A new optimization approach to the design of one-dimensional and two-dimensional finite impulse response digital filters

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A new optimization approach to the design of one-dimensional and two-dimensional finite impulse response digital filters

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A New Optimization Approach to the Design of
One-Dimensional and Two-Dimensional Finite Impulse Response Digital Filters

by

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abstract

The theory for designing finite impulse response (FIR) frequency sampling digital filters can be extended to two-dimensions. The linear phase frequency response can be represented as a linear combination of individual frequency responses corresponding to the filter’s bands. The design of two-dimensional frequency sampling filters (FSF) has been treated in the past by using the technique of linear programming to find the optimal values of the transition samples. Although in theory the method guarantees an optimal solution, convergence problems occurred.

This paper will introduce some detail of a one-dimensional FSF design technique and then extend these concepts to the two-dimensional problem. The mean of the squared error in both the stopband and the passband is minimized subject to constraints on the filter’s stopband. The filter’s coefficients
can be calculated by solving a linear system of equations.
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Chapter 1

Introduction

Many of the 2-D signals that require 2-D linear phase signal processing techniques include weather photos, medical X rays, seismic records, gravity and magnetic data, and electron micrographs. Some of the methods for the design of finite impulse response (FIR) two-dimensional digital filters are extensions of the one dimensional methods. FIR filters are capable of producing exact linear phase even if they were implemented recursively. Recursive filters are capable of approximating a desired amplitude response more efficiently than an nonrecursive filter, but they can't achieve linear phase. The 1-D frequency sampling FIR filter problem can serve as basis for the understanding of the 2-D problem. This paper will introduce some detail of a 1-D frequency sam-
plung filter design technique then extend these concepts to the 2-D problem.

The basic design of a 1-D frequency sampling filter is a direct procedure based primarily on the frequency response characteristics of the desired filters. Although most FIR filters are non-recursive which yield an impulse response from their design and then use the impulse response as coefficients in the filter implementation, frequency sampling filters (FSF) gain their advantage by recursive implementation in which specific frequency response values called frequency samples from the filter's frequency response are used in the implementation. For narrowband filters, a significant number of the frequency samples in the stopband are zero and will not require any operations in the recursive implementation. Some of the reasons why FIR filters are of importance include: (1) they can be easily designed to approximate an arbitrary magnitude frequency response with an exact linear phase characteristic. The effects of linear phase can be separated so that the amplitude can be approximated as a real valued function. This property is very useful for filter design. (2) they can be realized efficiently both nonrecursively (using FFT) and recursively (using a comb filter and a bank of resonators). (3) FIR filters realized nonrecursively are always stable. (4) they can be de-
Introduction

signed to have negligible quantization and round off problems when realized nonrecursively. (5) the coefficient accuracy problems inherent in sharp cutoff IIR filters can often be made less severe for realizations of equally sharp FIR filters[1].

Three error measures are generally used in FIR filter design: the average of the squared error in the frequency response approximation (least squared approximation), the maximum of the error over specified regions of the frequency response (chebyshev approximation) and a third approach which is based on a Taylor series approximation to the desired response (Butterworth or maximally flat approximation). The following 2-D techniques are direct extensions of techniques for designing 1-D nonrecursive digital filters: window functions, frequency sampling, the straightforward use of linear programming, and frequency transformation. Other techniques for designing optimal 1-D nonrecursive digital filters include: the second algorithm of REMES, and Parks and McClellan[11] which can't be extended to the 2-D case because the approximating function does not satisfy the Haar condition resulting in that the optimal solution is not necessarily unique and an extension of the 1-D REMES exchange algorithm may fail to converge. It has been shown[3] how the algorithm has to be complemented with a perturbation technique in
order to force convergence under all circumstances. It has been shown[4] that the 1-D frequency sampling technique is directly related to the Fourier series design and that the minimization of the transition band samples amounts to choosing an optimum smoothing window. It has also been shown[5] how to get a good 2-D windows from good 1-D ones.

For nonrecursive digital filters, the realizable approximations are trigonometric polynomials and the class of approximations is more constrained. Once the filter coefficients have been determined, linear filtering of a bandlimited waveform can be accomplished by using a digital computer to arithmetically combine samples of the waveform. The following two reasons distinguish the sampling method from other design methods[2]: (1) in implementing the filter, the designer need never concern himself with the impulse response which is appealing for filters with long impulse responses. (3) it can be realized more efficiently than a direct convolution structure for certain types of filters (narrow band filters). The following chapter will present some background on the 1-D and 2-D frequency sampling filters. Chapter 3 will present some detail of a new optimization approach to the design of 1-D FSF's and chapter 4 extends these concepts to the 2-D problem.
Chapter 2

1-D and 2-D Frequency Sampling Filters

2.1 One-Dimensional Digital Filters:

The impulse response of an FIR digital filter can be represented by its z-transform, $H(z)$ as follows:

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$$

(2.1)

The normalized sampling frequency (twice the Nyquist frequency) is $w = 2\pi$ rad./sec or $f = 1Hz$, therefore the maximum or Nyquist frequency is $w = \pi$
or \( f = 0.5 \text{Hz} \). The frequency response can be found by setting \( z = \exp(jw) \).

**Notation:** \( H(k) = H(e^{jw})|_{w = 2\pi k/N} \).

The discrete Fourier transform (DFT) can be used to evaluate the frequency response at certain frequencies. If an impulse response \( h(n) \) has \( N \) samples, then an \( N \) length DFT of \( h(n) \) is

\[
H(e^{j2\pi k/N}) = \sum_{n=0}^{N-1} h(n) e^{-j2\pi kn/N} \tag{2.2}
\]

where \( k = 0,1,\ldots,N - 1 \).

These \( N \) equally spaced frequency points of the frequency response may not give enough detail, therefore, any number \( L \) of equally spaced samples can be used by appending \( L - N \) zeros to \( h(n) \) and taking an \( L \)-length DFT. Theoretically, when the number of zeros goes to infinity, the DFT becomes the Fourier transform. The basic idea behind frequency sampling is that an \( N \)-length inverse DFT of a desired frequency response gives the filter coefficients \( h(n) \).

FIR filters could have even or odd length \( N \). For linear phase, \( h(n) \) is of the form

\[
h(n) = h^*(N - n - 1) \tag{2.3}
\]
where \( h^*(N - n - 1) \) is the complex conjugate of \( h(N - n - 1) \).

If we also constrain the filter to have a real impulse response then

\[
h(n) = h(N - n - 1)
\]

and by multiplying the right-hand side of equation (2.1) with \( z = \exp(jw) \) by \( e^{jwM} e^{-jw^M} \), the frequency response can be expressed as

\[
H(e^{jw}) = e^{-jw^M} \sum_{n=0}^{N-1} h(n) e^{jw(M - n)} = e^{j\theta} H(w)
\]

(2.4)

where \( M \) (not necessarily an integer) is defined by

\[
M = \frac{N - 1}{2}
\]

\( H(w) \) represents the frequency response without phase characteristics (also known as the zero-phase frequency response), therefore \( H(w) \) is a real function of \( w \). When \( N \) is odd

\[
H(w) = \sum_{n=0}^{M-1} 2h(n) \cos[w(M - n)] + h(M)
\]

(2.5)

When \( N \) is even

\[
H(w) = \sum_{n=0}^{\frac{N}{2}-1} 2h(n) \cos[w(M - n)]
\]

(2.6)

A frequency sampling filter's frequency samples can be obtained by the evaluation of the filter's frequency response at \( w = 2\pi k/N \) where \( N \) is the number
of frequency samples or the length of the impulse response of the filter. For very long filters, equations (2.5) and (2.6) are inefficient, and the FFT may be used to give $H(w)$ directly by circularly shifting $h(n)$ to achieve symmetry about $n = 0$, which means that the phase shift is zero.

The method of taking the DFT of $h(n)$ to obtain samples of the frequency response of an FIR filter also holds for general arbitrary phase filters. FIR filters may require a rather long length to achieve certain frequency response, so a large number of arithmetic operations per output value and a large number of coefficients have to be stored. The linear phase characteristic makes the time delay of the filter equal to half its length, which may be large. For a length $- N$ FIR filter and no transition region or transition band, the frequency sampling design procedure is straightforward: place $N$ equally spaced frequency response samples given by $H(2\pi k / N)$ and the filter coefficients are given by the IDFT. When $H(e^{j\omega})$ is linear phase, equation (2.4) gives

$$H(e^{j\omega}) = e^{-j\frac{2\pi k M}{N}} \sum_{n=0}^{N-1} h(n) e^{j(2\pi/N)k(M-n)}$$

(2.7)

IDFT gives

$$h(n) = \frac{1}{N} \sum_{k=0}^{N-1} H\left(\frac{2\pi k}{N}\right) e^{j(2\pi/N)(n-M)}$$

(2.8)
where \( H(\frac{2\pi k}{N}) \) is the value of the zero-phase frequency response at the kth sample. For \( h(n) \) real

\[
H\left(\frac{2\pi k}{N}\right) = H^*(\frac{2\pi(N - k)}{N})
\]

For \( N \) odd, (2.8) becomes

\[
h(n) = \frac{1}{N} \left[ H(0) + \sum_{k=1}^{M} 2H\left(\frac{2\pi k}{N}\right) \cos\left(\frac{2\pi(n - M)k}{N}\right) \right]
\]

(2.9)

For \( N \) even, equation (2.8) gives

\[
h(n) = \frac{1}{N} \left[ H(0) + \sum_{k=1}^{\frac{N}{2} - 1} 2H\left(\frac{2\pi k}{N}\right) \cos\left(\frac{2\pi(n - M)k}{N}\right) \right]
\]

(2.10)

which is of the same form as equation (2.9) except that the upper limit on the summation recognizes \( N \) as even and \( H\left(\frac{N}{2}\right) = 0 \).

