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Inverse and variable structure trajectory control of a flexible robotic manipulator

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Inverse and variable structure trajectory control of a flexible robotic manipulator

Madhavan, Sethu K., M.S.
University of Nevada, Las Vegas, 1991
INVERSE AND VARIABLE STRUCTURE TRAJECTORY CONTROL OF A FLEXIBLE ROBOTIC MANIPULATOR

by

Sethu K. Madhavan

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in

Electrical Engineering

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- The above paper was also accepted for presentation at The American Controls Conference (ACC), Boston, June 1991.


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Abstract

This thesis introduces two schemes that control the end effector trajectory and stabilize a two-link flexible robotic arm. They are (i) The Inverse Trajectory Control scheme and (ii) The Variable Structure System (VSS) scheme. Both schemes choose outputs as the sum of the joint angle and tip elastic deformation times a constant factor for each link.

The Inverse Trajectory Control scheme develops a control law based on the inversion of an input-output map. For the chosen output, the stable maneuver of the arm critically depends on the stability of the zero dynamics of the system. This scheme illustrates that this stability is sensitive to the choice of the constant multiplying factor, which explains the difficulty in controlling the actual tip position. A critical value of the constant factor for control is obtained and this corresponds to a coordinate in the neighbourhood of the actual tip position. Although the inverse controller accomplishes output control, this excites the rigid and elastic modes. A linear stabilizer is designed for the final capture of the terminal state and stabilization of the elastic modes. The stabilizer is designed using pole assignment technique, which includes a control logic that switches the stabilizer as soon as the output enters the specified neighbourhood of the terminal state.

The second scheme incorporates a Variable Structure Control law which includes robustness in its design. For the chosen output, a discontinuous output control law is derived based on the Variable Structure theory. The control law thus derived accomplishes the desired trajectory tracking of the output. Basically, this control
scheme involves two phases, namely the ‘reaching phase’ and the ‘sliding phase’. In the first phase, the trajectories are attracted towards a hypersurface in the state space. The next phase involves the ‘sliding’ of these trajectories on the surface. As in the first scheme, a linear stabilizer was designed (using pole assignment technique) for the final capture of the terminal state and stabilization of the elastic modes.

Simulation results are presented (for both schemes) to show that in the closed-loop system, large maneuvers can be performed in the presence of payload uncertainty, thereby exhibiting the excellent robustness of the controllers.
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Chapter 1

INTRODUCTION

Of late, researchers seem to be more interested in the design of flexible robotic systems. Their preference over the previously popular rigid arm can be attributed to a variety of reasons. It was probably the simplicity of calculation that motivated researchers to assume rigid conditions. However, the rigid arm analysis leads to poor performance due to structural flexibility. As a result, the controller performance is adversely affected. Apart from the complexity of analysis, the design of flexible robotic arms are further complicated by the presence of uncertainty in the system. Moreover, the equations of motion are highly nonlinear and this further aggravates the problem.

This thesis incorporates two controllers, namely the Inverse controller and the Variable Structure controller. At this point, it is worth mentioning a few recent research efforts in these areas. Various control systems for robotic systems with flexible links are presented in [1-10]. Articles [4-10] report on nonlinear control systems incorporating the principle of nonlinear inversion. An end effector trajectory control by inversion for a single link flexible manipulator is presented in [4-5]. The controller of [9] is based on the inverse control of the joint angles of a two link flexible arm. A singular perturbation strategy has been used to design controllers based on the
seperation of slow and fast modes in [11]. A few experimental results are presented in [12-15]. It is seen that the inverse control of a flexible robotic arm with joint angles as outputs gives stable elastic mode responses but the choice of the tip position (end effector position) as output for the control of a single link arm leads to unstable zero dynamics. This leads to difficulty in the tip trajectory control. In order to avoid this instability, the choice of reflected tip position as output for control has been suggested in [10]. Control of rigid manipulators based on the Variable Structure System theory are given in [19-25].

The dynamical model of the arm with two links is significantly complex to that of a single link flexible arm. In this thesis, the output is chosen as the sum of the joint angle and the elastic deflection at the tip multiplied by a constant factor (α/length of the link). From Fig.2.1, it can be seen that different values of this constant factor actually corresponds to different points on the beam. As an example, it can be observed from Fig.2.1 that the point ‘A*’ on the beam corresponds to the point ‘A’, which in turn corresponds to a particular value of the constant factor. Therefore, control of the point ‘A*’ accomplishes the control of the point ‘A’. The design approach presented in this paper can be easily extended to any multi-link elastic arm with rotational joints. In our case, one defines the controlled output variable for each link parametrized by ‘α’ for each link.

For the choice of α=1, the outputs are the tip angular positions ; α=0 gives the joint angles as outputs and for α=-1, the output is the reflected tip position. It has been suggested in [10] that the selection of the output corresponding to α=-1 may make the control of a single link arm easier. It is seen that such a choice yields a
transfer function with a well-defined relative degree for a single link manipulator.

In the first scheme, it is shown that there exists a positive critical value $\alpha^* < 1$ of $\alpha$ such that for $\alpha > \alpha^*$, significant unstable unobservable dynamics due to nonlinear inversion appear. The value of $\alpha^*$ depends on the terminal state. For $-1 \leq \alpha \leq \alpha^*$, almost stable or stable zero dynamics are obtained and an inverse controller can be designed to control the output. (Here a system is said to be almost stable if it has some unstable poles but the non-zero real parts of these poles are sufficiently small). Even though one must choose $\alpha = 1$ for tip position control, in order to avoid unstable zero dynamics, one must keep $\alpha \leq \alpha^* < 1$. Evidently this leads to the trajectory control of a coordinate close to the actual tip position.

In both the schemes considered in this thesis, synthesis of the controller lead to oscillatory responses of the rigid and flexible modes in the neighbourhood of the desired equilibrium state. Interestingly, these oscillations are larger in magnitude for a negative value of ‘$\alpha$’; even though the zero dynamics were almost stable. The computation of the critical value ‘$\alpha^*$’ for stability is useful. However, as will be indicated later, the derivation of the analytical expression of ‘$\alpha^*$’ is extremely difficult. In general, it is expected that its value will depend on the configuration of the robotic arm. However, numerical techniques can be rather easily used for specific problems for its computation.

Exploiting the asymptotic linear behavior of the closed-loop system, a stabilizer is designed using the pole assignment technique for regulating the trajectory to its terminal state. For both the schemes, in the closed loop system, the trajectory control is achieved in two phases. In the first phase, only the control law is used.
As the output variables enter a specified small neighbourhood of the terminal state, the trajectory is controlled by the combined action of the controller and the linear stabilizer. The controller presented in this thesis is found to tolerate a wide range of payload uncertainty compared to the inverse controller of [9] and the Variable Structure Controller of [19] both of which assume joint angles as output variables.

For the synthesis of the control law, it is assumed that all the state variables are available for feedback. In the practical situation, one has to obtain the estimate of elastic modes and their derivatives. The joint angles and their angular rates are measurable. For state estimation, sensors such as strain gauges and accelerometers can be used. The problem of state estimation is not treated in this thesis. Also the question of control and observation spillover is not treated here. For linear systems, several methods have been proposed for reduced order control systems to avoid spillover difficulties; but this is an open problem for nonlinear systems. For illustrative purposes, simple truncation of higher order modes is assumed permissible.

This thesis is organized as follows.

The Mathematical model is presented in Chapter 2 while Chapters 3 and 4 expound in detail the Inverse Control and the Variable Structure schemes respectively.
Chapter 2

MATHEMATICAL MODEL OF TWO LINK ROBOTIC ARM

2.1 Description of the Physical Model

Fig.2.1 illustrates a robotic arm with two elastic links. Both the joints are revolute and torque is applied at these joints. It is worth noting that the design approach for the two link case considered here can be extended to a general multi-link elastic arm.

