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## Feedback linearization, invertibility of map, zero dynamics, and nonlinear control of the space station

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FEEDBACK LINEARIZATION, INVERTIBILITY OF  
MAP, ZERO DYNAMICS, AND NONLINEAR  
CONTROL OF THE SPACE STATION

by

Theodore C. Bossart

A thesis submitted in partial fulfillment  
of the requirements for the degree of

Masters of Science

of

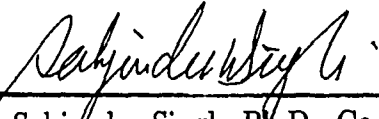
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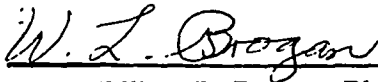
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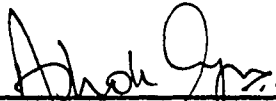
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## ABSTRACT

Based on feedback linearization and ultimate boundedness theory, a new approach to attitude control of the space station using control moment gyros (CMG's) is presented. A linearizing transformation is derived to obtain a simple linear representation of the nonlinear pitch axis dynamics. A feedback control law for trajectory tracking is derived when there is no disturbance torque acting on the space station. For attitude control in the presence of uncertain torque input, an additional control signal is superimposed such that in the closed-loop system, attitude responses are uniformly ultimately bounded and tend to a small set of ultimate boundedness. Extension of this approach to linearization of the coupled yaw and roll axis dynamics and control is presented. Simulation results for the pitch axis control are obtained to show that in the closed-loop system precise attitude control is accomplished.

Also, based on the nonlinear inversion technique, a control law is derived such that in the closed-loop system the input-output map is linear, and the selected output variables are independently controlled. The stability of the zero dynamics is examined and it is shown that in the closed-loop system



attitude angles and the CMG momenta converge to the origin. Simulation results are presented to show attitude and CMG momenta regulation capability of the controller.

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## CHAPTER ONE

### INTRODUCTION

By employing control moment gyro (CMG), this paper suggests methods to control the dynamics of the Space Station. The equations of motion of the space station are described by nonlinear differential equations. Often, attitude control system design using linear control theory [1-4] is obtained. Linear control systems are designed based on the assumption that the perturbations in attitude angles are small. For large changes in orientation of space vehicles employing moment exchange devices, nonlinear controllers have been designed in literature [5-11]. An adaptive control has been designed [10]. In recent papers [2, 4], an approach to CMG momentum management and attitude control of the space station has been presented. Control system design is based on a linearized model of the space station. Using feedback linearization theory [11], a linear representation of the nonlinear dynamics of the space station is derived. In the new state space, a feedback control law is derived for the control of pitch angle. In a realistic situation, unknown but bounded disturbance torque acts on the space station. A nonlinear control law is derived in the closed-loop system with the total control input, the

attitude trajectory is uniform ultimate bounded in a small neighborhood of the origin.

It is assumed that the attitude control system design of the space station employs control moment gyros (CMG's). CMG's are moment exchange devices that are considered to be ideal torquers in the mathematical model. For the derivation of control law, a new set of output vectors is defined. Each output variable is a linear combination of attitude angle, angular rate and CMG angular momentum.

The nonlinear inversion [14, 15] theory is applied to obtain a feedback control law for the linearization of the selected input-output map [12]. The parameters in the output vectors are chosen so that the relative degree of the system is two. Input-output feedback linearization allows independent regulation of each of the outputs to the origin. The zero dynamics [17, 18] are derived and are stable at the origin. In the closed-loop system, the attitude angles and the momentum tend to zero.

## CHAPTER TWO

### MATHEMATICAL MODEL

The space station is in a circular orbit. An orbital frame of reference (LVLH axis) with its origin at the center of mass of the space station is chosen. The axis of the reference frame are chosen with the roll axis is the flight direction, the pitch axis is perpendicular to the orbital plane, and the yaw axis points toward the Earth. The orientation of the space station with respect to the reference frame is obtained by a pitch-yaw-roll ( $\theta_2 - \theta_3 - \theta_1$ ) sequence of rotations, where  $\theta_1$ ,  $\theta_2$ , and  $\theta_3$  are the roll, pitch, and yaw angles. The nonlinear equations of motion can be written as [2].

Space Station Dynamics:

$$\begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix} = - \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

$$\begin{aligned}
& \times \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} + 3n^2 \begin{bmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{bmatrix} \\
& \times \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} + \begin{bmatrix} -u_1 + w_1 \\ -u_2 + w_2 \\ -u_3 + w_3 \end{bmatrix} \tag{1}
\end{aligned}$$

where

$$c_1 \triangleq -\sin\theta_2 \cos\theta_3$$

$$c_2 \triangleq \cos\theta_1 \sin\theta_2 \sin\theta_3 + \sin\theta_1 \cos\theta_2$$

$$c_3 \triangleq -\sin\theta_1 \sin\theta_2 \sin\theta_3 + \cos\theta_1 \cos\theta_2$$

Attitude kinematics:

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = \frac{1}{\cos\theta_3} \begin{bmatrix} \cos\theta_3 & -\cos\theta_1 \sin\theta_3 & \sin\theta_1 \sin\theta_3 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 \cos\theta_3 & \cos\theta_1 \cos\theta_3 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} + \begin{bmatrix} 0 \\ n \\ 0 \end{bmatrix} \tag{2}$$

CMG momentum:

$$\begin{bmatrix} \dot{h}_1 \\ \dot{h}_2 \\ \dot{h}_3 \end{bmatrix} + \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad (3)$$

where the orbital angular velocity is  $n=0.0011$  rad/s;  $(\omega_1, \omega_2, \omega_3)$  are the body-axis components of absolute angular velocity;  $(I_{11}, I_{22}, I_{33})$  are the moments of inertia;  $I_{ij}$  ( $i \neq j$ ) are the products of inertia;  $(h_1, h_2, h_3)$  are the body-axis components of CMG momentum;  $(u_1, u_2, u_3)$  are the body-axis components of control torque; and  $(w_1, w_2, w_3)$  are the body-axis components of disturbance torque. For the configuration of the space station (e.g., assembly flight 3), the complete equations of motion have been derived in the literature [2].

These are:

$$I_1 \ddot{\theta}_1 + (1 + 3\cos^2\theta_2)n^2(I_2 - I_3)\theta_1 - n(I_1 - I_2 + I_3)\dot{\theta}_3 \quad (4)$$

$$+ 3(I_2 - I_3)n^2(\sin\theta_2\cos\theta_2)\theta_3 = -u_1 + w_1$$

$$I_2 \ddot{\theta}_2 + 3n^2(I_1 - I_3)\sin\theta_2\cos\theta_2 = -u_2 + w_2$$

$$I_3\ddot{\theta}_3 + (1 + 3\sin^2\theta_2)n^2(I_2 - I_1)\theta_3 + n(I_1 - I_2 + I_3)\dot{\theta}_1$$

$$+ 3(I_2 - I_1)n^2(\sin\theta_2\cos\theta_2)\theta_1 = -u_3 + w_3$$

$$\dot{h}_1 - nh_3 = u_1$$

$$\dot{h}_2 = u_2$$

$$\dot{h}_3 + nh_1 = u_3$$

Equation (4) is derived from (1), (2), (3) assuming that  $\theta_2$  is large but the roll, and yaw attitude errors are small. It is assumed here that the products of inertia are small and these are neglected. The nonlinear functions of the pitch angle are retained in the model. Here  $I_i \triangleq I_{ii}; i=1,2,3$ . Observe from (4) that the pitch axis dynamics are decoupled from the roll and yaw dynamics. Uncoupling of pitch axis motion, simplifies the attitude control problem.

Define the state vector:

$$x = (\theta^T, \dot{\theta}^T, h^T)^T \in R^9$$

where

$$\theta = (\theta_1, \theta_2, \theta_3)^T$$



and

$$h = (h_1, h_2, h_3)^T$$

Here T denotes transposition. Let the control vector be

$$u = (u_1, u_2, u_3)^T$$

Using (4), one obtains the state variable representation of the system

$$\dot{x} = f(x) + Bu + Dw \quad (5)$$

where

$$f(x) = (\dot{\theta}^T, f_4(x), f_5(x), f_6(x), nh_3, 0, -nh_1)^T,$$

$$f_4(x) = [(1 + 3\cos^2\theta_2)n^2(I_3 - I_2)\theta_1 + n(I_1 - I_2 + I_3)\dot{\theta}_3$$

$$-1.5(I_2 - I_3)n^2\sin 2\theta_2\theta_3]/I_1$$

$$f_5(x) = 1.5n^2(I_3 - I_1)\sin 2\theta_2/I_2$$

$$f_6(x) = [(1 + 3\sin^2\theta_2)n^2(I_1 - I_2)\theta_3 - n(I_1 - I_2 + I_3)\dot{\theta}_1]/I_3$$

$$-1.5(I_2 - I_1)n^2\sin 2\theta_2\theta_1$$

$$B = \begin{bmatrix} 0 \\ -I^{-1} \\ I_d \end{bmatrix}, \quad I = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 \\ I^{-1} \\ 0 \end{bmatrix}$$

where  $I_d$  is a  $3 \times 3$  identity matrix and  $0$  denotes the null matrix.

## CHAPTER THREE

### FEEDBACK LINEARIZATION AND ULTIMATE BOUNDNESS CONTROL

#### 3.1 Linearizing Transformation for Pitch Dynamics

For the pitch axis control system design, consider the equation:

$$\ddot{\theta}_2 = -\frac{3n^2}{2I_2}(I_1 - I_3)\sin 2\theta_2 + \frac{1}{I_2}(-u_2 + w_2) \quad (6)$$

$$\dot{h}_2 = u_2$$

A nonlinear transformation from (5) is derived that has a linear representation by the theory of Hunt, Su, and Meyer[16]. For the derivation of linearizing transformation, the unknown disturbance torque  $w_2=0$ .

Equation (5) written in a compact notation is:

$$\dot{x} = f(x) + gu_2 \quad (7)$$

where

$$b_i = (1/I_i), \quad i = 1, 2, 3$$

$$x = (\theta_2, \dot{\theta}_2, h_2)^T, \quad a_2 = -\frac{3n^2}{2I^2}(I_1 - I_3),$$

$$f_{\theta_2} = -a_2 \sin 2\theta_2$$

$$f(x) = \begin{bmatrix} x_2 \\ -a_2 \sin 2\theta_2 \\ 0 \end{bmatrix}, \quad g = \begin{bmatrix} 0 \\ -b_2 \\ 1 \end{bmatrix}$$

Define the Lie bracket of two vector fields  $f$  and  $g$ , denoted as  $[f, g]$  or  $ad_f g$

by

$$ad_f g = [f, g] = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g$$

Repeated Lie brackets are denoted as

$$ad_f^0 g = g$$

$$ad_f^k = [f, ad_f^{k-1} g], \quad k > 0$$

**Definition:** A set of  $l$  vectors  $f_1(x), \dots, f_l(x)$  is involutive if there exist smooth

functions  $\gamma_{ijk}(x)$  such that

$$[f_i, f_j](x) = \sum_{k=1}^l \gamma_{ijk}(x) f_k(x), \forall i, j, k = 1, \dots, l$$

To verify the existence of linearizing transformation on an open set  $U$  containing the origin  $x=0$ , use the following theorem.

**Theorem 1:** The dynamics of the nonlinear system (6) are equivalent in a local region to the dynamics of a controllable linear system by change of state and input coordinates and state feedback if and only if

1. The distribution

$\text{span}\{g, ad_f g, ad_f^2 g\}$  has dimension 3 on  $U$ .

2. The distribution

$\text{span}\{g, ad_f g\}$  is involutive on  $U$ .

In the following conditions 1 and 2 are verified. Computing the Lie brackets yields:

$$ad_f g = [f, g] = \frac{\partial g(x)}{\partial x} f(x) - \frac{\partial f(x)}{\partial x} g(x)$$

$$= \begin{bmatrix} b_2 \\ 0 \\ 0 \end{bmatrix} \quad (8)$$

$$ad_f^2 g = [f, ad_f g] = - \begin{bmatrix} 0 & 1 & 0 \\ -2a_2 \cos 2\theta_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} ad_f g(x)$$

$$= \begin{bmatrix} 0 \\ 2a_2 b_2 \cos 2\theta_2 \\ 0 \end{bmatrix} \quad (9)$$

The vectors  $g$ ,  $ad_f g$ , and  $ad_f^2 g(x)$  are independent at  $x=0$ . Therefore, these vectors fields are independent on an open  $U$  containing the origin  $x=0$ . This verifies condition 1.

The vector fields  $g$ ,  $ad_f g$  are involutive on  $U$  since

$$[g, ad_f g] = 0 \in \text{span}\{g, ad_f g\}$$

and this verifies condition 2.

According to Theorem 1, a linearizing transformation exists. In order to

find this transformation [16], one needs to solve a system of linear partial differential equations. This can be accomplished by solving the following system of equations

$$\frac{dx}{ds} = \hat{f}(x), x(0) = 0 \quad \forall s \in R \quad (10)$$

$$\frac{dx}{dt_1} = ad_f g, x(s, 0) = x(s) \quad \forall t_1 \in R \quad (11)$$

$$\frac{dx}{dt_2} = g, x(s, t_1, 0) = x(s, t_1) \quad \forall t_2 \in R \quad (12)$$

where  $\hat{f}(x) = ad_f^2 g$  or  $\hat{f}(x)$  is any vector field that is linearly independent of  $g$  and  $[f, g]$ .

Choose

$$\hat{f}(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad (13)$$

The set of equations (9) to (11) are solved by first solving (9) with the initial condition  $x(0) = 0$ .

$$x(s) = (0, s, 0)^T. \quad (14)$$

Now solve (10). The solution of

$$\frac{dx}{dt_1} = \begin{bmatrix} b_2 \\ 0 \\ 0 \end{bmatrix}, x(s, 0) = (0, s, 0)^T \quad (15)$$

is

$$x(s, t_1) = (b_2 t_1, s, 0)^T$$

Solving (eqn. (11))

$$\frac{dx}{dt_2} = (0, -b_2, 1)^T, x(s, t_1, 0) = (b_2 t_1, s, 0)^T \quad (16)$$

one gets

$$x(s, t_1, t_2) = (b_2 t_1, -b_2 t_2 + s, t_2)^T \quad (17)$$

Solving (16) for  $s$

$$s = x_2 + b_2 x_3 \quad (18)$$



Set  $s=\xi_1$ . Then the linearizing transformation is by [11]

$$\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} \quad (19)$$

where

$$\xi_2 = \dot{\xi}_1, \quad \xi_3 = \dot{\xi}_2. \quad (20)$$

The nonlinear transformation is

$$\xi = \begin{bmatrix} x_2 + b_2 x_3 \\ -a_2 \sin 2\theta_2 \\ -2a_2 \cos 2\theta_2 \dot{\theta}_2 \end{bmatrix} \quad (21)$$

A linear representation of (6) is given by

$$\dot{\xi} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xi + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} v_2 \quad (22)$$

where

$$v_2 = [4a_2 \sin 2\theta_2 \dot{\theta}_2^2 + a_2^2 \sin 4\theta_2] + (2a_2 b_2 \cos 2\theta_2) u_2 \quad (23)$$

$$\triangleq a_2^*(x) + b_2^* u_2$$

A choice of coordinate transformation (20) and a feedback control

$$u_2 = b_2^{*-1}(x)[-a_2^* + v_2] \quad (24)$$

results in a controllable linear system representation (21) of the nonlinear pitch dynamics.

