distributed Lagrange multiplier, fictitious domain method, mixed finite element, an optimal error estimate, stability.

## 1 Introduction

Let $\Omega$ be an open bounded domain in $\mathbb{R}^d$ $(d = 2, 3)$ with a convex polygonal boundary $\partial\Omega$. Two subdomains, $\Omega_t^i := \Omega_i(t) \subset \Omega$ $(i = 1, 2)$, are separated by an interface $\Gamma_t := \Gamma(t)$ which may move/deform in time $t \in [0, T]$ $(T > 0)$, satisfying $\overline{\Omega} = \overline{\Omega_t^1} \cup \overline{\Omega_t^2}$, $\Omega_t^1 \cap \Omega_t^2 = \emptyset$, $\Gamma_t = \partial\Omega_t^1 \cap \partial\Omega_t^2$, as sketched in Fig. 1. Then, a transient Stokes interface problem with

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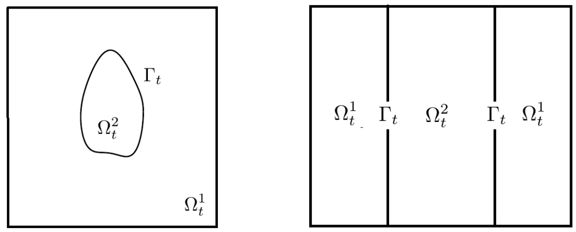


**Figure 1:** Two schematic domain decompositions divided by the interface $\Gamma_t$

jump coefficients can be defined as follows:

$$\rho_1 \frac{\partial \boldsymbol{u}_1}{\partial t} - \nabla \cdot (\beta_1 \nabla \boldsymbol{u}_1) + \nabla p_1 = \boldsymbol{f}_1, \quad \text{in } \Omega_t^1 \times (0, T], \tag{1}$$
$$\nabla \cdot \boldsymbol{u}_1 = 0, \quad \text{in } \Omega_t^1 \times (0, T], \tag{2}$$
$$\rho_2 \frac{\partial \boldsymbol{u}_2}{\partial t} - \nabla \cdot (\beta_2 \nabla \boldsymbol{u}_2) + \nabla p_2 = \boldsymbol{f}_2, \quad \text{in } \Omega_t^2 \times (0, T], \tag{3}$$
$$\nabla \cdot \boldsymbol{u}_2 = 0, \quad \text{in } \Omega_t^2 \times (0, T], \tag{4}$$
$$\boldsymbol{u}_1 = \boldsymbol{u}_2, \quad \text{on } \Gamma_t \times (0, T], \tag{5}$$
$$(\beta_1 \nabla \boldsymbol{u}_1 - p_1 \boldsymbol{I}) n_1 + (\beta_2 \nabla \boldsymbol{u}_2 - p_2 \boldsymbol{I}) n_2 = \boldsymbol{\tau}, \quad \text{on } \Gamma_t \times (0, T], \tag{6}$$
$$\boldsymbol{u}_1 = 0, \quad \text{on } \partial\Omega_t^1 \backslash \Gamma_t \times (0, T], \tag{7}$$
$$\boldsymbol{u}_2 = 0, \quad \text{on } \partial\Omega_t^2 \backslash \Gamma_t \times (0, T], \tag{8}$$
$$\boldsymbol{u}_1(\boldsymbol{x}, 0) = \boldsymbol{u}_1^0, \quad \text{in } \Omega_0^1, \tag{9}$$
$$\boldsymbol{u}_2(\boldsymbol{x}, 0) = \boldsymbol{u}_2^0, \quad \text{in } \Omega_0^2, \tag{10}$$

where the solution pair, $(\boldsymbol{u}, p)$ that is defined in $\Omega \times [0, T]$, satisfies $\boldsymbol{u}|_{\Omega_t^1} = \boldsymbol{u}_1,\ \boldsymbol{u}|_{\Omega_t^2} = \boldsymbol{u}_2,\ p|_{\Omega_t^1} = p_1,\ p|_{\Omega_t^2} = p_2$ which are associated with the source term $\boldsymbol{f} \in L^2(0, T; (L^2(\Omega))^d)$ such that $\boldsymbol{f}|_{\Omega_t^i} = \boldsymbol{f}_i \in L^2(0, T; (L^2(\Omega_t^i))^d),\ (i = 1, 2)$. The jump coefficients $\beta \in L^2(0, T; L^\infty(\Omega))$ and $\rho \in L^\infty(0, T; L^\infty(\Omega))$ satisfy $\beta|_{\Omega_t^i} = \beta_i \in L^2(0, T; W^{1,\infty}(\Omega_t^i))$, $\rho|_{\Omega_t^i} = \rho_i \in L^\infty(0, T; L^\infty(\Omega_t^i)),\ (i = 1, 2)$, and $\beta_1 \neq \beta_2,\ \rho_1 \neq \rho_2$. Due to the incompressibility properties (2) and (4), we know both $\rho_1$ and $\rho_2$ are constant.

It is well known that for the elliptic interface problem [Nicaise (1993); Bramble and King (1996); Boffi, Gastaldi and Ruggeri (2014); Auricchio, Boffi, Gastaldi et al. (2015)] and for the stationary Stokes interface problem [Shibataa and Shimizu (2003); Hansbo, Larson and Zahedi (2014); Olshanskii and Reusken (2006)] with jump coefficients across the interface $\Gamma_t$, the global regularity of solutions over the entire domain $\Omega$ are generally reduced from $(H^2(\Omega))^d$ down to $(H^1(\Omega))^d$, and the local regularity of solutions may be also deteriorated from $(H^2(\Omega_i))^d$ down to $(H^\sigma(\Omega_i))^d$ $(3/2 < \sigma \leq 2)$ in each subdomain $\Omega_i$ $(i = 1, 2)$ [Nicaise (1993)] due to a non-smooth interface $\Gamma_t$ which may be only Lipschitz continuous and on which the nonzero jump flux $\boldsymbol{\tau}$ may be only defined in $L^\infty(0, T; (H^{\sigma-3/2}(\Gamma_t))^d)$.

The regularity study for solutions to (1)-(10) is still open to the community of theoretical partial differential equations, especially when the interface $\Gamma_t$ deforms along the time, i.e., the shape of $\Gamma_t$ and $\Omega_t^i$ $(i = 1, 2)$ depend on the primary unknowns $(\boldsymbol{u}, p)$. In this paper, in order to show a certain amount of convergence rate during the numerical experiments process for validating the convergence theorem of the developed DLM/FD finite element method, in what follows we assume a reduced regularity result for the solution $(\boldsymbol{u}, p)$ to (1)-(10) which is similar with that of the stationary Stokes interface problem in space [Shibataa and Shimizu (2003); Hansbo, Larson and Zahedi (2014); Olshanskii and Reusken (2006)],

$$\begin{array}{rcl} \boldsymbol{u} & \in & (H^2 \cap L^\infty)(0, T; (H^\sigma(\Omega_t^1 \cup \Omega_t^2))^d \cap (H_0^1(\Omega))^d), \\ p & \in & L^\infty(0, T; H^1(\Omega_t^1 \cup \Omega_t^2) \cap L^2(\Omega)), \end{array} \tag{11}$$

where, $3/2 < \sigma \leq 2$. Without loss of generality, in this paper we only study the immersed interface case by assuming $\Omega_t^2 \subset \Omega$, as shown in the left of Fig. 1. The regularity property (11) is assumed to hold under the circumstance that $\Gamma_t$ does not deform but only rotates and/or translates with a prescribed domain velocity $\boldsymbol{w}(\boldsymbol{x}, t)$, as defined in Section 2.3. Thus the shape and position of $\Omega_i$ $(i = 1, 2)$ are prescribed and do not depend on the primary unknowns, and the regularity results (11) can still be hypothesized, accordingly. Numerical results shown in Section 5 also support the regularity property of solution $(\boldsymbol{u}, p)$ defined in (11).

In practice, (1)-(10) generally model a type of immiscible two-phase fluid flow problem, where two phases of the fluid are separated by an distinct interface, and both fluid phases are defined by Stokes/Navier–Stokes equations in terms of fluid velocity and pressure as sketched in (1)-(4). In this scenario, $\beta_i$ and $\rho_i$ $(i = 1, 2)$ may stand for the fluid viscosity and density of different phases. Hence, the essential characteristic of the immiscible two-phase fluid flow model is preserved in the transient Stokes interface problem (1)-(10), that is, two different types of fluid equations bearing with different viscosity and density are defined on either side of the moving interface $\Gamma_t$.

Some long existing body-unfitted mesh methods for interface problems such as the immersed interface method (IIM) [Deng, Ito and Li (2003); LeVeque and Li (1994); Li and Ito (2001)] and immersed finite element method (IFEM) [Li (1998); Ji, Chen and Li (2014)] are still far from satisfactory for solving the Stokes interface problem in either stationary or transient case. As for the representative body-fitted mesh method, the arbitrary Lagrangian–Eulerian (ALE) method [Hirth, Amsden and Cook (1974); Hughes, Liu and Zimmermann (1981); Huerta and Liu (1988); Nitikitpaiboon and Bathe (1993); Souli and Benson (2010)] is the most popular one for solving moving interface problems such fluid-structure interactions (FSI), where, the mesh on the interface is accommodated to be shared by both fluid and structure, and thus to automatically satisfy the interface conditions as s-ketched in (5) and (6). However, for large rotations and/or translations of the structure or inhomogeneous movements of the grid nodes, fluid elements tend to become ill-shaped, which reflects on the accuracy of the solution. In this case, re-meshing, in which the whole domain or part of the domain is spatially rediscretised, is then a common strategy. However, it could be very troublesome, time consuming and less accurate, and, the worst thing brought by the re-meshing is that the mesh connectivity is no longer preserved for ALE method and thus many properties of ALE method are lost.

To overcome the above problems and to deliver an efficient and accurate numerical method for the transient Stokes interface problem in which the immersed phase may be engaged in a large translational/rotational motion, in this paper we develop a body-unfitted mesh method based upon the framework of the distributed Lagrange multiplier/fictitious domain (DLM/FD) method [Glowinski, Pana, Hesla et al. (1999); Wachs (2007); Glowinski and Kuznetsov (2007); Boffi and Gastaldi (2017); Wang and Sun (2017)], where, one fluid phase is smoothly extended into the other phase that is defined in the immersed subdomain, then occupies the entire domain $\Omega$, and the Lagrange multiplier (physically a pseudo body force) is introduced to enforce the interior (fictitious) fluids in the immersed subdomain to satisfy the constraint of the immersed phase motion. The constraints are incorporated into the field equations to form an augmented matrix equation which involves the Lagrange multipliers as unknowns. Thus, the re-meshing in the fluid domain is no longer needed for DLM/FD method, and the possible failure of ALE method is completely avoided when the large translation/rotation occurs to the immersed phase motion [Auricchio, Boffi, Gastaldi et al. (2015); Shi and Phan-Thien (2005); Yu (2005); Glowinski, Pana, Hesla et al. (2001)].

The DLM/FD finite element method has been analyzed for the elliptic interface problem [Boffi, Gastaldi and Ruggeri (2014); Auricchio, Boffi, Gastaldi et al. (2015)], the parabolic interface problem [Wang and Sun (2017)], the stationary Stokes interface problem [Lundberg, Sun and Wang (2019)], but has not yet applied to the transient Stokes interface problem. As shown in Boffi et al. [Boffi, Gastaldi and Ruggeri (2014); Auricchio, Boffi, Gastaldi et al. (2015)], the DLM/FD method essentially produces a saddle-point problem in regard to the unknown of elliptic equation and Lagrange multiplier, so the existing Babuška-Brezzi's theory [Babuška (1971); Brezzi and Fortin (1991); Brezzi (1974); Brezzi and Pitkaranta (1984)] can be employed to analyze the well-posedness, stability and convergence properties of the corresponding saddle-point problem induced from the DLM/FD finite element method. However, for the stationary Stokes interface problem, which is the steady state of the transient Stokes interface problem (1)-(10), we can see that its corresponding DLM/FD formulation forms a nested saddle-point problem including two subproblems of saddle-point type: the inside one from Stokes equations regarding Stokes unknowns (velocity and pressure), and the outside one from the DLM/FD method itself regarding Lagrange multiplier and Stokes unknowns, of which the well-posedness, stability as well as convergence analyses are more sophisticated than those of the elliptic and the parabolic interface problems. In the authors' recent work [Lundberg, Sun and Wang (2019)], a modified DLM/FD finite element method is developed for a stationary Stokes interface problem that consists of a nested saddle-point problem, and its well-posedness, stability and optimal convergence properties are analyzed still by means of the Babuška–Brezzi's theory but a more complicated approach. So in this paper, we will be able to develop the DLM/FD finite element method for the transient Stokes interface problem (1)-(10) and analyze its stability and convergence properties based on our previous work.

The structure of the paper is the following: in Section 2 we introduce the fictitious fluid (Stokes) equations then derive weak formulations of a transient Stokes interface problem with and without the employment of DLM/FD method. Then we define the semi-discrete DLM/FD finite element approximation and analyze its stability and optimal convergence theorem in Section 3. The full discretization is defined and its stability and convergence properties are analyzed in Section 4. Numerical experiments are carried out in Section 5, where the theoretical convergence results are validated.

## 2 Weak formulations of DLM/FD method

Introduce Sobolev spaces $\boldsymbol{V} := (H_0^1(\Omega))^d,\ Q := L^2(\Omega)$, and their restrictions $\boldsymbol{V}_t^1 = \boldsymbol{V}|_{\Omega_t^1},\ \boldsymbol{V}_t^2 = \boldsymbol{V}|_{\Omega_t^2},\ Q_t^1 = Q|_{\Omega_t^1},\ Q_t^2 = Q|_{\Omega_t^2}$. Let $(\cdot,\cdot)_\omega$ stand for $L^2$- product in $\omega$. We also introduce the space $\boldsymbol{\Lambda}_t := [(H^1(\Omega_t^2))^d]^*$ that is the dual space of $\boldsymbol{V}_t^2$, and let $\langle\cdot,\cdot\rangle_{\Omega_t^2}$ denote the duality pairing between $\boldsymbol{\Lambda}_t$ and $\boldsymbol{V}_t^2$. In $\boldsymbol{\Lambda}_t$ we have the norm

$$\|\boldsymbol{\lambda}\|_{\boldsymbol{\Lambda}_t} = \sup_{\boldsymbol{v}_2\in\boldsymbol{V}_t^2} \frac{\langle\boldsymbol{\lambda},\boldsymbol{v}_2\rangle_{\Omega_t^2}}{\|\boldsymbol{v}_2\|_{\boldsymbol{V}_t^2}}. \tag{12}$$

### *2.1 Fictitious fluid (Stokes) equations*

We first define the following Stokes equations for the fictitious fluid in $\Omega_t^2$ in terms of $(\tilde{\boldsymbol{u}}_2,\tilde{p}_2)$

$$\tilde{\rho}_2\frac{\partial\tilde{\boldsymbol{u}}_2}{\partial t} - \nabla\cdot(\tilde{\beta}_2\nabla\tilde{\boldsymbol{u}}_2) + \nabla\tilde{p}_2 = \tilde{\boldsymbol{f}}_2, \quad \text{in } \Omega_t^2\times(0,T], \tag{13}$$

$$\nabla\cdot\tilde{\boldsymbol{u}}_2 = 0, \quad \text{in } \Omega_t^2\times(0,T], \tag{14}$$

$$\tilde{\boldsymbol{u}}_2 = \boldsymbol{u}_2, \quad \text{on } \Gamma_t\times(0,T], \tag{15}$$

$$\tilde{\boldsymbol{u}}_2 = 0, \quad \text{on } \partial\Omega_t^2\backslash\Gamma_t\times(0,T], \tag{16}$$

$$\tilde{\boldsymbol{u}}_2(\boldsymbol{x},0) = \tilde{\boldsymbol{u}}^0, \quad \text{in } \Omega_0^2, \tag{17}$$

where, we smoothly extend $\beta_1 \in L^2(0,T;W^{1,\infty}(\Omega_t^1))$ and $\rho_1 \in L^\infty(0,T;L^\infty(\Omega_t^1))$ into $\Omega_t^2$ and thus attain the continuous functions $\tilde{\beta} \in L^2(0,T;W^{1,\infty}(\Omega))$, $\tilde{\rho} \in L^\infty(0,T;L^\infty(\Omega))$, respectively, such that $\tilde{\beta}|_{\Omega_t^1} = \beta_1, \tilde{\beta}|_{\Omega_t^2} = \tilde{\beta}_2$, $\tilde{\rho}|_{\Omega_t^1} = \rho_1, \tilde{\rho}|_{\Omega_t^2} = \tilde{\rho}_2$. As a consequence, we attain a smooth function $\tilde{\boldsymbol{f}} \in L^2(0,T;(L^2(\Omega))^d)$ such that $\tilde{\boldsymbol{f}}|_{\Omega_t^1} = \boldsymbol{f}_1,\ \tilde{\boldsymbol{f}}|_{\Omega_t^2} = \tilde{\boldsymbol{f}}_2$. Because $\rho_1$ is a constant, we assume its extension, $\tilde{\rho}$, is a constant too. In general, $\tilde{\beta}_2 \neq \beta_2,\ \tilde{\rho}_2 \neq \rho_2,\ \tilde{\boldsymbol{f}}_2 \neq \boldsymbol{f}_2$. Further, we introduce the solution pair $(\tilde{\boldsymbol{u}},\ \tilde{p})$ such that $\tilde{\boldsymbol{u}}|_{\Omega_t^1} = \boldsymbol{u}_1,\ \tilde{\boldsymbol{u}}|_{\Omega_t^2} = \tilde{\boldsymbol{u}}_2,\ \tilde{p}|_{\Omega_t^1} = p_1,\ \tilde{p}|_{\Omega_t^2} = \tilde{p}_2$, then $\tilde{\boldsymbol{u}}|_{\partial\Omega} = 0$, and $\tilde{\boldsymbol{u}}|_{\Gamma_t} = \boldsymbol{u}_1|_{\Gamma_t} = \boldsymbol{u}_2|_{\Gamma_t}$. And, a similar regularity property with (11) is defined for $(\tilde{\boldsymbol{u}},\tilde{p})$ as well:

$$\tilde{\boldsymbol{u}} \in (H^2\cap L^\infty)(0,T;(H^\sigma(\Omega_t^1\cup\Omega_t^2))^d \cap (H_0^1(\Omega))^d),\ \tilde{p} \in L^\infty(0,T;H^1(\Omega_t^1\cup\Omega_t^2)). \tag{18}$$