In Fig.2.1, $OXY$ is a inertial frame with origin at joint 1, $OX_1Y_1$ is a reference frame with $X_1$ along link 1 and $O_2X_2Y_2$ is a reference frame with the origin at joint 2 with $X_2$ along link 2. ($OX$ is an axis that points vertically down). If the arms were rigid, the actual axis along which the arms would lie is depicted as $OO_{N1}O_{N2}$. $\theta_1$ and $\theta_2$ are the joint angles when rigid conditions are assumed. The elastic deformation causes the arm to lie along the axis $OO_2O_{E2}$.

2.2 Formulation of the Lagrangian Equations of Motion

By the assumed modes method, any arbitrary solution of flexible motions can be assumed to be composed of a linear combination of admissible functions multiplied
Figure 2.1: Physical Model of an Elastic Robotic arm with two links
by time-dependent generalized coordinates. Let the flexible displacement of link 1 be
\( \delta_1(x_1, t) \) at a distance \( x_1 \) from \( O \) along \( OX_1 \) and that of link 2 be \( \delta_2(x_2, t) \) at a distance \( x_2 \) from \( O_2 \) along \( O_2X_2 \). Also, \( \phi_{1i}(x_1) \) and \( \phi_{2i}(x_2) \) are the necessary admissible functions for clamped-free beams. For more details, the reader is referred to [1-2].

Let \( q_{1i}(t) \) and \( q_{2i}(t) \) be the generalized coordinates. Therefore, the flexible displacements of the two links can be assumed to be

\[
\delta_1(x_1, t) = \sum_{i=1}^{n} \phi_{1i}(x_1)q_{1i}(t) \\
\delta_2(x_2, t) = \sum_{i=1}^{n} \phi_{2i}(x_2)q_{2i}(t)
\]

For simplicity sake, we retain two elastic modes in this representation and therefore, in our case the value of 'n' is 2. It is assumed that longitudinal and torsional deformations are negligible.

Using the Lagrangian method, the equations of motion can be expressed by

\[
\frac{d}{dt}\left(\begin{array}{c}
\partial K \\
\partial \dot{z}_i
\end{array}\right) - \frac{\partial K}{\partial z_i} + \frac{\partial P}{\partial z_i} = B_1u
\]

where,

'\( K \)' is the kinetic energy of the arm

'\( P \)' is the potential energy of the arm

'\( z \)' is the column vector containing the generalized coordinates

'\( u \)' is the column vector containing the joint torques

Further we have,

(an upper subscript 'T' denotes matrix transposition)
\[ z = (\theta_1, \theta_2, q_{11}, ..., q_{1n}, q_{21}, ..., q_{2n})^T \in \mathbb{R}^{n_0}, \quad n_0 = (2n + 2). \]

\[ u = (u_1, u_2)^T \in \mathbb{R}^2 \]

\[ \theta = (\theta_1, \theta_2)^T \]

\[ B_1 = [I_{2 \times 2} \quad O_{2 \times 2n}]^T \] where I and O denote identity and null matrices of indicated dimensions.

In order to move the end effector of the manipulator to the desired final position, the corresponding torques that need to be applied at the joints need to be computed. For each link, the kinetic energy is given by

\[ K = \frac{1}{2} (\dot{z}^T M(z) \dot{z}) \]

where \( M(z) \) is a positive definite symmetric inertia matrix of dimension \((n_0 \times n_0)\) and is a nonlinear function of \( q \).

From (2.2), one obtains

\[ M(z) \ddot{z} + h_0(z, \dot{z}) + \frac{\partial P(z)}{\partial z} = B_1 u \] (2.3)

where

\[ h_0(z, \dot{z}) = \dot{M}(z) \dot{z} - \frac{1}{2} \frac{\partial (\dot{z}^T M(z) \dot{z})}{\partial z} \]

Let \( h(z, \dot{z}) = -h_0(z, \dot{z}) - \frac{\partial P(z)}{\partial z} \)

\[ M(z) = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \]

(Here \( M_{11} \) is a \( 2 \times 2 \) matrix)

If the state vector is defined as \( x = (z^T, \dot{z}^T)^T \in \mathbb{R}^{2n_0} \), then (2.2) can be expressed in the following state variable form.

\[ \dot{x} = A(x) + B(x)u \] (2.4)

8
where

\[
A(z) = \begin{pmatrix}
\dot{z} \\
M^{-1}(z)[h(z, \dot{z})]
\end{pmatrix}
\]

\[
B(x) = [O_{2 \times n_0}, (M^{-1}(z)B_1)^T]^T
\]

Let \(D(z) = M^{-1}(z)\) and \(D(z) = [D_1 \ D_2]\), where \(D_1\) has two columns.

2.3 Choosing the desired output vector

The output vector chosen for the purpose of control is given by

\[
y = \theta + \alpha[\tan^{-1}\left(\frac{\delta_1(l_1, t)}{l_1}\right), \tan^{-1}\left(\frac{\delta_2(l_2, t)}{l_2}\right)]^T \tag{2.5}
\]

where

\(-1 \leq \alpha \leq 1\) is a constant factor,

\(l_1\) and \(l_2\) are the lengths of link 1 and link 2 respectively.

The actual tip position is obtained when \(\alpha = 1\). The choice of \(\alpha = -1\) results in a transfer function with a well-defined relative degree [10]. For small elastic deformation, the output simplifies to

\[
y = \theta + \alpha[\frac{\delta_1(l_1, t)}{l_1}, \frac{\delta_2(l_2, t)}{l_2}]^T \tag{2.6}
\]

The control laws are derived such that in the closed-loop system, the output \(y(t) = [y_1(t), y_2(t)]^T\) follows the given reference trajectory \(y_c(t) = [y_{c1}(t), y_{c2}(t)]^T\).

Furthermore, a good range of \(\alpha\) is obtained, such that, the arm can be maneuvered efficiently. A linear stabilizer is designed to stabilize the rigid and elastic mode oscillations of the arm.
Chapter 3

THE INVERSE TRAJECTORY CONTROL SCHEME

3.1 Introduction

The term ‘Inverse’ results from the dynamics of the problem wherein one has to formulate a control input such that the chosen outputs follow a desired trajectory. This chapter elaborately discusses a control system design based on the principle of nonlinear inversion. Furthermore, a linear stabilizer is also designed to dampen the oscillations that result due to the excitation of the rigid and elastic modes. The inversion algorithm is applicable even if actuator dynamics are present in the system. However, in the following analysis, actuator transfer functions are ignored.

The primary aim is to determine a nonlinear control law ‘u’ based on the inversion of the input-output map; the input constituting of the joint (control) torques and the output is as given in (2.6). The inversion algorithm leads to a series of differentiation of the outputs until a term containing the input appears in the resulting expression. In order to increase the robustness of the controller, the control law ‘u’ includes an integral feedback of the output error. At this point, it is worth noting an important fact that in the terminal phase, although rigid and elastic modes continue to exist,
the resulting oscillations are quite small. They can be approximated to be linear in nature and can therefore be analysed by linear differential equations. As will be described in a subsequent section of this chapter, this fact plays a major role in the choice of a stabilizer that is linear.