The impulse response in equations (2.9) and (2.10) can also be expressed in terms of the magnitude and phase of the DFT.

\[
h(n) = \frac{1}{N} \sum_{k=0}^{N-1} |H(k)| \exp\left[j(\theta(k) + \frac{2\pi}{N}nk)\right]
\]

A purely real impulse response implies that

\[ H(k) = H^*(N - k) \]

where \( H^*(N - k) \) is the complex conjugate of \( H(N - k) \).

From equation (2.7), a linear phase FIR filter has the following phase constraint
\[ \theta(w) = -w\left(\frac{N-1}{2}\right) \quad k = 0, 1, \ldots, N - 1 \]

Applying these constraints, \( h(n) \) can be written as \[8\]
\[ h(n) = \frac{H(0)}{N} + \frac{1}{N} \sum_{k=1}^{K-1} (-1)^k 2 |H(k)| \cos \left[ \frac{2\pi}{N} (n + \frac{1}{2})k \right]_{N\text{even}} \]

and for \( N \) odd
\[ h(n) = \frac{H(0)}{N} + \frac{1}{N} \sum_{k=1}^{M} (-1)^k 2 |H(k)| \cos \left[ \frac{2\pi}{N} (n + \frac{1}{2})k \right]_{N\text{odd}} \]

The zero phase frequency response is periodic with period \( 2\pi \) for odd \( N \) and \( 4\pi \) for even \( N \). For a digital filter whose impulse response coefficients are real, the frequency response is characterized by an even magnitude symmetry and an odd phase symmetry,
\[ H(e^{jw}) = H^*(e^{-jw}) \]

where \( H^*(e^{-jw}) \) is the complex conjugate of \( H(e^{-jw}) \).

\[ |H(e^{jw})| = |H(e^{-jw})| \]

\[ \theta(w) = -\theta(-w) \]

Since ideal filters have discontinuities between the various stop and pass bands, sampling the ideal frequency response which is frequency sampling results in the Gibbs phenomenon\[2\], which is a fixed percentage overshoot and
ripple before and after an approximated discontinuity. It also occurs when truncating the infinite impulse response of an ideal filter to get an approximation of its frequency domain. It is a direct consequence of minimizing the squared error when approximating a discontinuity with no transition region. The filter's impulse response coefficients are the coefficients of the expanded Fourier series, and direct truncation of the Fourier series corresponds to least square minimization which is associated with the Gibbs phenomenon. To control the convergence of the Fourier series, a weighting function (window) is used to modify the Fourier coefficients. The design criteria for windows is to find a window with most of its energy in the main lobe of its Fourier transform. A disadvantage of the window design is that the Fourier series coefficients for the periodic frequency response being approximated must be computed. Generally, it is not trivial to obtain closed form expressions for these coefficients. An approximation can be obtained by sampling the frequency response at a number of frequencies larger than the number of Fourier series coefficients under the window.

Generally, for frequency sampling filters, the IDFT of samples taken over one period of the normalized frequency range which is $2\pi$ is
\[ h(n) = \frac{1}{N} \sum_{k=0}^{N-1} H(k) e^{i(2\pi/N)nk} \quad n = 0, 1, \ldots, N - 1 \]

Substituting this expression for \( h(n) \) into eq. (2.1) gives

\[ H(z) = \sum_{n=0}^{N-1} \left[ \frac{1}{N} \sum_{k=0}^{N-1} H(k) e^{i(2\pi/N)nk} \right] z^{-n} \quad (2.11) \]

interchanging the order of summation we get

\[ H(z) = \frac{1}{N} \sum_{k=0}^{N-1} H(k) \sum_{n=0}^{N-1} \left[ e^{i(2\pi/N)k} z^{-1} \right]^n \quad (2.12) \]

using a closed form of the geometric series, the right hand summation becomes

\[ \frac{1 - z^{-N}}{1 - e^{j2\pi k/N} z^{-1}} \quad (2.13) \]

Equation (2.12) becomes

\[ H(z) = \frac{1 - z^{-N}}{N} \sum_{k=0}^{N-1} \frac{H(k)}{1 - e^{j(2\pi/N)k} z^{-1}} \quad (2.14) \]

which can be thought of as the product of two transfer functions, \( \frac{1 - z^{-N}}{N} \) is the transfer function of a comb filter, a nonrecursive filter that has \( N \) poles at the origin and \( N \) zeros located on the unit circle. The rest of equation
Figure 2.1: Type 1 Frequency Sampling Structure.

(2.14) is a sum of $N$ transfer functions having a single pole which are referred to as complex resonators. It is shown in [6] that the zeros of the first transfer function cancels the poles of the second assuming infinite precision arithmetic. The structure used to realize this filter is shown in figure (2.1). Filters of this type are called frequency sampling filters because the filter’s coefficients are a function of the frequency samples. Evaluating (2.14) on the
unit circle gives the continuous interpolation formula

\[ H(\exp(j\omega)) = \exp\left[-j\left(\frac{N}{2}(1 - 1/N)\right)\right] \frac{\sum_{k=0}^{N-1} H(k) \exp[-j(\pi k/N)] \sin(\omega N/2)}{\sin(\omega/2 - \pi k/N)} \]  

(2.15)

which has the form of the Lagrange interpolating formula. Equation (2.15) assumes a frequency sample at \( f = 0 \). It is seen from (2.15) that the frequency response is linearly related to the frequency samples, thus linear optimization techniques can be used to select some or all values of the frequency samples to give the best approximation. By allowing variable frequency samples in a transition band, one can choose the frequency samples in this band to provide optimum ripple cancellation for either the out-of-band region, inband region, or a combination of the two. As the number of samples in the transition band increases, ever finer ripple cancellation is possible. For example, for a low pass filter, samples in the passband are preset to 1, a number of samples in the transition band (depending on the desired width) are left to be decided by a minimization algorithm, and the rest of samples, up to \( f = 0.5 \) because of symmetry, are set to zero. For each transition coefficient, the frequency response varies in a straight line with
different slope depending on the particular value of $w$. These lines do not intersect in a common point and their upper envelop forms a convex function. Since a convex function has a unique minimum, a procedure which searches for the minimum value of a maximum sidelobe must converge. This can be extended to more than one dimension. The search procedure is described in [1], which also shows that the frequency sampling technique is competitive with the standard window technique in that the number of terms needed to achieve a desired peak ripple in the stopband using frequency sampling is about 50% less than the number of terms using optimum windows described by Kaiser[12].

Helms[7] uses the basic frequency sampling method to obtain preliminary impulse response coefficients in which the number of frequency samples is larger than the final filter's coefficients $N$. The preliminary coefficients are truncated and then multiplied by the $P + 1$ window coefficients of a Dolph-Chebyshev function of the form:

$$\frac{\cos\{P \cos^{-1}(z_0 \cos \pi x)\}}{\cosh\{P \cosh^{-1}(z_0)\}}$$

where $x$ is frequency normalized by the sampling rate, $P$ and $z_0$ are chosen to make the frequency response fulfill the required band-
width of the transitions between discontinuities and the required ripple.

### 2.2 Two-Dimensional FIR Digital Filters:

Let $x(n_1, n_2)$ be a two dimensional sequence which is often shorthand for a sampled version of a continuous $2-D$ signal

$$x(n_1, n_2) = x(n_1T_1, n_2T_2) \quad (2.17)$$

The filter is said to be separable if its impulse response can be factored into a product of one-dimensional responses:

$$h(n_1, n_2) = g(n_1) f(n_2) \quad (2.18)$$

The advantage of separable filters is that the $2-D$ convolution can be carried out as a sequence of $1-D$ convolutions:

$$y(n_1, n_2) = \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} h(m_1, m_2) x(n_1 - m_1, n_2 - m_2) \quad (2.19)$$

$$y(n_1, n_2) = \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} g(m_1) f(m_2) x(n_1 - m_1, n_2 - m_2) \quad (2.20)$$

$$y(n_1, n_2) = \sum_{m_1=-\infty}^{\infty} g(m_1) \left[ \sum_{m_2=-\infty}^{\infty} f(m_2) x(n_1 - m_1, n_2 - m_2) \right] \quad (2.21)$$
2-D Digital Filters

If both the input sequence \(x(n_1, n_2)\) and the filter impulse response \(h(n_1, n_2)\) are separable, then the output sequence \(y(n_1, n_2)\) is separable. The set of points \(\{(m_1, m_2)\}\) which are included in the sum are referred to as the region of support.