The following assumptions are needed in this paper: there exist constants $\underline{\beta},\ \bar{\beta}$ and $\underline{\rho},\bar{\rho}$

such that

$$\infty > \bar{\beta} \geq \beta_2 > \tilde{\beta}_2 \geq \underline{\beta} > 0, \quad \beta_2 - \tilde{\beta}_2 \geq \underline{\beta} > 0, \tag{19}$$

$$\infty > \bar{\rho} \geq \rho_2 > \tilde{\rho}_2 \geq \underline{\rho} > 0, \quad \rho_2 - \tilde{\rho}_2 \geq \underline{\rho} \geq 0. \tag{20}$$

### *2.2 Weak formulations.*

If we add the fictitious fluid Eqs. (13)-(14), which are defined in $\Omega_t^2$, to the Stokes Eqs. (1)-(2), which are defined in $\Omega_t^1$, and integrate by parts, then

$$\begin{aligned}
&\left(\tilde{\rho}\frac{\partial \tilde{\boldsymbol{u}}}{\partial t}, \boldsymbol{v}\right)_\Omega + (\tilde{\beta}\nabla\tilde{\boldsymbol{u}}, \nabla\boldsymbol{v})_\Omega - (\tilde{p}, \nabla\cdot\boldsymbol{v})_\Omega \\
&= (\beta_1\nabla\boldsymbol{u}_1, \nabla\boldsymbol{v})_{\Omega_t^1} - (p_1, \nabla\cdot\boldsymbol{v})_{\Omega_t^1} + (\tilde{\beta}_2\nabla\tilde{\boldsymbol{u}}_2, \nabla\boldsymbol{v})_{\Omega_t^2} - (\tilde{p}_2, \nabla\cdot\boldsymbol{v})_{\Omega_t^2} \\
&= (-\nabla\cdot(\beta_1\nabla\boldsymbol{u}_1) + \nabla p_1, \boldsymbol{v})_{\Omega_t^1} + (-\nabla\cdot(\tilde{\beta}_2\nabla\tilde{\boldsymbol{u}}_2) + \nabla\tilde{p}_2, \boldsymbol{v})_{\Omega_t^2} \\
&+([\beta_1\nabla\boldsymbol{u}_1 - p_1\boldsymbol{I}]n_1 + [\tilde{\beta}_2\nabla\tilde{\boldsymbol{u}}_2 - \tilde{p}_2\boldsymbol{I}]n_2, \boldsymbol{v})_{\Gamma_t} \\
&= (\boldsymbol{f}_1, \boldsymbol{v})_{\Omega_t^1} + (\tilde{\boldsymbol{f}}_2, \boldsymbol{v})_{\Omega_t^2} + ([\beta_1\nabla\boldsymbol{u}_1 - p_1\boldsymbol{I}]n_1 + [\tilde{\beta}_2\nabla\tilde{\boldsymbol{u}}_2 - \tilde{p}_2\boldsymbol{I}]n_2, \boldsymbol{v})_{\Gamma_t} \\
&= (\tilde{\boldsymbol{f}}, \boldsymbol{v})_\Omega + ([\beta_1\nabla\boldsymbol{u}_1 - p_1\boldsymbol{I}]n_1 + [\tilde{\beta}_2\nabla\tilde{\boldsymbol{u}}_2 - \tilde{p}_2\boldsymbol{I}]n_2, \boldsymbol{v})_{\Gamma_t},
\end{aligned} \tag{21}$$

$$(\nabla\cdot\tilde{\boldsymbol{u}}, q)_\Omega = (\nabla\cdot\boldsymbol{u}_1, q)_{\Omega_t^1} + (\nabla\cdot\tilde{\boldsymbol{u}}_2, q)_{\Omega_t^2} = 0, \quad \forall(\boldsymbol{v}, q) \in \boldsymbol{V}\times Q. \tag{22}$$

On the other hand, if we subtract the fictitious fluid Eqs. (13) and (14) from the Stokes Eqs. (3) and (4), and integrate by parts in $\Omega_t^2$, then

$$\begin{aligned}
&\left(\rho_2\frac{\partial \boldsymbol{u}_2}{\partial t} - \tilde{\rho}_2\frac{\partial \tilde{\boldsymbol{u}}_2}{\partial t}, \boldsymbol{v}_2\right)_{\Omega_t^2} + (\beta_2\nabla\boldsymbol{u}_2 - \tilde{\beta}_2\nabla\tilde{\boldsymbol{u}}_2, \nabla\boldsymbol{v}_2)_{\Omega_t^2} - (p_2 - \tilde{p}_2, \nabla\cdot\boldsymbol{v}_2)_{\Omega_t^2} \\
&= (\boldsymbol{f}_2 - \tilde{\boldsymbol{f}}_2, \boldsymbol{v}_2)_{\Omega_t^2} + \Big([\beta_2\nabla\boldsymbol{u}_2 - (\tilde{\beta}_2\nabla\tilde{\boldsymbol{u}}_2 - \tilde{p}_2\boldsymbol{I})]n_2, \boldsymbol{v}_2\Big)_{\Gamma_t} \\
&= (\boldsymbol{f}_2 - \tilde{\boldsymbol{f}}_2, \boldsymbol{v}_2)_{\Omega_t^2} + (\boldsymbol{\tau}, \boldsymbol{v}_2)_{\Gamma_t} - \Big([\beta_1\nabla\boldsymbol{u}_1 - p_1\boldsymbol{I}]n_1 + [\tilde{\beta}_2\nabla\tilde{\boldsymbol{u}}_2 - \tilde{p}_2\boldsymbol{I}]n_2, \boldsymbol{v}_2\Big)_{\Gamma_t}
\end{aligned} \tag{23}$$

$$(\nabla\cdot(\boldsymbol{u}_2 - \tilde{\boldsymbol{u}}_2), q_2)_{\Omega_t^2} = 0, \quad \forall(\boldsymbol{v}_2, q_2) \in \boldsymbol{V}_t^2 \times Q_t^2 \tag{24}$$

If we add (21) to (23) and (22) to (24), then the terms of the fictitious Stokes equations and the normal derivative terms on $\Gamma_t$ are all cancelled out, resulting in the original weak formulation of (1)-(10) as follows.

**Weak Form I** Find $(\boldsymbol{u}, p) \in (H^1\cap L^\infty)(0,T;\boldsymbol{V})\times L^2(0,T;Q)$ with $\boldsymbol{u}|_{\Omega_t^1} = \boldsymbol{u}_1,\ \boldsymbol{u}|_{\Omega_t^2} = \boldsymbol{u}_2,\ p|_{\Omega_t^1} = p_1,\ p|_{\Omega_t^2} = p_2$ such that

$$\left(\rho\frac{\partial \boldsymbol{u}}{\partial t}, \boldsymbol{v}\right)_\Omega + (\beta\nabla\boldsymbol{u}, \nabla\boldsymbol{v})_\Omega - (p, \nabla\cdot\boldsymbol{v})_\Omega = (\boldsymbol{f}, \boldsymbol{v})_\Omega + (\boldsymbol{\tau}, \boldsymbol{v})_{\Gamma_t}, \tag{25}$$

$$(\nabla\cdot\boldsymbol{u}, q)_\Omega = 0, \quad \forall(\boldsymbol{v}, q) \in \boldsymbol{V}\times Q. \tag{26}$$

Now, we add a new constrain $\boldsymbol{u}_2 = \tilde{\boldsymbol{u}}_2 = \tilde{\boldsymbol{u}}|_{\Omega_t^2}$ enforced weakly in $\Omega_t^2$ by means of the Lagrange multiplier, defined by

$$< \boldsymbol{\xi}, \tilde{\boldsymbol{u}}|_{\Omega_t^2} - \boldsymbol{u}_2 >_{\Omega_t^2} = 0, \quad \forall \boldsymbol{\xi} \in \boldsymbol{\Lambda}_t. \tag{27}$$

By the Fréchet-Riesz representation theorem, for each $\boldsymbol{\xi} \in \boldsymbol{\Lambda}_t$, there exists a unique $\boldsymbol{\psi} \in \boldsymbol{V}_t^2$ such that

$$(\boldsymbol{\psi}, \boldsymbol{v})_{\boldsymbol{V}_t^2} = \langle \boldsymbol{\xi}, \boldsymbol{v} \rangle_{\Omega_t^2}, \quad \forall \boldsymbol{v} \in \boldsymbol{V}_t^2, \tag{28}$$

where $(\cdot,\cdot)_{\boldsymbol{V}_t^2}$ represents the $H^1$-inner product in $\boldsymbol{V}_t^2$, defined as

$$(\boldsymbol{\psi}, \boldsymbol{v})_{\boldsymbol{V}_t^2} = (\boldsymbol{\psi}, \boldsymbol{v})_{\Omega_t^2} + (\nabla \boldsymbol{\psi}, \nabla \boldsymbol{v})_{\Omega_t^2}. \tag{29}$$

In addition, (28) directly results in the following equality

$$\|\boldsymbol{\xi}\|_{\boldsymbol{\Lambda}_t} = \|\boldsymbol{\psi}\|_{\boldsymbol{V}_t^2}. \tag{30}$$

Thus, (27) is equivalent with the following equation by letting $\boldsymbol{v} = \tilde{\boldsymbol{u}}|_{\Omega_t^2} - \boldsymbol{u}_2$ in (28)

$$(\boldsymbol{\psi}, \tilde{\boldsymbol{u}}|_{\Omega_t^2} - \boldsymbol{u}_2)_{\boldsymbol{V}_t^2} = 0, \quad \forall \boldsymbol{\psi} \in \boldsymbol{V}_t^2. \tag{31}$$

**Lemma 2.1.** Let $(\tilde{\boldsymbol{u}}, \boldsymbol{u}_2) \in H^1(0,T;\boldsymbol{V}) \times H^1(0,T;\boldsymbol{V}_t^2)$ satisfy (27) or (31), then

$$\|\boldsymbol{u}_2 - \tilde{\boldsymbol{u}}|_{\Omega_t^2}\|_{\boldsymbol{V}_t^2} = 0, \tag{32}$$
$$(\nabla\cdot\boldsymbol{u}_2, q_2)_{\Omega_t^2} = \left(\nabla\cdot\tilde{\boldsymbol{u}}|_{\Omega_t^2}, q_2\right)_{\Omega_t^2}, \quad \forall q_2 \in Q_t^2, \tag{33}$$
$$\left(\tilde{\beta}_2(\nabla\boldsymbol{u}_2 - \nabla\tilde{\boldsymbol{u}}|_{\Omega_t^2}), \nabla\boldsymbol{v}_2\right)_{\Omega_t^2} = 0, \quad \forall \boldsymbol{v}_2 \in \boldsymbol{V}_t^2, \tag{34}$$
$$\left\|\frac{\partial\boldsymbol{u}_2}{\partial t} - \frac{\partial\tilde{\boldsymbol{u}}|_{\Omega_t^2}}{\partial t}\right\|_{0,\Omega_t^2} = 0, \tag{35}$$
$$\left(\tilde{\rho}_2\left(\frac{\partial\boldsymbol{u}_2}{\partial t} - \frac{\partial\tilde{\boldsymbol{u}}|_{\Omega_t^2}}{\partial t}\right), \boldsymbol{v}_2\right)_{\Omega_t^2} = 0, \quad \forall \boldsymbol{v}_2 \in \boldsymbol{V}_t^2. \tag{36}$$

**Proof.** (32) can be easily attained by letting $\boldsymbol{\psi} = \boldsymbol{u}_2 - \tilde{\boldsymbol{u}}|_{\Omega_t^2}$ in (31), then we also have

$$\|\boldsymbol{u}_2 - \tilde{\boldsymbol{u}}|_{\Omega_t^2}\|_{0,\Omega_t^2} = \|\nabla(\boldsymbol{u}_2 - \tilde{\boldsymbol{u}}|_{\Omega_t^2})\|_{0,\Omega_t^2} = 0. \tag{37}$$

In addition, because

$$\left|\left(\nabla\cdot\boldsymbol{u}_2 - \nabla\cdot\tilde{\boldsymbol{u}}|_{\Omega_t^2}, q_2\right)_{\Omega_t^2}\right| \le \|\nabla\cdot(\boldsymbol{u}_2 - \tilde{\boldsymbol{u}}|_{\Omega_t^2})\|_{L^2(\Omega_t^2)}\|q_2\|_{Q_t^2}$$
$$\le \sqrt{d}\|\boldsymbol{u}_2 - \tilde{\boldsymbol{u}}|_{\Omega_t^2}\|_{\boldsymbol{V}_t^2}\|q_2\|_{Q_t^2} = 0,$$

thus (33) is proved. Further, due to (32)

$$\left| \left(\tilde{\beta}_2(\nabla \boldsymbol{u}_2 - \nabla \tilde{\boldsymbol{u}}|_{\Omega_t^2}), \nabla \boldsymbol{v}_2\right)_{\Omega_t^2} \right| \leq \|\tilde{\beta}_2\|_{L^\infty} \|\boldsymbol{u}_2 - \tilde{\boldsymbol{u}}|_{\Omega_t^2}\|_{\boldsymbol{V}_t^2} \|\nabla \boldsymbol{v}_2\|_{0,\Omega_t^2} = 0, \tag{38}$$

then (34) is obtained. In addition,

$$(\boldsymbol{v}_2, \tilde{\boldsymbol{u}}|_{\Omega_t^2} - \boldsymbol{u}_2)_{\Omega_t^2} = 0, \quad \forall \boldsymbol{v}_2 \in \boldsymbol{V}_t^2, \tag{39}$$

because $|(\boldsymbol{v}_2, \tilde{\boldsymbol{u}}|_{\Omega_t^2} - \boldsymbol{u}_2)_{\Omega_t^2}| \leq \|\tilde{\boldsymbol{u}}|_{\Omega_t^2} - \boldsymbol{u}_2\|_{0,\Omega_t^2} \|\boldsymbol{v}_2\|_{0,\Omega_t^2} = 0$ due to (37).

We differentiate (39) in time and apply the Reynolds transport theorem, use the prescribed domain velocity of $\Omega_t^2$, $\boldsymbol{w}(\boldsymbol{x}, t)$, resulting in

$$\frac{d}{dt} \int_{\Omega_t^2} (\tilde{\boldsymbol{u}}|_{\Omega_t^2} - \boldsymbol{u}_2) \cdot \boldsymbol{v}_2 d\boldsymbol{x} = \int_{\Omega_t^2} \left( \frac{\partial((\tilde{\boldsymbol{u}}|_{\Omega_t^2} - \boldsymbol{u}_2) \cdot \boldsymbol{v}_2)}{\partial t} + \boldsymbol{w} \cdot \nabla((\tilde{\boldsymbol{u}}|_{\Omega_t^2} - \boldsymbol{u}_2) \cdot \boldsymbol{v}_2) \right.$$
$$\left. + ((\tilde{\boldsymbol{u}}|_{\Omega_t^2} - \boldsymbol{u}_2) \cdot \boldsymbol{v}_2) \nabla \cdot \boldsymbol{w} \right) d\boldsymbol{x} = 0,$$

where, with the identity

$$\boldsymbol{w} \cdot \nabla((\tilde{\boldsymbol{u}}|_{\Omega_t^2} - \boldsymbol{u}_2) \cdot \boldsymbol{v}_2) = \nabla(\tilde{\boldsymbol{u}}|_{\Omega_t^2} - \boldsymbol{u}_2) : (\boldsymbol{v}_2 \boldsymbol{w}^T) + (\tilde{\boldsymbol{u}}|_{\Omega_t^2} - \boldsymbol{u}_2) \cdot (\nabla \boldsymbol{v}_2 \boldsymbol{w}), \tag{40}$$

we can further have

$$\left( \frac{\partial(\tilde{\boldsymbol{u}}|_{\Omega_t^2} - \boldsymbol{u}_2)}{\partial t}, \boldsymbol{v}_2 \right)_{\Omega_t^2} + \left( \tilde{\boldsymbol{u}}|_{\Omega_t^2} - \boldsymbol{u}_2, \frac{\partial \boldsymbol{v}_2}{\partial t} \right)_{\Omega_t^2} + \left( \nabla(\tilde{\boldsymbol{u}}|_{\Omega_t^2} - \boldsymbol{u}_2), \boldsymbol{v}_2 \boldsymbol{w}^T \right)_{\Omega_t^2}$$
$$+ \left( \tilde{\boldsymbol{u}}|_{\Omega_t^2} - \boldsymbol{u}_2, \nabla \boldsymbol{v}_2 \boldsymbol{w} \right)_{\Omega_t^2} + \left( \tilde{\boldsymbol{u}}|_{\Omega_t^2} - \boldsymbol{u}_2, (\nabla \cdot \boldsymbol{w}) \, \boldsymbol{v}_2 \right)_{\Omega_t^2} = 0, \quad \forall \boldsymbol{v}_2 \in \boldsymbol{V}_t^2. \tag{41}$$

By the Cauchy-Schwartz inequality and (37), we obtain

$$\left| \left( \frac{\partial(\tilde{\boldsymbol{u}}|_{\Omega_t^2} - \boldsymbol{u}_2)}{\partial t}, \boldsymbol{v}_2 \right)_{\Omega_t^2} \right| \leq \|\tilde{\boldsymbol{u}}|_{\Omega_t^2} - \boldsymbol{u}_2\|_{0,\Omega_t^2} \left\| \frac{\partial \boldsymbol{v}_2}{\partial t} \right\|_{0,\Omega_t^2}$$
$$+ \|\nabla(\tilde{\boldsymbol{u}}|_{\Omega_t^2} - \boldsymbol{u}_2)\|_{0,\Omega_t^2} \left\| \boldsymbol{v}_2 \boldsymbol{w}^T \right\|_{0,\Omega_t^2} + \|\tilde{\boldsymbol{u}}|_{\Omega_t^2} - \boldsymbol{u}_2\|_{0,\Omega_t^2} \|\nabla \boldsymbol{v}_2 \boldsymbol{w}\|_{0,\Omega_t^2}$$
$$+ \|\tilde{\boldsymbol{u}}|_{\Omega_t^2} - \boldsymbol{u}_2\|_{0,\Omega_t^2} \|(\nabla \cdot \boldsymbol{w}) \boldsymbol{v}_2\|_{0,\Omega_t^2} = 0, \tag{42}$$

choose $\boldsymbol{v}_2 = \frac{\partial(\tilde{\boldsymbol{u}}|_{\Omega_t^2} - \boldsymbol{u}_2)}{\partial t}$, then (35), further, (36) are resulted, accordingly, because

$$\left| \left( \tilde{\rho}_2 \left( \frac{\partial \boldsymbol{u}_2}{\partial t} - \frac{\partial \tilde{\boldsymbol{u}}|_{\Omega_t^2}}{\partial t} \right), \boldsymbol{v}_2 \right)_{\Omega_t^2} \right| \leq \|\tilde{\rho}_2\|_{L^\infty} \left\| \frac{\partial \boldsymbol{u}_2}{\partial t} - \frac{\partial \tilde{\boldsymbol{u}}|_{\Omega_t^2}}{\partial t} \right\|_{(L^2(\Omega_t^2))^d} \|\boldsymbol{v}_2\|_{(L^2(\Omega_t^2))^d} = 0.$$

■

With (34) and (36) we can rewrite the first and the second term on the left hand side of (23) as

$$\left( \rho_2 \frac{\partial \boldsymbol{u}_2}{\partial t} - \tilde{\rho}_2 \frac{\partial \tilde{\boldsymbol{u}}_2}{\partial t}, \boldsymbol{v}_2 \right)_{\Omega_t^2} = \left( (\rho_2 - \tilde{\rho}_2) \frac{\partial \boldsymbol{u}_2}{\partial t}, \boldsymbol{v}_2 \right)_{\Omega_t^2}, \quad \forall \boldsymbol{v}_2 \in \boldsymbol{V}_t^2, \tag{43}$$

$$\left( \beta_2 \nabla \boldsymbol{u}_2 - \tilde{\beta}_2 \nabla \tilde{\boldsymbol{u}}|_{\Omega_t^2}, \nabla \boldsymbol{v}_2 \right)_{\Omega_t^2} = \left( (\beta_2 - \tilde{\beta}_2) \nabla \boldsymbol{u}_2, \nabla \boldsymbol{v}_2 \right)_{\Omega_t^2}, \quad \forall \boldsymbol{v}_2 \in \boldsymbol{V}_t^2. \tag{44}$$

Therefore, based upon (21), (23), (26), (43), (44) and (33), we can define a DLM/FD weak formulation for (1)-(10) as follows.