Basically, the control is accomplished in two stages. In the first stage, the linear stabilizer loop is left open while the control input ‘u’ acts on the system. It is seen that bounded elastic oscillations are present. The output follows the required smooth trajectory. The second stage involves the switching on of the stabilizer as soon as the output reaches a specified neighbourhood of the terminal state. Therefore the control input now includes a stabilization signal and the total control signal contribute to the capture of the terminal state and to dampen the elastic oscillations. Evidently, it will be seen later that the magnitude of the stabilization signal must be small so that the tracking ability of the inverse controller is not adversely affected.

An elaborate discussion on the inverse controller along with the linear stabilizer design is presented in the following sections of this chapter.

3.2 The Inverse Control System

In this section, a control law is derived using the inversion technique [16]. Inversion algorithm is applicable even if actuator dynamics are present in the system. However, in the following analysis, actuator transfer functions are ignored.

The inversion algorithm leads to a series of differentiation of the outputs. Listed below are two systems, System A and System B, which are obtained by following the inversion algorithm.
System A:

\[
\dot{x} = A(x) + B(x)u \quad (3.1)
\]

\[
\dot{y} = C_1(x) \equiv C_0 \dot{z}
\]

where

\[
C_0 = [I_{2 \times 2} \ E]
\]

\[
E = \begin{pmatrix} \phi_1(l_1) & \phi_2(l_1) \\ \phi_1(l_2) & \phi_2(l_2) \end{pmatrix}
\]

System B:

\[
\dot{x} = A(x) + B(x)u \quad (3.2)
\]

\[
\ddot{y} = C_2(x) + D_2(x)u
\]

where

\[
C_2(x) = C_0 M^{-1}(x) [-\dot{M}(z) \dot{z} + \frac{\delta K}{\delta z} - \frac{\delta P}{\delta z}]
\]

\[
D_2(x) = C_0 M^{-1}(z) B_1
\]

For the control of the arm, we are interested in a region of the state space ‘Ω’ where the matrix $D_2(x)$ is nonsingular. The tracking order of systems given by (3.1) and (3.2) is 2. From (3.2) we observe that System B is invertible.

We choose a control law of the form

\[
u = D_2^{-1}(x)[-C_2(x) - G_2 \ddot{y} - G_1 \dot{y} - G_0 w + \ddot{y}_e + v] \quad (3.4)
\]
\[ u_n(x) + D_2^{-1}(x)v \]

where \( \tilde{y} = y - y_c \) is the tracking error, \( G_2 = \text{diag}(g_{2i}) \); \( G_1 = \text{diag}(g_{1i}) \);
\( G_0 = \text{diag}(g_{0i}), i = 1, 2 \). 'v' is the stabilization signal to be determined later and the vector 'w' is the integral of the tracking error '\( \tilde{y} \)'. Thus
\[ \dot{w} = \tilde{y} \quad (3.5) \]

The signal 'w' is included in the control law in order to increase the robustness of the system.

The linear dynamics of the tracking error can be deduced by substituting (3.4) in (3.2) and is given by
\[ \ddot{y} + G_2 \dot{y} + G_1 \dot{y} + G_0w = v \quad (3.6) \]

With the stabilization signal \( v = 0 \), we differentiate (3.6) and substituting the value of 'w' from (3.5) we have
\[ \ddot{y} + G_2 \dot{y} + G_1 \dot{y} + G_0\tilde{y} = 0 \quad (3.7) \]

Independent and stable control of the output vector 'y' is obtained by a proper choice of matrices \( G_i \), \( i = 0, 1, 2 \). The required stable responses for the joint angles are then obtained. Since the tracking order of the system is 2, the control law employs the second derivative of the command input '\( y_c \)'. It is therefore appropriate to introduce a third order command generator of the form
\[ \ddot{y}_c + P_2 \dot{y} + P_1 \dot{y} + P_0y = P_0y^* \quad (3.8) \]

where '\( y^* \)' is the desired terminal output vector and matrices \( P_i \), \( i = 0, 1, 2 \) are appropriately chosen so as to obtain the required reference trajectory. In the closed-loop
system, a smooth output trajectory \( y_e \) in the state space region \( \Omega \) can be reproduced if

\[
\begin{aligned}
\ddot{y}(0) &= \ddot{\dot{y}}(0) = \dddot{y}(0) = 0, \\
v &= 0
\end{aligned}
\]  

(3.9)

It should be noted here that though the output follows the reference trajectory, oscillations result due to the excitation of the rigid and elastic modes. As a result, a stabilizer has to be designed in order to dampen these oscillations.

### 3.3 Zero Dynamic Stability

In this section, the stability of the closed loop system, including the inverse controller is examined. For stability analysis, we shall consider the motion of the arm in the neighbourhood of the equilibrium state \( x^* = (z^T, 0^T)^T \) and \( w^* = 0 \). It is worth noting here that \( z^* \) corresponds to the desired tip positions of the links.

Also, it will be assumed that \( y_c = y^*, \dot{y}_c = \ddot{y}_c = 0 \).

Noting that \( y = \theta + Eq \), one then has \( y^* = \theta^* + Eq^* \) where

\[
q = [q_{11}, q_{12}, q_{21}, q_{22}]^T, \\
\theta = [\theta_1, \theta_2]^T \text{ and } \\
z^* = [\theta^{*T}, q^{*T}]^T. \\
\]

The equilibrium value of \( z^* \) is the solution of the following equation

\[
\frac{\partial P(z^*)}{\partial q} = 0 
\]

(3.10)

Let \( \Delta y = y - y^*, \Delta \theta = \theta - \theta^*, \Delta q = q - q^* \) and \( \Delta w = w \). We then define \( \xi \) as the deviation in the state vector from the equilibrium point,

where \( \xi = [\Delta y^T \Delta \dot{y}^T \Delta q^T \Delta \dot{q}^T \Delta w^T]^T \) and \( \Delta y = \Delta \theta + E \Delta q \).
Neglecting the second order terms in the velocity of the generalized coordinates, the linearized q-responses are given by

\[ M_{21}(q^*) \Delta \ddot{\theta} + M_{22}(\theta^*) \Delta \ddot{q} + P_{\theta q}(\theta^*) \Delta \theta + P_{qq} \Delta q = 0 \] (3.11)

where

\[ P_{\theta q} = \frac{\partial^2 P}{\partial \theta \partial q} \]
\[ P_{qq} = \frac{\partial^2 P}{\partial q \partial q} \]

Using \( \Delta \theta = \Delta y - E \Delta q \) and \( \Delta \ddot{\theta} = \Delta \ddot{y} - E \Delta \ddot{q} \) in (3.11) yields

\[ M_{21}^* \Delta \ddot{y} + (M_{22}^* - M_{21}^* E) \Delta \ddot{q} + (P_{qq} - P_{\theta q}^* E) \Delta q + P_{\theta q}^* \Delta y = 0 \] (3.12)

Here \( M_{ij}^* \) and \( P_{ij}^* \) denote matrices computed at \( p = p^* \). The \( \Delta y \) - responses, obtained from (3.5) and (3.6), are given by

\[ \Delta \dot{y} = -G_2 \Delta \dot{y} - G_1 \Delta y - G_0 \Delta w \] (3.13)

\[ \Delta \dot{w} = \Delta y \]

To study the stability of the system, we consider the zero dynamics of the system given by (3.12) and (3.13) [17-18]. The zero dynamics of the system corresponds to the dynamics that describe the system when \( \Delta y(t) \equiv 0 \). By setting \( \Delta y \) and its derivatives to zero in (3.12), one can obtain the zero dynamics as

\[ \tilde{M}_{22} \Delta \ddot{q} + \tilde{P}_{qq} \Delta q = 0 \] (3.14)

where \( \tilde{M}_{22} = M_{22}^* - M_{21}^* E, \quad \tilde{P}_{qq} = P_{qq} - P_{\theta q}^* E \)
The stability of the equilibrium point \((z^*, 0)\) of (3.14) can be examined by solving for the roots of the polynomial

\[
C_p(\lambda) = \text{det}[\bar{M}_{22} \lambda^2 + \bar{P}_{qq}] = 0
\]  

(3.15)

It follows from the results of [17] that, if the zero dynamics described by (3.14) are stable, then the closed-loop system described by (3.7) and (3.13) is stable. One must note that these stability results are local in nature. The eigenvalues associated with the system (3.7) and (3.13) are solutions of

\[
C_r(\lambda)C_p(\lambda) = 0
\]  

(3.16)

where \(C_r(\lambda)\) is the characteristic polynomial of (3.7) and is given by

\[
C_r(\lambda) = (\lambda^3 I + G_2 \lambda^2 + G_1 \lambda + G_0) = 0
\]  

(3.17)

Let \(S_e\) and \(S_r\) denote the set of roots of (3.15) and (3.17), respectively.