### 3.2 Pitch Axis Control

The new state vector  $z_1$  is the integral of  $\xi_1$ . The introduction of integral term leads to robustness in the control system to uncertainty in system parameters.

$$z_1 = \theta_2 + z_s \quad (25)$$

where

$$\dot{z}_s = b_2 x_3. \quad (26)$$

Define a new state vector  $z = (z_1, z_2, z_3, z_4)^T = (z_1, \xi^T)^T \ z \in R^4$  given by

$$z = \begin{bmatrix} \theta_2 + z_s \\ x_2 + b_2 x_3 \\ -a_2 \sin 2\theta_2 \\ -2a_2 \cos 2\theta_2 \dot{\theta}_2 \end{bmatrix} \quad (27)$$

$$\dot{z} = \begin{bmatrix} z_2 \\ z_3 \\ z_4 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} v_2 + \begin{bmatrix} 0 \\ b_2 w_2 \\ b_2 \dot{w}_2 \\ -2a_2 b_2 \cos 2\theta_2 w_2 \end{bmatrix} \quad (28)$$

Suppose that it is desired to track the reference trajectory  $z_{c1}$  of a command generator

$$\Pi_c(s) z_{c1} = \lambda_c \omega_{nc1} \omega_{nc2} z_{c1}^* \quad (29)$$

$$\Pi_c(s) = (s + \lambda_c) \prod_{i=1}^2 (s^2 + 2\zeta_{ci} \omega_{nci} s + \omega_{nci}^2),$$

where  $s = \frac{d}{dt}$  and  $z_{c1}^*$  is the terminal value of  $z_{c1}$ . Define

$$\tilde{z}_i = z_i - z_{ci}, \quad i = 1, 2, 3, 4 \quad (30)$$

$$z_{c(i+1)} = \dot{z}_{ci}, \quad i = 1, 2, 3, 4$$

Then one has

$$\dot{\tilde{z}} = \begin{bmatrix} \tilde{z}_2 \\ \tilde{z}_3 \\ \tilde{z}_4 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} v_r + \begin{bmatrix} 0 \\ b_2 \omega_2 \\ b_2 \dot{\omega}_2 \\ -z_{c5} - 2a_2 b_2 \cos 2\theta_2 \omega_2 \end{bmatrix} \quad (31)$$

Choose a control law

$$v_2 = -p_0 \tilde{z}_1 - p_1 \tilde{z}_2 - p_2 \tilde{z}_3 - p_3 \tilde{z}_4 + z_{c5} + v_r \quad (32)$$

In the closed-loop system (30) and (31), one has

$$\dot{\tilde{z}} = A\tilde{z} + B(v_r - 2a_2 b_2 \cos 2\theta_2 \omega_2) + d(t) \quad (33)$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -p_0 & -p_1 & -p_2 & -p_3 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad (34)$$

$$d = (0, bw_2, b\dot{w}_2, 0)^T$$

Choose  $p_i$  such that  $A$  is a Hurwitz matrix.

Note that when  $w_2(t) \equiv 0$ ,  $\tilde{z}_1$  satisfies a linear differential equation given by

$$\Pi_e(s)\tilde{z}_1 = 0 \quad (35)$$

where

$$\Pi_e(s) = (s^4 + p_3s^3 + p_2s^2 + p_1s + p_0) \quad (36)$$

The tracking error  $\tilde{z}_1$  has a linear response. The parameters  $p_i$  are chosen by equating the polynomials

$$\Pi_e(s) = \prod_{i=1}^2 (s^2 + 2\zeta_{ei}\omega_{nei}s + \omega_{nei}^2) \quad (37)$$

where

$$\zeta_{ei} > 0, \omega_{nei} > 0.$$

The tracking error  $\tilde{z}_1(t) \rightarrow 0$ , as  $t \rightarrow \infty$  when  $w_2 \equiv 0$ . Bounded responses for  $\tilde{z}_i$  are obtained.

### 3.3 Ultimate Boundedness Control

In this section, based on the Lyapunov theory [19], a compensating control signal is derived which causes the closed-loop system the trajectories to tend to a small neighborhood of the origin in spite of the disturbance input  $w_2$ . It is assumed that the disturbance input  $w_2$  is bounded but unknown and is an arbitrary function of time. Since  $A$  is a Hurwitz matrix there exists a unique symmetric positive definite solution for  $P$  (denoted as  $P > 0$ ) of the Lyapunov equation

$$A^T P + P A = -Q \quad (38)$$

for any given  $Q > 0$ .

#### Assumption 1:

$$|2a_2 b_2 \cos 2\theta_2 w_2(t)| < \rho(\theta_2, w_2)$$

$$|2P d(t)| < \beta_1 \quad (39)$$

Choose a control law of the form

$$v_r = \frac{-K\alpha}{|\alpha| + \delta}, \quad \delta > 0 \quad (40)$$

$$\alpha = \tilde{z}^T P B$$

$$K \geq \frac{\dot{\rho}(\epsilon + \delta)}{\epsilon}, \quad \epsilon > 0$$

Consider a Lyapunov function

$$V(\tilde{z}) = \tilde{z}^T P \tilde{z} \quad (41)$$

Then along the solution of the system (32) and (39), the derivative of V is:

$$\begin{aligned} \dot{V} = & -\tilde{z}^T Q \tilde{z} + 2\tilde{z}^T P B v_r - 2\tilde{z}^T P B 2a_2 b_2 \cos 2\theta_2 w_2 \\ & + 2\tilde{z}^T P B d(t) \end{aligned} \quad (42)$$

Using the control law (39) in (41) gives

$$\dot{V} \leq -\lambda_m(Q) \|\tilde{z}\|^2 + \frac{-\alpha^2 K}{|\alpha| + \delta} + |\alpha| \rho + \|\tilde{z}\| \beta_1 \quad (43)$$

$$\dot{V} \leq -\lambda_m(Q) \|\tilde{z}\|^2 - \frac{|\alpha| P \delta}{\epsilon(|\alpha| + \delta)} \delta (|\alpha| - \epsilon) + \|\tilde{z}\| \beta_1$$

where  $\lambda_m(R)$  [ $\lambda_M(R)$ ] denotes the smallest [largest] eigenvalue of a matrix R.



$$\text{Let } \{ \max |\alpha|(-|\alpha| + \epsilon)/(|\alpha| + \delta), |\alpha| > 0 \} = c^* \quad (44)$$

Then

$$\dot{V} \leq -[\lambda_m(Q)\|\tilde{z}\|^2 - \|\tilde{z}\|\beta_1 - c^*] \quad (45)$$

let

$$\eta = \frac{\beta_1 + \sqrt{\beta_1^2 + 4c^*\lambda_m(Q)}}{2\lambda_m(Q)} \quad (46)$$

Define ellipsoids as

$$\tilde{Z}(r) = \left\{ \tilde{z} \in R^4 : \tilde{z}^T P \tilde{z} \leq r > 0 \right\} \quad (47)$$

Now let

$$r^* = \min\{ r : \tilde{Z}(r) \supseteq B(\eta) \} \quad (48)$$

where

$$B(\eta) = \{ \tilde{z} : \|\tilde{z}\| \leq \eta \} \quad (49)$$

Note that

$$r^* = \lambda_M(P)\eta^2$$

So that

$$\dot{V} < 0, \tilde{z} \notin \tilde{Z}(r^*) \quad (50)$$

Since  $V(\tilde{z}) > 0$ , for  $\tilde{z} \neq 0$ ,  $V(0) = 0$ , and the derivative of  $V$  is negative according to (49),  $V(\tilde{z})$  decreases along the trajectory of the system as long as  $\tilde{z}(t) \notin \tilde{Z}(\bar{r})$ ,  $\bar{r} > r^*$  and the trajectory approaches the ellipsoid  $\tilde{Z}(\bar{r})$  for any  $\bar{r} > r^*$ .

### 3.4 Yaw and Roll Linearization and Control Design

The yaw and roll dynamics are given by

$$\frac{d}{dt} \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \\ \theta_3 \\ \dot{\theta}_3 \\ \dot{h}_1 \\ \dot{h}_3 \end{bmatrix} = \begin{bmatrix} \dot{\theta}_1 \\ f_{\theta_1} \\ \dot{\theta}_3 \\ f_{\theta_3} \\ nh_3 \\ -nh_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -b_1 & 0 \\ 0 & 0 \\ 0 & -b_3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_3 \end{bmatrix} \quad (51)$$

$$+ \begin{bmatrix} 0 & 0 \\ b_1 & 0 \\ 0 & 0 \\ 0 & b_3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_3 \end{bmatrix}$$

$$\triangleq \bar{f}(\bar{x}, \theta_2) + g_1 u_1 + g_3 u_3 - g_1 w_1 - g_3 w_3$$

where

$$\bar{u} = [u_1, u_3]^T, \bar{w} = (w_1, w_3)^T, \bar{x} = (\theta_1, \dot{\theta}_1, \theta_3, \dot{\theta}_3, h_1, h_3)^T \quad (52)$$

$$f_{\theta_1} = \frac{-(1 + 3\cos^2\theta_2)n^2(I_2 - I_3)\theta_1}{I_1} + \frac{n(I_1 - I_2 + I_3)\dot{\theta}_3}{I_1}$$

$$\frac{-3(I_2 - I_3)n^2\sin 2\theta_2\theta_3}{2I_1}$$

$$f_{\theta_3} = \frac{-(1 + 3\sin^2\theta_2)n^2(I_2 - I_1)\theta_3}{I_3} + \frac{n(I_1 - I_2 + I_3)\dot{\theta}_1}{I_3}$$

$$\frac{-3(I_2 - I_3)n^2\sin 2\theta_2\theta_1}{2I_3}$$

A linearizing transformation for this system with  $\bar{w}(t)=0$ , is the Lie derivative of a scalar function  $h$  along a vector field  $f$ , denoted as  $L_f h(x)$ ; is given by

$$L_f h(x) = \left[ \frac{\partial h(x)}{\partial x} \right] f(x) \quad (53)$$

where

$$x = (\bar{x}^T, \theta_2, \dot{\theta}_2, h_2)^T, f = (\bar{f}^T, \dot{\theta}_2, f_{\theta_2}, u_2)^T \quad (54)$$

Let

$$L_f^2 h(x) = L_f(L_f h)(x) \quad (55)$$

$$L_{g_i} L_f(x) = L_{g_i}(L_f h)(x)$$

Define a transformation

$$\Phi = \begin{bmatrix} \phi_{11} \\ \phi_{12} \\ \phi_{13} \\ \phi_{31} \\ \phi_{32} \\ \phi_{33} \end{bmatrix} \quad (56)$$

where the coordinates  $\phi_{11}, \phi_{31}$  are chosen as

$$\phi_{11} = \dot{\theta}_1 + \frac{h_1}{I_1} + \frac{n(I_2 - I_1)}{I_1} \theta_3 \quad (57)$$

$$\phi_{31} = \dot{\theta}_3 + \frac{h_3}{I_3} + \frac{n(I_3 - I_2)}{I_3} \theta_1$$

By the choice of  $\phi_{11}$  and  $\phi_{31}$ , one has  $L_{g_k} L_f^j \phi_{i1} = 0$ ,  $i \in \{1, 3\}$ ;  $j=0,1$ ;  $k=\{1,3\}$ ,

$$\dot{\phi}_{12} = \dot{\phi}_{11} = L_f \phi_{11} \quad (58)$$

$$\phi_{13} = \dot{\phi}_{12} = L_f^2 \phi_{11}$$

$$\phi_{32} = \dot{\phi}_{31} = L_f \phi_{31}$$

$$\phi_{33} = \dot{\phi}_{32} = L_f \phi_{32} = L_f^2 \phi_{31}$$

Moreover

$$\begin{bmatrix} \phi_{11}^{(3)} \\ \phi_{31}^{(3)} \end{bmatrix} = \begin{bmatrix} L_f^3 \phi_{11} \\ L_f^3 \phi_{31} \end{bmatrix} + \begin{bmatrix} L_{g_1} L_f^2 \phi_{11} & L_{g_2} L_f^2 \phi_{11} \\ L_{g_1} L_f^2 \phi_{31} & L_{g_3} L_f^2 \phi_{31} \end{bmatrix} \bar{u} \quad (59)$$

$$\triangleq \bar{a}^*(x) + \bar{B}^*(x) \bar{u}$$

where  $\phi_{ij}^{(3)} = \frac{d^3 \phi_{ij}}{dt^3}$ .

Note that  $L_f^3 \phi_{11}$ , and  $L_f^3 \phi_{31}$  include input  $u_2$ , which has been already derived

in Section 3.1 and 3.2. Choose a linearizing feedback control law

$$\bar{u} = \bar{B}^{*-1}[-\bar{a}^*(x) + \bar{v}] \quad (60)$$

Then linear representation of the system is

$$\dot{\Phi}_i = A_1 \Phi_i + e_3 \bar{v}_i, \quad i = 1, 3 \quad (61)$$

$$\Phi = \begin{bmatrix} \Phi_1 \\ \Phi_3 \end{bmatrix}, \quad \Phi_1 = \begin{bmatrix} \phi_{i1} \\ \phi_{i3} \\ \phi_{i3} \end{bmatrix}, \quad \Phi_3 = \begin{bmatrix} \phi_{31} \\ \phi_{32} \\ \phi_{33} \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \bar{v} = \begin{bmatrix} \bar{v}_1 \\ \bar{v}_3 \end{bmatrix}$$

The coupled yaw and roll dynamics decompose into two third order linear subsystems. The integral of  $\phi_{11}$ , and  $\phi_{31}$  are additional state variables for robustness and carry out the design as done for controlling pitch angle. The coordinate transformation (55) leads to the additive unknown vector function

in (60) when  $w_i$  is included. The effect of the additive unknown function in (60) can be minimized by designing an ultimate boundedness controller following the steps of Section 3.2.



### 3.5 Simulation Results

Space Station Parameters are: (Inertia slug-ft<sup>2</sup>)

$$I_{11}=50.28E6,$$

$$I_{22}=10.80E6,$$

$$I_{33}=58.57E6,$$

The controller parameters are:

$$\lambda_c=.0005, \zeta_{ci}=1.0, \omega_{nci}=.0005, i=1,2$$

$$\omega_{nei}=.0015, \zeta_{ei}=1.0, i=1,2$$

Shown at the end of the paper in figs 1(a)-(e) are the simulation results for the control presented in this section. A 30° pitch angle was regulated. The complete closed-loop system (with  $w_2 = v_r=0$ ) was simulated for simplicity. The initial conditions were chosen to be  $\theta_2(0) = 30^\circ$ ,  $\dot{\theta}_2(0) = .005^\circ/sec.$ ,  $h_2(0) = 0$ , and  $z_s = 0$ . The initial conditions for the command generator were chosen such that  $z_{ci}(0) = z_i$ ,  $i=1,2,3,4$ , and  $z_{c5}(0)=0$ . Notice that with

this choice of  $z_{ci}(0)$ , one has  $\tilde{z}_i(0) = 0$ . Selected responses are shown in Fig. 2. It was observed that the pitch angle  $\theta_2$  and  $h_2$  converge to zero in about 500 minutes for the chosen command trajectory  $z_{c1}(t)$ . The maximum values of the torque input  $u_2$  and the angular momentum  $h_2$  were less than 7 ft·lb and 5900 ft·lb·sec, respectively, which are well within permissible limits. Here small torque is required, since a suitable command trajectory  $z_{c1}(t)$  for regulation was chosen. It is interesting to note that the tracking error  $\tilde{z}_1(t) \equiv 0$ , for all  $t$  as predicted.