**Weak Form II (DLM/FD Formulation)** Find $(\tilde{\boldsymbol{u}},\ \boldsymbol{u}_2,\ \tilde{p},\ \boldsymbol{\lambda}) \in (H^1 \cap L^\infty)(0,T;\boldsymbol{V}) \times (H^1 \cap L^\infty)(0,T;\boldsymbol{V}_t^2) \times L^2(0,T;Q) \times L^2(0,T;\boldsymbol{\Lambda}_t)$ such that

$$\left(\tilde{\rho}\frac{\partial \tilde{\boldsymbol{u}}}{\partial t}, \boldsymbol{v}\right)_\Omega + (\tilde{\beta}\nabla\tilde{\boldsymbol{u}}, \nabla\boldsymbol{v})_\Omega - (\tilde{p}, \nabla\cdot\boldsymbol{v})_\Omega + <\boldsymbol{\lambda}, \boldsymbol{v}|_{\Omega_t^2}>_{\Omega_t^2} = (\tilde{\boldsymbol{f}}, \boldsymbol{v})_\Omega, \tag{45}$$

$$(\nabla\cdot\tilde{\boldsymbol{u}}, q)_\Omega = 0, \tag{46}$$

$$\left((\rho_2 - \tilde{\rho}_2)\frac{\partial \boldsymbol{u}_2}{\partial t}, \boldsymbol{v}_2\right)_{\Omega_t^2} + \left((\beta_2 - \tilde{\beta}_2)\nabla\boldsymbol{u}_2, \nabla\boldsymbol{v}_2\right)_{\Omega_t^2} - <\boldsymbol{\lambda}, \boldsymbol{v}_2>_{\Omega_t^2}$$
$$= \left(\boldsymbol{f}_2 - \tilde{\boldsymbol{f}}_2, \boldsymbol{v}_2\right)_{\Omega_t^2} + (\boldsymbol{\tau}, \boldsymbol{v}_2)_{\Gamma_t}, \tag{47}$$

$$<\boldsymbol{\xi}, \tilde{\boldsymbol{u}}|_{\Omega_t^2} - \boldsymbol{u}_2>_{\Omega_t^2} = 0, \quad \forall(\boldsymbol{v}, \boldsymbol{v}_2, q, \boldsymbol{\xi}) \in \boldsymbol{V} \times \boldsymbol{V}_t^2 \times Q \times \boldsymbol{\Lambda}_t. \tag{48}$$

In the following theorem, we prove the equivalence between Weak Forms I and II.

**Theorem 2.2.** Given $\boldsymbol{f} \in L^2(0,T;(L^2(\Omega))^d)$ with $\boldsymbol{f}|_{\Omega_t^1} = \boldsymbol{f}_1,\ \boldsymbol{f}|_{\Omega_t^2} = \boldsymbol{f}_2$ , and $\beta \in L^2(0,T;L^\infty(\Omega))$ with $\beta|_{\Omega_t^1} = \beta_1$, $\beta|_{\Omega_t^2} = \beta_2$, let $\tilde{\boldsymbol{f}} \in L^2(0,T;(L^2(\Omega))^d)$ be any function that satisfies $\tilde{\boldsymbol{f}}|_{\Omega_1} = \boldsymbol{f}_1$, and let $\tilde{\beta} \in L^2(0,T;L^\infty(\Omega))$ be any function that satisfies $\tilde{\beta}|_{\Omega_1} = \beta_1$.

(i). Suppose $(\tilde{\boldsymbol{u}},\ \boldsymbol{u}_2,\ \tilde{p},\ \boldsymbol{\lambda}) \in (H^1\cap L^\infty)(0,T;\boldsymbol{V}) \times (H^1\cap L^\infty)(0,T;\boldsymbol{V}_t^2) \times L^2(0,T;Q) \times L^2(0,T;\boldsymbol{\Lambda}_t)$ is a solution to Weak Form II (45)-(48). Let $\boldsymbol{u}|_{\Omega_t^1} = \tilde{\boldsymbol{u}}|_{\Omega_t^1},\ \boldsymbol{u}|_{\Omega_t^2} = \boldsymbol{u}_2,\ \tilde{p}|_{\Omega_1} = p_1,\ \tilde{p}|_{\Omega_2} = p_2$. Then $(\boldsymbol{u}, p) \in (H^1 \cap L^\infty)(0,T;\boldsymbol{V}) \times L^2(0,T;Q)$ is a solution to Weak Form I (25) and (26).

(ii). Conversely, let $(\boldsymbol{u}, p) \in (H^1 \cap L^\infty)(0,T;\boldsymbol{V}) \times L^2(0,T;Q)$ be a solution to Weak Form I (25) and (26), and define $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}_t$ by

$$<\boldsymbol{\lambda}, \boldsymbol{v}|_{\Omega_t^2}>_{\Omega_t^2} = (\tilde{\boldsymbol{f}}, \boldsymbol{v})_\Omega - \left(\tilde{\rho}\frac{\partial\tilde{\boldsymbol{u}}}{\partial t}, \boldsymbol{v}\right)_\Omega - (\tilde{\beta}\nabla\tilde{\boldsymbol{u}}, \nabla\boldsymbol{v})_\Omega + (\tilde{p}, \nabla\cdot\boldsymbol{v})_\Omega,\ \forall\boldsymbol{v} \in \boldsymbol{V}, \tag{49}$$

where $\tilde{\boldsymbol{u}} := \boldsymbol{u}$, $\tilde{p} \in Q$ satisfies $\tilde{p}|_{\Omega_t^1} = p_1$ and $\tilde{p}|_{\Omega_t^2} = p_2$. Then, $(\tilde{\boldsymbol{u}} := \boldsymbol{u}, \boldsymbol{u}_2 := \boldsymbol{u}|_{\Omega_t^2}, \tilde{p}, \boldsymbol{\lambda}) \in \boldsymbol{V} \times \boldsymbol{V}_t^2 \times Q \times \boldsymbol{\Lambda}_t$ is a solution to Weak Form II (45)-(48).

**Proof.** (i). We have (44) due to (48) , then (25) can be easily proved by taking $\boldsymbol{v} \in \boldsymbol{V}$ in (45) with $\boldsymbol{v}|_{\Omega_t^2} = \boldsymbol{v}_2$, and simply adding (45) and (47) together to cancel all Lagrange multiplier terms. Due to (33) and (46), (26) is obvious.

(ii). The definition of $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}_t$ in (49) leads to (45), and

$$\left(\rho_1\frac{\partial\boldsymbol{u}_1}{\partial t}, \boldsymbol{v}\right)_\Omega + \left(\tilde{\rho}_2\frac{\partial\tilde{\boldsymbol{u}}|_{\Omega_t^2}}{\partial t}, \boldsymbol{v}\right)_\Omega + (\beta_1\nabla\boldsymbol{u}_1, \nabla\boldsymbol{v})_{\Omega_t^1} + (\tilde{\beta}_2\nabla\tilde{\boldsymbol{u}}|_{\Omega_t^2}, \nabla\boldsymbol{v})_{\Omega_t^2}$$
$$-(p_1, \nabla\cdot\boldsymbol{v})_{\Omega_t^1} - (p_2, \nabla\cdot\boldsymbol{v})_{\Omega_t^2} + <\boldsymbol{\lambda}, \boldsymbol{v}|_{\Omega_t^2}>_{\Omega_t^2} = (\tilde{\boldsymbol{f}}, \boldsymbol{v})_\Omega,\ \forall\boldsymbol{v} \in \boldsymbol{V}. \tag{50}$$

Subtract (50) from (25), (47) is then obtained because $\tilde{\boldsymbol{u}}|_{\Omega_t^2} = \boldsymbol{u}_2$ that is the selection of the solution, (48) is thus proved as well. Due to (33) and (26), (46) is obvious by taking $q \in Q$ with $q|_{\Omega_t^2} = q_2$. ■

According to the equivalence of (27) and (31), we can reformulate the weak forms of DLM/FD method as follows.

**Weak Form III (Equivalent DLM/FD Formulation)** Find $(\tilde{\boldsymbol{u}},\ \boldsymbol{u}_2,\ \tilde{p},\ \boldsymbol{\phi}) \in (H^1 \cap L^\infty)(0,T;\boldsymbol{V}) \times (H^1 \cap L^\infty)(0,T;\boldsymbol{V}_t^2) \times L^2(0,T;Q) \times L^2(0,T;\boldsymbol{V}_t^2)$ such that

$$(\tilde{\rho}\frac{\partial \tilde{\boldsymbol{u}}}{\partial t}, \boldsymbol{v})_\Omega + (\tilde{\beta}\nabla\tilde{\boldsymbol{u}}, \nabla\boldsymbol{v})_\Omega - (\tilde{p}, \nabla\cdot\boldsymbol{v})_\Omega + (\boldsymbol{\phi}, \boldsymbol{v}|_{\Omega_t^2})_{\boldsymbol{V}_t^2} = (\tilde{\boldsymbol{f}}, \boldsymbol{v})_\Omega, \tag{51}$$

$$(\nabla\cdot\tilde{\boldsymbol{u}}, q)_\Omega = 0, \tag{52}$$

$$\begin{aligned}&\left((\rho_2 - \tilde{\rho}_2)\frac{\partial \boldsymbol{u}_2}{\partial t}, \boldsymbol{v}_2\right)_{\Omega_t^2} + \left((\beta_2 - \tilde{\beta}_2)\nabla\boldsymbol{u}_2, \nabla\boldsymbol{v}_2\right)_{\Omega_t^2} - (\boldsymbol{\phi}, \boldsymbol{v}_2)_{\boldsymbol{V}_t^2}\\ &= \left(\boldsymbol{f}_2 - \tilde{\boldsymbol{f}}_2, \boldsymbol{v}_2\right)_{\Omega_t^2} + (\boldsymbol{\tau}, \boldsymbol{v}_2)_{\Gamma_t},\end{aligned} \tag{53}$$

$$(\boldsymbol{\psi}, \tilde{\boldsymbol{u}}|_{\Omega_t^2} - \boldsymbol{u}_2)_{\boldsymbol{V}_t^2} = 0, \quad \forall(\boldsymbol{v},\ \boldsymbol{v}_2,\ q,\ \boldsymbol{\psi}) \in \boldsymbol{V}\times\boldsymbol{V}_t^2\times Q\times\boldsymbol{V}_t^2. \tag{54}$$

### *2.3 The arbitrary Lagrange-Eulerian (ALE) formulation*

We assume that there exists a bijective mapping $\boldsymbol{X}_t \in H^1(0,T;(W^{2,\infty}(\Omega_0^2))^d)$ such that for each $t \in (0,T]$, the mapping [Martín, Smaranda and Takahashi (2009); Gastaldi (2001)]

$$\begin{aligned}\boldsymbol{X}_t : \Omega_0^2 &\to \Omega_t^2,\\ \boldsymbol{y} &\mapsto \boldsymbol{x}(\boldsymbol{y},t),\end{aligned} \tag{55}$$

is invertible and $\boldsymbol{X}_t^{-1} \in (W^{1,\infty}(\Omega_t^2))^d$. Here $\boldsymbol{y} \in \Omega_0^2$ is so called the arbitrary Lagrange-Eulerian (ALE) coordinate, and $\boldsymbol{x} \in \Omega_t^2$ is the spatial (or Eulerian) coordinate. We further introduce the domain velocity $\boldsymbol{w}$, defined by

$$\boldsymbol{w} : \Omega_t^2 \to \mathbb{R}^d, \qquad \boldsymbol{w}(\boldsymbol{x},t) = \frac{\partial \boldsymbol{X}_t}{\partial t}(\boldsymbol{X}_t^{-1}(\boldsymbol{x}),t). \tag{56}$$

In this paper we assume that $\Omega_t^2$ does not deform but only rotates/translates with a prescribed domain velocity, $\boldsymbol{w}$. Thus, $\boldsymbol{w}(\boldsymbol{x},t) \in H^1(0,T;(W^{1,\infty}(\Omega_t^2))^d)$ is a given velocity function with the assumption that

$$\max\{\|\boldsymbol{w}\|_{1,\infty,\Omega_t^2}, \|\frac{\partial \boldsymbol{w}}{\partial t}\|_{1,\infty,\Omega_t^2}\} \le c, \tag{57}$$

where $c$ denotes a constant independent of any discretization parameters in the rest of the paper. so the ALE mapping function $\boldsymbol{X}_t(\boldsymbol{y},t) \in H^1(0,T;(W^{2,\infty}(\Omega_0^2))^d)$, which can be considered as a prescribed displacement of domain motion for $\Omega_t^2$.

We use $\frac{d\boldsymbol{v}}{dt}|_{\boldsymbol{y}}$ to denote the temporal derivative on the ALE frame which is defined as follows: for any function $\boldsymbol{v} : \Omega_t^2 \to \mathbb{R}^d$ regular enough and defined on the Eulerian frame, we set [Gastaldi (2001); Martín, Smaranda and Takahashi (2009)]

$$\begin{aligned} \frac{d\boldsymbol{v}}{dt}|_{\boldsymbol{y}} : \Omega_t^2 \quad &\to \quad \mathbb{R}^d, \\ (\boldsymbol{x}, t) \quad &\mapsto \quad \frac{d\boldsymbol{v}}{dt}|_{\boldsymbol{y}}(\boldsymbol{x}, t) = \frac{\partial \boldsymbol{v}}{\partial t}(\boldsymbol{x}, t) + \boldsymbol{w}(\boldsymbol{x}, t) \cdot \nabla \boldsymbol{v}(\boldsymbol{x}, t). \end{aligned} \tag{58}$$

Now we need to redefine the space $\boldsymbol{V}_t^2$ on the ALE frame as follows: $\boldsymbol{V}_t^2 := \{\boldsymbol{v} : \Omega_t^2 \times [0, T] \to \mathbb{R}^d,\ \boldsymbol{v} = \hat{\boldsymbol{v}} \circ \boldsymbol{X}_t^{-1}$ for $\hat{\boldsymbol{v}} \in V_0^2\}$, where, $V_0^2 := (H^1(\Omega_0^2))^d$ is the reference (initial) domain of $\boldsymbol{V}_t^2$. Based on the above definitions, we can reformulate (51)-(54) as the following ALE-type weak formulation of the DLM/FD method.

**Weak Form IV (ALE-DLM/FD Formulation)** Find $(\tilde{\boldsymbol{u}},\ \boldsymbol{u}_2,\ \tilde{p},\ \boldsymbol{\phi}) \in (H^1 \cap L^\infty)(0, T; \boldsymbol{V}) \times (H^1 \cap L^\infty)(0, T; \boldsymbol{V}_t^2) \times L^2(0, T; Q) \times L^2(0, T; \boldsymbol{V}_t^2)$ such that

$$(\tilde{\rho}\frac{\partial \tilde{\boldsymbol{u}}}{\partial t}, \boldsymbol{v})_\Omega + (\tilde{\beta}\nabla\tilde{\boldsymbol{u}}, \nabla\boldsymbol{v})_\Omega - (\tilde{p}, \nabla \cdot \boldsymbol{v})_\Omega + (\boldsymbol{\phi}, \boldsymbol{v}|_{\Omega_t^2})_{\boldsymbol{V}_t^2} = (\tilde{\boldsymbol{f}}, \boldsymbol{v})_\Omega, \tag{59}$$

$$(\nabla \cdot \tilde{\boldsymbol{u}}, q)_\Omega = 0, \tag{60}$$

$$\begin{aligned} &\left((\rho_2 - \tilde{\rho}_2)\frac{d\boldsymbol{u}_2}{dt}|_{\boldsymbol{y}}, \boldsymbol{v}_2\right)_{\Omega_t^2} - ((\rho_2 - \tilde{\rho}_2)\boldsymbol{w} \cdot \nabla\boldsymbol{u}_2, \boldsymbol{v}_2)_{\Omega_t^2} \\ &+ \left((\beta_2 - \tilde{\beta}_2)\nabla\boldsymbol{u}_2, \nabla\boldsymbol{v}_2\right)_{\Omega_t^2} - (\boldsymbol{\phi}, \boldsymbol{v}_2)_{\boldsymbol{V}_t^2} = \left(\boldsymbol{f}_2 - \tilde{\boldsymbol{f}}_2, \boldsymbol{v}_2\right)_{\Omega_t^2} + (\boldsymbol{\tau}, \boldsymbol{v}_2)_{\Gamma_t}, \end{aligned} \tag{61}$$

$$(\boldsymbol{\psi}, \tilde{\boldsymbol{u}}|_{\Omega_t^2} - \boldsymbol{u}_2)_{\boldsymbol{V}_t^2} = 0, \quad \forall(\boldsymbol{v},\ \boldsymbol{v}_2,\ q,\ \boldsymbol{\psi}) \in \boldsymbol{V} \times \boldsymbol{V}_t^2 \times Q \times \boldsymbol{V}_t^2, \tag{62}$$

where, we introduce a convection term $((\rho_2-\tilde{\rho}_2)\boldsymbol{w}\cdot\nabla\boldsymbol{u}_2, \boldsymbol{v}_2)_{\Omega_t^2}$ in (61). The main technical reason to introduce this term is strictly numerical. Since the domain is time dependent, it is not possible to discretize directly the partial temporal derivative. In fact, if $\boldsymbol{x} \in \Omega_t^2$ and the time step size $\Delta t > 0$, the condition $\boldsymbol{x} \in \Omega_{t+\Delta t}^2$ is not always fulfilled. Therefore, the term $((\rho_2 - \tilde{\rho}_2)\boldsymbol{w} \cdot \nabla\boldsymbol{u}_2, \boldsymbol{v}_2)_{\Omega_t^2}$ could be seen as a numerical corrector term of the partial temporal derivative.