If the input to a 2 \(- D\) system is the sinusoidal signal

\[x(n_1, n_2) = e^{j(w_1 n_1 + w_2 n_2)}\] (2.22)

the output is

\[y(n_1, n_2) = x(n_1, n_2) H(e^{jw_1}, e^{jw_2})\] (2.23)

where

\[H(e^{jw_1}, e^{jw_2}) = \sum_{m_1=\infty}^{\infty} \sum_{m_2=\infty}^{\infty} h(m_1, m_2) e^{-j(w_1 m_1 + w_2 m_2)}\] (2.24)

which represents a Fourier series, thus

\[h(n_1, n_2) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} H(e^{jw_1}, e^{jw_2}) e^{j(w_1 n_1 + w_2 n_2)} dw_1 dw_2\] (2.25)

Notation: \(H(k_1, k_2) = H(e^{jw_1}, e^{jw_2})\big|_{w_1 = \frac{2\pi N_1}{N_1}, w_2 = \frac{2\pi N_2}{N_2}}\)

The frequency response \(H(e^{jw_1}, e^{jw_2})\) is doubly periodic in frequency.

For a real \(h(n_1, n_2)\),

\[H(e^{jw_1}, e^{jw_2}) = H^*(e^{-jw_1}, e^{-jw_2})\] (2.26)
Equation (2.26) implies that knowledge of $H(w_1, w_2)$ in the region $0 \leq w_1 \leq \pi, 0 \leq w_2 \leq \pi$, i.e., the first quadrant implies knowledge of its behavior for $-\pi \leq w_1 \leq 0, -\pi \leq w_2 \leq 0$, i.e., the third quadrant. If

$$h(-n_1, -n_2) = h^*(n_1, n_2)$$

(2.27)

then the frequency response $H(e^{jw_1}, e^{jw_2})$ will be a purely real function which results in zero phase filters that are symmetric with respect to rotations of 180 degrees about the origin of the $(n_1, n_2)$ plane. Moreover, if we assume a rectangular region of support with $2N + 1$ samples on each side, then the frequency response of the FIR filter can be written as

$$H(e^{jw_1}, e^{jw_2}) = \sum_{n_1=-N_1}^{N_1} \sum_{n_2=-N_2}^{N_2} h(n_1, n_2) e^{-j(n_1w_1 + n_2w_2)}$$

(2.28)

Assuming

$$h(-n_1, -n_2) = h^*(n_1, n_2)$$

(2.28) can be written in a number of different ways. One is to write the values along the horizontal $n_1$ axis

$$H(w_1, w_2) = h(0,0) + \sum_{n_1=-N_1, n_1 \neq 0}^{N_1} h(n_1, 0) \exp[-jw_1n_1] + \sum_{n_1=-N_1}^{N_1} \sum_{n_2=1}^{N_2} h(n_1, n_2) \exp[-j(w_1n_1 + w_2n_2)] +$$
\[ h(n_1, n_2) \exp[-j(w_1 n_1 + w_2 n_2)] \]

The second term can be written as

\[ \sum_{n_1=1}^{N_1} h(-n_1, 0) \exp[jw_1 n_1] + \sum_{n_1=1}^{N_1} h(n_1, 0) \exp[-jw_1 n_1] \]

Grouping the exponential terms into a cosine term results in

\[ 2 \sum_{n_1=1}^{N_1} h(n_1, 0) \cos(w_1 n_1) \]

The last two double summation terms can be written as

\[ \sum_{n_1=-N_1}^{N_1} \sum_{n_2=1}^{N_2} h(n_1, n_2) \exp[-j(w_1 n_1 + w_2 n_2)] + \]
\[ \sum_{n_1=-N_1}^{N_1} \sum_{n_2=1}^{N_2} h(n_1, -n_2) \exp[-j(w_1 n_1 - w_2 n_2)] \]

Since we assumed that

\[ h(n_1, n_2) = h(-n_1, -n_2) \]

we can group the exponential terms to result in

\[ H(w_1, w_2) = h(0, 0) + 2 \sum_{n_1=1}^{N_1} h(n_1, 0) \cos(w_1 n_1) \]
\[ + 2 \sum_{n_2=1}^{N_2} \sum_{n_1=-N_1}^{N_1} h(n_1, n_2) \cos(n_1 w_1 + n_2 w_2) \] (2.29)
Sometimes it is desirable to place further restrictions on the symmetry of the unit sample response. If it is assumed that

\[ h(n_1, n_2) = h(|n_1|, |n_2|) \]

then

\[ H(w_1, w_2) = H(|w_1|, |w_2|) \]

which corresponds to quadrature symmetry in both the unit sample and the frequency responses, and the frequency response can be written as:

\[
H(w_1, w_2) = h(0, 0) + \sum_{n_1 = -N_1}^{N_1} h(n_1, 0) \exp[-jw_1n_1] + \\
\sum_{n_1 = 1}^{N_1} h(n_1, 0) \exp[-jw_1n_1] + \\
\sum_{n_2 = -N_2}^{-1} h(0, n_2) \exp[-jw_2n_2] + \sum_{n_2 = 1}^{N_2} h(0, n_2) \exp[-jw_2n_2] + \\
\sum_{n_1 = -N_1}^{N_1} \sum_{n_2 = 1}^{N_2} h(n_1, n_2) \exp[-j(w_1n_1 + w_2n_2)] + \\
\sum_{n_1 = -N_1}^{N_1} \sum_{n_2 = -N_2}^{-1} h(n_1, n_2) \exp[-j(w_1n_1 + w_2n_2)] + \\
\sum_{n_1 = 1}^{N_1} \sum_{n_2 = -N_2}^{N_2} h(n_1, n_2) \exp[-j(w_1n_1 + w_2n_2)] + \\
\sum_{n_1 = -N_1}^{N_1} \sum_{n_2 = -N_2}^{N_2} h(n_1, n_2) \exp[-j(w_1n_1 + w_2n_2)]
\]
The previous equation can be written as

\[
H(w_1, w_2) = h(0, 0) + \sum_{n_1=1}^{N_1} h(-n_1, 0) \exp[jw_1n_1] + \\
\sum_{n_1=1}^{N_1} h(n_1, 0) \exp[-jw_1n_1] + \\
\sum_{n_2=1}^{N_2} h(0, -n_2) \exp[jw_2n_2] + \sum_{n_2=1}^{N_2} h(0, n_2) \exp[-jw_2n_2] + \\
\sum_{n_1=1}^{N_1} \sum_{n_2=1}^{N_2} h(n_1, n_2) \exp[-j(w_1n_1 + w_2n_2)] + \\
\sum_{n_1=1}^{N_1} \sum_{n_2=1}^{N_2} 2h(-n_1, n_2) \exp[j(w_1n_1 - w_2n_2)] + \\
\sum_{n_1=1}^{N_1} \sum_{n_2=1}^{N_2} h(n_1, -n_2) \exp[-jw_1n_1 + jw_2n_2] + \\
\sum_{n_1=1}^{N_1} \sum_{n_2=1}^{N_2} h(-n_1, -n_2) \exp[j(w_1n_1 + w_2n_2)]
\]

Using the assumed symmetry, the previous equation results in:

\[
H(w_1, w_2) = h(0, 0) + 2 \sum_{n_1=1}^{N} h(n_1, 0) \cos(w_1n_1) + \\
2 \sum_{n_2=1}^{N} h(0, n_2) \cos(w_2n_2) + \\
2 \sum_{n_2=1}^{N} \sum_{n_1=1}^{N} h(n_1, n_2) \cdot [\cos(n_1w_1 + n_2w_2) + \cos(n_1w_1 - n_2w_2)]
\]  

(2.30)
Differences between 1-D and 2-D nonrecursive filters are due primarily to three factors. First, multidimensional designs possess more degrees of freedom than 1-D ones and this can be exploited. Second, the available math for higher dimensional problems is more restrictive. Third, the 1-D filter design and implementation issues are distinct and decoupled while in the 2-D case is not. This is because multidimensional polynomials can't be factored in general. If an implementation can realize only factorable transfer functions, then the design algorithm must accommodate filters of this class.

An example of the frequency response of an ideal circular low pass filter is

\[
H(w_1, w_2) = \begin{cases} 
1 & \sqrt{(w_1^2 + w_2^2)} \leq w_p \\
0 & \text{otherwise}
\end{cases} \quad (2.31)
\]

over one period. The corresponding unit sample response is given by[9]

\[
h(n_1, n_2) = \frac{RJ_1(R\sqrt{(n_1^2 + m_2^2)})}{2\pi \sqrt{(n_1^2 + m_2^2)}} \quad (2.32)
\]

where \(J_1\) is the Bessel function of the first kind, order one. Equation (2.32) is of infinite extent, and can't be realized with (2.29) or (2.30) and it must be approximated. Some measure of closeness and some criterion for deciding on a best approximation is needed. Among these is the Tchebycheff
norm defined as [9]

\[ ||\text{Error}|| = \max |F(w_1, w_2) - H(w_1, w_2)| \] (2.33)

An optimization method known as linear programming which can be applied to the filter problem is as follows: [9]

minimize

\[ w = C_1^T X_1 + C_2^T X_2 + d \] (2.34)

subject to

\[ p_1 : A_{11}X_1 + A_{12}X_2 \geq B_1 \]
\[ p_2 : A_{21}X_1 + A_{22}X - 2 = B_2 \]
\[ n_1 : X_1 \geq 0 \]
\[ n_2 : X_2 \text{ free} \]

where

\[ p_1 \] is the number of inequality constraints
\[ p_2 \] is the number of equality constraints
\[ n_1 \] is the number of normal variables
\[ n_2 \] is the number of free variables
$w$ is the objective function made up of a constant cost term $d$ and a linear function of the variables.