For \(\alpha' = 0\), one has \(E\) in (3.1) equal to zero. In this case, \(C_p(\lambda)\) simplifies to \(C_p(\lambda) = M_{22}^* \lambda^2 + P_{qq} = 0\) which has purely imaginary roots since \(M_{22}^*\) and \(P_{qq}^*\) are positive definite matrices. However for nonzero value of \(\alpha\), the matrices \(\bar{M}_{22}\) and \(\bar{P}_{qq}\) are not symmetric and these are functions of \(\alpha\). Therefore, the roots of (3.15) depend nonlinearly on \(\alpha\) and the derivation of analytical expressions relating the roots of (3.15) to \(\alpha\) seems to be an extremely difficult problem.

Using (3.12) and (3.13), we obtain a new state variable representation of the
system as

\[
\dot{\xi} = \begin{pmatrix}
0 & I & 0 & 0 & 0 \\
-G_1 & -G_2 & 0 & 0 & -G_0 \\
0 & 0 & 0 & I & 0 \\
F_1 & F_2 & F_3 & 0 & F_4 \\
I & 0 & 0 & 0 & 0
\end{pmatrix} \xi + \begin{pmatrix}
0 \\
I \\
0 \\
0 \\
B_2
\end{pmatrix} v
\] (3.18)

\[\triangleq \ddot{\xi} + \tilde{B}v\]

where

\[F_1 = -\tilde{M}_{22}^{-1}(P_{q\theta}^* - M_{21}^* G_1)\]

\[F_2 = \tilde{M}_{22}^{-1}M_{21}^* G_2\]

\[F_3 = -\tilde{M}_{22}^{-1}\tilde{I}_{qq}\]

\[F_4 = \tilde{M}_{22}^{-1}M_{21}^* G_0\]

\[B_2 = -\tilde{M}_{22}^{-1}M_{21}^*\]

For any given value of \(\alpha\), the set of poles of (3.18) is given by \(S = S_rUSe\). Stable responses in the system (3.12) and (3.13) are obtained for those values of \(\alpha\) for which (3.15) gives stable roots. The analytical expression for the range of \(\alpha\) for stability is quite difficult to obtain for this complicated system. However, (3.15) has been numerically solved for the choice of manipulator parameters listed in the appendix. The previously mentioned critical value of \(\alpha\) (defined earlier as \(\alpha^*\)), was found to be 0.37. Apparently the value of \(\alpha^*\) depends on the terminal equilibrium point. Thus
for a given choice of $G_i$, the roots associated with the $\ddot{y}$ - response given by (3.17) are fixed, but the set of eigenvalues associated with the flexible modes, $\mathcal{S}_e$, continuously varies with the parameter $\alpha$. It was seen that for the values of $-1 \leq \alpha \leq 0.37$, the elements of $\mathcal{S}_e$ almost lie on the imaginary axis (The real part of the elements of $\mathcal{S}_e$ are less than $10^{-13}$). Evidently, for $\alpha=0$, the eigenvalues of $\tilde{A}$ are purely imaginary. But for values of $\alpha > \alpha^*$, significant instability of flexible modes were found to occur.

Table(1) shows only the unstable roots from the set $\mathcal{S}_e$ for different values of ‘$\alpha$’. It is seen that for $\alpha > \alpha^*$, the dynamic behaviour of the zero dynamics is highly sensitive to the value of $\alpha$, since the positive real part of the unstable eigenvalue rapidly increases as the value of $\alpha$ approaches 1.

At this point, a question arises as to what the value of $\alpha$ should be. In order to control the actual tip position, the value of $\alpha$ must be set to 1. However for this value of $\alpha$, the corresponding zero dynamics are highly unstable. As a consequence the inverse controller leads to unbounded responses. Evidently, eventhough it is necessary to set $\alpha = 1$ for actual tip position control, one has to choose $\alpha \in [-1, \alpha^*]$ so as to ensure almost stability or stability in the closed-loop system, including the inverse controller. All the same, larger the value of $\alpha$, closer we are to the actual tip position and hence better the tip trajectory control. The controlled variable tends to the actual tip position as $\alpha \to 1$. Thus a good choice of $\alpha$ is $\alpha = \alpha^*$ or $\alpha > \alpha^*$ such that the zero dynamics is very lightly unstable.
3.4 Linear Stabilization

In this section, the design of the stabilizer is presented. Here we assume that the reference trajectory $y_c(t)$ is such that $\dot{y}_c(t) \to 0$, $\ddot{y}_c(t) \to 0$ and $y_c(t) \to y^*$ as $t \to \infty$. For a choice of $\alpha \in [-1, \alpha^*]$, the closed-loop system gives stable responses for $y(t)$ and $y(t)$ tends to $y^*$. Although the trajectory tracking error converges to zero, bounded periodic oscillations of the joint angles along with the elastic modes are observed. This is a result of the maneuver of the arm in view of the critical stability of the zero dynamics. However, the asymptotic motion of the arm remains in a small neighbourhood of the equilibrium point and in the terminal phase, the dynamics of the arm is almost linear. As such, a linear stabilizer is adequate for the stabilization and capture of the terminal state.

The pole assignment technique is employed to design the stabilizer. A proper selection of poles of the closed-loop system is essential for obtaining good responses. It is worth noting here that once the stabilizing signal 'v' is superimposed on the inverse control signal, the tracking ability of the control law is affected. All the same, the stabilizing signal is required to dampen the elastic oscillations and its magnitude must be small such that the tracking ability of the inverse controller is not adversely affected. But it should be of sufficient magnitude so that fast damping of the elastic oscillations can be achieved. A set of desirable eigen values of the closed-loop system is taken as

$$S_{cl} = S_r U S_e$$

where

$$S_e = \{\mu : \mu = -c + j \Re(\lambda), \lambda \in S_e, c > 0\}$$
In the set $S_d$, the eigenvalues associated with the $y$-responses are retained, but $\lambda_i \in S_e$ are shifted to the left by ‘c’ units in the complex plane. The control law for the stabilization is of the form

$$v = -L\xi,$$  \hspace{1cm} (3.19)

such that the closed-loop system matrix $[F_d] = [\hat{A} - \hat{B}L]$ has the desirable eigenvalues.

The synthesis of the controller is as follows. First, the inverse controller is used for large maneuvers of the arm. When the trajectory reaches the neighbourhood of the terminal value, the linear stabilizer loop is closed for the capture of the terminal state and to dampen the vibrations. The stabilizer is effective if the $\theta$ and $q$ - responses remain in a small neighbourhood of the equilibrium state at the instant when the stabilizer is switched. Apparently, $\alpha > \alpha^*$ can also be chosen as long as the zero dynamics is lightly unstable and the trajectory remains close to the equilibrium point at the instant of switching of the stabilizer.