## 3 Semi-discretization of DLM/FD finite element method

Let $T_h(\Omega)$ be a partition of $\Omega$ with the mesh size $h$ that is independent of the location of the interface $\Gamma_t$, and let $T_H(\Omega_t^2)$ be a partition of $\Omega_t^2$ with the mesh size $H$, where, $H$ can be different from $h$. Based on these two meshes, we can introduce the conforming finite element space to each continuous space as: $\boldsymbol{V}_h \subset \boldsymbol{V},\ \boldsymbol{V}_{H,t}^2 \subset \boldsymbol{V}_t^2,\ Q_h \subset Q,\ \boldsymbol{\Lambda}_{H,t} \subset \boldsymbol{\Lambda}_t$. Considering the limited regularity results (18), one possible choice of finite element spaces is the following

$$\begin{aligned}
&\boldsymbol{V}_h = \{\boldsymbol{v} \in \boldsymbol{V} : \boldsymbol{v}|_K \in P^2(K),\ \forall K \in T_h(\Omega)\},\\
&\boldsymbol{V}^2_{H,t} = \{\boldsymbol{v}_2 \in \boldsymbol{V}^2_t : \boldsymbol{v}_2|_K \in P^2(K),\ \forall K \in T_H(\Omega^2_t)\},\\
&Q_h = \{q \in Q : q|_K \in P^1(K),\ \forall K \in T_h(\Omega)\},\\
&\boldsymbol{\Lambda}_{H,t} = \{\boldsymbol{\xi} \in \boldsymbol{\Lambda}_t : \exists\, \boldsymbol{u}_{2,H} \in \boldsymbol{V}^2_{H,t}, \langle \boldsymbol{\xi}, \boldsymbol{v}_2\rangle_{\Omega^2_t} = (\boldsymbol{u}_{2,H}, \boldsymbol{v}_2)_{\boldsymbol{V}^2_t}, \forall \boldsymbol{v}_2 \in \boldsymbol{V}^2_t\}.
\end{aligned} \tag{63}$$

where, $\boldsymbol{V}_h \times Q_h$ is a stable pair of $P^2P^1$ mixed (Taylor-Hood) finite element space for Stokes equations. In fact, it is proved in [Lundberg, Sun and Wang (2019)] that such chosen mixed finite element spaces $\boldsymbol{V}_h \times \boldsymbol{V}^2_{H,t} \times Q_h \times \boldsymbol{\Lambda}_{H,t}$ or $\boldsymbol{V}_h \times \boldsymbol{V}^2_{H,t} \times Q_h \times \boldsymbol{V}^2_{H,t}$ for a fixed time $t$ is stable for the developed DLM/FD method for the stationary Stokes interface problem – the steady state case of the transient Stokes interface problem (1)-(10). Thus, based upon the DLM/FD weak form III (51)-(54), $\boldsymbol{V}_h \times \boldsymbol{V}^2_{H,t} \times Q_h \times \boldsymbol{V}^2_{H,t}$ is still adopted as the stable mixed finite element spaces for the DLM/FD finite element method of (1)-(10), as defined below. Find $(\boldsymbol{u}_h,\ \boldsymbol{u}_{2,H},\ p_h,\ \boldsymbol{\phi}_H) \in (H^1 \cap L^\infty)(0,T;\boldsymbol{V}_h) \times (H^1 \cap L^\infty)(0,T;\boldsymbol{V}^2_{H,t}) \times L^2(0,T;Q_h) \times L^2(0,T;\boldsymbol{V}^2_{H,t})$ such that

$$(\tilde{\rho}\frac{\partial \boldsymbol{u}_h}{\partial t}, \boldsymbol{v}_h)_\Omega + (\tilde{\beta}\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h)_\Omega - (p_h, \nabla\cdot \boldsymbol{v}_h)_\Omega + (\boldsymbol{\phi}_H, \boldsymbol{v}_h)_{\boldsymbol{V}^2_t} = (\tilde{\boldsymbol{f}}, \boldsymbol{v}_h)_\Omega, \tag{64}$$

$$(\nabla\cdot \boldsymbol{u}_h, q_h)_\Omega = 0, \tag{65}$$

$$\begin{aligned}
&\left((\rho_2 - \tilde{\rho}_2)\frac{d\boldsymbol{u}_{2,H}}{dt}|_{\boldsymbol{y}}, \boldsymbol{v}_{2,H}\right)_{\Omega^2_t} - ((\rho_2 - \tilde{\rho}_2)\boldsymbol{w}\cdot\nabla \boldsymbol{u}_{2,H}, \boldsymbol{v}_{2,H})_{\Omega^2_t}\\
&+\left((\beta_2 - \tilde{\beta}_2)\nabla \boldsymbol{u}_{2,H}, \nabla \boldsymbol{v}_{2,H}\right)_{\Omega^2_t} - (\boldsymbol{\phi}_H, \boldsymbol{v}_{2,H})_{\boldsymbol{V}^2_t} = (\boldsymbol{f}_2 - \tilde{\boldsymbol{f}}|_{\Omega^2_t}, \boldsymbol{v}_{2,H})_{\Omega^2_t}\\
&+(\boldsymbol{\tau}, \boldsymbol{v}_{2,H})_{\Gamma_t},
\end{aligned} \tag{66}$$

$$(\boldsymbol{\psi}_H, \boldsymbol{u}_h - \boldsymbol{u}_{2,H})_{\boldsymbol{V}^2_t} = 0, \quad \forall (\boldsymbol{v}_h, \boldsymbol{v}_{2,H}, q_h, \boldsymbol{\psi}_H) \in \boldsymbol{V}_h \times \boldsymbol{V}^2_{H,t} \times Q_h \times \boldsymbol{V}^2_{H,t}. \tag{67}$$

To analyze the convergence and stability properties of the semi-discrete finite element approximation (64)-(67), we first introduce the following bilinear forms for an ease of deduction:

$$a(\tilde{\boldsymbol{u}}, \boldsymbol{u}_2; \boldsymbol{v}, \boldsymbol{v}_2) = (\tilde{\beta}\nabla\tilde{\boldsymbol{u}}, \nabla \boldsymbol{v})_\Omega + \left((\beta_2 - \tilde{\beta}_2)\nabla \boldsymbol{u}_2, \nabla \boldsymbol{v}_2\right)_{\Omega^2_t}, \tag{68}$$

$$b(\boldsymbol{v}, \boldsymbol{v}_2; q, \boldsymbol{\psi}) = -(q, \nabla\cdot \boldsymbol{v})_\Omega + (\boldsymbol{\psi}, \boldsymbol{v}|_{\Omega^2_t} - \boldsymbol{v}_2)_{\boldsymbol{V}^2_t}, \tag{69}$$

and define the discrete divergence-free space as

$$\tilde{\boldsymbol{V}}_h \times \tilde{\boldsymbol{V}}^2_{H,t} := \{(\boldsymbol{v}_h, \boldsymbol{v}_{2,H}) \in \boldsymbol{V}_h \times \boldsymbol{V}^2_{H,t} : b(\boldsymbol{v}_h, \boldsymbol{v}_{2,H}; q_h, \boldsymbol{\psi}_H) = 0,\ \forall (q_h, \boldsymbol{\psi}_H) \in Q_h \times \boldsymbol{V}^2_{H,t}\}.$$

Then we know $(\boldsymbol{u}_h, \boldsymbol{u}_{2,H}) \in \tilde{\boldsymbol{V}}_h \times \tilde{\boldsymbol{V}}^2_{H,t}$.

Let $(\boldsymbol{z}_h,\ \boldsymbol{z}_{2,H},\ \chi_h,\ \boldsymbol{\theta}_H)$ be arbitrary functions in $\tilde{\boldsymbol{V}}_h \times \tilde{\boldsymbol{V}}^2_{H,t} \times Q_h \times \boldsymbol{V}^2_{H,t}$, and let $\boldsymbol{\eta} = \tilde{\boldsymbol{u}} - \boldsymbol{z}_h,\ \boldsymbol{\eta}_2 = \boldsymbol{u}_2 - \boldsymbol{z}_{2,H},\ \boldsymbol{\mu} = \boldsymbol{z}_h - \boldsymbol{u}_h,\ \boldsymbol{\mu}_2 = \boldsymbol{z}_{2,H} - \boldsymbol{u}_{2,H},\ \zeta = \tilde{p} - \chi_h,\ \xi = \chi_h - p_h,\ \boldsymbol{\delta} =$

$\boldsymbol{\phi} - \boldsymbol{\theta}_H,\ \boldsymbol{\gamma} = \boldsymbol{\theta}_H - \boldsymbol{\phi}_H$. Take $\boldsymbol{v}_h = \boldsymbol{\mu},\ \boldsymbol{v}_{2,H} = \boldsymbol{\mu}_2$ in (64)-(67), subtract (64)-(67) from (59)-(62) and consider (52), (54), (65), (67) and (58), yields

$$
\begin{aligned}
&\tilde{\rho}(\frac{\partial \boldsymbol{\mu}}{\partial t}, \boldsymbol{\mu})_\Omega + (\rho_2 - \tilde{\rho}_2)\left(\frac{d\boldsymbol{\mu}_2}{dt}|_{\boldsymbol{y}}, \boldsymbol{\mu}_2\right)_{\Omega_t^2} + a(\boldsymbol{\mu}, \boldsymbol{\mu}_2; \boldsymbol{\mu}, \boldsymbol{\mu}_2) + b(\boldsymbol{\mu}, \boldsymbol{\mu}_2; \xi, \boldsymbol{\gamma}) \\
&= -\tilde{\rho}(\frac{\partial \boldsymbol{\eta}}{\partial t}, \boldsymbol{\mu})_\Omega - (\rho_2 - \tilde{\rho}_2)(\frac{\partial \boldsymbol{\eta}_2}{\partial t}, \boldsymbol{\mu}_2)_{\Omega_t^2} - a(\boldsymbol{\eta}, \boldsymbol{\eta}_2; \boldsymbol{\mu}, \boldsymbol{\mu}_2) - b(\boldsymbol{\mu}, \boldsymbol{\mu}_2; \zeta, \boldsymbol{\delta}) \\
&+(\rho_2 - \tilde{\rho}_2)(\boldsymbol{w} \cdot (\nabla \boldsymbol{\mu}_2 + \nabla \boldsymbol{\eta}_2), \boldsymbol{\mu}_2)_{\Omega_t^2}. \qquad (70)
\end{aligned}
$$

Reynolds transport theorem leads to the following formula

$$
\begin{aligned}
&\left(\frac{d\boldsymbol{\mu}_2}{dt}|_{\boldsymbol{y}}, \boldsymbol{\mu}_2\right)_{\Omega_t^2} = \frac{1}{2}\left(\frac{d(\boldsymbol{\mu}_2)^2}{dt}|_{\boldsymbol{y}}, 1\right)_{\Omega_t^2} = \frac{1}{2}\Big[\frac{\partial}{\partial t}\left((\boldsymbol{\mu}_2)^2, 1\right)_{\Omega_t^2} - \\
&((\boldsymbol{\mu}_2)^2 \nabla \cdot \boldsymbol{w}, 1)_{\Omega_t^2}\Big] = \frac{1}{2}\frac{\partial}{\partial t}\|\boldsymbol{\mu}_2\|^2_{0,\Omega_t^2} - \frac{1}{2}(\boldsymbol{\mu}_2 \nabla \cdot \boldsymbol{w}, \boldsymbol{\mu}_2)_{\Omega_t^2}.
\end{aligned}
$$

Further, apply the fact $b(\boldsymbol{\mu}, \boldsymbol{\mu}_2; \xi, \boldsymbol{\gamma}) = 0$ and Young's $\varepsilon-$inequality to (70), we obtain

$$
\begin{aligned}
&\frac{\partial}{\partial t}\|\boldsymbol{\mu}\|^2_{0,\Omega} + \frac{\partial}{\partial t}\|\boldsymbol{\mu}_2\|^2_{0,\Omega_t^2} + \left(\|\nabla \boldsymbol{\mu}\|^2_{0,\Omega} + \|\nabla \boldsymbol{\mu}_2\|^2_{0,\Omega_t^2}\right) \\
&\le c\left(\|\frac{\partial \boldsymbol{\eta}}{\partial t}\|^2_{0,\Omega} + \|\frac{\partial \boldsymbol{\eta}_2}{\partial t}\|^2_{0,\Omega_t^2} + \|\nabla \boldsymbol{\eta}\|^2_{0,\Omega} + \|\nabla \boldsymbol{\eta}_2\|^2_{0,\Omega_t^2}\right. \\
&\left.+\|\boldsymbol{\mu}\|^2_{0,\Omega} + \|\boldsymbol{\mu}_2\|^2_{0,\Omega_t^2} + \|\zeta\|^2_{0,\Omega} + \|\boldsymbol{\delta}\|^2_{\boldsymbol{V}_t^2}\right) + \varepsilon\left(\|\boldsymbol{\mu}\|^2_{\boldsymbol{V}} + \|\boldsymbol{\mu}_2\|^2_{\boldsymbol{V}_t^2}\right), \qquad (71)
\end{aligned}
$$

where, we use conditions (19), (20) and (57). Add $\|\boldsymbol{\mu}\|^2_{0,\Omega} + \|\boldsymbol{\mu}_2\|^2_{0,\Omega_t^2}$ to both sides of (71), choose a sufficiently small $\varepsilon$, then take integral on both sides of (71) with respect to time from 0 to $t$, yields

$$
\begin{aligned}
&\|\boldsymbol{\mu}\|^2_{0,\Omega} + \|\boldsymbol{\mu}_2\|^2_{0,\Omega_t^2} + \int_0^t \left(\|\boldsymbol{\mu}\|^2_{\boldsymbol{V}} + \|\boldsymbol{\mu}_2\|^2_{\boldsymbol{V}_t^2}\right) d\tilde{t} \le \|\boldsymbol{\mu}(0)\|^2_{0,\Omega} + \|\boldsymbol{\mu}_2(0)\|^2_{0,\Omega_0^2} \\
&+c\int_0^t \left(\|\frac{\partial \boldsymbol{\eta}}{\partial t}\|^2_{0,\Omega} + \|\frac{\partial \boldsymbol{\eta}_2}{\partial t}\|^2_{0,\Omega_t^2} + \|\nabla \boldsymbol{\eta}\|^2_{0,\Omega} + \|\nabla \boldsymbol{\eta}_2\|^2_{0,\Omega_t^2} + \|\zeta\|^2_{0,\Omega} + \|\boldsymbol{\delta}\|^2_{\boldsymbol{V}_t^2}\right. \\
&\left.+\|\boldsymbol{\mu}\|^2_{0,\Omega} + \|\boldsymbol{\mu}_2\|^2_{0,\Omega_t^2}\right) d\tilde{t}. \qquad (72)
\end{aligned}
$$

Apply the Grönwall's inequality, leads to

$$
\begin{aligned}
&\|\boldsymbol{\mu}\|_{L^\infty(0,T;(L^2(\Omega))^d)} + \|\boldsymbol{\mu}_2\|_{L^\infty(0,T;(L^2(\Omega_t^2))^d)} + \|\boldsymbol{\mu}\|_{L^2(0,T;\boldsymbol{V})} + \|\boldsymbol{\mu}_2\|_{L^2(0,T;\boldsymbol{V}_t^2)} \\
&\le c\Big[\|\boldsymbol{\mu}(0)\|_{0,\Omega} + \|\boldsymbol{\mu}_2(0)\|_{0,\Omega_0^2} + \|\frac{\partial \boldsymbol{\eta}}{\partial t}\|_{L^2(0,T;(L^2(\Omega))^d)} + \|\frac{\partial \boldsymbol{\eta}_2}{\partial t}\|_{L^2(0,T;(L^2(\Omega_t^2))^d)} \\
&+\|\boldsymbol{\eta}\|_{L^2(0,T;\boldsymbol{V})} + \|\boldsymbol{\eta}_2\|_{L^2(0,T;\boldsymbol{V}_t^2)} + \|\zeta\|_{L^2(0,T;L^2(\Omega))} + \|\boldsymbol{\delta}\|_{L^2(0,T;\boldsymbol{V}_t^2)}\Big]. \qquad (73)
\end{aligned}
$$

Then, we have the following convergence theorem for the semi-discretization (64)-(67).