## CHAPTER FOUR

### NONLINEAR INVERSION

#### 4.1 Input Output Linearization

In the previous chapter, a controller is designed for the attitude control of the space station based on the theory of exact linearization. The exact linearization technique has a linear representation of the original nonlinear model, it was observed that the pitch axis control system has a singularity at  $\theta_2 = \pm 45^\circ$ . In this chapter a controller is designed based on (i-o) linearization which is effective over a larger region in the state space than the previous method. By solving with nonlinear inversion techniques [14, 15], one obtains a linear input-output map for the nonlinear dynamics of the system. In the closed-loop system, independent control of each output is achieved.

The inversion approach is dependent on the choice of the output variables. The output vector must be chosen so the desired variables are controlled. When the output variables are nulled, it may happen that residual internal dynamics (zero dynamics) may be unstable. The choice of controlled output variables which lead to stable zero dynamics are required in the design.

**4.2 Control Law** The attitude control system designed using nonlinear inversion depends on the choice of controlled output variables. The choice of these variables is essential for the attitude and CMG momenta regulation of the space station. In this study the output vector  $y = (y_1, y_2, y_3)^T$  to be controlled is chosen as

$$y = \begin{bmatrix} \alpha_1 \theta_1 + \dot{\theta}_1 + I_1^{-1} h_1 \\ \alpha_2 \theta_2 + \dot{\theta}_2 + I_2^{-1} h_2 \\ \alpha_3 \theta_3 + \dot{\theta}_3 + I_3^{-1} h_3 \end{bmatrix} \quad (62)$$

$$\triangleq C(x)$$

where  $\alpha_i > 0$ ,  $C(x) = C_0 x$ ,

$$C_0 = [\alpha, I_d, I^{-1}]$$

$$\alpha = \text{diag}(\alpha_i), \quad i = 1, 2, 3$$

The input-output (i-0) map of the system (5) and (62) is nonlinear. The nonlinear feedback control law is derived. This law is based on nonlinear

inversion theory in the closed-loop system the input-output map. Differentiation of the output and nonlinear transformation beginning with the original system (5) and (62) when certain rank condition for the solvability of the control input vector is satisfied. For the derivation of the inverse control law, assume that  $w = 0$ . In the sequel, these definitions will be useful.

$$L_f(C)(x) = \left[ \frac{\partial C(x)}{\partial x} \right] f(x)$$

$$L_f^j(C)(x) = L_f(L_f^{j-1}(C))(x)$$

$$L_B L_f(C)(x) = \left[ \frac{\partial L_f(C)(x)}{\partial x} \right] B$$

Successively differentiating  $y$  and using (5), one obtains System 1 and System 2 by the inversion algorithm of [14, 15]

System 1:

$$\dot{x} = f(x) + Bu \tag{63}$$

$$z_1 = L_f(C)(x)$$

where

$$z_1 = \dot{y}$$

$$L_f(C)(x) = \alpha \dot{\theta} + (f_4(x), f_5(x), f_6(x))^T + n(I_1^{-1}h_3, 0, -I_3^{-1}h_1)^T$$

System 2:

$$\dot{x} = f(x) + Bu \quad (64)$$

$$z_2 = L_f^2(C)(x) + [L_B L_f(C)(x)]u$$

where

$$z_2 = \ddot{y},$$

$$\hat{f}(x) = [f_4(x), f_5(x), f_6(x)]^T,$$

$$g(x) = [g_1(x), g_2(x), g_3(x)]^T,$$

$$L_f^2(C)(x) = \alpha \hat{f}(x) + g(x)$$

The analytical expression for  $g(x)$  is in the appendix. The matrix  $D^* \triangleq$

$L_B L_f(C)(x)$  is

$$D^* = \begin{bmatrix} -\alpha_1 I_1^{-1} & 0 & n(I_1 I_3)^{-1}(I_2 - I_1) \\ 0 & -\alpha_2 I_2^{-1} & 0 \\ n(I_1 I_3)^{-1}(I_3 - I_2) & 0 & -\alpha_3 I_3^{-1} \end{bmatrix} \quad (65)$$

Since each row of  $D_2$  is nonzero, the relative degree of each output component is 2. The determinant of  $D_2$  is nonzero provided that

$$\alpha_1 \alpha_3 - n^2 (I_2 I_3)^{-1} (I_2 - I_1)(I_3 - I_2) \neq 0 \quad (66)$$

For the Space Station  $I_2 < I_1$  and  $I_3 > I_2$ .  $D_2$  is nonsingular when  $\alpha_1 > 0$ , and  $\alpha_3 > 0$ . The inversion algorithm terminates here. System 2 is invertible and the tracking order of the system is 2.

An input-output linearizing feedback control law is derived from (64).

Choose a control law of the form

$$u = D^*^{-1} [-L_f^2(C)(x) - P_2 \dot{\tilde{y}} - P_1 \tilde{y} - P_0 x_s + \ddot{y}_r] \quad (67)$$

where  $P_i = \text{diag}(p_{ij})$ ,  $i=0,1,2$ ,  $j=1,2,3$ , and  $y_r = (y_{r1}, y_{r2}, y_{r3})^T$  is a reference

trajectory to be tracked,  $\tilde{y} = (\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)$  is the tracking error and

$$\dot{x}_s = \tilde{y} \quad (68)$$

where  $\tilde{y}_i = (y_i - y_{r,i})$ . The integral of the tracking error is  $x_s$ . Feedback of  $x_s$  is used to obtain robustness of the control system.

Substituting control law (67) in the output equation of System 2

$$\ddot{\tilde{y}} = -P_2 \dot{\tilde{y}} - P_1 \tilde{y} - P_0 x_s \quad (69)$$

Differentiating (69) and using (68), a linear differential equation is found for  $\tilde{y}$  of the form

$$\ddot{\tilde{y}} + P_2 \dot{\tilde{y}} + P_1 \tilde{y} + P_0 \tilde{y} = 0 \quad (70)$$

The gains  $p_{i,j}$  are chosen such that (70) is asymptotically stable. From (70) that each component  $\tilde{y}_i$  of the tracking error vector is independently controlled and  $y_i$  follows  $y_{r,i}$  provided that the initial state matching conditions

$$y(0) = y_r(0), \quad \dot{y}(0) = \dot{y}_r(0)$$



$$\ddot{y}(0) = \ddot{y}_r(0) \quad (71)$$

are satisfied.

For the purpose of attitude regulation to zero, a command generator of the form below is added.

$$\ddot{y}_r + F_2 \dot{y}_r + F_1 \dot{y}_r + F_0 y_r = 0 \quad (72)$$

where  $F_i = \text{diag}(f_{ij})$ ,  $i=0,1,2$ ; and  $j=1,2,3$ . The gains  $f_{ij}$  are properly chosen to obtain desirable reference trajectories for smooth regulation. The reference trajectory (16) is asymptotically stable so that  $y_r(t) \rightarrow 0$  as  $t \rightarrow \infty$ .

In the closed-loop system  $y(t) \rightarrow 0$  as  $t \rightarrow \infty$ . To regulate  $\theta$  and  $h$  to the origin, it is necessary to derive the zero dynamics of the system and examine their stability property.

### 4.3 Stability of Zero Dynamics

Zero dynamics of the closed-loop system (5) and (67) (with disturbance  $w = 0$ ) represent the internal dynamics when the trajectory  $x(t)$  is such that  $y(t) = C(x) \equiv 0$  and  $\dot{y} = L_f(C)(x) \equiv 0$ . Using (62) and (3); one has

$$y = \alpha\theta + \dot{\theta} + I^{-1}h = 0 \quad (73)$$

$$\dot{y} = \alpha\dot{\theta} + \hat{f}(x) + n(I^{-1}h_3, 0, -I_3^{-1}h_1)^T = 0 \quad (74)$$

#### 4.3a Pitch Zero Dynamics

To consider stability of pitch axis zero dynamics, use (74) and

$$\alpha_2\dot{\theta}_2 = -f_5(x) = -1.5n^2 I_2^{-1}(I_3 - I_1)\sin 2\theta_2 = 0 \quad (75)$$

which yields

$$\dot{\theta}_2 = -\beta_2 \sin 2\theta_2 \quad (76)$$

where  $\beta_2 = 1.5n^2(I_3 - I_1)(I_2\alpha_2)^{-1}$ . When  $y_2(t) \equiv 0$ ,  $\theta_2$  trajectory evolves

according to (76).

The stability of the equilibrium point is found for  $\theta_2 = 0$  of (75) by the Lyapunov approach. Choose a Lyapunov function

$$V(\theta_2) = \int_0^{\theta_2} \sin(2\xi) d\xi = \frac{1}{2}(1 - \cos 2\theta_2) \quad (77)$$

Note that  $V(0) = 0$ , and  $V(\theta_2) > 0$  if  $\theta_2 \neq 0$  and  $\theta_2 \in \Omega$ , where  $\Omega = \{\theta_2 : |\theta_2| < \pi/2\}$ . For studying the stability of  $\theta_2 = 0$ , compute the derivative of  $V(\theta_2)$  which is

$$\dot{V}(\theta_2) = -\beta_2 \sin^2 2\theta_2 < 0$$

provided that  $\theta_2 \neq \frac{n\pi}{2}$ ,  $n = 0, \pm 1, \pm 2, \dots$ . For inertia parameters of the space station  $I_3 > I_1$ , and  $\beta_2 > 0$  if  $\alpha_2 > 0$  so that  $\dot{V}(\theta_2)$  is negative for all  $\theta_2 \in \Omega$  if  $\theta_2 \neq 0$ . Using a theorem of Lyapunov [19],  $\theta_2 = 0$  of (75) is asymptotically stable. Then the closed-loop system for any trajectory beginning in the region  $\Omega$ ,  $\theta_2 \rightarrow 0$  as  $t \rightarrow \infty$ , and from (20),  $\dot{\theta}_2 \rightarrow 0$ , at  $t \rightarrow \infty$ . In view of the relation  $y_2 = \alpha_2 \theta_2 + \dot{\theta}_2 + I_2^{-1} h_2$ , and since  $y_2(t) \rightarrow 0$  as  $t \rightarrow \infty$ , then the angular momentum  $h_2 \rightarrow 0$  as  $t \rightarrow \infty$ .

### 4.3b Yaw and Roll Zero Dynamics

The zero dynamics representing yaw and roll motion is  $y(t) \equiv 0$ . Since  $\theta_2(t)$  converges to zero,  $\theta_2 = 0$  in (73) and (74). Using equations for  $y_1 = 0$ ,  $y_3 = 0$ ,  $\dot{y}_1 = 0$ , and  $\dot{y}_3$  from (73) and (74),

$$\alpha_1 \theta_1 + \dot{\theta}_1 + I_1^{-1} h_1 = 0 \quad (78)$$

$$\alpha_3 \theta_3 + \dot{\theta}_3 + I_3^{-1} h_3 = 0 \quad (79)$$

$$\alpha_1 \dot{\theta}_1 - 4I_1^{-1} n^2 (I_2 - I_3) \theta_1 + nI_1^{-1} (I_1 - I_2 + I_3) \dot{\theta}_3 + nI_1^{-1} h_3 = 0 \quad (80)$$

$$\alpha_3 \dot{\theta}_3 - I_3^{-1} n^2 (I_2 - I_1) \theta_3 - nI_3^{-1} (I_1 - I_2 + I_3) \dot{\theta}_1 - nI_3^{-1} h_1 = 0 \quad (81)$$

Solving for  $h_3$  and  $h_1$  from (80) and (81) and then substituting in (78) and (79),

$$(I_2 - I_3) n \dot{\theta}_1 + n I_1 \alpha_1 \theta_1 + I_3 \alpha_3 \dot{\theta}_3 - n^2 (I_2 - I_1) \theta_3 = 0 \quad (82)$$

$$-I_1 \alpha_1 \dot{\theta}_1 + 4n^2 (I_2 - I_3) \theta_1 + n (I_2 - I_1) \dot{\theta}_3 + n I_3 \alpha_3 \theta_3 = 0$$

The stability of the equilibrium state ( $\theta_1 = 0$ ,  $\theta_3 = 0$ ) of (82) can be examined by determining the characteristic polynomial  $p(\lambda)$  associated with

(82) which is the determinant of the matrix L, where

$$L = \begin{bmatrix} \lambda(I_2 - I_3)n + nI_1\alpha_1 & I_3\alpha_3\lambda + n^2(I_1 - I_2) \\ -I_1\alpha_1\lambda - 4n^2(I_3 - I_2) & n(I_2 - I_1)\lambda + nI_3\alpha_3 \end{bmatrix} \quad (83)$$

The characteristic polynomial is

$$\begin{aligned} p(\lambda) &= [\alpha_1\alpha_3 + n^2(I_2 - I_3)(I_2 - I_1)]\lambda^2 + 3I_3(I_3 - I_2)\alpha_3n^2\lambda \\ &\quad + 4n^4(I_1 - I_2)(I_3 - I_2) + \alpha_1\alpha_3n^2I_1I_3 \\ &\triangleq a_2\lambda^2 + a_1\lambda + a_0 = 0 \end{aligned} \quad (84)$$

where  $a_i$  is determined from (84). The second order polynomial is a Hurwitz polynomial if and only if  $a_i > 0$ ,  $i = 0, 1, 2$ . For the inertia parameters of the space station, one has  $I_3 > I_1$ ,  $I_3 > I_2$  and  $I_1 > I_2$ . Choosing  $\alpha_1 > 0$  and  $\alpha_3 > 0$  then  $a_i > 0$ ,  $i = 0, 1, 2$ , and the equilibrium point  $(\theta_1 = 0, \theta_3 = 0)$  of (82) is globally asymptotically stable and in the closed-loop system  $(\theta_1(t), \theta_3(t)) \rightarrow 0$  as  $t \rightarrow \infty$ .

In view of (82) this implies that  $(\dot{\theta}_1(t), \dot{\theta}_3(t)) \rightarrow 0$  at  $t \rightarrow \infty$  since  $n^2(I_2 - I_3)(I_2 - I_1) + I_1I_3\alpha_1\alpha_3 \neq 0$  and (26) is uniquely solvable for  $\dot{\theta}_1$  and  $\dot{\theta}_3$ . Using

(80) and (81), then  $(h_1(t), h_3(t)) \rightarrow 0$  as  $t \rightarrow \infty$ .