### 3.5 Simulation Results

#### 3.5.1 Introduction to the Digital Simulations

This section explores the results of the digital simulations. The mathematical model of the arm has been taken from [1,2]. This model is highly nonlinear and includes the functions causing rigid and elastic mode interactions. The parameters assume the nominal values that are listed in the appendix. The mode shapes $\phi_{ij}$ are selected as clamped-free modes [1]. Assuming that the amplitude of higher modes of the flexible links are very small when compared to the first ones, we have illustrated
the case with \( n=2 \) in the expression for elastic deflection given by (2.1). For the
derivation of equations of motion using the Lagrangian approach, the expression for
the potential energy is obtained which includes the effect of elasticity and gravity.
With the choice of \( n=2 \), one has the state vector \( \mathbf{x}' \) of dimension 12 where
\[
\mathbf{x} = [\mathbf{z}^T, \mathbf{z}'^T]^T
\]
and \( \mathbf{z} = [\theta_1, \theta_2, q_{11}, q_{12}, q_{21}, q_{22}]^T \). Also, the following initial conditions are assumed.
\[
y_c(0) = \dot{y}_c(0) = \ddot{y}_c(0) = 0, \ x(0) = 0 \text{ and } w(0) = 0.
\]
A command trajectory \( y_c(t) \) was generated to control \( y(0) = 0 \) to \( y^* \). It was
assumed that the given tip position corresponds to \( \theta^* = (90^\circ, 60^\circ)^T \). The terminal
value \( y^* \) was set to \( y^* = \theta^* + \alpha \mathbf{q}^* \) with \( \alpha = 0.34 \).

Let \( y_t = (y_{1t}, y_{2t})^T \) denote the angular positions of the tip of the two links and
\( \theta = (\theta_1, \theta_2)^T \), then we have
\[
y_t = \theta + \left( \frac{\mathbf{E} \mathbf{q}(t)}{\alpha} \right) = \theta + \left( \frac{D_{1e}}{l_1} \frac{D_{2e}}{l_2} \right)^T.
\]
where \( D_{1e} \) and \( D_{2e} \) are the tip elastic deflections for link 1 and link 2 respectively.
The matrices \( P_i \) of the command generator are taken as \( P_i = p_i I_{2 \times 2} \), \( i=0,1,2 \) and
are selected such that the poles associated with \( y_{ci}(t) \), the \( i^{th} \) component of \( y_c(t) \), are
at \( \{-2, -2 \pm i2\} \). The feedback matrices \( G_i \) are selected and are set to yield poles
associated with \( \tilde{y}_i \) in (2.4), of values \( \{-10, -10 \pm i10\} \), where
\[
\tilde{y} = (\tilde{y}_1, \tilde{y}_2)^T.
\]
These poles are chosen to obtain fast tracking error responses.

For the chosen feedback gains, the sets \( S_r \) and \( S_e \) are
\[
S_r = [-10, -10, -10 \pm 10i, -10 \pm 10i] \quad (3.20)
\]
\[
S_e = [-2.043\beta \pm 20.80i, 2.842\beta \pm 23.69i, 1.421\beta \pm 107.31i, 1.421\beta \pm 229.30i]
\]

where \(\beta = 10^{-14}\).

The feedback matrix 'L' of the stabilizer was chosen such that the set of eigenvalues of the matrix '\(S_{cl}\)' of the closed-loop system is taken as

\[
S_{cl} = S_r \cup S_e \quad (3.21)
\]

where \(S_c = \{ -2 + Im(\lambda), \lambda \in S_e \}\) and \(Im(\lambda)\) denotes the imaginary part of \(\lambda\).

Notice that in the closed-loop system the set of eigenvalues of \(S_r\) is retained, and the elements of \(S_e\) are simply moved to the left by nearly 2 units in the complex plane.

In the simulations that follow, \(y_1\) and \(y_2\) denote the chosen outputs; \(e_1 = \tilde{y}_1\) and \(e_2 = \tilde{y}_2\) are the trajectory errors; \(\theta_1\) and \(\theta_2\) are the joint angles; \(u_1\) and \(u_2\) denote the control inputs; \(D_{1e}\) and \(D_{2e}\) are the tip deflections for the two links and \(y_{t1}\) and \(y_{t2}\) give the actual end effector positions. It should be noted here that the outputs \(y_1\) and \(y_2\) are those that actually correspond to the point on the beam we have chosen to control (a point close to the actual tip). The simulation results show that these outputs follow the actual end effector positions \(y_{t1}\) and \(y_{t2}\), thus validating our choice.

3.5.2 Trajectory tracking : Control without stabilization

In this case, simulation was carried out for nominal payload and the stabilizer loop was left open. The value of the constant factor '\(\alpha\)' is chosen to be 0.34. Digital simulations were performed using (2.4), \(u_n(x)\) from (3.4) and (3.5) with the stabilizer loop left open. The trajectory errors were found to be equal to zero as predicted and
efficient tracking of the chosen outputs ($y_1$ and $y_2$) and joint angles ($\theta_1$ and $\theta_2$) is evident from Fig.3.1 and Fig.3.2 respectively. The absence of the stabilizer leads to persistent elastic mode oscillations of the tips of the two links (Fig.3.3). It can be seen that the $\theta$ responses (Fig.3.2) depict oscillations of extremely small magnitudes.

Simulation was also done to show that severe instability of the zero dynamics when the outputs are chosen close to the actual tip angular positions i.e. $\alpha^* \leq \alpha \leq 1$. Divergent responses were obtained and Fig.3.4 illustrates the divergence of the elastic oscillations for each link with the stabilizer loop left open for $\alpha = 0.39$ ($\alpha > \alpha^*$).
Figure 3.2: Trajectory tracking without stabilization; Joint angles

Figure 3.3: Trajectory tracking without stabilization; Elastic deflections
3.5.3 Trajectory tracking : Control with stabilization

Nominal payload

The closed-loop system defined by (2.4), $u_n(x)$ from (3.4), (3.5) and (3.19) were digitally simulated and the switching logic closes the stabilizer loop when the trajectory enters the neighbourhood of the terminal value (in about 2.5 seconds). It can be observed from Fig.3.6 that though the tracking error was zero prior to switching of the stabilizer, the error tends to increase at and after the instant of switching but dies down subsequently in about two seconds. The stabilizer dampens out the elastic mode oscillations (Fig.3.9) and the system quickly settles down to its steady state values (Fig.3.6, Fig.3.7 and Fig.3.8). It can be noted that the chosen outputs $y_1$ and $y_2$ (Fig.3.5) very closely follow the actual end effector outputs $y_{t1}$ and $y_{t2}$ (Fig.3.10).
Figure 3.5: Nominal payload with stabilization; Chosen outputs

Figure 3.6: Nominal payload with stabilization; Tracking error
Figure 3.7: Nominal payload with stabilization; Joint angles

Figure 3.8: Nominal payload with stabilization; Control inputs
Figure 3.9: Nominal payload with stabilization; Elastic deflections

Figure 3.10: Nominal payload with stabilization; Actual end effector outputs
Higher payload uncertainty

Simulations were carried for a higher payload uncertainty of 125% (\( \alpha = 0.34 \)). The controller used here is the one that was designed with nominal parameters and the initial conditions were assumed to be zero. In comparison to the nominal payload case, it can be observed that the stabilizer takes a longer time to dampen the oscillations (Fig. 3.13) and therefore the system takes a longer time to settle down to its steady state values. It can also be noted that the control torque required in this case (Fig. 3.12) is larger than the control torque required for the nominal case (Fig. 3.8) and the total elastic mode deflection in this case (Fig. 3.13) is more when compared to the nominal payload case (Fig. 3.9). Also, the chosen outputs \( y_1 \) and \( y_2 \) (Fig. 3.11) very closely follow the actual end effector outputs \( y_{t1} \) and \( y_{t2} \) (Fig. 3.14).