$C_1, C_2$ are the cost vectors

$B_1, B_2$ are the constraint vectors

$X_1, X_2$ are the program vectors

$A_{11}, A_{12}, A_{21}, A_{22}$ are the coefficient matrices.

An algorithm called the simplex algorithm is used to solve linear programming problems. The algorithm produces one of three mutually exclusive results: first, the optimal solution if the problem has a finite one; second, it indicates if the problem is feasible but the optimal solution is unbounded; third, it indicates if the problem is infeasible. Feasible means that all constraints can be simultaneously satisfied for some choice of the variables. It is infeasible if all constraints cannot be simultaneously satisfied for any choice of the variables.

The $z-$transform of a 2-$D$ sequence is defined as

$$X(z_1, z_2) = \sum_{n_1 = -\infty}^{\infty} \sum_{n_2 = -\infty}^{\infty} x(n_1, n_2) z_1^{-n_1} z_2^{-n_2}$$

Convergence of the $z-$transform is guaranteed for the class of finite duration sequences that are bounded. This guarantees that filters designed from finite
sequences will be stable.

A periodic signal in two dimensions is

\[ x_p(n_1, n_2) = x_p(n_1 + m_1 N_1, n_2 + m_2 N_2) \]  

(2.36)

the subscript \( p \) indicates that the signal is periodic. The two dimensional discrete Fourier transform is given by:

\[ X_p(k_1, k_2) = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x_p(n_1, n_2) \exp \left[ -j \left( \frac{2\pi}{N_1} n_1 k_1 + \frac{2\pi}{N_2} n_2 k_2 \right) \right] \]  

(2.37)

and the inverse Discrete Fourier transform is

\[ x_p(n_1, n_2) = \frac{1}{N_1 N_2} \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} X_p(k_1, k_2) \exp \left[ j \left( \frac{2\pi}{N_1} n_1 k_1 + \frac{2\pi}{N_2} n_2 k_2 \right) \right] \]  

(2.38)

Equation (2.37) can be written as

\[ X_p(k_1, k_2) = \sum_{n_1=0}^{N_1-1} e^{-j \left( \frac{2\pi}{N_1} n_1 k_1 \right)} \left[ \sum_{n_2=0}^{N_2-1} x_p(n_1, n_2) e^{-j \left( \frac{2\pi}{N_2} n_2 k_2 \right)} \right] \]  

(2.39)

The term in the bracket is a series of \( N_1 \) \(-D\) DFT's obtained by varying \( n_1 \) from 0 to \( N_1 - 1 \) and the outer summation is another series of \( N_2 \) \(-D\) DFT's on the resultants from the bracket. The \( 2-D \) DFT can be used to evaluate linear convolutions if care is taken to avoid spatial aliasing.

An FIR digital filter is completely characterized by its unit sample response
or by an equivalent number of uniformly spaced samples of its frequency response given by the discrete Fourier transform. Consider a filter with $h(n_1, n_2)$ defined over $0 \leq n_1 \leq N_1 - 1, 0 \leq n_2 \leq N_2 - 1$ with $z-$transform

$$H(z_1, z_2) = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} h(n_1, n_2) z^{-n_1} z^{-n_2} \quad (2.40)$$

The frequency response is obtained by evaluating (2.40) for $z_1 = e^{jw_1}$ and $z_2 = e^{jw_2}$ giving

$$H(w_1, w_2) = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} h(n_1, n_2) e^{-j(w_1 n_1 + w_2 n_2)} \quad (2.41)$$

The DFT is obtained by evaluating (2.41) at

$$w_1 = \frac{2\pi}{N_1} k_1 \quad k_1 = 0, 1, \ldots, N_1 - 1$$

$$w_2 = \frac{2\pi}{N_2} k_2 \quad k_2 = 0, 1, \ldots, N_2 - 1$$

The inverse DFT is

$$h(n_1, n_2) = \frac{1}{N_1 N_2} \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} H(k_1, k_2) \exp \left[ j2\pi \left( \frac{k_1 n_1}{N_1} + \frac{k_2 n_2}{N_2} \right) \right] \quad (2.42)$$

By inserting (2.42) into (2.41), we get

$$H(w_1, w_2) = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} \left[ \frac{1}{N_1 N_2} \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} H(k_1, k_2) \exp \left[ j2\pi \left( \frac{k_1 n_1}{N_1} + \frac{k_2 n_2}{N_2} \right) \right] \right]$$

$$\exp[-j(w_1 n_1 + w_2 n_2)] \quad (2.43)$$
Interchanging order of summation gives

\[ H(w_1, w_2) = \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} H(k_1, k_2) A(k_1, k_2, w_1, w_2) \]  \hspace{1cm} (2.44)

where

\[ A(k_1, k_2, w_1, w_2) = \frac{1}{N_1 N_2} \left[ \frac{1 - e^{-jN_1 w_1}}{1 - e^{j(2\pi(k_1/N_1)-w_1)}} \right] \left[ \frac{1 - e^{-jN_2 w_2}}{1 - e^{j(2\pi(k_2/N_2)-w_2)}} \right] \]  \hspace{1cm} (2.45)

Equation (2.44) is the basic design equation for 2-D frequency sampling filters. It shows that the continuous frequency response is a linear combination of shifted interpolating functions weighted by the DFT coefficients \( H(k_1, k_2) \) which are the frequency samples because they exactly specify the value of the frequency response at uniform spaced frequencies.

If certain symmetries are maintained in \( h(n_1, n_2) \) (similar statements can be made about the DFT coefficients), the filter's frequency response becomes purely real to within a linear phase shift in each of the two dimensions. The symmetries are:

\[ h(n_1, n_2) = h(N_1 - 1 - n_1, n_2) = h(n_1, N_2 - 1 - n_2) \]  \hspace{1cm} (2.46)

the resulting response when \( N_1 \) and \( N_2 \) are even is

\[ H(w_1, w_2) = \exp[-j(w_1 M_1) + (w_2 M_2)]. \]
2-D Digital Filters

\[
\sum_{n_1=0}^{(N_1/2)-1} \sum_{n_2=0}^{(N_2/2)-1} h(n_1, n_2) \cdot \cos(M_1 - n_1) \cos(M_2 - n_2)
\]

(2.47)

where

\[
M_1 = \frac{N_1 - 1}{2}
\]

\[
M_2 = \frac{N_2 - 1}{2}
\]

To design linear phase frequency sampling filters, equation (2.44) must be modified through the use of symmetry relations on the DFT coefficients

\[
H(k_1, k_2) = |H(k_1, k_2)| \exp[j\theta(k_1, k_2)]
\]

(2.48)

where

\[
|H(k_1, k_2)| = |H(k_1, N_2 - k_2)| = |H(N_1 - k_1, k_2)|
\]

(2.49)

\[
\theta(k_1) = \begin{cases} 
\frac{2\pi}{N_1} (M_1 k_1) & k_1 = 0, 1, \ldots, NU \\
\frac{2\pi}{N_1} M_1 (N_1 - k_1) & k_1 = NU + 1, \ldots, N_1 - 1
\end{cases}
\]

(2.50)

\(\theta(k_2)\) is the same as \(\theta(k_1)\) with \(N_2, M_2, k_2\) replacing \(N_1, M_1, k_1\) respectively, and

\[
NU = \begin{cases} 
\frac{N_1}{2} & N_1 \text{ even} \\
M_1 & N_1 \text{ odd}
\end{cases}
\]

(2.51)
Whenever \( N_1 \) or \( N_2 \) is even, the additional constraint

\[
\theta(k_1, N_2/2) = \theta(N_1/2, k_2) = 0 \quad (2.52)
\]

\[
H(k_1, N_2/2) = H(N_1/2, k_2) = 0 \quad (2.53)
\]

must be maintained.

With these conditions in equation (2.44), we get

\[
H(w_1, w_2) = \exp[-j(M_1 w_1 + M_2 w_2)] \frac{1}{N_1 N_2} \left[ H(0, 0) \alpha(w_1, N_1) \alpha(w_2, N_2) + \sum_{k_1=1}^{NU} |H(k_1, 0)| \alpha(w_2, N_2) \beta(w_1, k_1, N_1) + \sum_{k_2=1}^{NV} |H(0, k_2)| \alpha(w_1, N_1) \beta(w_2, k_2, N_2) + \sum_{k_1=1}^{NU} \sum_{k_2=1}^{NV} |H(k_1, k_2)| \beta(w_1, k_1, N_1) \beta(w_2, k_2, N_2) \right] \quad (2.54)
\]

where

\[
\alpha = \frac{\sin(w N/2)}{\sin(w/2)}
\]

\[
\beta(w, k, N) = \frac{\sin[(w/2 - \pi k/N)N]}{\sin[w/2 - \pi k/N]} + \frac{\sin[(w/2 + \pi k/N)N]}{\sin[w/2 + \pi k/N]}
\]

\[
NU = \begin{cases} 
    N_1/2 & \text{if } N_1 \text{ even} \\
    M_1 & \text{if } N_2 \text{ odd}
\end{cases}
\]
Equation (2.54) is the basic design equation for 2-\(D\) frequency sampling filters.