The equilibrium point  $(\theta = 0, \dot{\theta} = 0, h = 0, x_s = 0)$  of the closed-loop system (5) and (67) is asymptotically stable and  $(\theta(t), h(t)) \rightarrow 0$  as  $t \rightarrow \infty$ . Since the closed-loop system is asymptotically stable, for small disturbance, the trajectory of the closed-loop system remains bounded for  $t \geq 0$ .

#### 4.4 Simulation Results

The Space Station Parameters are: (Inertia slug-ft<sup>2</sup>)

$$I_{11}=23.22E6,$$

$$I_{22}=1.30E6,$$

$$I_{33}=23.23E6,$$

The elements  $f_{ij}$  of the command generator are chosen such that the characteristic polynomials associated with  $y_{r,i}$  in (16) are ( $i = 1, 2, 3$ )

$$(s + \lambda_{ci})(s^2 + 2\zeta_{ci}\omega_{nci}s + \omega_{nci}^2) = 0$$

with the parameters.

$$\lambda_{c1}=0.0008, \zeta_{c1}=0.707$$

$$\lambda_{c2}=0.008, \zeta_{c2}=0.707$$

$$\lambda_{c3}=0.0008, \zeta_{c3}=0.707$$

$$\omega_{nci} = \frac{\lambda_{ci}}{\zeta_{ci}}$$

The elements of the diagonal matrices  $P_i$  in (70) are chosen that the characteristic polynomials associated with  $\tilde{y}_i$  in (74) are

$$(s + \lambda_{ei})(s^2 + 2\zeta_{ei}\omega_{nei}s + \omega_{ei}^2) = 0$$

with  $\zeta_{ei} = 0.707$ , and  $\omega_{nei} = \lambda_{ei}/\zeta_{ei}$ ,  $\lambda_{ei} = 0.2$ ,  $i = 1, 2, 3$ .

In order to meet the control requirements for the simulation 4.4a the parameters  $\alpha_1$ ,  $\alpha_2$ , and  $\alpha_3$  were all chosen to equal unity. The purpose of  $\alpha_1$ ,  $\alpha_2$ , and  $\alpha_3$  gains are to control the convergence rate. This was verified by computer simulation. When a disturbance was added to the system, the best performance was achieved for the above  $\alpha$  gains equal to unity.

#### 4.4a Attitude Regulation: Small Pitch Angle

The complete closed-loop system was simulated with the initial condition  $\theta(0) = (10^\circ, 30^\circ, 8^\circ)$   $\dot{\theta}(0) = 0$ ,  $h(0) = 0$ ,  $x_s(0) = 0$ . The matched initial conditions for the command generator according to (71) were set. Selected responses are in Figures 2(a)-(i). Smooth trajectory tracking and  $\tilde{y}(t) \equiv 0$  occurred. The attitude angles and the CMG momenta converge to the origin. For the chosen control parameters convergence of  $h_2$  and  $\theta_2$ .



#### 4.4b Attitude Regulation with Large Pitch Angle

A large pitch angle perturbation of  $\theta_2(0) = 60^\circ$  instead of  $\theta_2(0) = 30^\circ$  of case 4.4a was considered. The reference trajectory, the feedback gains and the remaining initial conditions of case 4.4a were retained. Selected responses are in Fig. 3(a)-(c). In spite of large perturbation in the pitch angle, there was smooth regulation to the origin.

#### 4.4c Attitude Regulation: Effect of Disturbance

To examine the sensitivity of the control system to a disturbance input,  $w_1 = \sin(nt) + .5\sin(2nt)$ ,  $w_2 = \sin(nt) + .5\sin(2nt)$ , and  $w_3 = \sin(nt) + .5\sin(2nt)$  (lb-ft) were applied at each axis. The command generator parameters were set the same as in 4.4a. Attitude angles were set to  $\theta(0) = (10^\circ, 30^\circ, 8^\circ)$  and the remaining initial conditions and feedback gains were the same as in case 4.4a. A small effect of disturbance was observed. As expected bounded oscillations for  $\theta$  and  $h$  were observed. Oscillatory responses for  $\theta$  remained in the neighborhood of  $\theta = 0$ . A decaying average value of  $\theta_2$  is observed in the figure that oscillates about the time axis given a longer simulation. The figures are shown in 4(a)-(c).

## CONCLUSION

Based on feedback linearization and ultimate boundedness theory, a new attitude control system was designed for controlling the orientation of the space station using CMG's. Feedback linearization gave rise to three decoupled controllable linear systems which describe the nonlinear, coupled pitch, yaw and roll dynamics. The control law is derived based on the linear system representation in the new state space. An ultimate boundedness controller was designed to compensate for the unknown torques acting on the space station. In the closed-loop system, the trajectories are uniform and ultimate bounded in a small neighborhood of the origin.

Using feedback linearization of an input-output map, a second attitude control system was designed for controlling the orientation of the space station using CMG's. A controlled output vector is made for the derivation of the control law. This output vector is a linear function of the attitude angles, angular rates and the CMG momenta. Nonlinear inversion theory was used to obtain a linear input-output map and the independent control of each output variable. Zero dynamics of the closed-loop system were derived and their stability property was examined. For the space station parameters, the

origin of the zero dynamics are stable, and in the closed-loop system attitude angles and the CMG momenta converge to the origin.

Both methods were able to provide simulations that could regulate the pitch axis. The second method was able to handle a 3 axis maneuver with a disturbance. The theoretical limit of an angle that could be regulated by the first method, was a  $45^\circ$  angle. The reason for this limit was that the denominator of the control law was equal to zero at  $45^\circ$ . On the computer simulation for the first method, the limit of an angle which could be regulated was  $30^\circ$ . At that  $30^\circ$  limit the momentum and control torque were within the limits allowed (20,000 ft $\times$ lbs $\times$ sec and 150 ft $\times$ lb). At approximately  $33^\circ$  the system could not converge. The theoretical limit of the second method was  $90^\circ$ . This is because at that point the  $\sin 2\theta$  term was equal to 0. Because of this a large pitch angle maneuver was simulated at  $\frac{2}{3}$  the theoretical limit. The maneuver converged, but the control torque and momentum requirements were exceeded. For a small maneuver convergence to the origin occurred, and the control requirements were met. Because of the robustness of the second method, a disturbance was simulated. The disturbance was regulated to a neighborhood of the equilibrium point. Further studies of this paper include: disturbance rejection, and trajectory planning.

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## APPENDIX

Functions  $g_i(x)$ :

$$I_1 g_1(x) = \{3 \sin 2\theta_2 \dot{\theta}_2 \theta_1 - (1 + 3 \cos^2 \theta_2) \dot{\theta}_1 \times$$

$$(-2 \cos 2\theta_2 \dot{\theta}_2 \theta_3 - \sin 2\theta_2 \dot{\theta}_3) 1.5\} n^2 (I_2 - I_3)$$

$$+ n(I_1 - I_2 + I_3) f_6(x) - n^2 h_1$$

$$I_2 g_2(x) = 3n^2 (I_3 - I_1) \cos 2\theta_2 \dot{\theta}_2$$

$$I_3 g_3(x) = -\{3 \sin 2\theta_2 \dot{\theta}_2 \theta_3 + (1 + 3 \sin^2 \theta_2) \dot{\theta}_3$$

$$+ 1.5(2 \cos 2\theta_2 \dot{\theta}_2 \theta_1 + \sin 2\theta_2 \dot{\theta}_1)\} n^2 (I_2 - I_1)$$

$$- n(I_1 - I_2 + I_3) f_4(x) - n^2 h_3$$



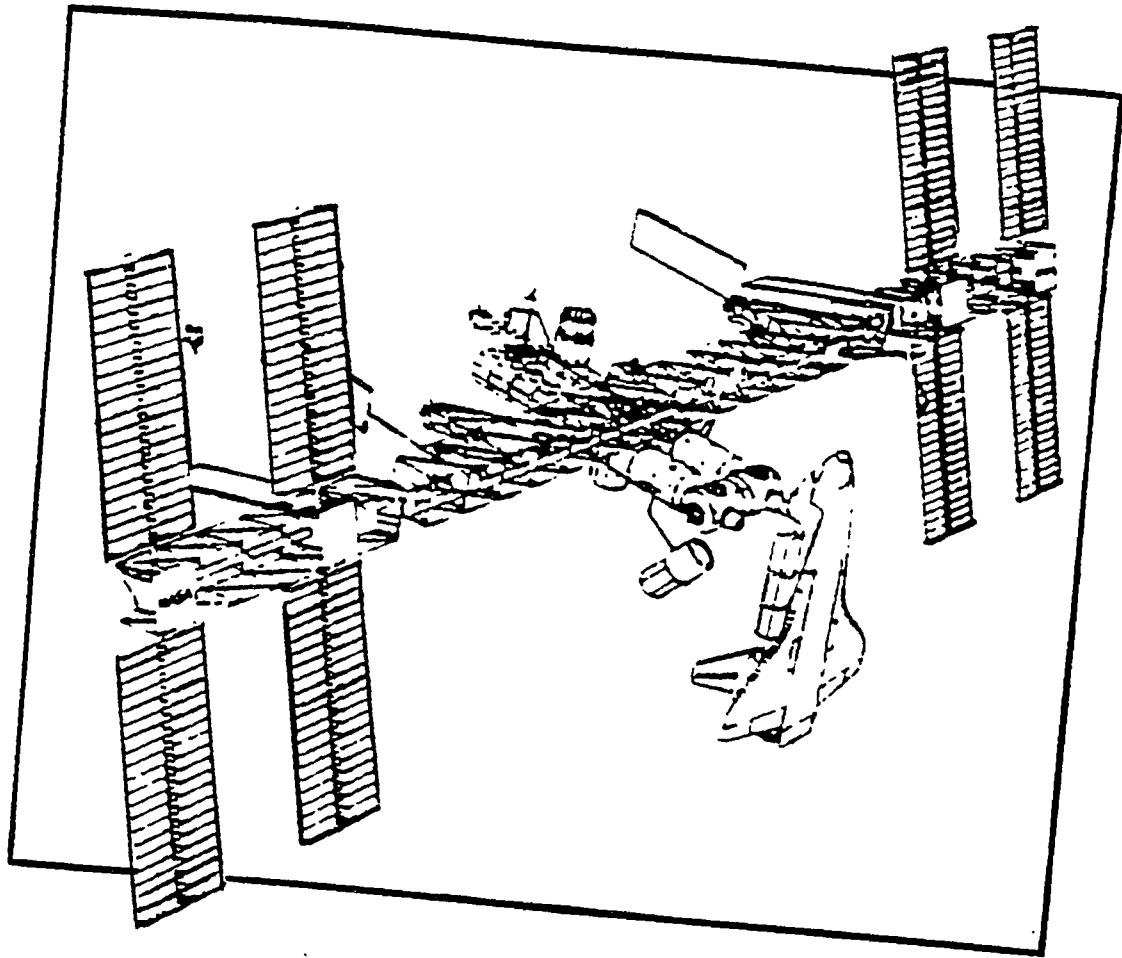


Fig. 1 Space Station

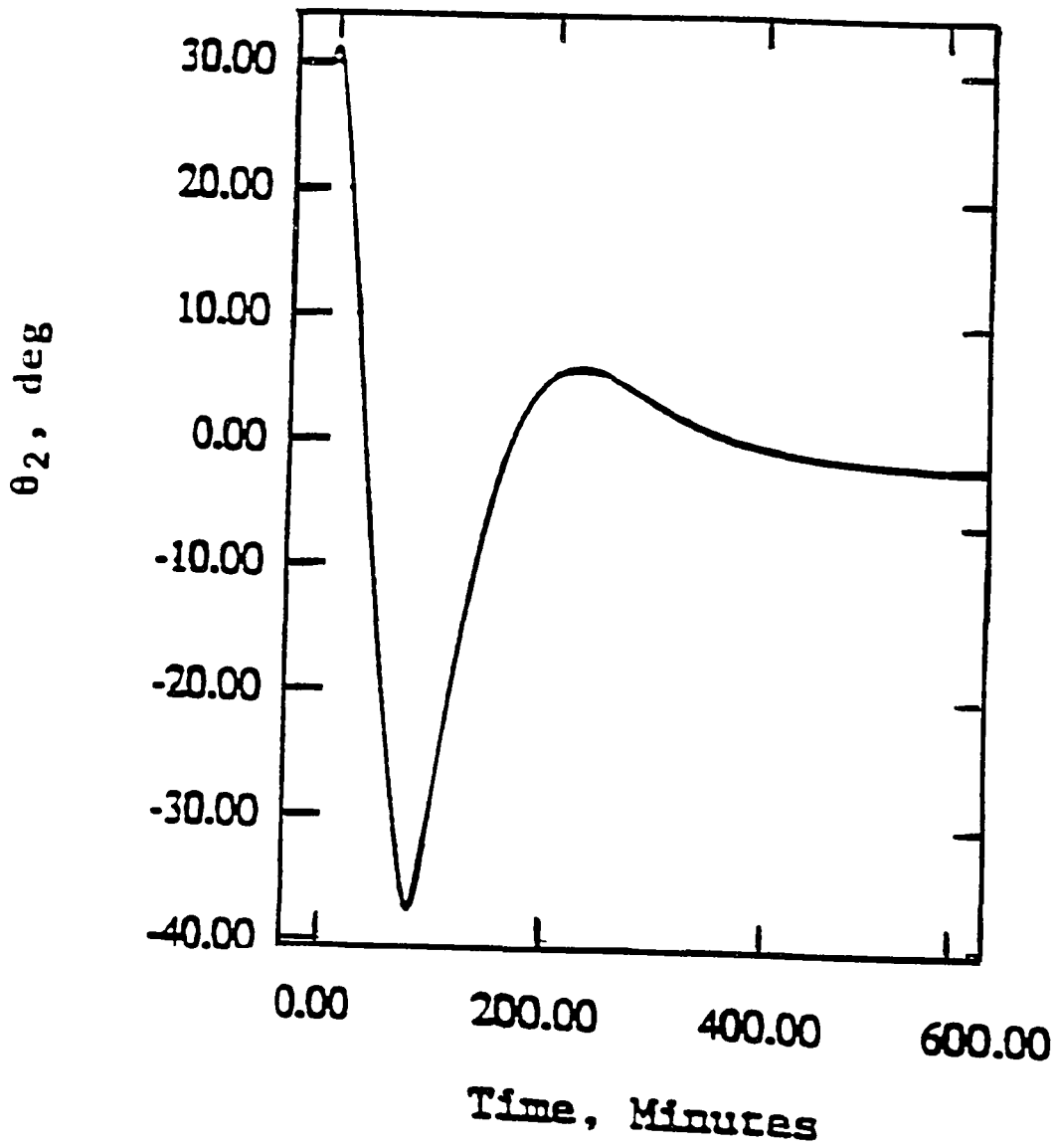


Fig. 1 (a)