Similar simulations for \( \alpha = 0 \) showed that the closed-loop system was unstable for
Figure 3.12: Higher payload uncertainty; Control inputs

Figure 3.13: Higher payload uncertainty; Elastic deflections
Figure 3.14: Higher payload uncertainty; Actual end effector outputs

this payload uncertainty. (It should be noted that when $\alpha=0$, the controller structure is similar to the controller structure presented in [9]).

**Lower payload uncertainty**

Simulation results for a lower payload uncertainty of 50% ($\alpha = 0.34$) employing the controller that was designed for the nominal case are presented in this section. When compared to the higher payload uncertainty case, it can be noted that though the time taken by the system to attain steady state conditions is more or less the same, the frequency of the oscillations in the lower payload case is much higher, which is also a physical reality (Fig.3.13 and Fig.3.17). Moreover, the values of the control inputs (Fig.3.12 and Fig.3.16) and the magnitude of the elastic deflections (Fig.3.13 and Fig.3.17) are lower than the higher payload case.
Figure 3.15: Lower payload uncertainty; Chosen outputs

Figure 3.16: Lower payload uncertainty; Control inputs
Figure 3.17: Lower payload uncertainty; Elastic deflections

Figure 3.18: Lower payload uncertainty; Actual end effector outputs
Similar results for $\alpha=0$ showed that the closed-loop system was unstable for this 
payload uncertainty (when $\alpha = 0$, the controller is similar to the one in [9]).

3.5.4 **Nominal payload with stabilization ; $\alpha = 0.39$**

As described in a previous section, for a value of the constant factor $'\alpha \geq \alpha^*$' ( in our 
case $'\alpha^* = 0.37$ ), the system tends to exhibit unstable unobservable dynamics due 
to nonlinear inversion (Table 1). Simulation was done to show that the arm can be 
controlled provided the zero dynamics remains only lightly unstable. Interestingly, 
the combination of the nonlinear inverse controller and the linear stabilizer effectively 
controls the arm since the trajectory remains in the region of stability about the 
equilibrium point at the instant of switching of the stabilizer. Hence for the case 
when $'\alpha' = 0.39$, it can be observed from Fig.3.21 that the elastic oscillations tends to 
increase in magnitude quite drastically until the stabilizer comes into effect. Moreover, 
the chosen outputs $y_1$ and $y_2$ (Fig.3.19) very closely follow the actual end effector 
outputs $y_{t1}$ and $y_{t2}$ (Fig.3.20).

3.5.5 **Nominal payload with stabilization ; $\alpha = -1$**

Here the value of $'\alpha'$ is taken as -1 ( This has been suggested in [10] ). Though 
the system is stable, it can be observed from Fig.3.23 that the magnitude of the 
oscillations is larger when compared to the other nominal case discussed earlier with 
$\alpha = 0.34$. Though the chosen outputs reaches the required terminal values (Fig.3.22), 
this case is found to be less efficient when compared to the case when $\alpha = 0.34$. 

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Figure 3.19: Nominal payload with $\alpha = 0.39$; Chosen outputs

Figure 3.20: Nominal payload with $\alpha = 0.39$; Actual end effector outputs
Figure 3.21: Nominal payload with $\alpha = 0.39$; Elastic deflections

Figure 3.22: Nominal payload with $\alpha = -1$; Chosen outputs
3.6 Conclusion

Trajectory control of the tip position of an elastic manipulator based on nonlinear inversion and linear stabilization was considered. It was seen that exact tip trajectory tracking lead to unstable zero dynamics in the closed-loop system. This instability was avoided by controlling a point on the beam at a close vicinity of the actual tip point. Simulation results showed that our choice was valid. The accurate control of the actual tip positions were accomplished by designing an inverse controller for the point in the near vicinity of the actual tip position. A stabilizer was designed to suppress the oscillatory responses of the zero dynamics that result from the use of the inverse control law. The control law presented here is found to be extremely robust.
Chapter 4

VARIABLE STRUCTURE SYSTEM CONTROL SCHEME

4.1 Introduction

In this scheme, the robotic manipulator is controlled using a sliding controller based on the principles of Variable Structure Systems theory. For the class of systems to which it applies, the sliding controller design provides a systematic approach to the problem of maintaining stability and consistent performance in the face of modeling imprecisions. Variable Structure System theory involves the choice of a hypersurface in the state space. A discontinuous control law is derived which switches when the system trajectories cross this chosen hypersurface. This surface is commonly called as the 'sliding surface'.

Basically, the motion in the closed-loop system consists of two stages; namely the 'reaching mode' and the 'sliding mode'. The reaching mode constitutes the phase of operation when the trajectory reaches the defined switching surface from an arbitrary initial state. Once on the switching surface, the system trajectories are confined to this surface. Any subsequent motion of the trajectories involve the 'sliding' of the trajectories on this surface. This phase is called the sliding mode.
In the sliding mode, with the system trajectories already on the switching surface, any further motion of the trajectories corresponds to the sliding of the trajectories along the switching surface.

In our case, the sliding surface is chosen as a function of the output trajectory error, its derivative and the integral of the output trajectory error. The integral term is useful to improve the performance of the system. A discontinuous control law is derived in order to accomplish the tracking of the desired output trajectory. This control law excites the rigid and elastic modes. In the terminal phase, the oscillations that result from these modes are small and can be approximated to be linear. Thus the closed-loop system follows linear characteristics. This implies that a linear stabilizer is sufficient to dampen the elastic oscillations and capture the final state. A linear stabilizer is designed using the pole assignment technique.

Basically, the control is accomplished in two stages. In the first stage, the linear stabilizer loop is left open while the control input ‘u’ acts on the system. It is seen that bounded rigid and elastic mode oscillations are present. The output follows the required smooth trajectory. The second stage involves the switching on of the stabilizer as soon as the output reaches a specified neighbourhood of the terminal state. Therefore the control input now includes a stabilization signal and the total control signal contribute to the capture of the terminal state and to dampen the elastic oscillations. Evidently, it will be seen later that the magnitude of the stabilization signal must be small so that the tracking ability of the Variable Structure controller is not adversely affected.

An elaborate discussion of the Variable Structure controller along with the linear
stabilizer design is presented in the following sections of this chapter.

4.2 Choosing the Hypersurface

As discussed earlier, this control scheme involves the derivation of a discontinuous control law that switches when the desired output trajectories cross a hypersurface in the state space. First the system trajectories have to be attracted towards this surface and subsequent motion of the trajectories involves the sliding of the trajectories along this surface.

We choose a switching surface 'S' that is a function of the output trajectory error, its derivative and the integral of the output trajectory error.

\[ S(E_s, W_s) = \dot{\gamma} + 2\zeta\omega_n e + \omega_n^2 W_s \]  \hspace{1cm} (4.1)

where

'\dot{\gamma}' is the output trajectory error

\[ E_s = (\dot{\gamma}^T, \ddot{\gamma}^T)^T \]

\[ S = (s_1, s_2)^T, \zeta > 0, \omega_n > 0. \]

'\zeta' is the damping coefficient

'\omega_n' is the natural frequency

\[ W_s = (W_{s1}, W_{s2})^T \] is the integral of the tracking error, implying

\[ \dot{W}_s = \ddot{\gamma} \]  \hspace{1cm} (4.2)
4.3 Analysis of the Sliding phase

As mentioned earlier, once the system trajectories reach the switching surface, subsequent motion of the trajectories involve the sliding of the trajectories on the surface.