Figure 2.2 shows the desired set of frequency bands in the \((w_1, w_2)\) plane for a circularly symmetric lowpass filter. The passband is the region where

\[
\rho(w_1, w_2) = \left( w_1^2 + w_2^2 \right)^{1/2} \leq R_1
\]

The transition band is the region where

\[
R_1 < \rho(w_1, w_2) < R_2
\]

and the stop band is

\[
\rho(w_1, w_2) \geq R_2
\]

Figure 2.3 shows the frequency samples which are located on an \((N_1 \text{ by } N_2)\) grid of points in the \((w_1, w_2)\) plane, where the \(X\)'s stand for variable samples to be determined according to some optimization criterion while the dots are known samples.
Figure 2.2: Frequency bands for a circularly symmetric two-dimensional lowpass filter

Figure 2.3: Frequency samples on a (9 x 9) grid of points for a circularly symmetric two-dimensional lowpass filter
Chapter 3

A New Optimization Approach to The Design of One Dimensional Frequency Sampling Filters

As was mentioned briefly in the introduction, frequency sampling filters use $N$ (where $N$ is the length of the filter’s impulse response) specific values from the filter’s frequency response in their implementation. This is an advantage
because a filter's frequency response is more sensitive to small coefficient perturbations when a filter is implemented using its impulse response values as filter coefficients than when implemented using frequency response samples as filter coefficients. The design technique interpolates a frequency response from a set of $N$ evenly spaced samples from the filter's desired frequency response. The resulting frequency response will pass through these frequency samples, but there is no guarantee to the behavior between these samples. If the frequency response of the filter we are trying to approximate has relatively smooth response (no discontinuities), then the basic frequency sampling method (Ch. 2) typically results in a well behaved response between the frequency samples. On the other hand, if the desired response has abrupt changes, then an overshoot will typically result between the frequency samples at the changes. The approximation of the desired frequency response by the frequency sampling filter can be improved by various design methods. The filter design technique developed in this chapter minimizes the mean squared error in both the passband and the stopband leaving a number of samples in the transition band as don't cares while constraining the frequency samples in the stopband to zero. Depending on the specifications of the desired frequency response and the filter's length $N$, the number of don't
care samples in the transition band is determined. For example, using two
don't care samples in the transition band will result in less ripple in the pass-
band than using one sample, but the width of the transition band is longer.
A weighting factor can be used in either the passband or the stopband to
put more weight in minimizing that cost function.

The implementation of a FSF uses a recursive structure to implement
a finite impulse response linear phase filter. Under certain conditions[8],
these filters can be implemented more efficiently than the well known direct
convolution structure. However, because it uses a recursive structure, finite
word length effects may cause the recursive structure to yield an infinite
impulse response resulting in nonlinear phase for the filter. There is a way
around this problem which might result in some degradation of the frequency
response. The solution is to pull the poles and zeros a little bit towards the
origin by replacing $z$ with $z = re^{j\omega}$ in which $r$ is a number a little bit less
than one. The design technique in this chapter is based on $r = 1$.

We will present the details for designing Type 1 frequency sampling filters
which interpolates a frequency response through the points $H(e^{j2\pi/N}k)$ for
$k \in I$ (integers). The same principles work for Type 2 FSF's which inter-
polate a frequency response through the points $H(e^{j(2\pi/N)(k+1/2)})$ for $k \in I$
Optimization Approach for 1-D Filters

(Integers). Let $H(k)$, $k \in I$ represent a discrete set of evenly spaced values from an arbitrary frequency response, $H(e^{jw})$. One example of an application that requires a narrowband lowpass filter to limit the effects of aliasing would be a sampling rate conversion system in which minimizing the mean squared error in the stopband is essential. For $N$ even, and if we constrain the filter to have linear phase and real impulse response, then the frequency response term is given by

$$H(e^{jw}) = e^{-jw \frac{N-1}{2}} H(w)$$

where

$$H(w) = \sum_{n=0}^{\frac{N-1}{2}} 2h(n) \cos[w(M - n)] \quad (3.1)$$

and

$$M = \frac{N - 1}{2}$$

$H(w)$ is called the zero phase frequency response. Equation (3.1) can be written in vector notation. Defining the filter coefficient vector

$$x = \left[2h(0) \ 2h(1) \ 2h(2) \ldots \ 2h(\frac{N}{2} - 1)\right]^T$$

and the transform kernel vector
Optimization Approach for 1-D Filters

\[ s(w) = [\cos Mw \cos(M-1)w \ldots \cos(0.5w)]^T \]

the frequency response can be written as a vector inner product

\[ H(w) = s^T(w)x \] (3.2)

The mean squared error is given by

\[ E = \frac{1}{2\pi} \int_{-\pi}^{\pi} |H(w) - H_d(w)|^2 \, dw \] (3.3)

where \( H_d(w) \) is the desired frequency response.

Exploiting the symmetric characteristics of the zero phase frequency response \( H(w) \) and keeping in mind that \( h(n) \) is real, the mean squared error reduces to

\[ E = \frac{1}{\pi} \int_0^{\pi} |H(w) - H_d(w)|^2 \, dw \] (3.4)

Assuming that the desired frequency response is zero in the stopband, we have

\[ E = \frac{1}{w_r} \int_{w\text{sb}}^{\pi} |H(w)|^2 \, dw \] (3.5)

where

\[ w_r = \pi - w\text{sb} \]
Optimization Approach for 1-D Filters

\[ w_{sb} = \text{beginning of stopband frequency.} \]

Writing the integrand in vector-matrix notation

\[ H^2(w) = \left[ s^T(w) x \right]^2 = x^T s(w) s^T(w) x \]

and defining the matrix

\[ Q = \int_{w_{sb}}^{w_r} s(w) s^T(w) \, dw \]

the stopband cost function can be written as

\[ J_{sb} = \frac{1}{w_r} x^T Q x \]  \hspace{1cm} (3.6)

The stopband frequency samples are constrained to zero. The frequency response is set to zero at the discrete stopband samples. That is

\[ H(w) \big|_{w_k = \frac{2\pi k}{N}} = 0 \]

\[ k = k_{sb}, k_{sb+1}, \ldots, M \]

at each sample point in the stopband where the value of the frequency response can be calculated by

\[ H(w_k) = \sum_{n=0}^{N/2-1} 2h(n) \cos[w_k(M - n)] \]  \hspace{1cm} (3.7)

where

\[ w_k = \frac{2\pi}{N} k \]
Setting these values to zero results in

\[
\begin{bmatrix}
\cos[w_{k,0}(M - 0)] & \cos[w_{k,0}(M - 1)] & \ldots & \cos[w_{k,0}(0.5)] \\
\cos[w_{k,1}(M - 0)] & \cos[w_{k,1}(M - 1)] & \ldots & \cos[w_{k,1}(0.5)] \\
\vdots & \vdots & \ddots & \vdots \\
\cos[w_{k,M}(M - 0)] & \cos[w_{k,M}(M - 1)] & \ldots & \cos[w_{k,M}(0.5)]
\end{bmatrix}
\begin{bmatrix}
2h(0) \\
2h(1) \\
\vdots \\
2h(M)
\end{bmatrix} = \begin{bmatrix} 0 \\
0 \\
\vdots \\
0 \end{bmatrix}
\]

This equation can be written as

\[ F \cdot x = 0 \]

where \( F \) is the constraint matrix. The dimensions of the \( F \)-matrix are \((\eta \text{ by } M1 + 1)\), \( x \) is an \((M1 + 1 \text{ by } 1)\) and the zero vector is an \( \eta \) by one vector, where \( \eta \) is the number of zero constraints in the stopband and \( M1 = (N/2) - 1 \) for \( N \) even and \( N-1 \) for \( N \) odd. To minimize the mean squared error in the passband, let the passband cost function be given by

\[
J_{pb} = \frac{1}{w_{pb}} \int_{0}^{w_{pb}} |H_d(w) - H(w)|^2 \, dw \tag{3.8}
\]

where \( w_{pb} \) is the passband frequency.