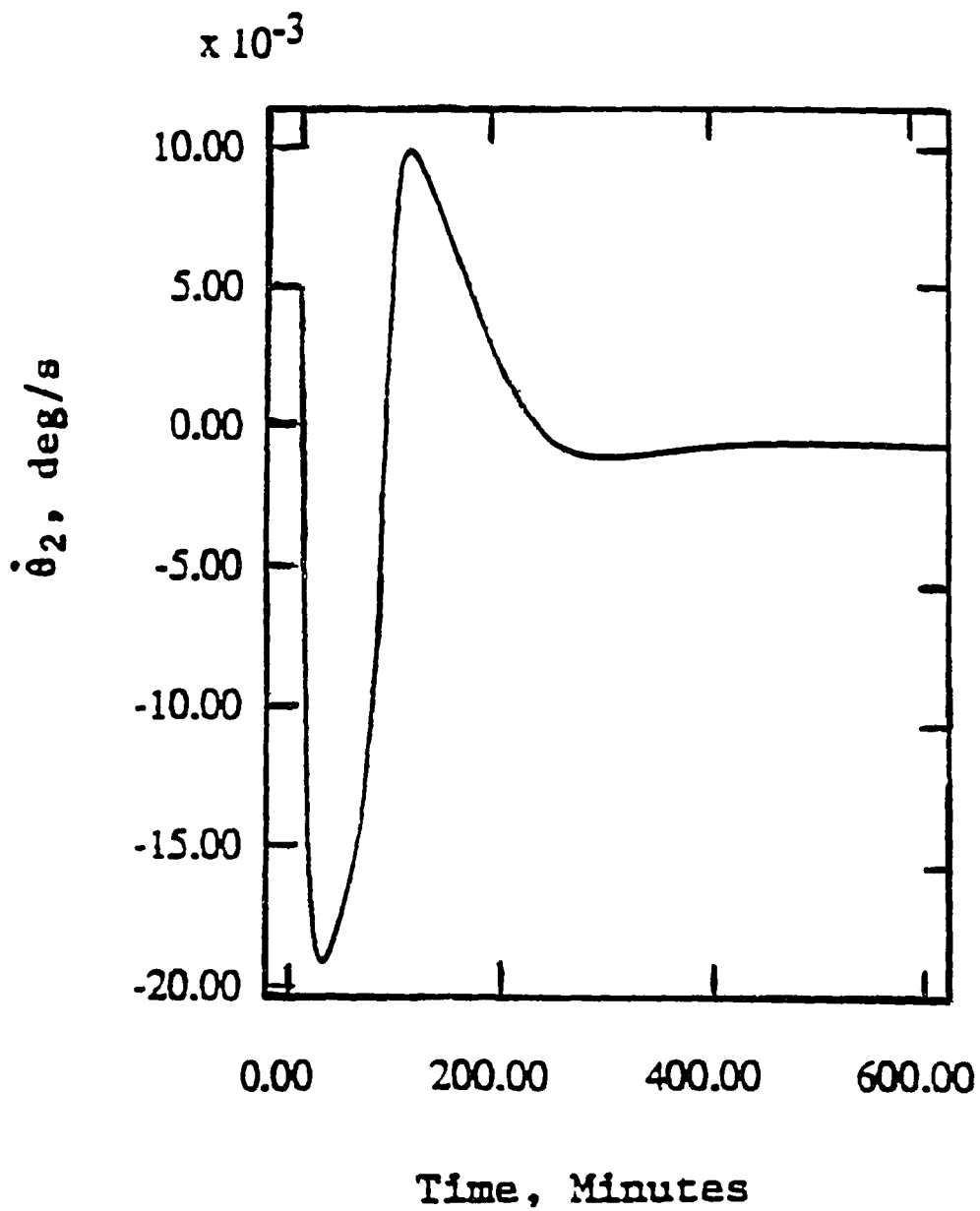


Fig. 1 (b)

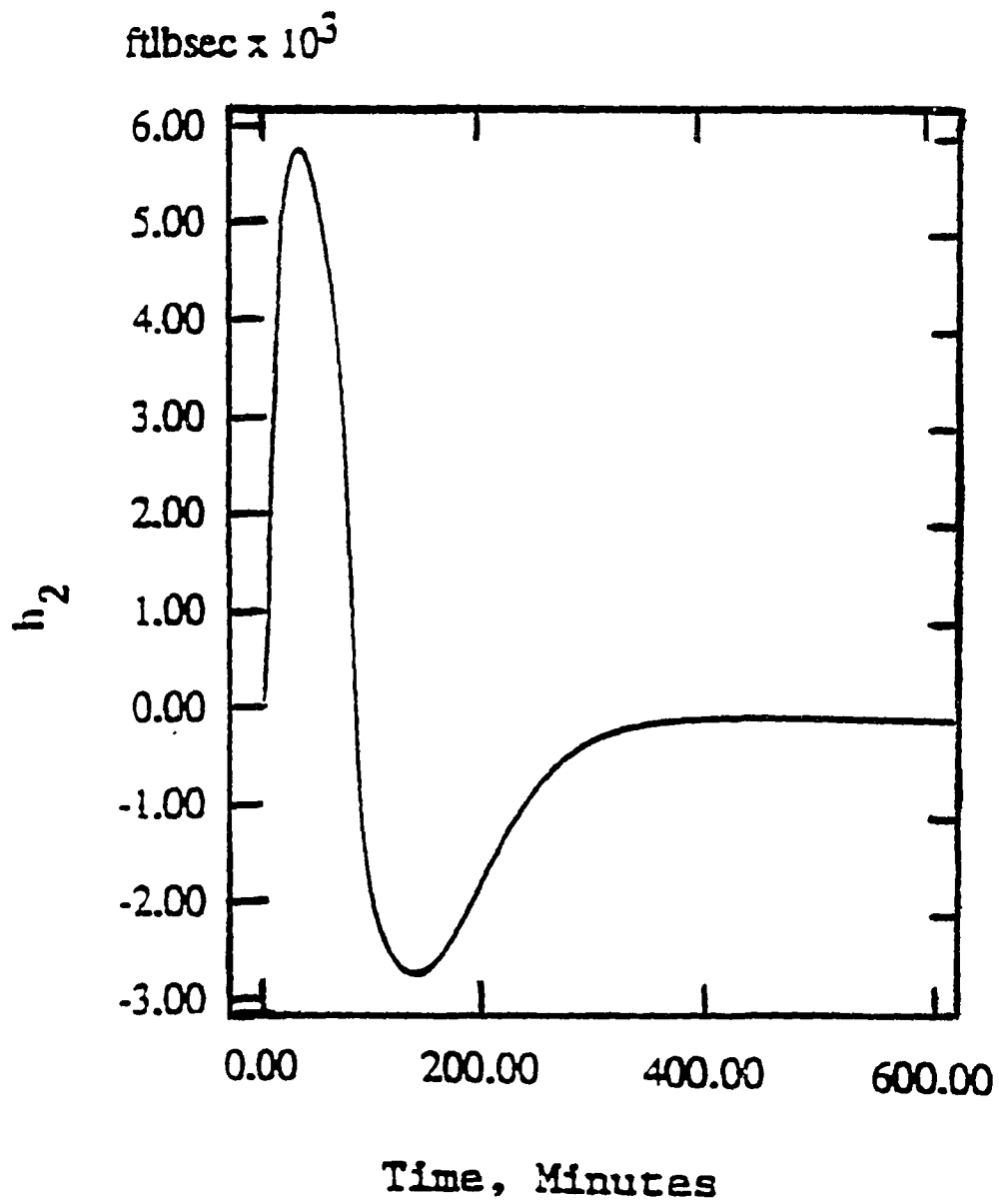


Fig. 1 (c)

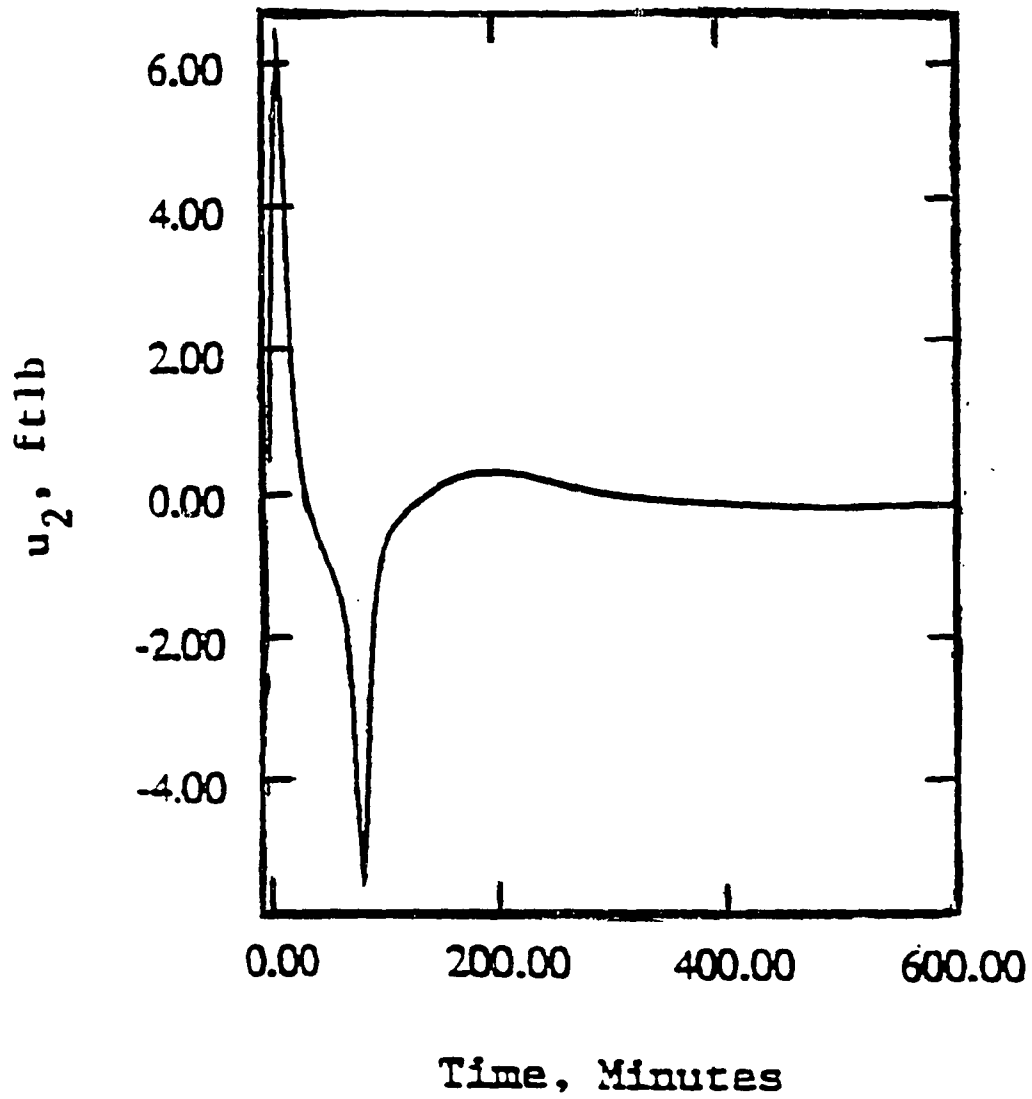


Fig. 1 (d)