**Theorem 3.1.** Suppose $(\tilde{\boldsymbol{u}}, \boldsymbol{u}_2, p, \boldsymbol{\phi})$ is the solution to (59)-(62), $(\boldsymbol{u}_h, \boldsymbol{u}_{2,H}, p_h, \boldsymbol{\phi}_H)$ is the solution to (64)-(67). With $P^2$-$P^2$-$P^1$-$P^2$ mixed finite element to respectively discretize $\boldsymbol{u}_h,\ \boldsymbol{u}_{2,H},\ p_h,\ \boldsymbol{\phi}_H$, we have the following error estimate

$$\begin{aligned}
&\|\tilde{\boldsymbol{u}} - \boldsymbol{u}_h\|_{L^\infty(0,T;(L^2(\Omega))^d)} + \|\boldsymbol{u}_2 - \boldsymbol{u}_{2,H}\|_{L^\infty(0,T;(L^2(\Omega_t^2))^d)} + \|\tilde{\boldsymbol{u}} - \boldsymbol{u}_h\|_{L^2(0,T;\boldsymbol{V})} \\
&+\|\boldsymbol{u}_2 - \boldsymbol{u}_{2,H}\|_{L^2(0,T;\boldsymbol{V}_t^2)} \le c\Bigg[\|\tilde{\boldsymbol{u}}^0 - \boldsymbol{u}_h(0)\|_{0,\Omega} + \|\boldsymbol{u}_2^0 - \boldsymbol{u}_{2,H}(0)\|_{0,\Omega_0^2} \\
&+ \inf_{(\boldsymbol{z}_h, \boldsymbol{z}_{2,H}, \chi_h, \boldsymbol{\theta}_H) \in \tilde{\boldsymbol{V}}_h \times \tilde{\boldsymbol{V}}_{H,t}^2 \times Q_h \times \boldsymbol{V}_{H,t}^2} \Bigg( \|\frac{\partial(\tilde{\boldsymbol{u}} - \boldsymbol{z}_h)}{\partial t}\|_{L^2(0,T;(L^2(\Omega))^d)} \\
&+\|\frac{\partial(\boldsymbol{u}_2 - \boldsymbol{z}_{2,H})}{\partial t}\|_{L^2(0,T;(L^2(\Omega_t^2))^d)} + \|\tilde{\boldsymbol{u}} - \boldsymbol{z}_h\|_{L^2(0,T;\boldsymbol{V})} + \|\boldsymbol{u}_2 - \boldsymbol{z}_{2,H}\|_{L^2(0,T;\boldsymbol{V}_t^2)} \\
&+\|\tilde{p} - \chi_h\|_{L^2(0,T;L^2(\Omega))} + \|\boldsymbol{\phi} - \boldsymbol{\theta}_H\|_{L^2(0,T;\boldsymbol{V}_t^2)} \Bigg)\Bigg]. \qquad (74)
\end{aligned}$$

Note that on the right hand side of (74), all terms but $\inf_{\boldsymbol{\theta}_H \in \boldsymbol{V}_{H,t}^2} \|\boldsymbol{\phi} - \boldsymbol{\theta}_H\|_{\boldsymbol{V}_t^2}$ can be directly estimated based on a priori interpolation error estimates and the regularity assumptions (18). To find an error estimate for $\inf_{\boldsymbol{\theta}_H \in \boldsymbol{V}_{H,t}^2} \|\boldsymbol{\phi} - \boldsymbol{\theta}_H\|_{\boldsymbol{V}_t^2}$, we first pick any $\boldsymbol{v} \in \boldsymbol{V}$ in (51) such that $\boldsymbol{v} = 0$ outside $\Omega_t^1$ including $\partial\Omega_t^1$. Integrating by parts gives

$$\tilde{\boldsymbol{f}} \;=\; \tilde{\rho}\frac{\partial \tilde{\boldsymbol{u}}}{\partial t} - \nabla\cdot(\tilde{\beta}\nabla\tilde{\boldsymbol{u}}) + \nabla\tilde{p}, \quad \text{in } \Omega_t^1. \qquad (75)$$

Similarly, we can pick $(\boldsymbol{v}, \boldsymbol{v}_2) \in \boldsymbol{V} \times \boldsymbol{V}_t^2$ such that $\boldsymbol{v}|_{\Omega_t^2} = \boldsymbol{v}_2$ and $\boldsymbol{v} = 0$ outside $\Omega_t^2$ including $\partial\Omega_t^2$. Add (51) to (53) and integrate by parts, yields

$$\boldsymbol{f}_2 \;=\; \rho_2\frac{\partial \boldsymbol{u}_2}{\partial t} - \nabla\cdot(\beta_2\nabla\boldsymbol{u}_2) + \nabla\tilde{p}, \quad \text{in } \Omega_t^2. \qquad (76)$$

Now, let $\boldsymbol{v} \in \boldsymbol{V}$ and take $\boldsymbol{v}_2 = \boldsymbol{v}|_{\Omega_t^2}$ in $\Omega_t^2$. Add (51) to (53), we have

$$\begin{aligned}
&(\tilde{\rho}\frac{\partial\tilde{\boldsymbol{u}}}{\partial t}, \boldsymbol{v})_{\Omega_t^1} + (\rho_2\frac{\partial\boldsymbol{u}_2}{\partial t}, \boldsymbol{v})_{\Omega_t^2} + (\tilde{\beta}\nabla\tilde{\boldsymbol{u}}, \nabla\boldsymbol{v})_{\Omega_t^1} - (\tilde{p}, \nabla\cdot\boldsymbol{v})_{\Omega_t^1} \\
&-(\tilde{p}, \nabla\cdot\boldsymbol{v})_{\Omega_t^2} + (\beta_2\nabla\boldsymbol{u}_2, \nabla\boldsymbol{v})_{\Omega_t^2} = (\tilde{\boldsymbol{f}}, \boldsymbol{v})_{\Omega_t^1} + (\boldsymbol{f}_2, \boldsymbol{v})_{\Omega_t^2} + (\boldsymbol{\tau}, \boldsymbol{v})_{\Gamma_t}, \qquad (77)
\end{aligned}$$

where the integrals over $\Omega$ are split into $\Omega_t^1$ and $\Omega_t^2$. Integrating by parts, yields

$$\begin{aligned}
&(\tilde{\rho}\frac{\partial\tilde{\boldsymbol{u}}}{\partial t}, \boldsymbol{v})_{\Omega_t^1} + (\rho_2\frac{\partial\boldsymbol{u}_2}{\partial t}, \boldsymbol{v})_{\Omega_t^2} - (\nabla\cdot(\tilde{\beta}\nabla\tilde{\boldsymbol{u}}), \boldsymbol{v})_{\Omega_t^1} + (\tilde{\beta}\nabla\tilde{\boldsymbol{u}}\,\boldsymbol{n}_1, \boldsymbol{v})_{\Gamma_t} + (\nabla\tilde{p}, \boldsymbol{v})_{\Omega_t^1} \\
&-(\tilde{p}, \boldsymbol{v}\cdot\boldsymbol{n}_1)_{\Gamma_t} + (\nabla\tilde{p}, \boldsymbol{v})_{\Omega_t^2} - (\tilde{p}, \boldsymbol{v}\cdot\boldsymbol{n}_2)_{\Gamma_t} - (\nabla\cdot(\beta_2\nabla\boldsymbol{u}_2), \boldsymbol{v})_{\Omega_t^2} + (\beta_2\nabla\boldsymbol{u}_2\,\boldsymbol{n}_2, \boldsymbol{v})_{\Gamma_t} \\
&= (\tilde{\boldsymbol{f}}, \boldsymbol{v})_{\Omega_t^1} + (\boldsymbol{f}_2, \boldsymbol{v})_{\Omega_t^2} + (\boldsymbol{\tau}, \boldsymbol{v})_{\Gamma_t}, \qquad (78)
\end{aligned}$$

then apply (75) and (76), we attain

$$(\tilde{\beta}\nabla\tilde{\boldsymbol{u}}\,\boldsymbol{n}_1,\boldsymbol{v})_{\Gamma_t}+(\beta_2\nabla\boldsymbol{u}_2\,\boldsymbol{n}_2,\boldsymbol{v})_{\Gamma_t}=(\boldsymbol{\tau},\boldsymbol{v})_{\Gamma_t}. \tag{79}$$

Further, apply (76) and (79) to (53), and note that $\boldsymbol{n}_1=-\boldsymbol{n}_2$, we have

$$\begin{aligned}
&(\boldsymbol{\phi},\boldsymbol{v}_2)_{\boldsymbol{V}_t^2}=-(\tilde{\rho}_2\tfrac{\partial\boldsymbol{u}_2}{\partial t},\boldsymbol{v}_2)_{\Omega_t^2}+(\rho_2\tfrac{\partial\boldsymbol{u}_2}{\partial t},\boldsymbol{v}_2)_{\Omega_t^2}-(\tilde{\beta}\nabla\boldsymbol{u}_2,\nabla\boldsymbol{v}_2)_{\Omega_t^2}+(\beta_2\nabla\boldsymbol{u}_2,\boldsymbol{v}_2)_{\Omega_t^2}\\
&-(\boldsymbol{f}_2,\boldsymbol{v}_2)_{\Omega_t^2}+(\tilde{\boldsymbol{f}},\boldsymbol{v}_2)_{\Omega_t^2}-(\boldsymbol{\tau},\boldsymbol{v}_2)_{\Gamma_t}\\
&=-(\tilde{\rho}_2\tfrac{\partial\boldsymbol{u}_2}{\partial t},\boldsymbol{v}_2)_{\Omega_t^2}+(\rho_2\tfrac{\partial\boldsymbol{u}_2}{\partial t},\boldsymbol{v}_2)_{\Omega_t^2}+(\nabla\cdot(\tfrac{\tilde{\beta}}{\beta_2}\beta_2\nabla\boldsymbol{u}_2),\boldsymbol{v}_2)_{\Omega_t^2}-(\tilde{\beta}\nabla\boldsymbol{u}_2\,\boldsymbol{n}_2,\boldsymbol{v}_2)_{\Gamma_t}\\
&-(\nabla\cdot(\beta_2\nabla\boldsymbol{u}_2),\boldsymbol{v}_2)_{\Omega_t^2}+(\beta_2\nabla\boldsymbol{u}_2\,\boldsymbol{n}_2,\boldsymbol{v}_2)_{\Gamma_t}-(\nabla\tilde{p}|_{\Omega_t^2},\boldsymbol{v}_2)_{\Omega_t^2}+(\nabla\tilde{p}|_{\Omega_t^2},\boldsymbol{v}_2)_{\Omega_t^2}\\
&-(\boldsymbol{f}_2,\boldsymbol{v}_2)_{\Omega_t^2}+(\tilde{\boldsymbol{f}},\boldsymbol{v}_2)_{\Omega_t^2}-(\boldsymbol{\tau},\boldsymbol{v}_2)_{\Gamma_t}\\
&=-(\tilde{\rho}_2\tfrac{\partial\boldsymbol{u}_2}{\partial t},\boldsymbol{v}_2)_{\Omega_t^2}+(\beta_2\nabla\boldsymbol{u}_2\nabla\tfrac{\tilde{\beta}}{\beta_2},\boldsymbol{v}_2)_{\Omega_t^2}+(\tfrac{\tilde{\beta}}{\beta_2}\nabla\cdot(\beta_2\nabla\boldsymbol{u}_2),\boldsymbol{v}_2)_{\Omega_t^2}-(\tilde{\beta}\nabla\boldsymbol{u}_2\,\boldsymbol{n}_2,\boldsymbol{v}_2)_{\Gamma_t}\\
&+(\beta_2\nabla\boldsymbol{u}_2\,\boldsymbol{n}_2,\boldsymbol{v}_2)_{\Gamma_t}-(\nabla\tilde{p}|_{\Omega_t^2},\boldsymbol{v}_2)_{\Omega_t^2}+(\tilde{\boldsymbol{f}},\boldsymbol{v}_2)_{\Omega_t^2}\\
&=-(\tilde{\rho}_2\tfrac{\partial\boldsymbol{u}_2}{\partial t},\boldsymbol{v}_2)_{\Omega_t^2}+(\beta_2\nabla\boldsymbol{u}_2\nabla\tfrac{\tilde{\beta}}{\beta_2},\boldsymbol{v}_2)_{\Omega_t^2}+(\tfrac{\tilde{\beta}}{\beta_2}\nabla\cdot(\beta_2\nabla\boldsymbol{u}_2),\boldsymbol{v}_2)_{\Omega_t^2}-(\tilde{\beta}\nabla\boldsymbol{u}_2\,\boldsymbol{n}_2,\boldsymbol{v}_2)_{\Gamma_t}\\
&-(\tilde{\beta}\nabla\tilde{\boldsymbol{u}}\,\boldsymbol{n}_1,\boldsymbol{v}_2)_{\Gamma_t}-(\nabla\tilde{p}|_{\Omega_t^2},\boldsymbol{v}_2)_{\Omega_t^2}+(\tilde{\boldsymbol{f}},\boldsymbol{v}_2)_{\Omega_t^2}\\
&=\left[-(\tfrac{\tilde{\beta}}{\beta_2}\boldsymbol{f}_2-\tilde{\boldsymbol{f}},\boldsymbol{v}_2)_{\Omega_t^2}\right]+\left[-(\tilde{\beta}(\nabla\boldsymbol{u}_2-\nabla\tilde{\boldsymbol{u}})\boldsymbol{n}_2,\boldsymbol{v}_2)_{\Gamma_t}\right]\\
&+\left[(\beta_2\nabla\boldsymbol{u}_2\nabla\tfrac{\tilde{\beta}}{\beta_2},\boldsymbol{v}_2)_{\Omega_t^2}-((1-\tfrac{\tilde{\beta}}{\beta_2})\nabla\tilde{p},\boldsymbol{v}_2)_{\Omega_t^2}\right]+\left[\left((\tfrac{\tilde{\beta}}{\beta_2}\rho_2-\tilde{\rho}_2)\tfrac{\partial\boldsymbol{u}_2}{\partial t},\boldsymbol{v}_2\right)_{\Omega_t^2}\right].
\end{aligned}$$

We can write $\boldsymbol{\phi}=\boldsymbol{\phi}_1+\boldsymbol{\phi}_2+\boldsymbol{\phi}_3+\boldsymbol{\phi}_4$, where

$$(\boldsymbol{\phi}_1,\boldsymbol{v}_2)_{\boldsymbol{V}_t^2}=-(\frac{\tilde{\beta}}{\beta_2}\boldsymbol{f}_2-\tilde{\boldsymbol{f}},\boldsymbol{v}_2)_{\Omega_t^2}, \tag{80}$$

$$(\boldsymbol{\phi}_2,\boldsymbol{v}_2)_{\boldsymbol{V}_t^2}=-(\tilde{\beta}(\nabla\boldsymbol{u}_2-\nabla\tilde{\boldsymbol{u}})\boldsymbol{n}_2,\boldsymbol{v}_2)_{\Gamma_t}, \tag{81}$$

$$(\boldsymbol{\phi}_3,\boldsymbol{v}_2)_{\boldsymbol{V}_t^2}=(\beta_2\nabla\boldsymbol{u}_2\nabla\frac{\tilde{\beta}}{\beta_2},\boldsymbol{v}_2)_{\Omega_t^2}-((1-\frac{\tilde{\beta}}{\beta_2})\nabla\tilde{p},\boldsymbol{v}_2)_{\Omega_t^2}, \tag{82}$$

$$(\boldsymbol{\phi}_4,\boldsymbol{v}_2)_{\boldsymbol{V}_t^2}=\left((\frac{\tilde{\beta}}{\beta_2}\rho_2-\tilde{\rho}_2)\frac{\partial\boldsymbol{u}_2}{\partial t},\boldsymbol{v}_2\right)_{\Omega_t^2}. \tag{83}$$

With the regularity assumptions (18), we can obtain the following estimates by doing an analogous analysis with [Auricchio, Boffi, Gastaldi et al. (2015); Lundberg, Sun and Wang (2019)] while $t\in(0,T]$ is temporarily fixed,

$$\inf_{\boldsymbol{\theta}_H\in\boldsymbol{V}_{H,t}^2}\|\boldsymbol{\phi}_1-\boldsymbol{\theta}_H\|_{\boldsymbol{V}_t^2}\le cH\|(\tilde{\beta}/\beta_2)\boldsymbol{f}_2-\tilde{\boldsymbol{f}}\|_{(L^2(\Omega_t^2))^d}, \tag{84}$$

$$\inf_{\boldsymbol{\theta}_H\in\boldsymbol{V}_{H,t}^2}\|\boldsymbol{\phi}_2-\boldsymbol{\theta}_H\|_{\boldsymbol{V}_t^2}\le cH^{\sigma-1}\left(\|\tilde{\boldsymbol{u}}\|_{(H^\sigma(\Omega_t^1\cup\Omega_t^2))^d}\right), \tag{85}$$

$$\inf_{\boldsymbol{\theta}_H\in\boldsymbol{V}_{H,t}^2}\|\boldsymbol{\phi}_3-\boldsymbol{\theta}_H\|_{\boldsymbol{V}_t^2}\le cH\left(\|\tilde{p}\|_{H^1(\Omega_t^2)}+\|\boldsymbol{u}_2\|_{\boldsymbol{V}_t^2}\right). \tag{86}$$

To get an estimate for $\inf_{\boldsymbol{\theta}_H\in\boldsymbol{V}_{H,t}^2}\|\boldsymbol{\phi}_4-\boldsymbol{\theta}_H\|_{\boldsymbol{V}_t^2}$, due to (28) and (30), we just need to find an error estimate for $\inf_{\boldsymbol{\xi}_H\in\boldsymbol{\Lambda}_{H,t}}\|\boldsymbol{\lambda}_4-\boldsymbol{\xi}_H\|_{\boldsymbol{\Lambda}_t}$ instead, where, $(\boldsymbol{\phi}_4,\boldsymbol{v}_2)_{\boldsymbol{V}_t^2}=\langle\boldsymbol{\lambda}_4,\boldsymbol{v}_2\rangle_{\boldsymbol{\Lambda}_t}$, $\forall\boldsymbol{v}_2\in$

$\boldsymbol{V}_t^2$. To that end, we first let $\pi_H$ be the $L^2$ projection of $\boldsymbol{V}_t^2$ into $\boldsymbol{V}_{H,t}^2$, that is, for any $\boldsymbol{w}_2 \in \boldsymbol{V}_t^2$,

$$(\pi_H \boldsymbol{w}_2, \boldsymbol{v}_2)_{\Omega_t^2} \quad = \quad (\boldsymbol{w}_2, \boldsymbol{v}_2)_{\Omega_t^2} \quad \forall \boldsymbol{v}_2 \in \boldsymbol{V}_{H,t}^2. \tag{87}$$

It is easy to see that $\boldsymbol{\lambda}_4 \in L^2(\Omega_t^2)$ and $\|\boldsymbol{\lambda}_4\|_0 \leq \|\big(\frac{\tilde{\beta}}{\beta_2}\rho_2 - \tilde{\rho}_2\big)\frac{\partial \boldsymbol{u}_2}{\partial t}\|_0$. On the other hand, because of the choice of finite element space (63), our finite elements $\boldsymbol{V}_{H,t}^2$ and $\boldsymbol{\Lambda}_{H,t}$ are contained in $L^2(\Omega_t^2)$, we can interpret the duality pairing as scalar product in $L^2(\Omega_t^2)$ [Auricchio, Boffi, Gastaldi et al. (2015); Boffi, Gastaldi and Ruggeri (2014)]. Thus, we can define $P_H\boldsymbol{\lambda}_4 \in \boldsymbol{\Lambda}_{H,t}$ be the $L^2$-projection of $\boldsymbol{\lambda}_4$ onto $\boldsymbol{\Lambda}_{H,t}$ such that

$$\langle P_H\boldsymbol{\lambda}_4, \boldsymbol{v}_2\rangle_{\Omega_t^2} = (P_H\boldsymbol{\lambda}_4, \boldsymbol{v}_2)_{\Omega_t^2} = \left(\pi_H\big((\frac{\tilde{\beta}}{\beta_2}\rho_2 - \tilde{\rho}_2)\frac{\partial \boldsymbol{u}_2}{\partial t}\big), \boldsymbol{v}_2\right)_{\Omega_t^2}, \forall \boldsymbol{v}_2 \in \boldsymbol{V}_t^2. \tag{88}$$