Since the desired frequency response equals one in the passband, the integrand reduces to

\[
|1 - s^T(w) x|^2
\]
and equation (3.8) becomes

$$J_{pb} = \frac{1}{w_{pb}} \left[ \int_0^{w_{pb}} dw - 2 \int_0^{w_{pb}} s^T(w) dw x + x^T \int_0^{w_{pb}} s(w) s^T(w) dw x \right]$$

letting the row-vector

$$R = \int_0^{w_{pb}} s^T(w) dw$$

and the matrix

$$Q_p = \int_0^{w_{pb}} s(w) s^T(w) dw$$

Equation (3.8) can be written as

$$J_{pb} = 1 - \frac{2}{w_{pb}} R x + \frac{1}{w_{pb}} x^T Q_p x$$

Now the problem can be stated as follows:

minimize

$$J = \alpha J_{sb} + \beta J_{pb}$$

subject to

$$F^T x = 0$$
where alpha and beta are weighting factors for the stopband and the passband respectively. The above problem can be solved using the method of the Lagrange multipliers. Define the Lagrange multiplier vector as

\[ \lambda = [\lambda_1 \lambda_2 \ldots \lambda_L]^T \]

where \( L \) is calculated by \( N/2 - \) (number of samples in the passband) - (number of samples in the transition band) - 1 for \( N \) even, and \( N-1 \) - (number of samples in passband) - (number of samples in transition band) + 1 for \( N \) odd. Form the augmented cost function

\[ J_a = \frac{\alpha}{w_r} x^T Q x + \beta \frac{2\beta}{w_{pb}} R x + \beta \frac{x^T Q_p x + \lambda^T [F x]}{w_{pb}} \]

The necessary conditions for an optimal solution are

\[ \frac{\partial J_a}{\partial x} = 0 \quad (3.10) \]

\[ \frac{\partial J_a}{\partial \lambda} = 0 \quad (3.11) \]

Carrying out the partial differentiation, equation (3.10) gives

\[ \frac{2\alpha}{w_r} Q x - \frac{2\beta}{w_{pb}} R x + \frac{2\beta}{w_{pb}} Q_p x + F^T \lambda = 0 \quad (3.12) \]

and equation (3.11) gives

\[ F x = 0 \quad (3.13) \]
Equations (3.12) and (3.13) can be written in the following matrix form

\[
\begin{bmatrix}
\frac{2\alpha}{w_r} Q + \frac{2\beta}{w_{pb}} Q_p & F^T \\
F & 0
\end{bmatrix}
\begin{bmatrix}
x \\
\lambda
\end{bmatrix} =
\begin{bmatrix}
\frac{2\beta}{w_{pb}} R \\
0
\end{bmatrix}
\]  

(3.14)

Equation (3.14) can be solved to yield

\[
x = -Q_0^{-1} F^T (F Q_0^{-1} F^T)^{-1} F Q_0^{-1} R_p + Q_0^{-1} R_p
\]  

(3.15)

and

\[
\lambda = (F Q_0^{-1} F^T)^{-1} F Q_0^{-1} R_p
\]  

(3.16)

where

\[
Q_0 = \frac{2\alpha}{w_r} Q + \frac{2\beta}{w_{pb}} Q_p
\]

and

\[
R_p = \frac{2\beta}{w_{pb}} R
\]

Since the majority of the filter’s frequency samples are constrained to zero and require no operations in the recursive realization of the filter (efficient realization), the remaining degrees of freedom are used to minimize the mean squared error in both the passband and the stopband. The design method
first finds the impulse response values and then uses them to calculate the frequency response. The non-zero frequency samples to be used in the implementation can then be found by evaluating the frequency response at the desired discrete values, i.e. for Type 1 FSF $H(k) = H(e^{j\frac{2\pi}{N}k})$ and for Type 2 FSF $H(k) = H(e^{j\frac{2\pi}{N}(k+\frac{1}{2})})$.

3.1 Example 3.1: A Lowpass Type 1 FSF for $N$ even

For this example, we will design a linear phase Type 1 lowpass frequency sampling filter which has an impulse response of length $N$, where $N = 128$. The mean squared error in the passband and stopband will be minimized equally. The number of frequency samples in the passband will be set to 5 with 3 samples in the transition band and 56 in the stopband. Since for $N$ even the frequency response must be zero at $w = \pi[s]$, only 55 constraints are needed in the stopband, so that the design technique has seven degrees of freedom to carry out the minimization.
In a real design situation, the specifications are most likely to be given in terms of the cutoff frequency, the width of the transition band, and the maximum allowed ripple in both the stop and pass bands. These specifications give the values of $w_{pb}$ and $w_{sh}$ so that the number of samples in the passband and transition band can be calculated by

$$w_k = \frac{2\pi}{N}(k + 0.5)$$

where $k$ is the number of samples.

A compromise might be necessary between the width of the transition band and the maximum allowable ripple in both the stop and the pass bands. The frequency response without the linear phase term is given by

$$H(w) = \sum_{n=0}^{63} 2h(n) \cos[w(63.5 - n)]$$

The filter coefficient vector is

$$x = [2h(0) \ 2h(1) \ \ldots \ 2h(63)]^T$$

and the transform kernel vector is

$$s(w) = [\cos(63.5w) \ \cos(62.5w) \ \ldots \ \cos(0.5w)]^T$$

The frequency response is constrained to zero at the following discrete stop-band samples
Optimization Approach for 1-D Filters

$$H(w)\big|_{w = \frac{2\pi}{N}k} = 0$$

where $k = 8, 9, \ldots, 62$

The constraining equation is then given by

$$\begin{bmatrix}
\cos[8\mu(62.5)] & \cos[8\mu(61.5)] & \ldots & \cos[8\mu(0.5)] \\
\cos[9\mu(62.5)] & \cos[9\mu(61.5)] & \ldots & \cos[9\mu(0.5)] \\
\vdots & \vdots & \ddots & \vdots \\
\cos[62\mu(62.5)] & \cos[62\mu(61.5)] & \ldots & \cos[62\mu(0.5)]
\end{bmatrix}
\begin{bmatrix}
h(0) \\
h(1) \\
\vdots \\
h(62)
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
\ddots \\
0
\end{bmatrix}$$

where $\mu = 2\pi/N = \pi/64$.

The dimensions of the $Q$ and $Q_F$ matrices are 64 by 64. The dimensions of the $R$ vector is 1 by 64 since it is defined as the transpose of the transform kernel vector $s(w)$. The dimensions of the Lagrange multiplier vector and the zero vector are 55 by 1. The values of $\alpha$ and $\beta$ are set to one in this example.

Now the impulse response coefficients vector $x$ can be calculated from equation (3.15). The zero-phase frequency response $H(w)$ can be calculated using equation (3.2). If this frequency response meets the desired specifications, then the non-zero frequency samples can be determined from
Optimization Approach for 1-D Filters

\[ H(k) = H(\omega) \bigg|_{\omega_k = \frac{2\pi k}{N}} \]

for \( k = 0, 1, \ldots, k_t \)

where \( k_t \) is the last sample in the transition band. In this example the non-zero frequency samples are:

0.999383
1.00061
0.999425
1.00034
1.00099
0.941753
0.545603
0.0925135
3.2 Example 3.2: A Lowpass FSF With Stopband Weighting

In this example, we will design the same filter as in example 3.1 with a stopband weighting factor of 1000. This means that the value of $\alpha$ in the stopband cost function will be set to 1000 instead of 1 as in the previous example. In this example, the non-zero frequency samples are:

1.00381
0.99607
1.00436
0.994447
1.00888
0.843905
0.35023
0.0354357
3.3 Example 3.3: A lowpass Type 2 FSF for $N$ odd

For this example, we will design a linear phase Type 2 lowpass frequency sampling filter which has an impulse response of length $N$, where $N = 129$. The mean squared error in the passband and stopband will be minimized with more weight on the passband. For applications where it is more important to have a minimal aliasing error, more weight can be put on the stopband to achieve a particular filter specifications. The number of frequency samples in the passband will be arbitrarily set to five with two samples in the transition band and 58 in the stop band. The frequency response will not go to zero at $w = \pi$ for $N$ odd and therefore 58 constraints are needed in the stopband. The number of degrees of freedom available to carry out the minimization will be seven.

The frequency response is given by

$$H(w) = \sum_{n=0}^{63} 2h(n) \cos[w(64 - n)] + h(64)$$

The filter coefficients vector is

$$x = [2h(0) \ 2h(1) \ldots \ 2h(63) \ h(64)]^T$$
The frequency response is constrained to zero at the following discrete stop-band samples

\[ H(w) \bigg|_{\omega_k = \frac{2\pi}{129}(k+0.5)} = 0 \]

where \( k = 7, 8, \ldots, 64 \)

The constraining equation is then given by

\[
\begin{bmatrix}
\cos[7.5\mu(64)] & \cos[7.5\mu(63)] & \ldots & \cos[7.5\mu] & 1 \\
\cos[8.5\mu(64)] & \cos[8.5\mu(63)] & \ldots & \cos[8.5\mu] & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\cos[64.5\mu(64)] & \cos[64.5\mu(63)] & \ldots & \cos[64.5\mu] & 1
\end{bmatrix}
\begin{bmatrix}
2h(0) \\
2h(1) \\
\vdots \\
2h(63) \\
h(64)
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\]

where \( \mu = 2\pi/129 \).

The value of \( \beta \) in equation (3.9) is set arbitrarily to five to put more weight on minimizing the mean squared error in the pass band.