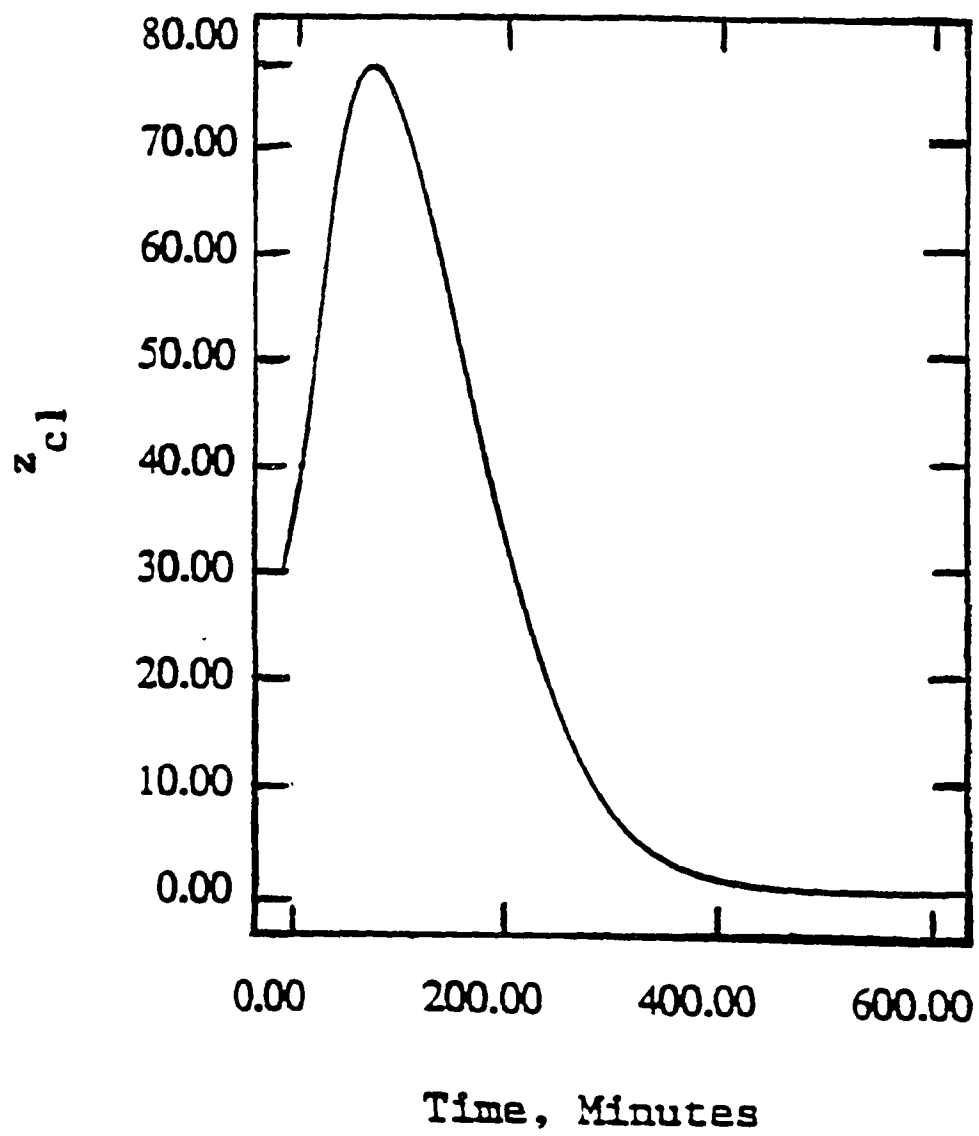


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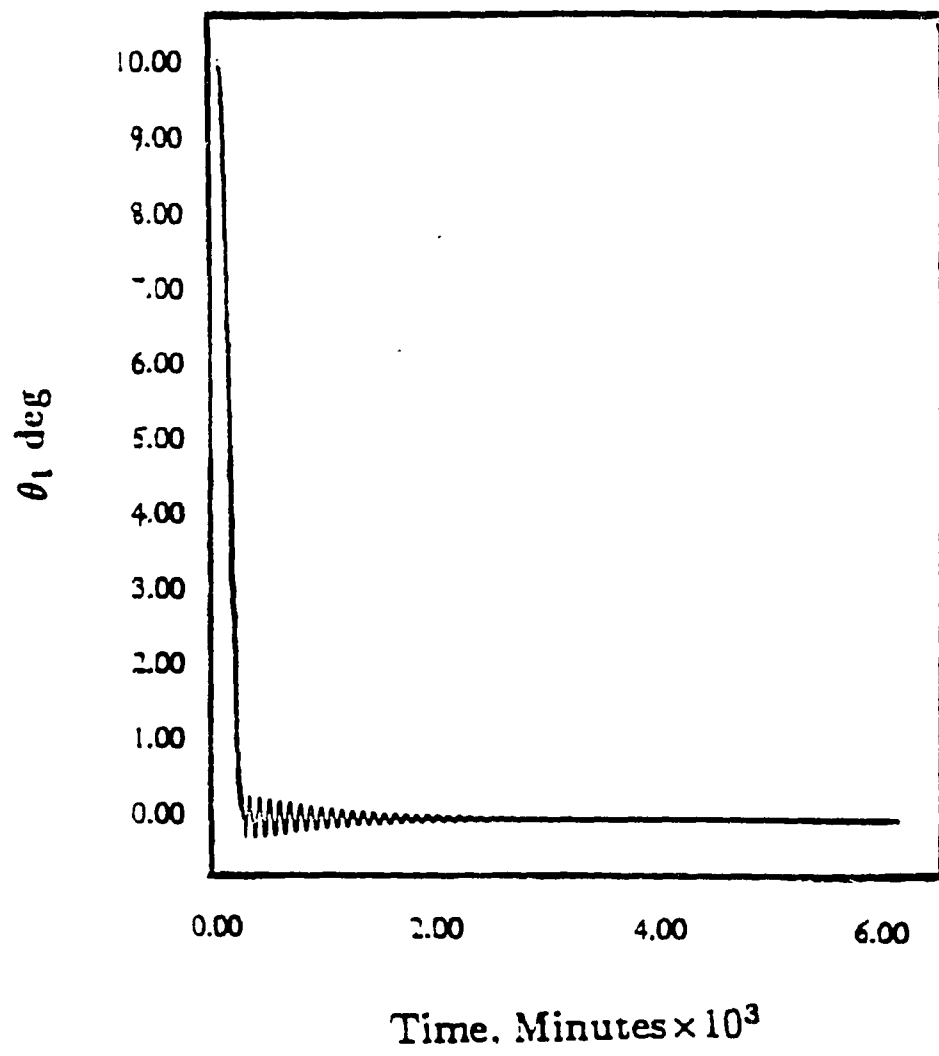
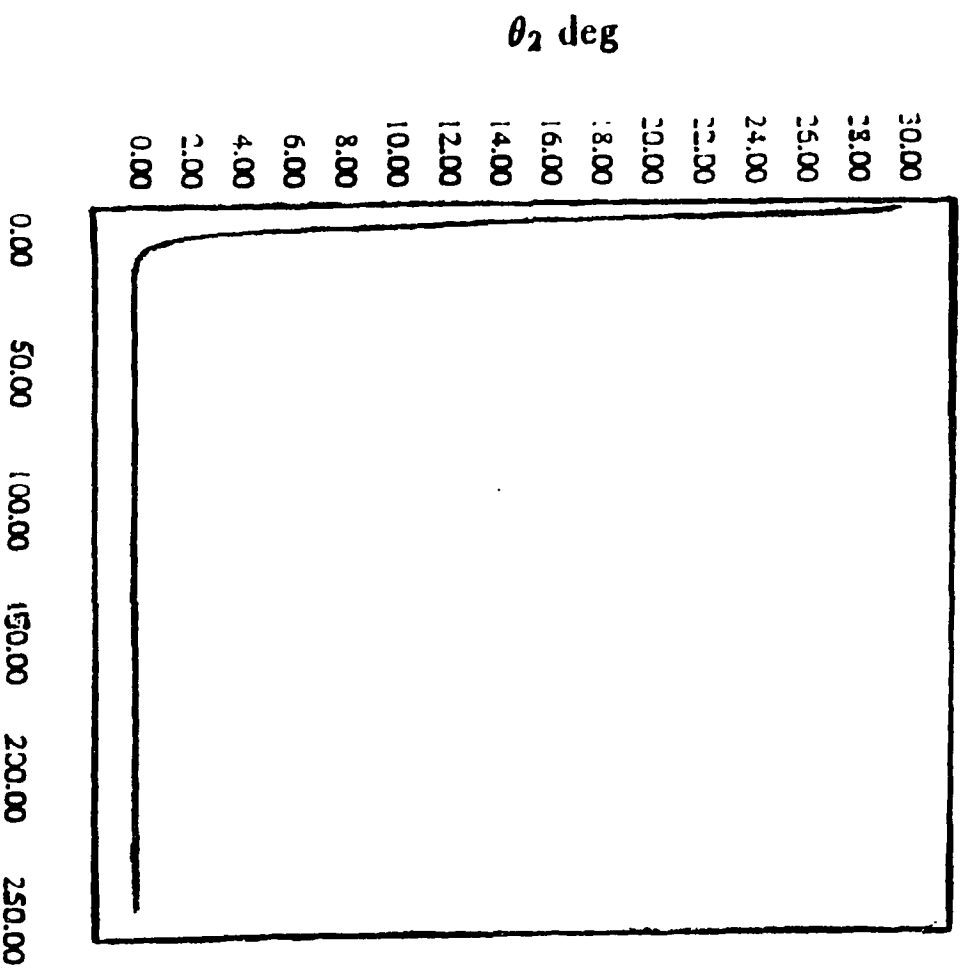


Fig. 2 (a)



Time, Minutes

Fig. 2 (b)



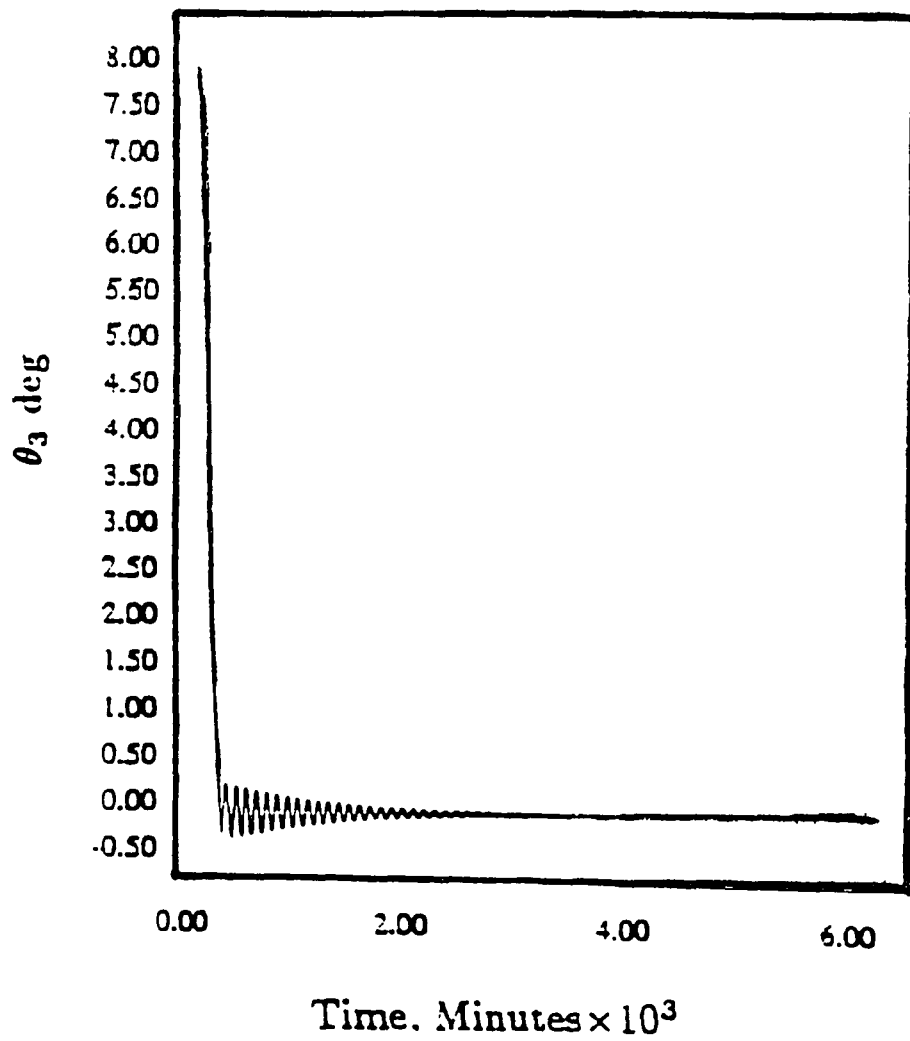


Fig. 2 (c)

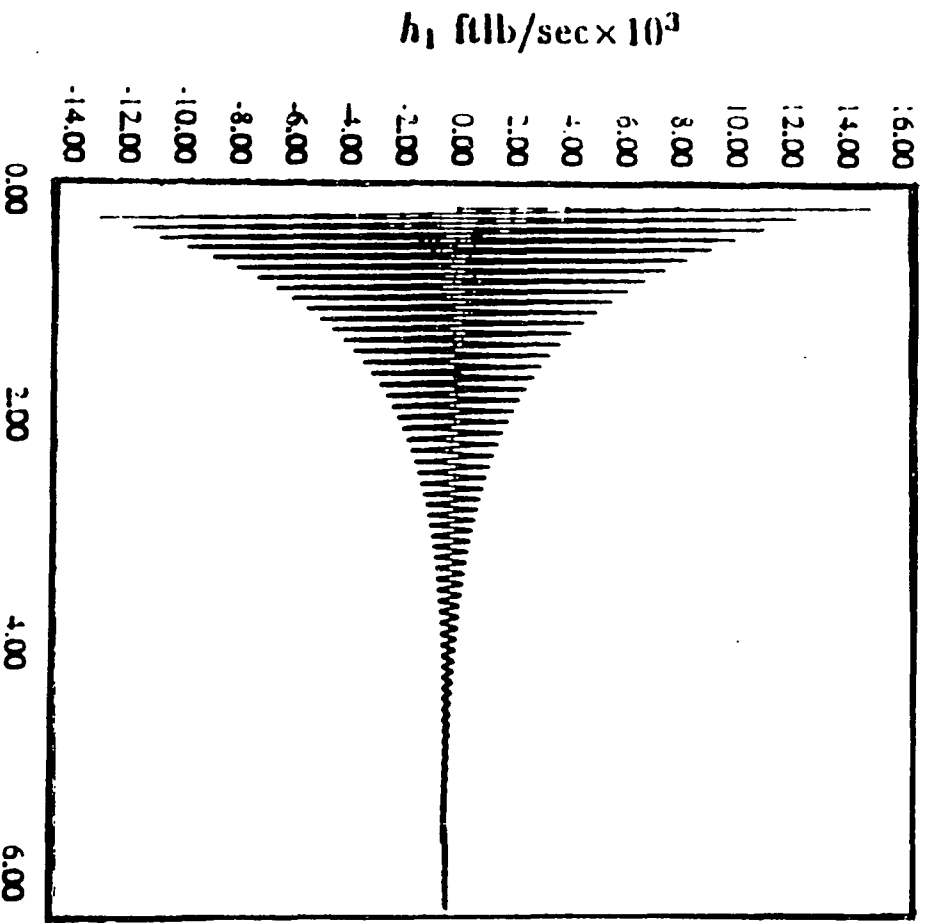


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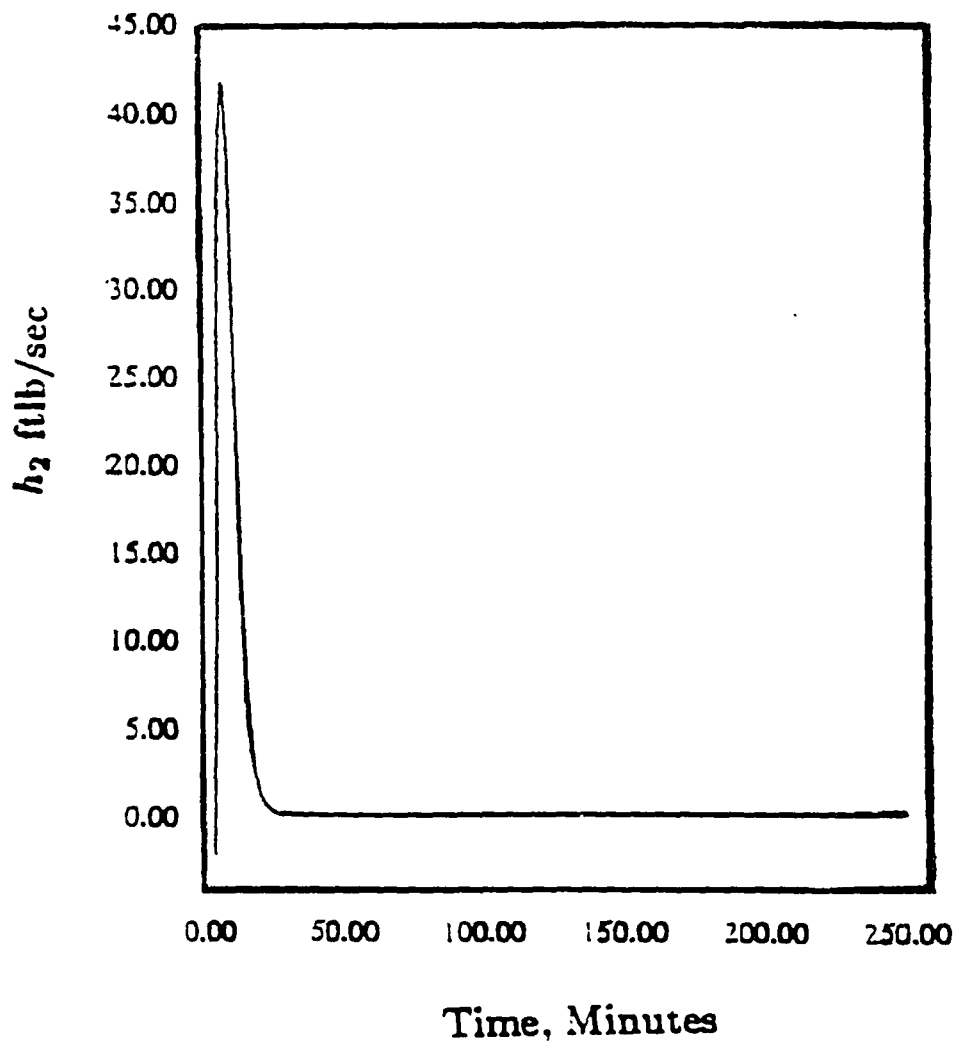