So by (87) we have $\langle \boldsymbol{\lambda}_4 - P_H\boldsymbol{\lambda}_4, \boldsymbol{v}_2\rangle_{\Omega_t^2} = 0$ for all $\boldsymbol{v}_2 \in \boldsymbol{V}_{H,t}^2$. Then,

$$\begin{aligned}
&\|\boldsymbol{\lambda}_4 - P_H\boldsymbol{\lambda}_4\|_{\boldsymbol{\Lambda}_t} = \sup_{\boldsymbol{v}_2\in\boldsymbol{V}_t^2} \frac{\langle \boldsymbol{\lambda}_4 - P_H\boldsymbol{\lambda}_4, \boldsymbol{v}_2\rangle_{\Omega_t^2}}{\|\boldsymbol{v}_2\|_{\boldsymbol{V}_t^2}} \\
&= \sup_{\boldsymbol{v}_2\in\boldsymbol{V}_t^2} \frac{\langle \boldsymbol{\lambda}_4 - P_H\boldsymbol{\lambda}_4, \boldsymbol{v}_2 - \pi_H\boldsymbol{v}_2\rangle_{\Omega_t^2}}{\|\boldsymbol{v}_2\|_{\boldsymbol{V}_t^2}} \\
&= \sup_{\boldsymbol{v}_2\in\boldsymbol{V}_t^2} \frac{\langle \boldsymbol{\lambda}_4, \boldsymbol{v}_2 - \pi_H\boldsymbol{v}_2\rangle_{\Omega_t^2} - (\pi_H\big((\frac{\tilde{\beta}}{\beta_2}\rho_2 - \tilde{\rho}_2)\frac{\partial \boldsymbol{u}_2}{\partial t}\big), \boldsymbol{v}_2 - \pi_H\boldsymbol{v}_2)_{\Omega_t^2}}{\|\boldsymbol{v}_2\|_{\boldsymbol{V}_t^2}} \\
&= \sup_{\boldsymbol{v}_2\in\boldsymbol{V}_t^2} \frac{\Big(\big(\frac{\tilde{\beta}}{\beta_2}\rho_2 - \tilde{\rho}_2\big)\frac{\partial \boldsymbol{u}_2}{\partial t}, \boldsymbol{v}_2 - \pi_H\boldsymbol{v}_2\Big)_{\Omega_t^2}}{\|\boldsymbol{v}_2\|_{\boldsymbol{V}_t^2}},
\end{aligned} \tag{89}$$

where, we apply (87) and (83). By applying the Cauchy–Schwartz inequality and the a priori interpolation error estimate for $\pi_H$, we obtain

$$\begin{aligned}
&\left(\big(\frac{\tilde{\beta}}{\beta_2}\rho_2 - \tilde{\rho}_2\big)\frac{\partial \boldsymbol{u}_2}{\partial t}, \boldsymbol{v}_2 - \pi_H\boldsymbol{v}_2\right)_{\Omega_t^2} \\
&\leq c\|\frac{\tilde{\beta}}{\beta_2}\rho_2 - \tilde{\rho}_2\|_{L^\infty(\Omega_t^2)}\|\frac{\partial \boldsymbol{u}_2}{\partial t}\|_{0,\Omega_t^2}\|\boldsymbol{v}_2 - \pi_H\boldsymbol{v}_2\|_{0,\Omega_t^2} \leq cH\|\frac{\partial \boldsymbol{u}_2}{\partial t}\|_{0,\Omega_t^2}\|\boldsymbol{v}_2\|_{\boldsymbol{V}_t^2} \quad .
\end{aligned} \tag{90}$$

Then there exists a constant $c > 0$ such that

$$\inf_{\boldsymbol{\xi}_H\in\boldsymbol{\Lambda}_{H,t}} \|\boldsymbol{\lambda}_4 - \boldsymbol{\xi}_H\|_{\boldsymbol{\Lambda}_t} \leq cH\|\frac{\partial \boldsymbol{u}_2}{\partial t}\|_{0,\Omega_t^2}. \tag{91}$$

Combine (84)-(86) and (91), we obtain the error estimate of $\inf\limits_{\boldsymbol{\theta}_H\in\boldsymbol{V}^2_{H,t}}\|\boldsymbol{\phi}-\boldsymbol{\theta}_H\|_{\boldsymbol{V}_t^2}$, displayed as

$$\inf_{\boldsymbol{\theta}_H\in\boldsymbol{V}^2_{H,t}}\|\boldsymbol{\phi}-\boldsymbol{\theta}_H\|_{\boldsymbol{V}_t^2}\le cH^{\sigma-1}\left(\|\tilde{\boldsymbol{u}}\|_{(H^\sigma(\Omega_t^1\cup\Omega_t^2))^d}+\|\frac{\partial\boldsymbol{u}_2}{\partial t}\|_{(L^2(\Omega_t^2))^d}+\|\tilde{p}\|_{H^1(\Omega_t^2)}\right.$$
$$\left.+\|(\tilde{\beta}/\beta_2)\boldsymbol{f}_2-\tilde{\boldsymbol{f}}\|_{(L^2(\Omega_t^2))^d}\right). \quad (92)$$

Together with the a priori interpolation error estimates for $\inf\limits_{\boldsymbol{z}_h\in\boldsymbol{V}_h}\|\tilde{\boldsymbol{u}}-\boldsymbol{z}_h\|_{\boldsymbol{V}}$, $\inf\limits_{\boldsymbol{z}_{2,H}\in\boldsymbol{V}^2_{H,t}}\|\boldsymbol{u}_2-\boldsymbol{z}_{2,H}\|_{\boldsymbol{V}_t^2}$, $\inf\limits_{\chi_h\in Q_h}\|\tilde{p}-\chi_h\|_Q$, $\inf\limits_{\boldsymbol{z}_{2,H}\in\boldsymbol{V}^2_{H,t}}\|\frac{\partial(\boldsymbol{u}_2-\boldsymbol{z}_{2,H})}{\partial t}\|_{(L^2(\Omega_t^2))^d}$ and $\inf\limits_{\boldsymbol{z}_h\in\boldsymbol{V}_h}\|\frac{\partial(\tilde{\boldsymbol{u}}-\boldsymbol{z}_h)}{\partial t}\|_{(L^2(\Omega))^d}$, and take $\boldsymbol{u}_h(0)=\pi_h\tilde{\boldsymbol{u}}^0$, $\boldsymbol{u}_{2,H}(0)=\pi_H\boldsymbol{u}_2^0$ where $\pi_h:\boldsymbol{V}\to\boldsymbol{V}_h$ and $\pi_H:\boldsymbol{V}_t^2\to\boldsymbol{V}^2_{H,t}$ are appropriately defined interpolation operators . Then we attain the following a priori error estimates for the semi-discrete DLM/FD finite element method of the transient Stokes interface problem.

**Theorem 3.2.** Let $(\tilde{\boldsymbol{u}},\boldsymbol{u}_2,\tilde{p},\boldsymbol{\phi})\in\boldsymbol{V}\times\boldsymbol{V}_t^2\times Q\times\boldsymbol{V}_t^2$ be the solution to (59)-(62), and let $(\boldsymbol{u}_h,\boldsymbol{u}_{2,H},p_h,\boldsymbol{\phi}_H)\in\boldsymbol{V}_h\times\boldsymbol{V}^2_{H,t}\times Q_h\times\boldsymbol{V}^2_{H,t}$ be the solution to (64)-(67). If (19), (20) and (57) hold, then there exists a constant $c>0$ independent of $h$ and $H$ such that

$$\|\tilde{\boldsymbol{u}}-\boldsymbol{u}_h\|_{L^2(0,T;\boldsymbol{V})}+\|\boldsymbol{u}_2-\boldsymbol{u}_{2,H}\|_{L^2(0,T;\boldsymbol{V}_t^2)}\le c(h^{\sigma-1}+H^{\sigma-1})$$
$$(\|\tilde{\boldsymbol{u}}\|_{H^1(0,T;(H^\sigma(\Omega_t^1\cup\Omega_t^2))^d)}+\|\tilde{p}\|_{L^2(0,T;H^1(\Omega_t^1\cup\Omega_t^2))}+\|(\tilde{\beta}/\beta_2)\boldsymbol{f}_2-\tilde{\boldsymbol{f}}\|_{L^2(0,T;(L^2(\Omega))^d)}),$$

where, $\frac{3}{2}<\sigma\le 2$.

Now we analyze the stability property of the semi-discrete scheme. Take $\boldsymbol{v}_h=\boldsymbol{u}_h,\ \boldsymbol{v}_{2,H}=\boldsymbol{u}_{2,H}$ in (64)-(67), yields

$$\tilde{\rho}(\frac{\partial\boldsymbol{u}_h}{\partial t},\boldsymbol{u}_h)_\Omega+(\rho_2-\tilde{\rho}_2)\left(\frac{d\boldsymbol{u}_{2,H}}{dt}|_{\boldsymbol{y}},\boldsymbol{u}_{2,H}\right)_{\Omega_t^2}+a(\boldsymbol{u}_h,\boldsymbol{u}_{2,H};\boldsymbol{u}_h,\boldsymbol{u}_{2,H})$$
$$+b(\boldsymbol{u}_h,\boldsymbol{u}_{2,H};p_h,\boldsymbol{\phi}_H)=(\rho_2-\tilde{\rho}_2)(\boldsymbol{w}\cdot\nabla\boldsymbol{u}_{2,H},\boldsymbol{u}_{2,H})_{\Omega_t^2}+(\tilde{\boldsymbol{f}},\boldsymbol{u}_h)_\Omega$$
$$+(\boldsymbol{f}_2-\tilde{\boldsymbol{f}}|_{\Omega_t^2},\boldsymbol{u}_{2,H})_{\Omega_t^2}+(\boldsymbol{\tau},\boldsymbol{u}_{2,H})_{\Gamma_t}.$$

Note that $b(\boldsymbol{u}_h,\boldsymbol{u}_{2,H};p_h,\boldsymbol{\phi}_H)=0$, apply Reynolds transport theorem and Young's $\varepsilon-$inequality, and use (19), (20) and (57), results

$$\frac{\partial}{\partial t}\|\boldsymbol{u}_h\|^2_{0,\Omega}+\frac{\partial}{\partial t}\|\boldsymbol{u}_{2,H}\|^2_{0,\Omega_t^2}+\left(\|\nabla\boldsymbol{u}_h\|^2_{0,\Omega}+\|\nabla\boldsymbol{u}_{2,H}\|^2_{0,\Omega_t^2}\right)$$
$$\le c\left(\|\boldsymbol{u}_h\|^2_{0,\Omega}+\|\boldsymbol{u}_{2,H}\|^2_{0,\Omega_t^2}+\|\tilde{\boldsymbol{f}}\|^2_{0,\Omega}+\|\boldsymbol{f}_2-\tilde{\boldsymbol{f}}|_{\Omega_t^2}\|^2_{0,\Omega_t^2}+\|\boldsymbol{\tau}\|^2_{0,\Gamma_t}\right)$$
$$+\varepsilon\|\boldsymbol{u}_{2,H}\|^2_{\boldsymbol{V}_t^2}, \quad (93)$$

where, the trace estimate is applied to get $\|\boldsymbol{u}_{2,H}\|_{0,\Gamma_t} \leq c\|\boldsymbol{u}_{2,H}\|_{\boldsymbol{V}_t^2}$. Integrate both sides of (93) in time from 0 to $t$, add $\|\boldsymbol{u}_{2,H}\|^2_{0,\Omega_t^2}$ to both sides, choose a sufficiently small $\varepsilon$, and apply Poincaré inequality in $\Omega$ and Grönwall's inequality to (93), reads

$$
\begin{aligned}
&\|\boldsymbol{u}_h\|_{L^\infty(0,T;(L^2(\Omega))^d)} + \|\boldsymbol{u}_{2,H}\|_{L^\infty(0,T;(L^2(\Omega_t^2))^d)} + \|\boldsymbol{u}_h\|_{L^2(0,T;\boldsymbol{V})} + \|\boldsymbol{u}_{2,H}\|_{L^2(0,T;\boldsymbol{V}_t^2)} \\
&\leq c\bigg( \|\boldsymbol{u}_h(0)\|_{0,\Omega} + \|\boldsymbol{u}_{2,H}(0)\|_{0,\Omega^2{}_0} + \|\boldsymbol{f}^{\tilde{}}\|_{L^2(0,T;(L^2(\Omega))^d)} \\
&+\|\boldsymbol{f}_2 - \tilde{\boldsymbol{f}}|_{\Omega_t^2}\|_{L^2(0,T;(L^2(\Omega))^d)} + \|\boldsymbol{\tau}\|_{L^2(0,T;(L^2(\Gamma_t))^d)}\bigg).
\end{aligned}
\tag{94}
$$

Then, we have the following stability theorem for the semi-discrete scheme.

**Theorem 3.3.** Suppose all hypotheses of Theorem 3.1 are held, then the stability result (94) exists for (64)-(67).

## 4 Full discretization of DLM/FD finite element method

Introduce a uniform partition $0 = t_0 < t_1 < \cdots < t_N = T$ with the time-step size $\Delta t = T/N$, then set $t^n = n\Delta t$ where $n \geq 0$ is an integer, and

$$
\begin{aligned}
&\varphi^n = \varphi(\boldsymbol{x}^n, t^n), \quad d_t\varphi^n = \tfrac{\varphi(\boldsymbol{x},t^{n+1})-\varphi(\boldsymbol{x},t^n)}{\Delta t}, \\
&d_t^{\boldsymbol{X}_t}\varphi^n = \tfrac{1}{\Delta t}\left[\varphi^{n+1} - \varphi\left(\boldsymbol{X}_{t^n} \circ \boldsymbol{X}_{t^{n+1}}^{-1}(\boldsymbol{x}^{n+1}), t^n\right)\right].
\end{aligned}
$$

At $t^n$, we particularly let $T_{2,H}^n$ be a partition of $\Omega_n^2 := \Omega_{t^n}^2$ with the mesh size $H$, and let $\boldsymbol{V}_{2,H}^n \subset \boldsymbol{V}_n^2 := \boldsymbol{V}_{t^n}^2$ be the conforming finite element space defined on $T_{2,H}^n$. And, we still let $T_h$ be a partition of $\Omega$ with the mesh size $h$ that is independent of the location of the interface $\Gamma_n := \Gamma_{t^n}$. Then, the full discretization of DLM/FD finite element approximation for (59)-(62) can be defined as: for $n = 0, 1, \cdots, N-1$, suppose $\boldsymbol{u}_h^n \in \boldsymbol{V}$ and $\boldsymbol{u}_{2,H}^n \in \boldsymbol{V}_{2,H}^n$ are known, find $(\boldsymbol{u}_h^{n+1},\ \boldsymbol{u}_{2,H}^{n+1},\ p_h^{n+1},\ \boldsymbol{\phi}_H^{n+1}) \in \boldsymbol{V}_h \times \boldsymbol{V}_{2,H}^{n+1} \times Q_h \times \boldsymbol{V}_{2,H}^{n+1}$ such that

$$
\begin{aligned}
&\tilde{\rho}(d_t\boldsymbol{u}_h^n, \boldsymbol{v}_h)_\Omega + (\tilde{\beta}^{n+1}\nabla\boldsymbol{u}_h^{n+1}, \nabla\boldsymbol{v}_h)_\Omega - (p_h^{n+1}, \nabla\cdot\boldsymbol{v}_h)_\Omega + \left(\boldsymbol{\phi}_H^{n+1}, \boldsymbol{v}_h\right)_{\boldsymbol{V}_{n+1}^2} \\
&= (\tilde{\boldsymbol{f}}^{n+1}, \boldsymbol{v}_h)_\Omega, &(95)\\
&(\nabla\cdot\boldsymbol{u}_h^{n+1}, q_h)_\Omega = 0, &(96)\\
&(\rho_2 - \tilde{\rho}_2)\left(d_t^{\boldsymbol{X}_t}\boldsymbol{u}_{2,H}^n, \boldsymbol{v}_{2,H}\right)_{\Omega_{n+1}^2} - ((\rho_2 - \tilde{\rho}_2)\boldsymbol{w}^{n+1}\cdot\nabla\boldsymbol{u}_{2,H}^{n+1}, \boldsymbol{v}_{2,H})_{\Omega_{n+1}^2} \\
&+ \left((\beta_2 - \tilde{\beta}_2)^{n+1}\nabla\boldsymbol{u}_{2,H}^{n+1}, \nabla\boldsymbol{v}_{2,H}\right)_{\Omega_{n+1}^2} - \left(\boldsymbol{\phi}_H^{n+1}, \boldsymbol{v}_{2,H}\right)_{\boldsymbol{V}_{n+1}^2} \\
&= (\boldsymbol{f}_2^{n+1} - \tilde{\boldsymbol{f}}^{n+1}|_{\Omega_{n+1}^2}, \boldsymbol{v}_{2,H})_{\Omega_{n+1}^2} + (\boldsymbol{\tau}^{n+1}, \boldsymbol{v}_{2,H})_{\Gamma_{n+1}}, &(97)\\
&\left(\boldsymbol{\psi}_H, \boldsymbol{u}_h^{n+1} - \boldsymbol{u}_{2,H}^{n+1}\right)_{\boldsymbol{V}_{n+1}^2} = 0, &(98)\\
&\forall(\boldsymbol{v}_h, \boldsymbol{v}_{2,H}, q_h, \boldsymbol{\psi}_H) \in \boldsymbol{V}_h \times \boldsymbol{V}_{2,H}^{n+1} \times Q_h \times \boldsymbol{V}_{2,H}^{n+1}.
\end{aligned}
$$

Introduce the following bilinear forms at $t^n$:

$$\begin{aligned} a^n(\tilde{\boldsymbol{u}}^n, \boldsymbol{u}_2^n; \boldsymbol{v}, \boldsymbol{v}_2) &= (\tilde{\beta}^n \nabla \tilde{\boldsymbol{u}}^n, \nabla \boldsymbol{v})_\Omega + \left( (\beta_2^n - \tilde{\beta}_2^n) \nabla \boldsymbol{u}_2^n, \nabla \boldsymbol{v}_2 \right)_{\Omega_n^2}, \\ b^n(\boldsymbol{v}^n, \boldsymbol{v}_2^n; q, \boldsymbol{\psi}) &= -(q, \nabla \cdot \boldsymbol{v}^n)_\Omega + (\boldsymbol{\psi}, \boldsymbol{v}^n|_{\Omega_n^2} - \boldsymbol{v}_2^n)_{\boldsymbol{V}_n^2}. \end{aligned}$$