As in the previous example, the impulse response coefficient vector can be calculated from equation (3.15). The zero-phase frequency response \( H(w) \)
can be calculated from the following equation

\[ H(w) = \sum_{n=0}^{M-1} 2h(n) \cos[w(M - n)] + h(M) \]

where \( M = \frac{(N - 1)}{2} = 64 \). In this example, the non-zero frequency samples are:

0.997351
1.00266
0.99735
1.00235
0.999761
0.948059
0.478347
3.4 Example 3.4: A lowpass Type 2 FSF for $N \text{ odd}$

In this example, we will design the same filter as in example 3.3 with a stopband weighting factor of 1000. This means that the value of $\alpha$ in the stopband cost function will be set to 1000 instead of 1 as in the previous example. Also the number of samples in the transition band will be increased to 3. The non-zero frequency samples are:

0.999897
1.00012
0.999873
0.999921
1.0019
0.951482
0.547118
0.102921
Figure 3.1: A Length 128 Lowpass Filter of Type 1 with no Weighting
Figure 3.2: A Length 128 Lowpass Filter of Type 1 with a Stopband Weight of 1000
Figure 3.3: A Length 129 Lowpass Filter of Type 2 with a Passband Weight of 5
Figure 3.4: A Length 129 Lowpass Filter of Type 2 with a Stopband Weight of 1000
Chapter 4

A New Optimization Approach to The Design of Two-Dimensional Frequency Sampling Filters

The development in this chapter extends the same idea in chapter three to the design of two-dimensional FIR filters. The mean squared error in the passband and the stopband will be minimized in both dimensions leaving
a number of samples in the two transition bands as don't cares. The de-
sign technique will find optimal values for these transition band samples in
the mean squared sense. The frequency response values that are used as
filter coefficients will be constrained to zero in both stopbands. The two-
dimensional frequency response is periodic with period $2\pi$ and we will use
an evenly spaced frequency samples along $2\pi$ in both dimensions.

The design procedure is simplified by requiring that the two-dimensional
filter be four-quadrant symmetric, that is

$$H(w_1, w_2) = H(-w_1, w_2) = H(w_1, -w_2)$$

This symmetry is not necessary and the design technique permits the design
of arbitrary zero-phase FIR filters.

We will use the quadrature symmetry equation for the frequency response,
namely, equation (2.30) written as

$$H(w_1, w_2) = h(0, 0) + 2 \sum_{n_1=1}^{M} h(n_1, 0) \cos(w_1 n_1) + 2 \sum_{n_2=1}^{M} h(0, n_2) \cos(w_2 n_2) + 4 \sum_{n_1=1}^{M} \sum_{n_2=1}^{M} h(n_1, n_2) \cos(w_1 n_1) \cos(w_2 n_2)$$  (4.1)
where

\[ M = \frac{N - 1}{2} \]

and assuming a square region of support.

Equation (4.1) can be written in vector-matrix notation. Defining the transform kernel vector

\[ s(w) = [1 \ 2\cos(w) \ 2\cos(2w) \ldots 2\cos(M)w]^T \]

and the filter coefficient matrix

\[ M = \begin{bmatrix}
    h_{0,0} & h_{0,1} & \ldots & h_{0,M-1} \\
    h_{1,0} & h_{1,1} & \ldots & h_{1,M-1} \\
    \vdots & \vdots & \ddots & \vdots \\
    h_{M-1,0} & h_{M-1,1} & \ldots & h_{M-1,M-1}
\end{bmatrix} \]

the frequency response can be written as

\[ H(w_1, w_2) = s^T(w_1) M s(w_2) \quad (4.2) \]

The mean squared error is given by

\[ E = \frac{1}{4\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |H(w_1, w_2) - H_d(w_1, w_2)|^2 \, dw_1 \, dw_2 \quad (4.3) \]
Optimization Approach for 2-D Filters

where $H_d$ is the desired frequency response.

For the stopband, the desired frequency response is zero. The area in the stopband corresponds to the total area of the first quadrant minus the area of the passband and the transition band, so the stopband cost function can be written as

$$ J_{sb} = \frac{1}{\pi^2 - w_{s1} w_{s2}} \left[ \int_{w_{s1}}^{\pi} \int_{w_{s1}}^{\pi} |H(w_1, w_2)|^2 \, dw_1 \, dw_2 + \right. \\
\left. \int_{0}^{w_{s1}} \int_{w_{s2}}^{\pi} |H(w_1, w_2)|^2 \, dw_1 \, dw_2 \right] $$  \hspace{1cm} (4.4)

Another way to represent equation (4.2) is to vectorize[13] the filter coefficient matrix $M$ by stacking the rows together into a column vector $h$, where

$$ h = [h_{0,0} \ h_{0,1} \ldots h_{0,M-1} \ h_{1,0} \ h_{1,1} \ldots h_{1,M-1} \ldots]^T $$

To generate a transform kernel vector that contains values in both dimensions, let the matrix

$$ s_1(w_1, w_2) = s(w_1) \ s^T(w_2) $$

and let $s(w_1, w_2)$ represent a column vector obtained by vectorizing $s_1(w_1, w_2)$.

Now the zero phase frequency response can be written as
Optimization Approach for 2-D Filters

\[ H(w_1, w_2) = s^T(w_1, w_2) \ h \]  \hspace{1cm} (4.5)

and

\[ |H(w_1, w_2)|^2 = h^T s(w_1, w_2) \ s^T(w_1, w_2) \ h \]  \hspace{1cm} (4.6)

using this expression, the stopband cost function can be written as

\[ J_{sb} = \frac{1}{\pi^2 - w_{sb1}w_{sb2}} \left[ h^T Q_1 \ h + h^T Q_2 \ h \right] \]  \hspace{1cm} (4.7)

where

\[ Q_1 = \int_{w_{sb1}}^{w_{sb1}} \int_{0}^{\pi} s(w_1, w_2) \ s^T(w_1, w_2) \ dw_1 \ dw_2 \]

and

\[ Q_2 = \int_{0}^{w_{sb2}} \int_{w_{sb2}}^{\pi} s(w_1, w_2) \ s^T(w_1, w_2) \ dw_1 \ dw_2 \]

The stopband frequency samples are constrained to zero at the discrete values

\[ H(w_1, w_2) \bigg|_{w_1 = \frac{\pi}{N_1}, k_1, w_2 = \frac{\pi}{N_2}, k_2} \]

\[ k_1 = k_{sb1}, k_{sb1+1} \ldots, M_1 \]

\[ k_2 = k_{sb2}, k_{sb2+1} \ldots, M_2 \]

where
Optimization Approach for 2-D Filters

\[ M_1 = \frac{N_1 - 1}{2} \]
\[ M_2 = \frac{N_2 - 1}{2} \]

these values will be set to zero in the following form

\[ F \cdot h = 0 \]

where \( F \) is the constraint matrix.

Setting a stopband frequency sample \((k_1, k_2)\) to zero requires

\[ H\left(\frac{2\pi}{N_1}k_1, \frac{2\pi}{N_2}k_2\right) = 0 \]

Therefore the \( F \) matrix will have \( s\left(\frac{2\pi}{N_1}k_1, \frac{2\pi}{N_2}k_2\right) \) as its rows with \( k_1 \) and \( k_2 \) set to the coordinates which are made up of integer values corresponding to the appropriate sample. So
Optimization Approach for 2-D Filters

\[
F = \begin{bmatrix}
\phi\left(\frac{2\pi}{N_1} k_{sb1}, \frac{2\pi}{N_2} 0 \right) \\
\phi\left(\frac{2\pi}{N_1} k_{sb1}, \frac{2\pi}{N_2} 1 \right) \\
\vdots \\
\phi\left(\frac{2\pi}{N_1} k_{sb1}, \frac{2\pi}{N_2} M_2 \right) \\
\phi\left(\frac{2\pi}{N_1} k_{sb2}, \frac{2\pi}{N_2} 0 \right) \\
\phi\left(\frac{2\pi}{N_1} k_{sb2}, \frac{2\pi}{N_2} 1 \right) \\
\vdots \\
\phi\left(\frac{2\pi}{N_1} k_{sb2}, \frac{2\pi}{N_2} M_2 \right) \\
\phi\left(\frac{2\pi}{N_1} M_1, \frac{2\pi}{N_2} M_2 \right) \\
\phi\left(\frac{2\pi}{N_1} 0, \frac{2\pi}{N_2} k_{sb2} \right) \\
\phi\left(\frac{2\pi}{N_1} 0, \frac{2\pi}{N_2} k_{sb2+1} \right) \\
\vdots \\
\phi\left(\frac{2\pi}{N_1} 0, \frac{2\pi}{N_2} M_2 \right) \\
\phi\left(\frac{2\pi}{N_1} \frac{2\pi}{N_2} k_{sb2} \right) \\
\phi\left(\frac{2\pi}{N_1} \frac{2\pi}{N_2} k_{sb2+1} \right) \\
\vdots \\
\phi\left(\frac{2\pi}{N_1} \frac{2\pi}{N_2} M_2 \right)
\end{bmatrix}
\]
The dimensions of the F matrix are $\eta (M_1 + 1)(M_2 + 1)$ where $\eta$ is the number of constraints in the stopband grid. To minimize the mean squared error in the passband, let the passband cost function be given by

$$J_{pb} = \frac{1}{w_{pb1}w_{pb2}} \int_0^{w_{pb1}} \int_0^{w_{pb2}} |1 - s^T(w_1, w_2) h|^2 dw_1 dw_2$$