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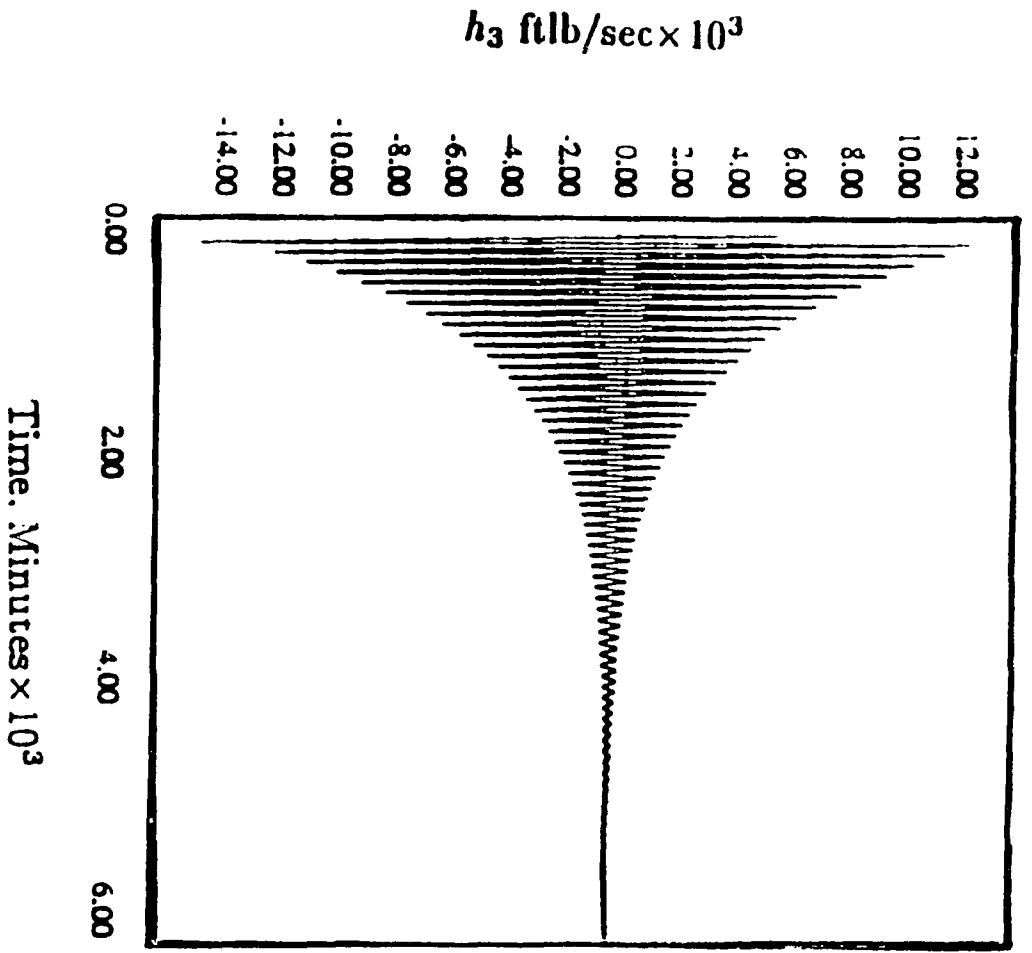


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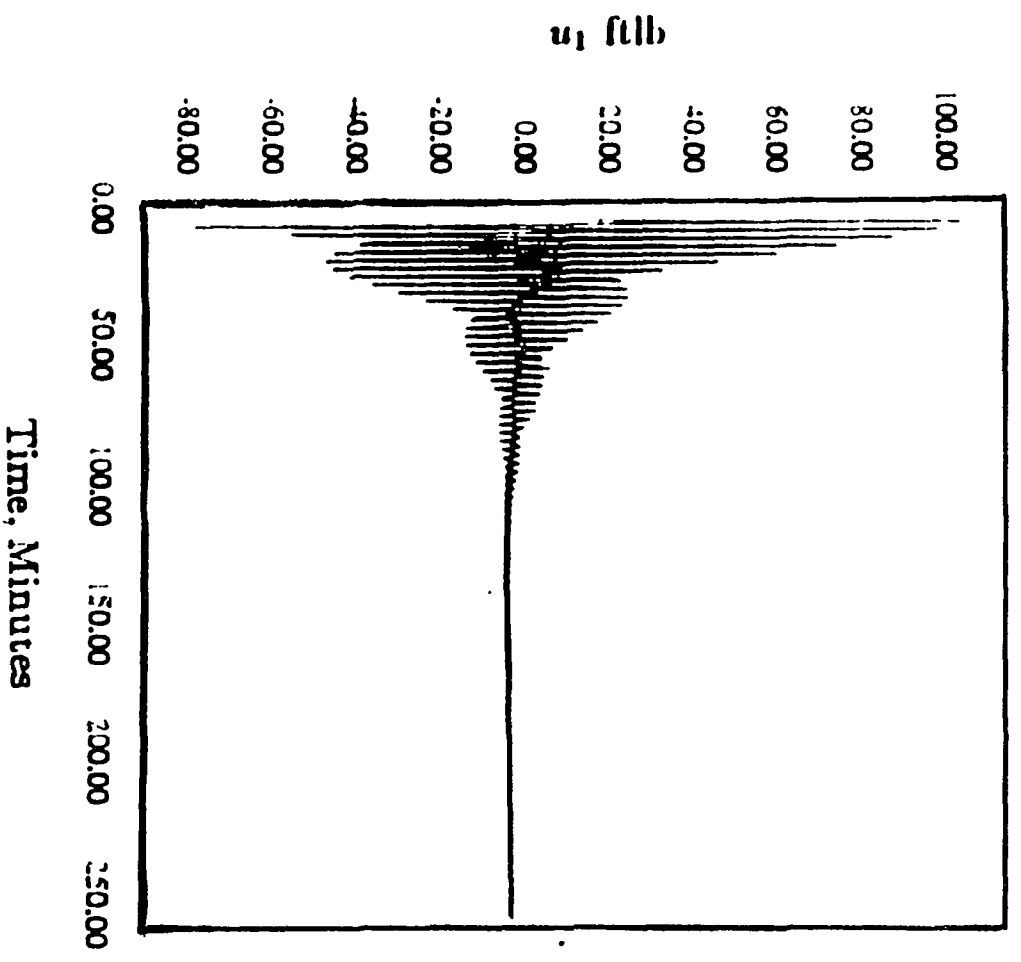


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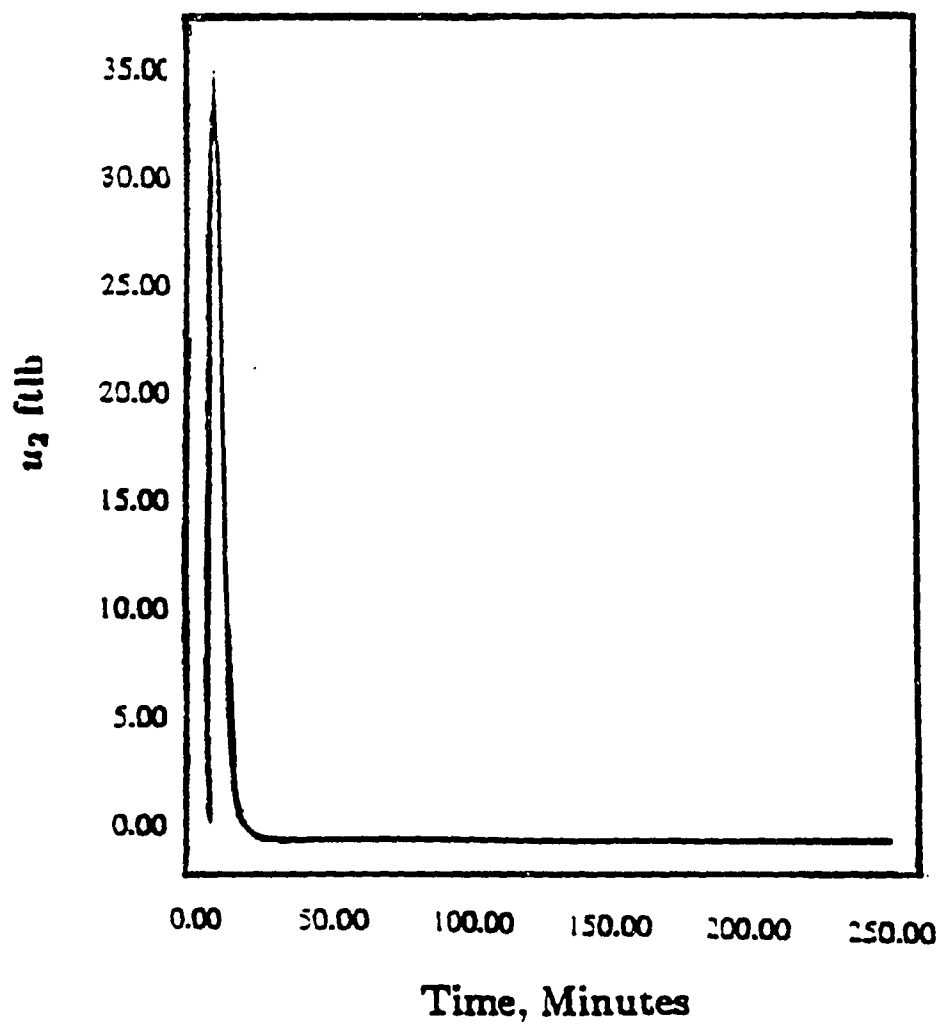


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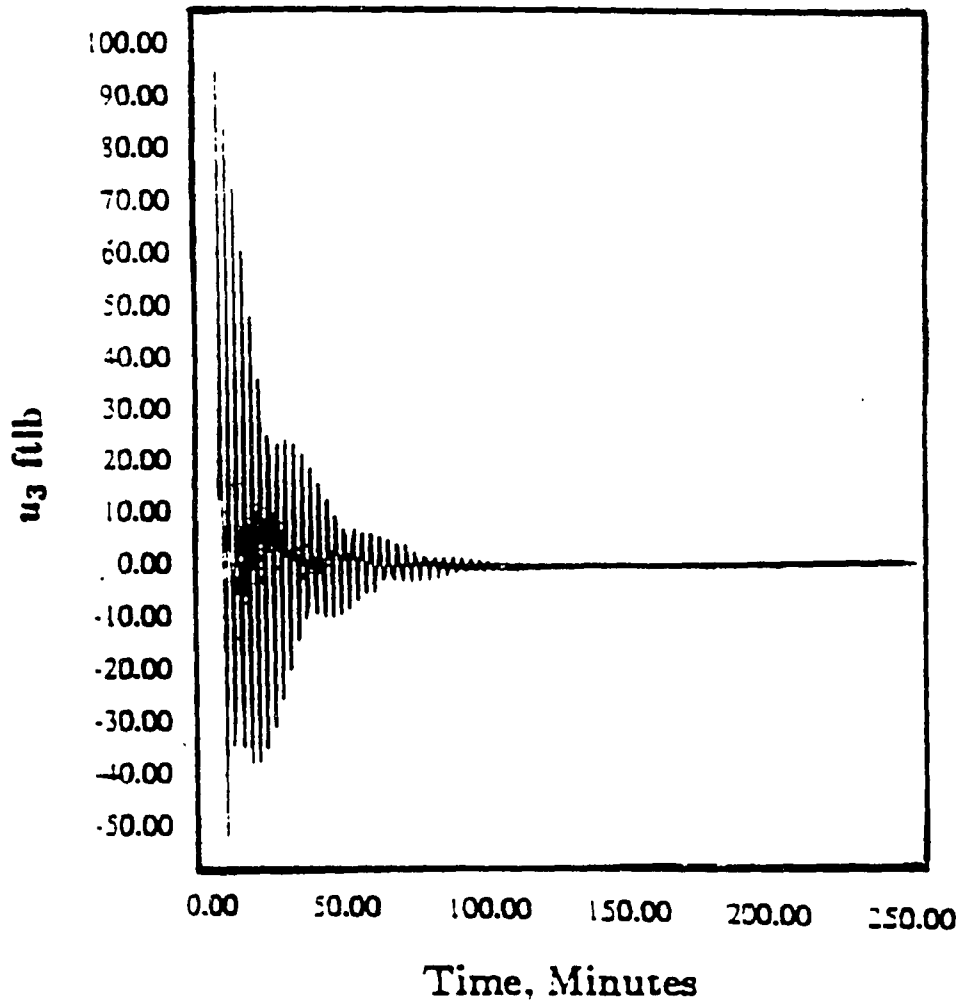


Fig. 2 (i)

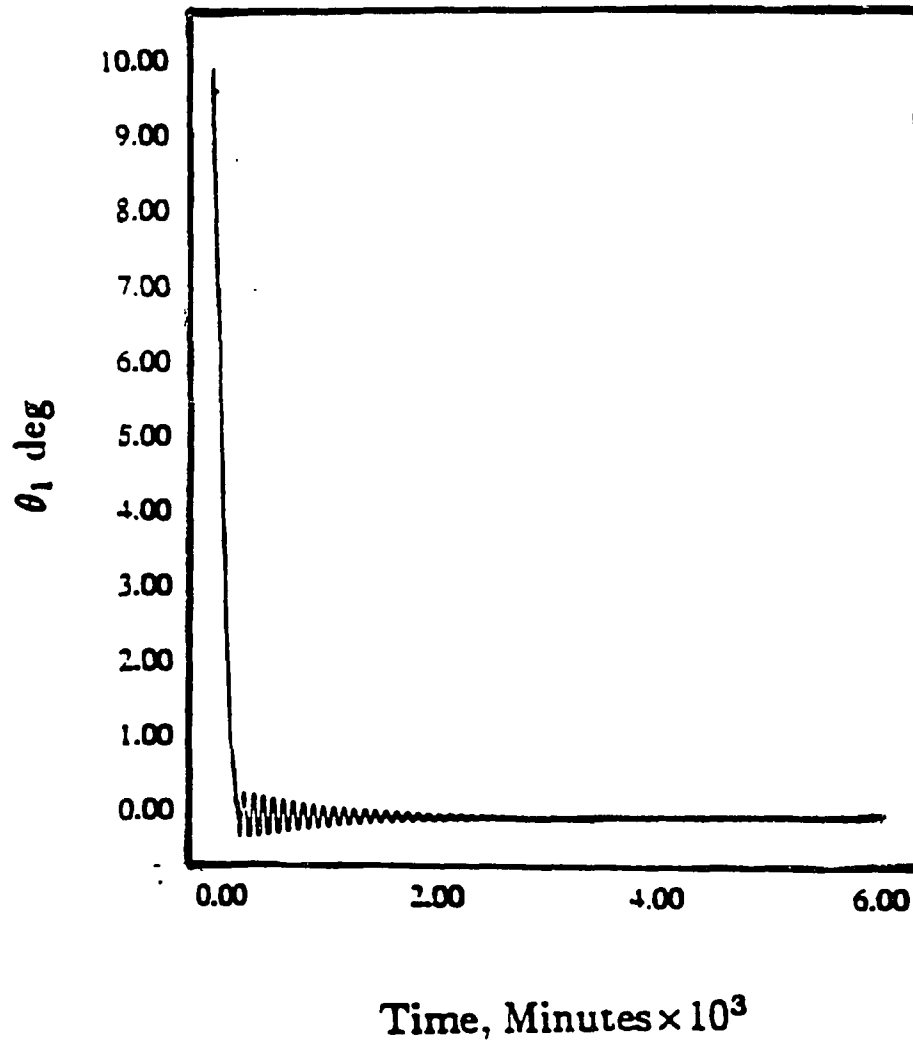
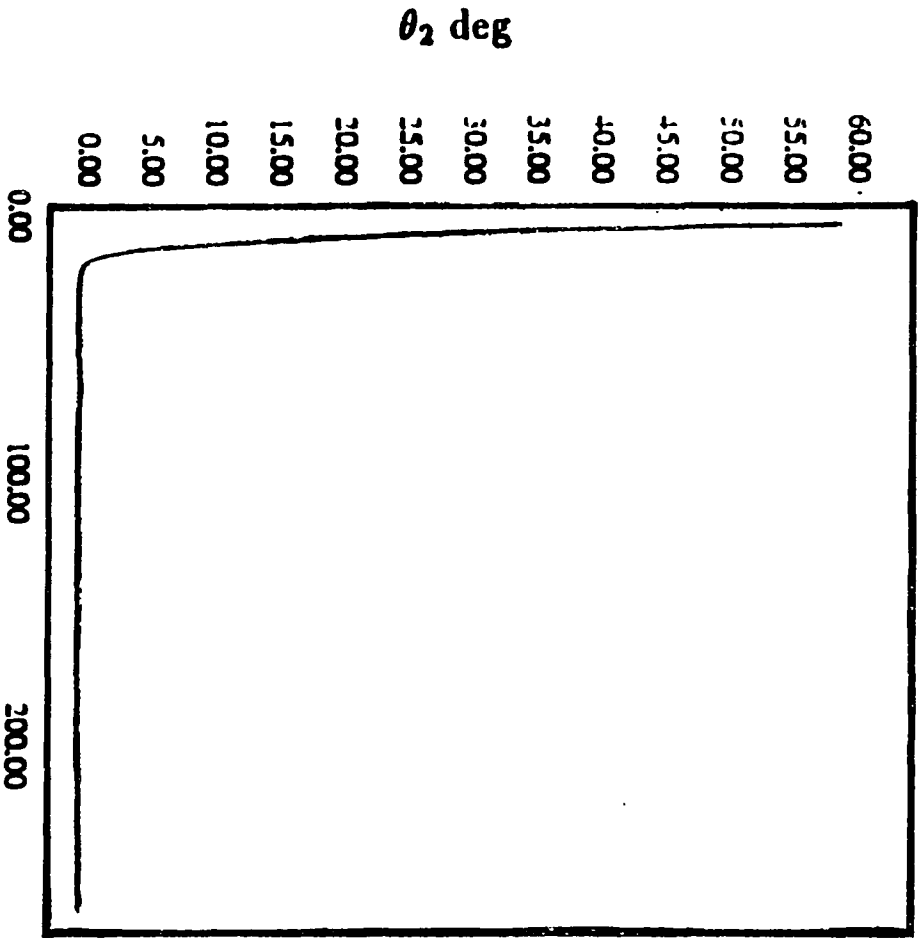