Now we analyze the error estimate of the full discretization (95)-(98) by letting $(\boldsymbol{z}_h, \boldsymbol{z}_{2,H}, \chi_h, \boldsymbol{\theta}_H)$ be arbitrary functions in $\tilde{\boldsymbol{V}}_h \times \tilde{\boldsymbol{V}}_{2,H}^{n+1} \times Q_h \times \boldsymbol{V}_{2,H}^{n+1}$, where $\tilde{\boldsymbol{V}}_h \times \tilde{\boldsymbol{V}}_{2,H}^{n+1}$ is the discrete divergence-free space at $t^{n+1}$. With the same notations in Section 3, we have the following error equation by subtracting (95)-(98) from (59)-(62),

$$\begin{aligned} &\tilde{\rho}(d_t \boldsymbol{\mu}^n, \boldsymbol{v}_h)_\Omega + (\rho_2 - \tilde{\rho}_2) \left( d_t^{\boldsymbol{X}_t} \boldsymbol{\mu}_2^n, \boldsymbol{v}_{2,H} \right)_{\Omega_{n+1}^2} + a^{n+1}(\boldsymbol{\mu}^{n+1}, \boldsymbol{\mu}_2^{n+1}; \boldsymbol{v}_h, \boldsymbol{v}_{2,H}) \\ &+ b^{n+1}(\boldsymbol{v}_h, \boldsymbol{v}_{2,H}; \xi^{n+1}, \boldsymbol{\gamma}^{n+1}) = -\tilde{\rho}(d_t \boldsymbol{\eta}^n, \boldsymbol{v}_h)_\Omega - \tilde{\rho} \left( \left( \frac{\partial \tilde{\boldsymbol{u}}}{\partial t} \right)^{n+1} - d_t \tilde{\boldsymbol{u}}^n, \boldsymbol{v}_h \right)_\Omega \\ &- (\rho_2 - \tilde{\rho}_2) \left( d_t^{\boldsymbol{X}_t} \boldsymbol{\eta}_2^n, \boldsymbol{v}_{2,H} \right)_{\Omega_{n+1}^2} - (\rho_2 - \tilde{\rho}_2) \left( \frac{d\boldsymbol{u}_2}{dt}\Big|_{\boldsymbol{y}}^{n+1} - d_t^{\boldsymbol{X}_t} \boldsymbol{u}_2^n, \boldsymbol{v}_{2,H} \right)_{\Omega_{n+1}^2} \\ &- a^{n+1}(\boldsymbol{\eta}^{n+1}, \boldsymbol{\eta}_2^{n+1}; \boldsymbol{v}_h, \boldsymbol{v}_{2,H}) - b^{n+1}(\boldsymbol{v}_h, \boldsymbol{v}_{2,H}; \zeta^{n+1}, \boldsymbol{\delta}^{n+1}) \\ &+ (\rho_2 - \tilde{\rho}_2)(\boldsymbol{w}^{n+1} \cdot (\nabla \boldsymbol{\mu}_2^{n+1} + \nabla \boldsymbol{\eta}_2^{n+1}), \boldsymbol{v}_{2,H})_{\Omega_{n+1}^2}, \\ &\forall (\boldsymbol{v}_h, \boldsymbol{v}_{2,H}) \in \boldsymbol{V}_h \times {\boldsymbol{V}^{n+1}}_{2,H} \end{aligned} \tag{99}$$

Take $\boldsymbol{v}_h = \boldsymbol{\mu}^{n+1}$, $\boldsymbol{v}_{2,H} = \boldsymbol{\mu}_2^{n+1}$ in (99), notice $b^{n+1}(\boldsymbol{\mu}^{n+1}, \boldsymbol{\mu}_2^{n+1}; \xi^{n+1}, \boldsymbol{\gamma}^{n+1}) = 0$, and apply (19), (20), (57) and Young's $\varepsilon-$inequality, we obtain

$$\begin{aligned} &\frac{\tilde{\rho}}{2}(\|\boldsymbol{\mu}^{n+1}\|_{0,\Omega}^2 - \|\boldsymbol{\mu}^n\|_{0,\Omega}^2) + \frac{\rho_2 - \tilde{\rho}_2}{2} \Big( \|\boldsymbol{\mu}_2^{n+1}\|_{0,\Omega_{n+1}^2}^2 \\ &- \|\boldsymbol{\mu}_2 \left( \boldsymbol{X}_{t^n} \circ \boldsymbol{X}_{t^{n+1}}^{-1}(\boldsymbol{x}^{n+1}), t^n \right)\|_{0,\Omega_{n+1}^2}^2 \Big) + \Delta t \left( \|\nabla \boldsymbol{\mu}^{n+1}\|_{0,\Omega}^2 + \|\nabla \boldsymbol{\mu}_2^{n+1}\|_{0,\Omega_{n+1}^2}^2 \right) \\ &\le c\Delta t \Big[ -\tilde{\rho}(d_t \boldsymbol{\eta}^n, \boldsymbol{\mu}^{n+1})_\Omega - (\rho_2 - \tilde{\rho}_2) \left( d_t^{\boldsymbol{X}_t} \boldsymbol{\eta}_2^n, \boldsymbol{\mu}_2^{n+1} \right)_{\Omega_{n+1}^2} \\ &- \tilde{\rho} \left( \left( \frac{\partial \tilde{\boldsymbol{u}}}{\partial t} \right)^{n+1} - d_t \tilde{\boldsymbol{u}}^n, \boldsymbol{\mu}^{n+1} \right)_\Omega - (\rho_2 - \tilde{\rho}_2) \left( \frac{d\boldsymbol{u}_2}{dt}\Big|_{\boldsymbol{y}}^{n+1} - d_t^{\boldsymbol{X}_t} \boldsymbol{u}_2^n, \boldsymbol{\mu}_2^{n+1} \right)_{\Omega_{n+1}^2} \\ &+ \|\nabla \boldsymbol{\eta}^{n+1}\|_{0,\Omega}^2 + \|\nabla \boldsymbol{\eta}_2^{n+1}\|_{0,\Omega_{n+1}^2}^2 + \|\boldsymbol{\mu}^{n+1}\|_{0,\Omega}^2 + \|\boldsymbol{\mu}_2^{n+1}\|_{0,\Omega_{n+1}^2}^2 \\ &+ \|\zeta^{n+1}\|_{0,\Omega}^2 + \|\boldsymbol{\delta}^{n+1}\|_{\boldsymbol{V}_{n+1}^2}^2 \Big] + \varepsilon \Delta t \left( \|\boldsymbol{\mu}^{n+1}\|_{\boldsymbol{V}}^2 + \|\boldsymbol{\mu}_2^{n+1}\|_{\boldsymbol{V}_{n+1}^2}^2 \right), \end{aligned} \tag{100}$$

where, we need to further analyze the term $\|\boldsymbol{\mu}_2 \left( \boldsymbol{X}_{t^n} \circ \boldsymbol{X}_{t^{n+1}}^{-1}(\boldsymbol{x}^{n+1}), t^n \right)\|_{0,\Omega_{n+1}^2}^2$ on the left hand side of (100). By the Reynolds transport theorem, we know

$$\frac{d}{dt} \int_{\Omega_t^2} \left[ \boldsymbol{\mu}_2 \left( \boldsymbol{X}_{t^n} \circ \boldsymbol{X}_t^{-1}(\boldsymbol{x}), t^n \right) \right]^2 d\boldsymbol{x} = \int_{\Omega_t^2} \left[ \boldsymbol{\mu}_2 \left( \boldsymbol{X}_{t^n} \circ \boldsymbol{X}_t^{-1}(\boldsymbol{x}), t^n \right) \right]^2 \nabla \cdot \boldsymbol{w} d\boldsymbol{x}.$$

Integrate from $t^n$ to $t^{n+1}$, yields

$$\int_{\Omega^2_{n+1}} \left[\boldsymbol{\mu}_2\left(\boldsymbol{X}_{t^n}\circ \boldsymbol{X}^{-1}_{t^{n+1}}(\boldsymbol{x}^{n+1}),t^n\right)\right]^2 d\boldsymbol{x}^{n+1} - \int_{\Omega^2_n} \left[\boldsymbol{\mu}_2\left(\boldsymbol{X}_{t^n}\circ (\boldsymbol{X}^{-1}_{t^n}(\boldsymbol{x}^n),t^n\right)\right]^2 d\boldsymbol{x}^n$$
$$= \int_{t^n}^{t^{n+1}} \int_{\Omega^2_t} \left[\boldsymbol{\mu}_2\left(\boldsymbol{X}_{t^n}\circ \boldsymbol{X}^{-1}_{t}(\boldsymbol{x}),t^n\right)\right]^2 \nabla\cdot\boldsymbol{w} d\boldsymbol{x}dt,$$

namely,

$$\begin{aligned}&\|\boldsymbol{\mu}_2\left(\boldsymbol{X}_{t^n}\circ \boldsymbol{X}^{-1}_{t^{n+1}}(\boldsymbol{x}^{n+1}),t^n\right)\|^2_{0,\Omega^2_{n+1}} - \|\boldsymbol{\mu}^n_2\|^2_{0,\Omega^2_n}\\ &\leq \sup_{t\in[t^n,t^{n+1}]}\|\nabla\cdot\boldsymbol{w}\|_{\infty,\Omega^2_t}\int_{t^n}^{t^{n+1}}\|\boldsymbol{\mu}_2\left(\boldsymbol{X}_{t^n}\circ \boldsymbol{X}^{-1}_{t}(\boldsymbol{x}),t^n\right)\|^2_{0,\Omega^2_t}dt.\end{aligned} \tag{101}$$

In order to bound the last temporal integral, due to the change of variable $\boldsymbol{z}=\boldsymbol{X}_{t^n}(\boldsymbol{X}^{-1}_t(\boldsymbol{x}))$, we have[Martín, Smaranda and Takahashi (2009)]

$$\|\boldsymbol{\mu}_2\left(\boldsymbol{X}_{t^n}\circ \boldsymbol{X}^{-1}_{t}(\boldsymbol{x}),t^n\right)\|^2_{0,\Omega^2_t} \leq \|\boldsymbol{J}_{\boldsymbol{X}_t}\|_{\infty,\Omega^2_0}\|\boldsymbol{J}_{\boldsymbol{X}^{-1}_{t^n}}\|_{\infty,\Omega^2_n}\|\boldsymbol{\mu}^n_2\|^2_{0,\Omega^2_n} \leq c\|\boldsymbol{\mu}^n_2\|^2_{0,\Omega^2_n}, \tag{102}$$

where, $\boldsymbol{J}$ denotes the determinant of Jacobian matrix which are bounded since the one-to-one function $\boldsymbol{X}_t$ is prescribed. Together with (57), (101) leads to

$$\|\boldsymbol{\mu}_2\left(\boldsymbol{X}_{t^n}\circ \boldsymbol{X}^{-1}_{t^{n+1}}(\boldsymbol{x}^{n+1}),t^n\right)\|^2_{0,\Omega^2_{n+1}} \leq \|\boldsymbol{\mu}^n_2\|^2_{0,\Omega^2_n} + c\Delta t\|\boldsymbol{\mu}^n_2\|^2_{0,\Omega^2_n}. \tag{103}$$

Then, (100) can be further rewritten as

$$\begin{aligned}&(\|\boldsymbol{\mu}^{n+1}\|^2_{0,\Omega} - \|\boldsymbol{\mu}^n\|^2_{0,\Omega}) + \left(\|\boldsymbol{\mu}^{n+1}_2\|^2_{0,\Omega^2_{n+1}} - \|\boldsymbol{\mu}^n_2\|^2_{0,\Omega^2_n}\right)\\ &+\Delta t\left(\|\boldsymbol{\mu}^{n+1}\|^2_{\boldsymbol{V}} + \|\boldsymbol{\mu}^{n+1}_2\|^2_{\boldsymbol{V}^2_{n+1}}\right) \leq c\Delta t\big\{\|\boldsymbol{\mu}^n_2\|^2_{0,\Omega^2_n} + \|d_t\boldsymbol{\eta}^n\|^2_{0,\Omega} + \|\boldsymbol{\mu}^{n+1}\|^2_{0,\Omega}\\ &+\|d_t^{\boldsymbol{X}_t}\boldsymbol{\eta}^n_2\|^2_{0,\Omega^2_{n+1}} + \|\boldsymbol{\mu}^{n+1}_2\|^2_{0,\Omega^2_{n+1}} + \left\|\left(\frac{\partial\tilde{\boldsymbol{u}}}{\partial t}\right)^{n+1} - d_t\tilde{\boldsymbol{u}}^n\right\|^2_{0,\Omega}\\ &+\left\|\frac{d\boldsymbol{u}_2}{dt}\Big|^{n+1}_{\boldsymbol{y}} - d_t^{\boldsymbol{X}_t}\boldsymbol{u}^n_2\right\|^2_{0,\Omega^2_{n+1}} + \|\nabla\boldsymbol{\eta}^{n+1}\|^2_{0,\Omega} + \|\nabla\boldsymbol{\eta}^{n+1}_2\|^2_{0,\Omega^2_{n+1}}\\ &+\|\boldsymbol{\mu}^{n+1}\|^2_{0,\Omega} + \|\boldsymbol{\mu}^{n+1}_2\|^2_{0,\Omega^2_{n+1}} + \|\zeta^{n+1}\|^2_{0,\Omega} + \|\boldsymbol{\delta}^{n+1}\|^2_{\boldsymbol{V}^2_{t^{n+1}}}\big\}\\ &+\varepsilon\Delta t\left(\|\boldsymbol{\mu}^{n+1}\|^2_{\boldsymbol{V}} + \|\boldsymbol{\mu}^{n+1}_2\|^2_{\boldsymbol{V}^2_{n+1}}\right).\end{aligned} \tag{104}$$

Based on Taylor's expansions, the following inequalities can be derived:

$$\|d_t\varphi^n\|_{0,\Omega} \leq c\|\left(\frac{\partial\varphi}{\partial t}\right)^n\|_{0,\Omega}, \tag{105}$$

$$\left\|\left(\frac{\partial\varphi}{\partial t}\right)^{n+1} - d_t\varphi^n\right\|_{0,\Omega} \leq c\Delta t\left\|\frac{\partial^2\varphi}{\partial t^2}\right\|_{0,\Omega}, \tag{106}$$

$$\left\|d_t^{\boldsymbol{X}_t}\varphi^n\right\|_{0,\Omega^2_{n+1}} \leq c\|\frac{d\varphi}{dt}\Big|_{\boldsymbol{y}}^{n+1}\|_{0,\Omega^2_{n+1}}, \tag{107}$$

$$\left\|\frac{d\varphi}{dt}\Big|_{\boldsymbol{y}}^{n+1} - d_t^{\boldsymbol{X}_t}\varphi^n\right\|_{0,\Omega^2_{n+1}} \leq c\Delta t\left(\left\|\frac{d^2\varphi}{dt^2}\Big|_{\boldsymbol{y}}\right\|_{0,\Omega^2_{n+1}} + \left\|\nabla\varphi^{n+1}\right\|_{0,\Omega^2_{n+1}}\right), \tag{108}$$

where, $\frac{d^2\varphi}{dt^2}\big|_{\boldsymbol{y}}$ denotes the second partial temporal derivative on the ALE frame, and the assumption (57) is used.
Sum up both sides of (104) over $n$ from 0 to $M-1$ ($M = 1, \cdots, N$), apply Taylor's expansions (105)-(108) and choose sufficiently small $\varepsilon$, yield

$$\|\boldsymbol{\mu}^M\|^2_{0,\Omega} + \|\boldsymbol{\mu}_2^M\|^2_{0,\Omega^2_M} + \Delta t\sum_{n=0}^{M}\left(\|\boldsymbol{\mu}^n\|^2_{\boldsymbol{V}} + \|\boldsymbol{\mu}_2^n\|^2_{\boldsymbol{V}^2_n}\right) \leq c\Big[\|\boldsymbol{\mu}^0\|^2_{0,\Omega} + \|\boldsymbol{\mu}_2^0\|^2_{0,\Omega^2_0}$$
$$+(\Delta t)^2 + \Delta t\sum_{n=0}^{M}\left(\|\nabla\boldsymbol{\eta}^n\|^2_{0,\Omega} + \|\nabla\boldsymbol{\eta}_2^n\|^2_{0,\Omega^2_n} + \|\left(\frac{\partial\boldsymbol{\eta}}{\partial t}\right)^n\|^2_{0,\Omega} + \|\frac{d\boldsymbol{\eta}_2}{dt}\Big|_{\boldsymbol{y}}^n\|^2_{0,\Omega^2_n}\right.$$
$$\left.+\|\zeta^n\|^2_{0,\Omega} + \|\boldsymbol{\delta}^n\|^2_{\boldsymbol{V}^2_n} + \|\boldsymbol{\mu}^n\|^2_{0,\Omega} + \|\boldsymbol{\mu}_2^n\|^2_{0,\Omega^2_n}\right)\Big].$$

Apply the discrete Grönwall's inequality, results

$$\|\boldsymbol{\mu}^M\|_{0,\Omega} + \|\boldsymbol{\mu}_2^M\|_{0,\Omega^2_M} + \left(\Delta t\sum_{n=0}^{M}\left(\|\boldsymbol{\mu}^n\|^2_{\boldsymbol{V}} + \|\boldsymbol{\mu}_2^n\|^2_{\boldsymbol{V}^2_n}\right)\right)^{1/2}$$
$$\leq c\Big[\|\boldsymbol{\mu}^0\|_{0,\Omega} + \|\boldsymbol{\mu}_2^0\|_{0,\Omega^2_0} + \Delta t + \Delta t\sum_{n=0}^{M}\left(\|\nabla\boldsymbol{\eta}^n\|_{0,\Omega} + \|\nabla\boldsymbol{\eta}_2^n\|_{0,\Omega^2_n} + \|\left(\frac{\partial\boldsymbol{\eta}}{\partial t}\right)^n\|_{0,\Omega}\right.$$
$$\left.+\|\frac{d\boldsymbol{\eta}_2}{dt}\Big|_{\boldsymbol{y}}^n\|_{0,\Omega^2_n} + \|\zeta^n\|_{0,\Omega} + \|\boldsymbol{\delta}^n\|_{\boldsymbol{V}^2_n}\right)\Big].$$

Then we have the following error estimate

$$
\begin{aligned}
&\|\tilde{\boldsymbol{u}}^M-\boldsymbol{u}_h^M\|_{0,\Omega}+\|\boldsymbol{u}_2^M-\boldsymbol{u}_{2,H}^M\|_{0,\Omega_M^2}+\left(\Delta t\sum_{n=0}^{M}\left(\|\tilde{\boldsymbol{u}}^n-\boldsymbol{u}_h^n\|_{\boldsymbol{V}}^2+\|\boldsymbol{u}_2^n-\boldsymbol{u}_{2,H}^n\|_{\boldsymbol{V}_n^2}^2\right)\right)^{1/2}\\
&\le c\Bigg[\|\tilde{\boldsymbol{u}}^0-\boldsymbol{u}_h(0)\|_{0,\Omega}+\|\boldsymbol{u}_2^0-\boldsymbol{u}_{2,H}(0)\|_{0,\Omega_0^2}+\Delta t\\
&+\inf_{(\boldsymbol{z}_h,\boldsymbol{z}_{2,H},\chi_h,\boldsymbol{\theta}_H)\in\tilde{\boldsymbol{V}}_h\times\tilde{\boldsymbol{V}}_{H,t}^2\times Q_h\times\boldsymbol{V}_{H,t}^2}\Delta t\sum_{n=0}^{M}\Bigg(\|\boldsymbol{u}^n-\boldsymbol{z}_h^n\|_{\boldsymbol{V}}+\|\boldsymbol{u}_2^n-\boldsymbol{z}_{2,H}^n\|_{\boldsymbol{V}_n^2}\\
&+\left\|\left(\frac{\partial(\tilde{\boldsymbol{u}}-\boldsymbol{z}_h)}{\partial t}\right)^n\right\|_{0,\Omega}+\left\|\frac{d(\boldsymbol{u}_2-\boldsymbol{z}_{2,H})}{dt}\Big|_{\boldsymbol{y}}^n\right\|_{0,\Omega_{t^n}^2}+\|p^n-\chi_h^n\|_{0,\Omega}+\|\boldsymbol{\phi}^n-\boldsymbol{\theta}_H^n\|_{\boldsymbol{V}_n^2}\Bigg)\Bigg].
\end{aligned}
$$

Consider the regularity assumptions (18), adopt the same approximation to the initial values $\tilde{\boldsymbol{u}}^0$ and $\boldsymbol{u}_2^0$ as done in Section 3, and apply (92), then the following convergence theorem is derived for the fully discrete DLM/FD scheme (95)-(98).