(4.8)

where $w_{pb1}$ and $w_{pb2}$ are the cutoff frequencies in the first and second dimension respectively. The value of one in the integrand signifies the desired frequency response in the passband. Expanding the integrand, equation (4.8) becomes

$$J_{pb} = \frac{1}{w_{pb1}w_{pb2}} \left[ \int_0^{w_{pb1}} \int_0^{w_{pb2}} dw_1 dw_2 - 2 \int_0^{w_{pb1}} \int_0^{w_{pb2}} s^T(w_1, w_2) dw_1 dw_2 h^T \int_0^{w_{pb1}} \int_0^{w_{pb2}} s(w_1, w_2) s^T(w_1, w_2) dw_1 dw_2 h \right]$$

Letting the row-vector

$$R = \int_0^{w_{pb1}} \int_0^{w_{pb2}} s^T(w_1, w_2) dw_1 dw_2$$

and the matrix

$$Q_p = \int_0^{w_{pb1}} \int_0^{w_{pb2}} s(w_1, w_2) s^T(w_1, w_2) dw_1 dw_2$$

equation (4.8) can written as
Now the problem can be stated as follows: minimize

\[ J = \alpha J_{sb} + \beta J_{pb} \]

that is

\[ J = \frac{\alpha}{w_r} \left[ h^T Q_1 h + h^T Q_2 h \right] + \frac{\beta}{w_{pb1} w_{pb2}} \left[ h^T Q_p h - 2 R h \right] + \beta \quad (4.9) \]

where \( w_r = \pi^2 - w_{pb1} w_{pb2} \) subject to

\[ F h = 0 \]

where alpha and beta are weighting factors for the stopband and passband respectively. The dimensions of the zero vector is equal to the number of stopband zero samples which is \( \eta \). Following the solution method of chapter 3 and defining the Lagrange multiplier vector as

\[ \lambda = [\lambda_1 \; \lambda_2 \; \ldots \; \lambda_\eta]^T \]

The augmented cost function is given by

\[ J_a = J + \lambda^T [F \, h] \]

The necessary conditions for an optimal solution are
Optimization Approach for 2-D Filters

\[
\frac{\partial J_a}{\partial h} = 0 \quad (4.10)
\]

\[
\frac{\partial J_a}{\partial \lambda} = 0 \quad (4.11)
\]

Carrying out the partial differentiation, equation (4.10) gives

\[
\frac{2\alpha}{w_r} [Q_1 h + Q_2 h] + \frac{2\beta}{w_{pb1}w_{pb2}} [Q_p h - R] + F^T \lambda = 0
\]

and equation (4.11) gives

\[
F h = 0 \quad (4.13)
\]

Equations (4.12) and (4.13) can be written in the following matrix form

\[
\begin{bmatrix}
  A & F^T \\
  F & 0
\end{bmatrix}
\begin{bmatrix}
  h \\
  \lambda
\end{bmatrix} =
\begin{bmatrix}
  \frac{2\beta}{w_{pb1}w_{pb2}} R \\
  0
\end{bmatrix} \quad (4.14)
\]

where

\[
A = \frac{2\alpha}{w_r} [Q_1 + Q_2] + \frac{2\beta}{w_{pb1}w_{pb2}} Q_p
\]

Equation (4.14) can be solved to yield

\[
h = -A^{-1}F^T (FA^{-1}F^T)^{-1} FA^{-1} R_p + A^{-1} R_p \quad (4.15)
\]

and
\[ \lambda = (FA^{-1}F^T)^{-1} FA^{-1} R_p \] (4.16)

where

\[ R_p = \frac{2\beta}{w_{p1}w_{p2}} R \]

From equation (4.15), the impulse response values are used to calculate the frequency response. That is

\[ H(w_1, w_2) = s^T(w_1, w_2) h \]

The non-zero frequency samples to be used in the implementation can then be found by evaluating the frequency response at the desired discrete values, that is

\[ H(k_1, k_2) = H\left( \frac{2\pi}{N_1} k_1, \frac{2\pi}{N_2} k_2 \right) \]
4.1 Example 4.1: A Two-Dimensional Low Pass Frequency Sampling Filter for $N_1 = N_2$ Odd

We will design a 2-D zero phase lowpass FSF with length 17 in both dimensions, that is, $N_1 = N_2 = 17$. The mean squared error in the passband and stopband will be minimized equally. The filter will be constrained to have quadrature symmetry. Looking at the first quadrature, the passband will have 3 samples in both dimensions with one transition band sample in both dimensions. The stopband grid will have 65 constraints so that 16 degrees of freedom remains to carry out the minimization. The zero-phase frequency response is given by

$$H(w_1, w_2) = h(0, 0) + 2 \sum_{n_1=1}^{8} h(n_1, 0) \cos(w_1 n_1) +$$
$$2 \sum_{n_2=1}^{8} h(0, n_2) \cos(w_2 n_2) +$$
$$4 \sum_{n_1=1}^{8} \sum_{n_2=1}^{8} \cos(w_1 n_1) \cos(w_2 n_2)$$
The filter coefficient vector $h$ is the vectorized version of the 9 by 9 filter coefficient matrix. The transform kernel vector is

$$s(w) = [1 \ 2 \cos(w) \ 2 \cos(2w) \ldots 2 \cos(8w)]^T$$

The 2-D transform kernel vector is obtained by vectorizing $s_1$, where $s_1$ is given by

$$s_1 = s(w_1) s^T(w_2)$$

In this example, the values of the passband and stopband frequencies are given by

$$w_{pb1} = w_{pb2} = \left(\frac{2\pi}{17}\right)(2.5)$$

$$w_{sb1} = w_{sb2} = \left(\frac{2\pi}{17}\right)(3.5)$$

The constraint matrix is given by
Optimization Approach for 2-D Filters

\[
F = \begin{bmatrix}
\sigma(\frac{4}{17}, \frac{0}{17}) \\
\sigma(\frac{4}{17}, \frac{1}{17}) \\
\vdots \\
\sigma(\frac{4}{17}, \frac{8}{17}) \\
\sigma(\frac{5}{17}, \frac{0}{17}) \\
\sigma(\frac{5}{17}, \frac{1}{17}) \\
\vdots \\
\sigma(\frac{5}{17}, \frac{8}{17}) \\
\vdots \\
\sigma(\frac{8}{17}, \frac{0}{17}) \\
\sigma(\frac{0}{17}, \frac{4}{17}) \\
\sigma(\frac{0}{17}, \frac{5}{17}) \\
\vdots \\
\sigma(\frac{0}{17}, \frac{8}{17}) \\
\sigma(\frac{1}{17}, \frac{4}{17}) \\
\sigma(\frac{1}{17}, \frac{5}{17}) \\
\vdots \\
\sigma(\frac{1}{17}, \frac{8}{17}) \\
\vdots 
\end{bmatrix}
\]
The dimensions of the constraint matrix are 65 by 81. Now all the quantities in equation 4.14 can be formulated for this example. Since the mean squared error in both the passband and stopband are minimized equally, the values of alpha and beta are set to one. Equation (4.14) can be solved to yield the impulse response coefficients of the filter for the first quadrant in vector form. The two-dimensional frequency response can then be found from

\[ H(w_1, w_2) = s^T(w_1, w_2) h \]

The non-zero frequency samples are found by evaluating the frequency response at the desired discrete values

\[ H(k_1, k_2) = H\left(\frac{2\pi}{17}k_1, \frac{2\pi}{17}k_2\right) \]

where

\[ k_1 = k_2 = 0, 1, \ldots, 3 \]

They are:
Optimization Approach for 2-D Filters

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Figure 4.1: A Square Symmetric Two-Dimensional Lowpass Filter of Length (17 x 17). a) Front View b) Side View
Chapter 5

Conclusions

The previous chapters show a new optimization technique for designing one dimensional digital filters and the extension of the technique to the two-dimensional problem. The filter's impulse response coefficients were calculated by solving a set of linear equations. Although a respectable computational cost was needed to do integration of big matrices, an analytical formula can be derived in which the computer is used only to evaluate the limits which reduces the computational costs considerably. The recursive implementation exploits the fact that the stopband samples were constrained to zero which means that the method works best for narrowband filters. The results show that as the width of the passband gets narrower, less degrees of freedom are
Conclusions

available to carry out the minimization in both the stopband and the passband. Also, increasing the number of transition band samples results in less ripple. A great deal of flexibility is available to the designer through the use of the weighting factors in both the passband and stopband cost functions. A compromise between the width of the transition band, the filter length and the maximum allowable ripple in both bands might be necessary in a real problem. A comparison between this method and the famous minimax method was not done. Future efforts might be aimed at constraining the stopband of lowpass filters to zero at equally spaced points and use different design techniques so that they can be realized more efficiently. Also for the same technique used here, different functions can be used in the transition band instead of leaving them as don’t cares.
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