Fig. 3 (a)





**Time, Minutes**

**Fig. 3 (b)**

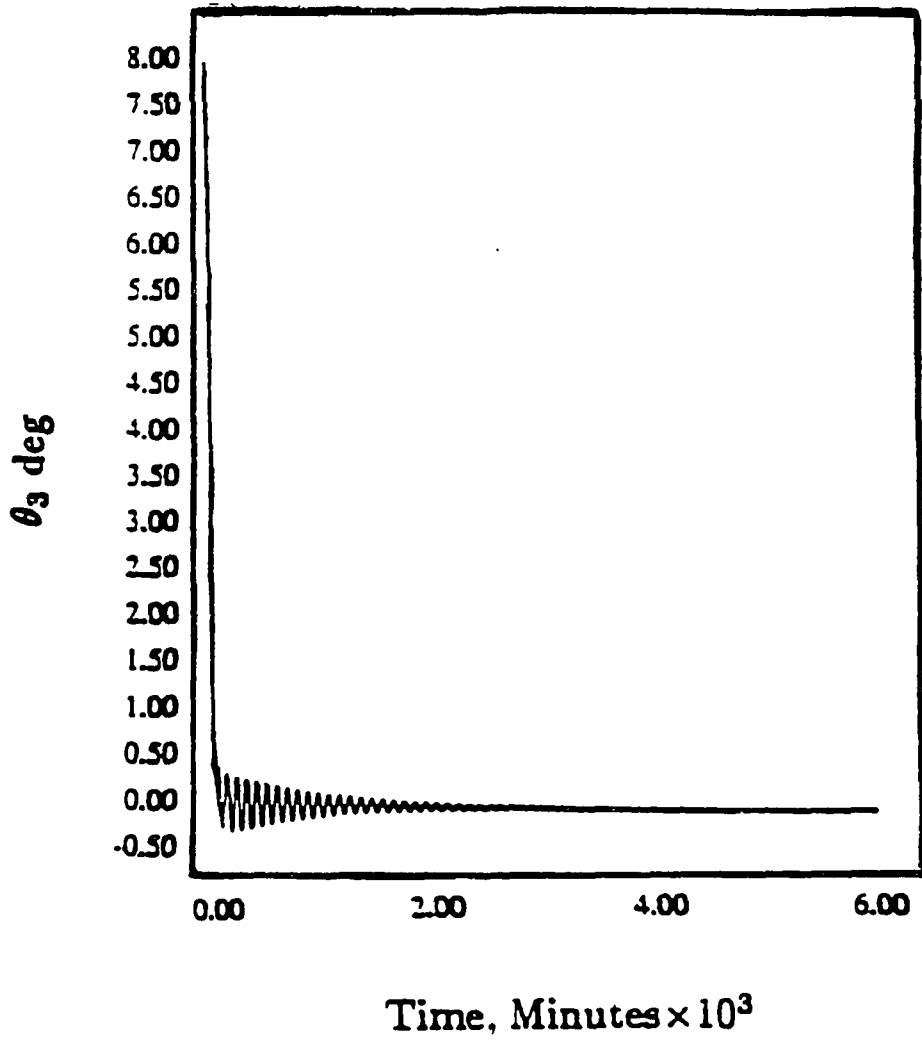


Fig. 3 (c)

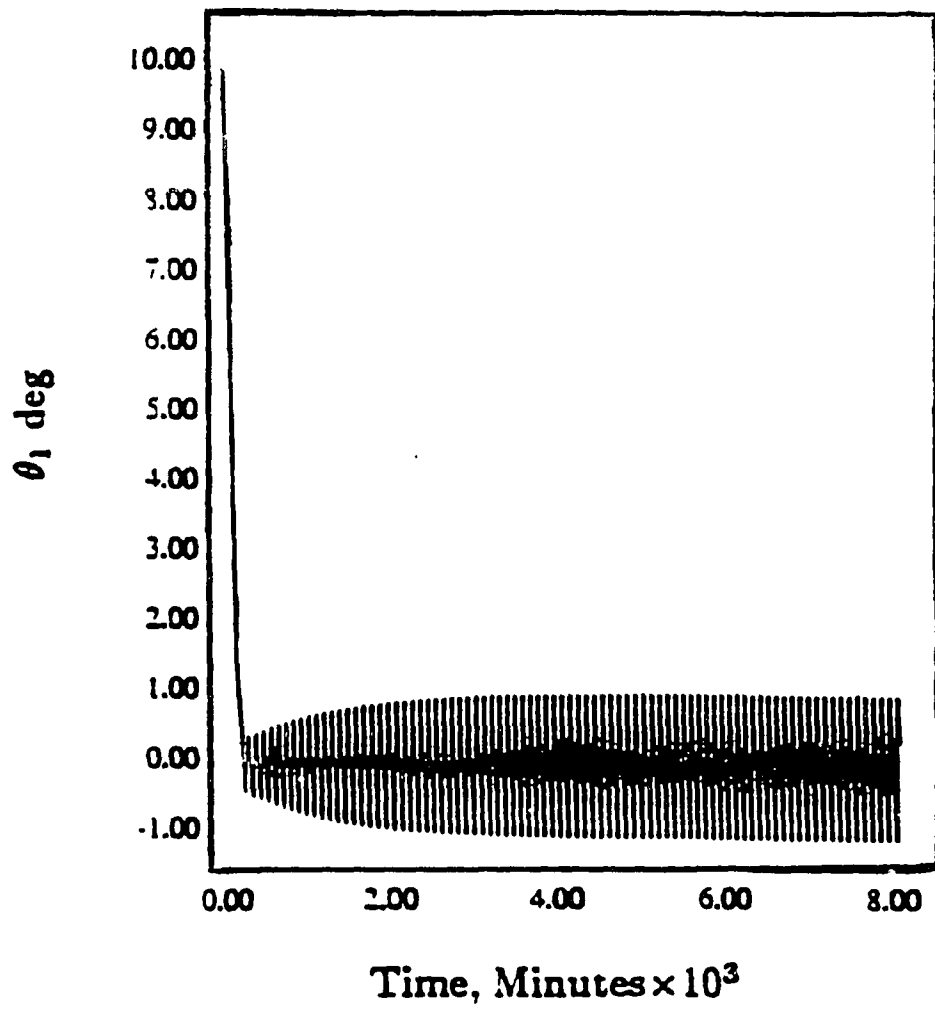


Fig. 4 (a)

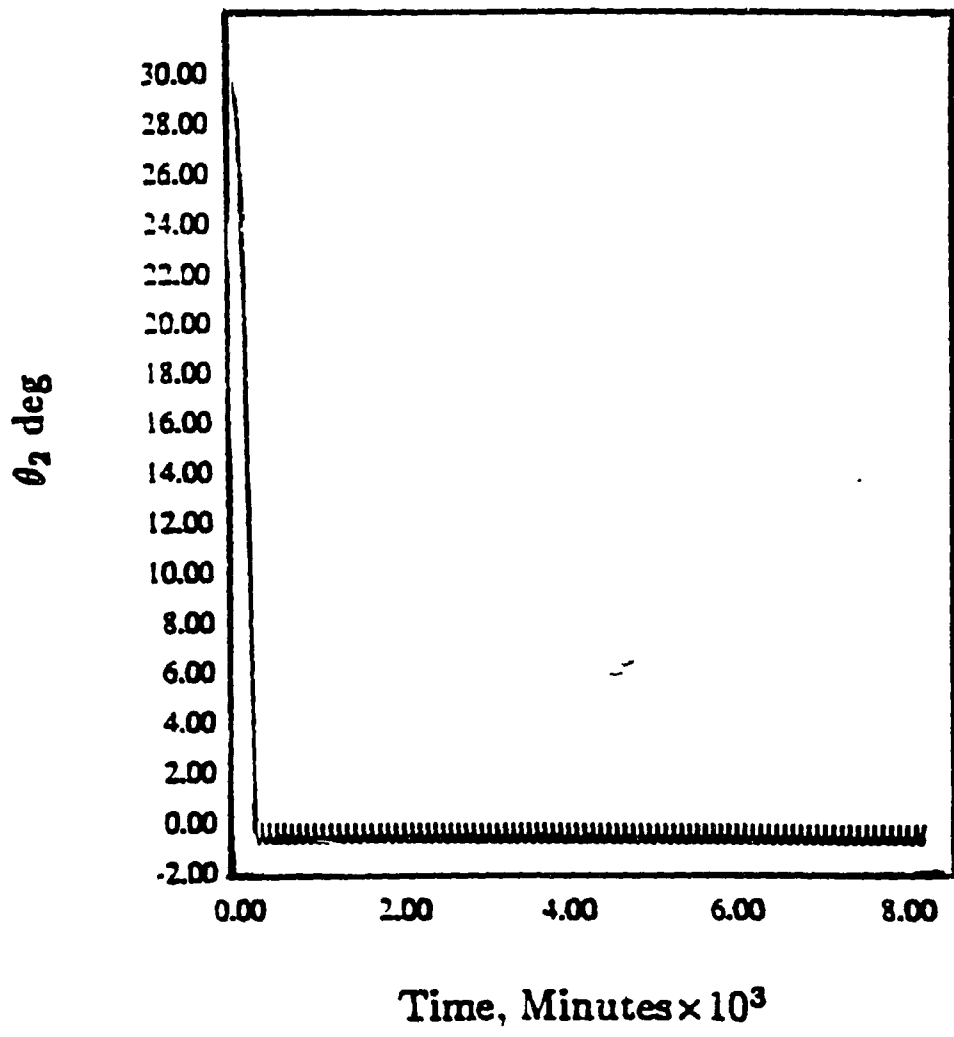


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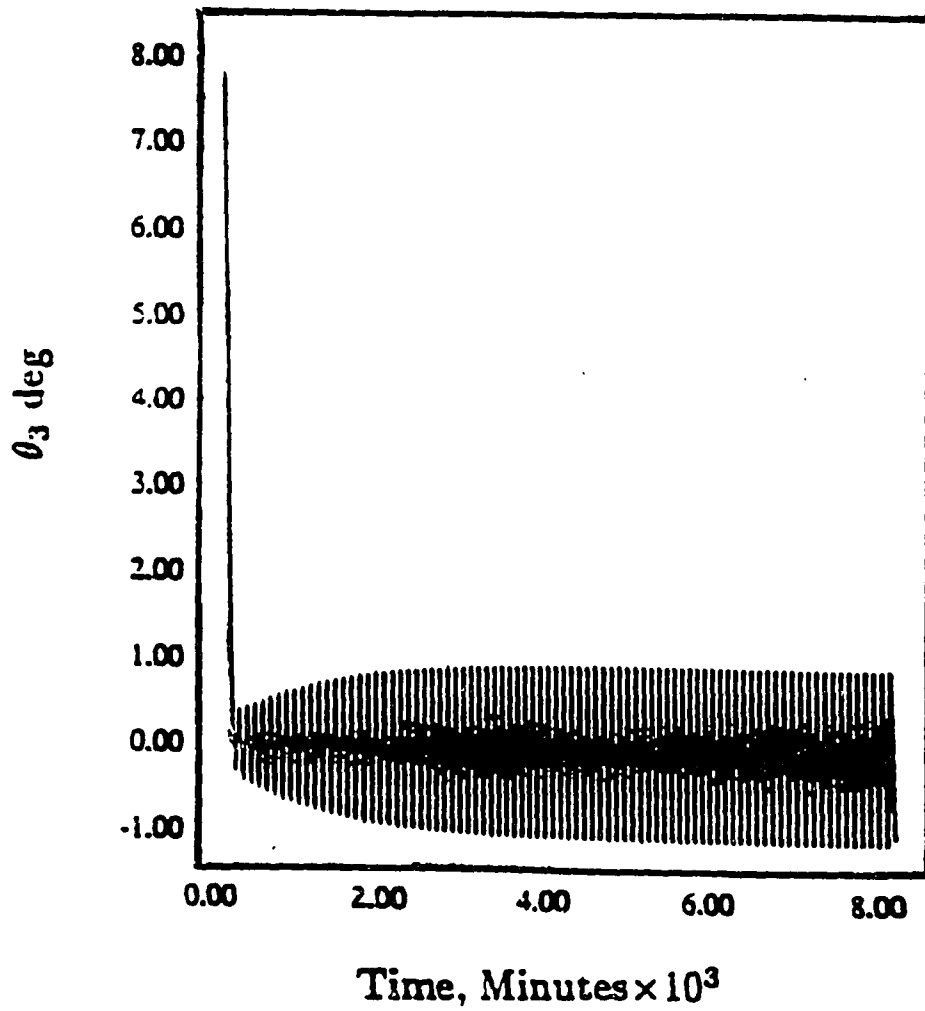


Fig. 4 (c)