**Theorem 4.1.** Let $(\tilde{\boldsymbol{u}},\boldsymbol{u}_2,\tilde{p},\boldsymbol{\phi})\in\boldsymbol{V}\times\boldsymbol{V}_t^2\times Q\times\boldsymbol{V}_t^2$ be the solution to (59)-(62), and let $(\boldsymbol{u}_h^M,\ \boldsymbol{u}_{2,H}^M,\ p_h^M,\ \boldsymbol{\phi}_H^M)\in\boldsymbol{V}_h\times\boldsymbol{V}_{2,H}^M\times Q_h\times\boldsymbol{V}_{2,H}^M$, $1\le M\le N$, be the solution to (95)-(98). If (19), (20) and (57) hold, then there exists a constant $c>0$ independent of $h$ and $H$ such that

$$
\begin{aligned}
&\|\tilde{\boldsymbol{u}}^M-\boldsymbol{u}_h^M\|_{0,\Omega}+\|\boldsymbol{u}_2^M-\boldsymbol{u}_{2,H}^M\|_{0,\Omega_M^2}+\left(\Delta t\sum_{n=0}^{M}\left(\|\tilde{\boldsymbol{u}}^n-\boldsymbol{u}_h^n\|_{\boldsymbol{V}}^2+\|\boldsymbol{u}_2^n-\boldsymbol{u}_{2,H}^n\|_{\boldsymbol{V}_n^2}^2\right)\right)^{1/2}\\
&\le c(h^{\sigma-1}+H^{\sigma-1}+\Delta t).
\end{aligned}
$$

The stability of the full discretization is studied as follows. Take $\boldsymbol{v}_h=\boldsymbol{u}_h^{n+1},\ \boldsymbol{v}_{2,H}=\boldsymbol{u}_{2,H}^{n+1}$ in (95)-(98), yields

$$
\begin{aligned}
&\tilde{\rho}(d_t\boldsymbol{u}_h^n,\boldsymbol{u}_h^{n+1})_\Omega+(\rho_2-\tilde{\rho}_2)\left(d_t^{\boldsymbol{X}_t}\boldsymbol{u}_{2,H}^n,\boldsymbol{u}_{2,H}^{n+1}\right)_{\Omega_{n+1}^2}+a^{n+1}(\boldsymbol{u}_h^{n+1},\boldsymbol{u}_{2,H}^{n+1};\boldsymbol{u}_h^{n+1},\boldsymbol{u}_{2,H}^{n+1})\\
&+b^{n+1}(\boldsymbol{u}_h^{n+1},\boldsymbol{u}_{2,H}^{n+1};p_h^{n+1},\boldsymbol{\phi}_H^{n+1})=(\rho_2-\tilde{\rho}_2)\boldsymbol{w}^{n+1}\cdot\nabla\boldsymbol{u}_{2,H}^{n+1},\boldsymbol{u}_{2,H}^{n+1})_{\Omega_{n+1}^2}\\
&+(\tilde{\boldsymbol{f}}^{n+1},\boldsymbol{u}_h^{n+1})_\Omega+(\boldsymbol{f}_2^{n+1}-\tilde{\boldsymbol{f}}^{n+1}|_{\Omega_{n+1}^2},\boldsymbol{u}_{2,H}^{n+1})_{\Omega_{n+1}^2}+(\boldsymbol{\tau}^{n+1},\boldsymbol{u}_{2,H}^{n+1})_{\Gamma_{n+1}}.
\end{aligned}
$$

Note that $b^{n+1}(\boldsymbol{u}_h^{n+1},\boldsymbol{u}_{2,H}^{n+1};p_h^{n+1},\boldsymbol{\phi}_H^{n+1})=0$, apply (19), (20), (57) and Young's $\varepsilon-$inequality, we obtain

$$\frac{\tilde{\rho}}{2}(\|\boldsymbol{u}_h^{n+1}\|_{0,\Omega}^2 - \|\boldsymbol{u}_h^n\|_{0,\Omega}^2) + \frac{\rho_2 - \tilde{\rho}_2}{2}\Bigg(\|\boldsymbol{u}_{2,H}^{n+1}\|_{0,\Omega_{n+1}^2}^2 -$$
$$\|\boldsymbol{u}_{2,H}\left(\boldsymbol{X}_{t^n} \circ \boldsymbol{X}_{t^{n+1}}^{-1}(\boldsymbol{x}^{n+1}), t^n\right)\|_{0,\Omega_{n+1}^2}^2\Bigg) + \Delta t\left(\|\boldsymbol{u}_h^{n+1}\|_{\boldsymbol{V}}^2 + \|\boldsymbol{u}_{2,H}^{n+1}\|_{\boldsymbol{V}_{n+1}^2}^2\right)$$
$$\leq c\Delta t\Bigg[\|\boldsymbol{u}_h^{n+1}\|_{0,\Omega}^2 + \|\boldsymbol{u}_{2,H}^{n+1}\|_{0,\Omega_{n+1}^2}^2 + \|\tilde{\boldsymbol{f}}^{n+1}\|_{0,\Omega}^2 + \|\boldsymbol{f}_2^{n+1} - \tilde{\boldsymbol{f}}^{n+1}|_{\Omega_{n+1}^2}\|_{0,\Omega_{n+1}^2}^2$$
$$+\|\boldsymbol{\tau}^{n+1}\|_{0,\Gamma_{n+1}}^2\Bigg] + \varepsilon\Delta t\|\boldsymbol{u}_{2,H}^{n+1}\|_{\boldsymbol{V}_{n+1}^2}^2. \tag{109}$$

Apply (103) to the term $\|\boldsymbol{u}_{2,H}\left(\boldsymbol{X}_{t^n} \circ \boldsymbol{X}_{t^{n+1}}^{-1}(\boldsymbol{x}^{n+1}), t^n\right)\|_{0,\Omega_{n+1}^2}^2$ on the left hand side of (109), then sum up both sides of (109) over $n$ from 0 to $M-1$ ($M = 1, \cdots, N$), apply Taylor's expansions (105)-(108), choose sufficiently small $\varepsilon$, yield

$$\|\boldsymbol{u}_h^M\|_{0,\Omega}^2 + \|\boldsymbol{u}_{2,H}^M\|_{0,\Omega_M^2}^2 + \Delta t\sum_{n=0}^{M}\left(\|\boldsymbol{u}_h^n\|_{\boldsymbol{V}}^2 + \|\boldsymbol{u}_{2,H}^n\|_{\boldsymbol{V}_n^2}^2\right) \leq c\left[\|\boldsymbol{u}_h^0\|_{0,\Omega}^2 + \|\boldsymbol{u}_{2,H}^0\|_{0,\Omega_0^2}^2\right.$$
$$\left.+\Delta t\sum_{n=0}^{M}\left(\|\boldsymbol{u}_h^n\|_{0,\Omega}^2 + \|\boldsymbol{u}_{2,H}^n\|_{0,\Omega_n^2}^2 + \|\tilde{\boldsymbol{f}}^n\|_{0,\Omega}^2 + \|\boldsymbol{f}_2^n - \tilde{\boldsymbol{f}}^n|_{\Omega_n^2}\|_{0,\Omega_n^2}^2 + \|\boldsymbol{\tau}^n\|_{0,\Gamma_n}^2\right)\right].$$

Apply the discrete Grönwall's inequality, results

$$\|\boldsymbol{u}_h^M\|_{0,\Omega} + \|\boldsymbol{u}_{2,H}^M\|_{0,\Omega_M^2} + \left(\Delta t\sum_{n=0}^{M}\left(\|\boldsymbol{u}_h^n\|_{\boldsymbol{V}}^2 + \|\boldsymbol{u}_{2,H}^n\|_{\boldsymbol{V}_n^2}^2\right)\right)^{1/2} \leq c\Bigg[\|\boldsymbol{u}_h^0\|_{0,\Omega}$$
$$+\|\boldsymbol{u}_{2,H}^0\|_{0,\Omega_0^2} + \Delta t\sum_{n=0}^{M}\left(\|\tilde{\boldsymbol{f}}^n\|_{0,\Omega} + \|\boldsymbol{f}_2^n - \tilde{\boldsymbol{f}}^n|_{\Omega_n^2}\|_{0,\Omega_n^2} + \|\boldsymbol{\tau}^n\|_{0,\Gamma_n}\right)\Bigg]. \tag{110}$$

Then, we have the following stability theorem for the fully discrete scheme.

**Theorem 4.2.** Suppose all hypotheses of Theorem 4.1 are held, then the stability result (110) exists for (95)-(98).

## 5 Numerical experiments

In this section, we study the numerical performance of the developed DLM/FD finite element method for an example of the transient Stokes interface problem (1)-(10) defined in $\Omega = [0,1] \times [0,1]$, where the circular subdomain $\Omega_t^2$ makes a translational motion and the position of $\partial\Omega_t^2$, which is the interface $\Gamma_t$, satisfies

$(x - 0.3 - w_1 t)^2 + (y - 0.3 - w_2 t)^2 = 0.01,$

where, $\boldsymbol{w} = (w_1, w_2)^T$ denotes the moving velocity of $\Gamma_t$.

We properly choose the functions of coefficients, source terms and jump flux of (1)-(10), i.e., $\beta_i,\ \rho_i,\ \boldsymbol{f}_i,\ (i = 1, 2)$ and $\boldsymbol{\tau}$ such that the true solution $(\boldsymbol{u}, p)$ to (1)-(10), where $\boldsymbol{u} = (u, v)^T$, is defined by

$$
\begin{aligned}
&u = (y - 0.3 - w_2 t)((x - 0.3 - w_1 t)^2 + (y - 0.3 - w_2 t)^2 - 0.01)t/\beta,\\
&v = -(x - 0.3 - w_1 t)((x - 0.3 - w_1 t)^2 + (y - 0.3 - w_2 t)^2 - 0.01)t/\beta,\\
&p = \sin(\pi x)\sin(\pi y)t,
\end{aligned}
$$

where, $\beta = \beta_i(\boldsymbol{x}),\ \forall \boldsymbol{x} \in \Omega_i\ (i = 1, 2)$ is chosen as a piecewise constant depending on the location of $\boldsymbol{x}$. Clearly, such chosen solution $(\boldsymbol{u}, p)$ satisfies the following regularity property:

$$
\boldsymbol{u} \in (H^1(\Omega))^2 \cap (H^2(\Omega_t^1 \cup \Omega_t^2))^2,\ \ p \in H^1(\Omega_t^1), \quad \forall t \in [0, T].
$$

In what follows, we take a constant moving velocity $\boldsymbol{w} = (0.1,\ 0.2)^T$, and let $T = 1$. The meshes $T_h(\Omega)$ and $T_H(\Omega_t^2)$ are constructed independently and thus mismatched with each other.

Convergence results of the velocity vector at the time $t = T$ in its $H^1$- and $L^2$ norm, i.e., $\|\boldsymbol{u}-\boldsymbol{u}_h\|_{(H^1(\Omega))^2}$ and $\|\boldsymbol{u}-\boldsymbol{u}_h\|_{(L^2(\Omega))^2}$ which are displayed in their component forms, and of the pressure in its $L^2$ norm, $\|p-p_h\|_{L^2(\Omega_T^1)}$, are illustrated in Tabs. 1 and 2 for large jump coefficient cases. We can observe that: (1) the developed DLM/FD–mixed finite element discretization is stable and converges in all cases, little influence from the choice of the time step size; (2) the convergence results are relatively more sensitive to $\beta_2/\beta_1$, comparing with the jump $\rho_2/\rho_1$, noting that the exact solution $\boldsymbol{u}$ depends on $\beta$, but independent of $\rho$; (3) due to the reduced regularity property of the solution, and the discontinuity of the normal derivative of $\boldsymbol{u}$ across $\Gamma_t$, the convergence rates of velocity errors in $H^1$- and $L^2$-norm decrease to $0.55 \sim 0.9$ and $1.0 \sim 1.3$, respectively, and the convergence rates of pressure errors in $L^2$-norm keeps around $1.0 \sim 2.0$, which validate our theoretical conclusions, and also match with the convergence rates of other types of interface problems when the DLM/FD method is applied [Boffi, Gastaldi and Ruggeri (2014); Auricchio, Boffi, Gastaldi et al. (2015); Wang and Sun (2017); Lundberg, Sun and Wang (2019); Sun (2019)].

Next, we investigate the influence of time step size on the convergence rate of the developed DLM/FD finite element method. In order to let $O(\Delta t)$ be the main part of the error in comparison with the part $O(h^{\sigma-1} + H^{\sigma-1})$, we particularly pick up the case of $\beta_2/\beta_1 = 2$, $\rho_2/\rho_1 = 2$, and take $\boldsymbol{f}_i,\ (i = 1, 2)$ and $\boldsymbol{\tau}$ such that the true solution $(\boldsymbol{u}, p)$ to (1)-(10), where $\boldsymbol{u} = (u, v)^T$, is defined by

$$
\begin{aligned}
&u = (y - 0.3 - w_2 t)((x - 0.3 - w_1 t)^2 + (y - 0.3 - w_2 t)^2 - 0.01)(2t^9 - t^5)/\beta,\\
&v = -(x - 0.3 - w_1 t)((x - 0.3 - w_1 t)^2 + (y - 0.3 - w_2 t)^2 - 0.01)(2t^9 - t^5)/\beta,\\
&p = \sin(\pi x)\sin(\pi y)(2t^9 - t^5).
\end{aligned}
$$

Numerical results of this test are reported in Tab. 3, from which we can observe the first-order convergent for all errors with respect to $\Delta t$, as predicted by the theoretical result.

**Table 1:** Convergence results of the case: $\beta_2/\beta_1 = 100$, $\rho_2/\rho_1 = 1000$, $\Delta t = 1/128$

| $h$ | $H$ | $\|u-u_h\|_1$ | $\|v-v_h\|_1$ | $\|u-u_h\|_0$ | $\|v-v_h\|_0$ | $\|p-p_h\|_0$ |
|---|---|---|---|---|---|---|
| $1/10$ | $1/40$ | 2.7928e-03 | 2.7963e-03 | 1.0744e-04 | 1.0792e-04 | 7.1102e-03 |
| $1/16$ | $1/64$ | 2.6581e-03 | 1.7827e-03 | 7.8270e-05 | 5.8261e-05 | 3.2093e-03 |
| $1/20$ | $1/80$ | 1.1572e-03 | 1.1592e-03 | 2.9076e-05 | 2.9160e-05 | 2.0570e-03 |
| $1/24$ | $1/96$ | 1.5134e-03 | 1.2372e-03 | 3.7764e-05 | 3.3477e-05 | 1.4763e-03 |
| $1/28$ | $1/112$ | 1.2882e-03 | 1.2571e-03 | 3.2113e-05 | 3.0158e-05 | 1.1366e-03 |
| $1/32$ | $1/128$ | 1.0039e-03 | 1.0510e-03 | 2.4317e-05 | 2.3925e-05 | 6.7959e-04 |
| rate | | 0.89 | 0.81 | 1.29 | 1.27 | 1.94 |

**Table 2:** Convergence results of the case: $\beta_2/\beta_1 = 10000$, $\rho_2/\rho_1 = 1000$, $\Delta t = 1/128$

| $h$ | $H$ | $\|u-u_h\|_1$ | $\|v-v_h\|_1$ | $\|u-u_h\|_0$ | $\|v-v_h\|_0$ | $\|p-p_h\|_0$ |
|---|---|---|---|---|---|---|
| $1/10$ | $1/40$ | 4.1171e-03 | 4.1292e-03 | 1.6189e-04 | 1.6359e-04 | 6.9973e-03 |
| $1/16$ | $1/64$ | 3.6978e-03 | 3.7803e-03 | 1.3492e-04 | 1.2738e-04 | 4.5674e-03 |
| $1/20$ | $1/80$ | 1.2353e-03 | 1.2373e-03 | 3.4987e-05 | 3.4662e-05 | 1.9919e-03 |
| $1/24$ | $1/96$ | 3.2329e-03 | 1.7224e-03 | 7.9301e-05 | 5.3506e-05 | 3.1686e-03 |
| $1/28$ | $1/112$ | 1.9472e-03 | 2.1853e-03 | 5.3129e-05 | 5.5084e-05 | 1.8631e-03 |
| $1/32$ | $1/128$ | 2.4036e-03 | 2.4490e-03 | 6.0265e-05 | 5.9451e-05 | 2.5130e-03 |
| rate | | 0.55 | 0.62 | 0.97 | 1.04 | 1.02 |

**Table 3:** Convergence results of the case: $\beta_2/\beta_1 = 2$, $\rho_2/\rho_1 = 2$, $h = 1/32$, $H = 1/128$

| $\Delta t$ | $\|u-u_h\|_1$ | $\|v-v_h\|_1$ | $\|p-p_h\|_0$ |
|---|---|---|---|
| $1/8$ | 3.7155e-02 | 3.8232e-02 | 1.9715e-01 |
| $1/16$ | 2.1289e-02 | 2.1852e-02 | 1.1324e-01 |
| $1/32$ | 1.1418e-02 | 1.1703e-02 | 6.0909e-02 |
| $1/64$ | 5.9152e-03 | 6.0588e-03 | 3.1714e-02 |
| $1/128$ | 3.0114e-03 | 3.0837e-03 | 1.6299e-02 |
| $1/256$ | 1.5213e-03 | 1.5583e-03 | 8.3803e-03 |
| rate | 0.93 | 0.93 | 0.92 |

## 6 Conclusion and future work

We develop the DLM/FD–mixed finite element method for a generic transient Stokes interface problem and carry out numerical analyses for both semi- and fully discrete scheme on the convergence and stability properties. By using the Taylor-Hood ($P^2P^1$) mixed finite element space, we are able to obtain a nearly optimal convergence rate for both the velocity and the pressure in their respective norms, subjecting to the reduced regularity assumption for the solution to the transient Stokes interface problem. Numerical experiments validate the theoretical results, showing that the convergence rates of the velocity with respect to the mesh size is the 0.5th in $H^1$-norm, and the first order in $L^2$-norm, at least, which is true even for larger jump coefficient cases up to 1:10000, relatively insensitive to different choices of jump coefficients and time step sizes. And, the first order convergence rate with respect to the time step size is also validated.

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