Therefore it can be seen that \( S(E_s, W_s) \equiv 0. \)

Differentiating (4.1) and using (4.2), we have

\[
\dot{S} = \ddot{y} + 2\zeta_n \omega_n \dot{y} + \omega_n^2 y = 0 \tag{4.3}
\]

It is worth noting two important results from the above equation. During the sliding phase, the system is asymptotically stable implying that \( \dot{y}(t) \to 0 \) as \( t \to \infty. \) Furthermore, it can be seen that once in the sliding phase, the motion of the system trajectories is insensitive to parameter uncertainty.

4.4 Analysis of the Reaching phase

In this phase, the system trajectories are to be attracted towards the switching surface which implies that one has to derive a suitable law that governs the control input.

As given in (2.6), for small elastic deformations, the output simplifies to

\[
y_1(t) = \theta_1 + \alpha\frac{\delta_1(l_1, t)}{l_1}
\]

\[
y_2(t) = \theta_2 + \alpha\frac{\delta_2(l_2, t)}{l_2}
\]

where the output \( y(t) = [y_1(t), y_2(t)]^T \) follows the given reference trajectory \( y_c(t) = [y_{c1}(t), y_{c2}(t)]^T. \) ‘\( \alpha \)’ in this case is set at 0.34.

At this point, we refer back to Chapter 2. Using the Lagrangian method, the equations of motion can be expressed by
\[ \frac{d}{dt} \left( \frac{\partial K}{\partial \dot{z}_i} \right) - \frac{\partial K}{\partial z_i} + \frac{\partial P}{\partial z_i} = B_1 u \]  

(4.4)

where,

'K' is the kinetic energy of the arm

'P' is the potential energy of the arm

'z' is the column vector containing the generalized co-ordinates

'u' is the column vector containing the joint torques

Further we have,

(an upper subscript 'T' denotes matrix transposition)

\[ z = (\theta_1, \theta_2, q_{11}, ..., q_{1n}, q_{21}, ..., q_{2n})^T \in R^{n_0}, \quad n_0 = (2n + 2). \]

\[ u = (u_1, u_2)^T \in R^2 \]

\[ \theta = (\theta_1, \theta_2)^T \]

\[ B_1 = [I_{2 \times 2} \quad O_{2 \times 2n}]^T \] where I and O denote identity and null matrices of indicated dimensions.

\[ K = (\dot{z}^T M(z) \dot{z})/2 \]

where \( M(z) \) is a positive definite symmetric inertia matrix of dimension \( (n_0 \times n_0) \)

and is a nonlinear function of \( z \).

From (2.2), one obtains

\[ M(z) \ddot{z} + h_0(z, \dot{z}) + \frac{\partial P(z)}{\partial z} = B_1 u \]  

(4.5)

where

\[ h_0(z, \dot{z}) = \dot{M}(z) \dot{z} - \frac{1}{2} \frac{\partial (\dot{z}^T M(z) \dot{z})}{\partial z} \]

Let \( h(z, \dot{z}) = -h_0(z, \dot{z}) - \frac{\partial P(z)}{\partial z} \)
\[ M(z) = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \]

Here \( M_{11} \) is a \((2 \times 2)\) matrix.

If the state vector is defined as \( x = (z^T, \dot{z}^T)^T \in \mathbb{R}^{2n_0} \), then (2.2) can be expressed in the following state variable form.

\[
\dot{x} = A(x) + B(x)u \quad (4.6)
\]

where

\[
A(x) = \begin{pmatrix} \dot{z} \\ M^{-1}(z)[h(z, \dot{z})] \end{pmatrix}
\]

\[
B(x) = [O_{2\times n_0}, (M^{-1}(z)B_1)^T]^T
\]

Let \( D(z) = M^{-1}(z) \) and \( D(z) = [D_1 \ D_2] \), where \( D_1 \) has two columns.

In the reaching phase, the system trajectories are to be attracted towards the switching surface and one has to derive a suitable law that governs control input. To this end, it is assumed that

\[
D(z) = D^*(z) + \Delta D(z)
\]

\[
D_i(z) = D_i^*(z) + \Delta D_i(z)
\]

\[
h(z, \dot{z}) = h^*(z, \dot{z}) + \Delta h(z, \dot{z})
\]

\[
D(z)h(z, \dot{z}) = D^*(z)h^*(z, \dot{z}) + \Delta F(z, \dot{z})
\]

where \( \Delta F(z, \dot{z}) = D^*(z)\Delta h(z, \dot{z}) + D(z)\Delta h(z, \dot{z}) \);

\( D^*(z), h^*(z, \dot{z}), D_i^*(z) \) are known functions and \( \Delta D(z), \Delta D_i(z) \) and \( \Delta h(z, \dot{z}) \) represent the uncertainty in the robot arm dynamics.
Choosing \( q = [q_{11}, q_{12}, q_{21}, q_{22}]^T \) and \( \theta = [\theta_1, \theta_2]^T \), we have a nonlinear function of \( z \).

\[
y = \theta + Eq \triangleq C_0 z = [I_{2 \times 2} E] z 
\]  

where

\[
E = \begin{pmatrix} \frac{\alpha \phi_1(l_1)}{l_1} & \frac{\alpha \phi_2(l_1)}{l_1} \\ \frac{\alpha \phi_1(l_2)}{l_2} & \frac{\alpha \phi_2(l_2)}{l_2} \end{pmatrix}
\]

Subsequent differentiation of the output vector (4.7) yields

\[
\ddot{y} = C_0 [D(z)h(z, \dot{z}) + B_1 u] 
\]  

which implies

\[
\ddot{y} = C_0 [D^*(z)h^*(z, \dot{z}) + \Delta F(z, \dot{z}) + (D_1^*(z) + \Delta D_1(z))u] 
\]

If \( y \) is the actual trajectory and \( y_c \) is the reference trajectory, then the trajectory error \( \ddot{y} \) is given by \( y - y_c \).

Using this in (4.3), we obtain

\[
\dot{S} = \ddot{y} - \dot{y}_c + 2\zeta \omega _{ne} \dot{y} + \omega _{ne}^2 \ddot{y} 
\]  

Substituting for \( \ddot{y} \) from (4.8) in (4.3), we have

\[
\dot{S} = C_0 [D^*(z)h^*(z, \dot{z}) + \Delta F + (D_1^*(z) + \Delta D_1(z))u] - \ddot{y}_c + 2\zeta \omega _{ne} \dot{y} + \omega _{ne}^2 \ddot{y} 
\]  

which results in
\[ \dot{S} \triangleq C^*(x) + \Delta C(x) + (B^*(x) + \Delta B(x))u \]

where
\[
x = (z^T, \dot{z}^T)^T,
\]
\[
\Delta C(x) = C_0\Delta F(z, \dot{z}), \quad B^*(x) = C_0D_1^*(z)
\]
\[
\Delta B(x) = C_0\Delta D_1(z)
\]
\[
C^*(x, t) = C_0D^*(z)h^*(z, \dot{z}) - \gamma_c + 2\zeta_\omega \omega_\omega \dot{\gamma} + \omega_\omega^2 \gamma
\]

We now choose a control law of the form
\[
u = (B^*)^{-1}(x)[-C^*(x) - k\text{sgn}(S) + v] \quad (4.11)
\]
where gain \( k > 0 \) is determined later, \( v \) is the stabilization signal and
\[
\text{sgn}\{S\} = [\text{sgn}(s_1), \text{sgn}(s_2)]^T
\]
Substituting (4.11) in (4.10) gives
\[
\dot{S} = \Delta C(x) + \Delta B(x)u(x, t) - k\text{sgn}(S)
\]
When there is no uncertainty in the system, \( \Delta C(x) = 0 \) and \( \Delta B(x) = 0 \).

Therefore, we have
\[
\dot{S} = -k\{\text{sgn}(S)\} + v \quad (4.12)
\]
\[
\text{sgn}\{S_i\} = \begin{cases} 
1, & s_i > 0 \\
0, & s_i = 0 \\
-1, & s_i < 0 
\end{cases}
\]
45
Thus when \( v=0, \dot{S} < 0 \) if \( S \neq 0 \) and the trajectory reaches the surface \( S=0 \) in finite time. We can also show that the surface is reached even in the presence of uncertainty by suitably choosing the gain ‘k’ using Lyaponov theory. It has been suggested in [21] that a Lyaponov function can be chosen of the form

\[
V(s) = |s_1| + |s_2|
\]

which can be alternately expressed as

\[
V(s) = \begin{bmatrix} Sgn(S_1) & Sgn(S_2) \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}
\]

For the design of the VSC law, we assume that the stabilization signal is zero. Therefore with \( v = 0 \), one has

\[
\dot{V}(s) = [Sgn(S)]^T[-kSgn(S) + \Delta C + \Delta Bu]
\]

To make \( \dot{V} < 0 \) in the presence of uncertainty, one needs certain bounds on the uncertain functions.

**ASSUMPTION 1 :** There exists functions \( \gamma_0, \gamma_1(x) \) and \( \gamma_2(x) \) such that

\[
\| \Delta B(x)(B^*)^{-1}(x) \| \leq \gamma_1(x) < \gamma_0 < 1
\]

\[
\| \Delta C(x) - \Delta B(x)(B^*)^{-1}(x)C^*(x) \| \leq \gamma_2(x, t)
\]

We choose ‘ \( k \) ’ such that

\[
k \geq (1 - \gamma_1(x))^{-1}(\delta + \gamma_2(x, t))
\]

where ‘\( \delta \)’ is a positive real number.

**THEOREM :** Consider the closed-loop system. For a given reference trajectory, the solution \( x(t) \) with given initial conditions is assumed to be bounded. Therefore
in the closed-loop system, $S$ converges to 0 in finite time and continues to remain in that state thereafter.

From [22], for all $S \neq 0$ and $t \in [0, \infty)$

$$\dot{V}(t) \leq -\delta$$

The theorem can be proved by invoking the results of Lyaponov stability.

Furthermore, we have

$$\dot{V} = Sgn(S)^T[-kSgn(S) + v]$$

$$= -k[(Sgn(S_1))^2 + (Sgn(S_2))^2] + vSgn(S)$$

For the reaching phase; before the stabilizer is switched on, $v = 0$

Therefore, we have

$$\dot{V} = -k[(Sgn(S_1))^2 + (Sgn(S_2))^2]$$

which implies that $\dot{V} < 0$ if $S \neq 0$.

This also shows that the system trajectories can reach the switching surface from an arbitrary initial state. In the above analysis, we have not considered parameter uncertainty, in which case, additional nonlinear terms appear in the above equation for $\dot{V}$. However, it can be shown that by choosing the gain ‘$k$’ to be large enough, one can still make $\dot{V} < 0$ when $S \neq 0$.

Obviously, the value of ‘$k$’ must be chosen such that the trajectories should reach the switching surface ‘$S$’ even in presence of uncertainty. Although analytical bounds similar to [19] can be obtained, its computation is extremely difficult. In our case, the value of ‘$k$’ is chosen by observation of the simulated responses.
4.5 Stabilizer Design

4.5.1 A General Overview

As discussed earlier, the design of a stabilizer is inevitable for the damping of the elastic oscillations and capture of the terminal state. In the neighbourhood of the terminal state, since the rigid and elastic modes result in small oscillations, the system can be well approximated by linear differential equations. Therefore, a linear stabilizer is found to be adequate.

The linear stabilizer is designed using the pole assignment technique. Fairly good responses are obtained by a proper selection of poles for the closed-loop system. Initially, the stabilizer loop is left open implying that only the control input is applied to the system. When the system trajectories reach the specified neighbourhood of the terminal values, the linear stabilizer loop is closed for the capture of the terminal state and to dampen the oscillations. Thereby the stabilization signal is superimposed on the control signal. In order to make sure that the stabilization signal does not adversely affect the tracking ability of the control law, the magnitude of the stabilization signal is kept as small as possible while efficient damping is still accomplished.

4.5.2 Defining the State Vector and a Quantitative Review of the Model

Let $y^*$ be the desired terminal output vector. Noting that $y = \theta + Eq$, one then has $y^* = \theta^* + Eq^*$ where $q = [q_{11} q_{12} q_{21} q_{22}]^T$, $\theta = [\theta_1 \theta_2]^T$ and $z^* = [\theta^* q^*]^T$.

The equilibrium value of $z^*$ is the solution of the following equation

$$\frac{\partial P(z^*)}{\partial q} = 0$$

(4.14)
Let $\Delta q = q - q^*$. 

We now define the deviation in the state vector from the equilibrium point as the vector $\xi$, where $\xi = [y^T \; S^T \; \Delta q^T \; \Delta q^T \; W^T]$ and 

$$\Delta y = \Delta \theta + E \Delta q$$  \hspace{1cm} (4.15)

Neglecting the second order terms in the velocity of the generalized co-ordinates, the linearized $q$-responses are given by

$$M_{21}(q^*) \Delta \hat{\theta} + M_{22}(\theta^*) \Delta \ddot{q} + P_{\dot{q}q}(\theta^*) \Delta \dot{\theta} + P_{qq} \Delta q = 0$$  \hspace{1cm} (4.16)

where 

$$P_{\dot{q}q} = \frac{\partial^2 P}{\partial \dot{\theta} \partial q}$$

$$P_{qq} = \frac{\partial^2 P}{\partial q \partial q}$$

where, the inertia matrix $M$ is partitioned as

$$M(z) = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

Here $M_{11}$ is a $(2 \times 2)$ matrix.

Substituting for $\Delta \hat{\theta}$ in (4.16) in terms of $\Delta \ddot{q}$ and $\Delta \ddot{y}$ (from (4.15)), we have

$$M_{21}^* \Delta \ddot{y} + (M_{22}^* - M_{21}^* E) \Delta \ddot{q} + (P_{\dot{q}q} - P_{\dot{q}q}^* E) \Delta \dot{q} + P_{\dot{q}q}^* \Delta y = 0$$  \hspace{1cm} (4.17)

Here $M_{ij}^*$ and $P_{\dot{q}q}^*$ denote matrices computed at $z = z^*$. 

4.5.3 Elimination of Chattering

The control law we derive is discontinuous across the switching surface $S \equiv 0$. Theoretically, the switching of the control must be instantaneous but in actual practice, this
switching is not instantaneous. This imperfection in the control switching gives rise to the 'chattering' of the trajectory, which is an undesirable phenomenon. The control law discussed earlier is slightly modified to counter this adverse effect of chattering. The Sgn(S) function is replaced by the Sat(S) function, where, the Sat(S) function is defined as follows:

\[ sat\{s_i\} = \begin{cases} 
1 & s_i > \epsilon_1 \\
\frac{s_i}{\epsilon_1} & |s_i| \leq \epsilon_1 \\
-1 & s_i < -\epsilon_1 
\end{cases} \]

Therefore, to prevent chattering, (4.12) is modified as

\[ \dot{S} = -\left(\frac{k S}{\epsilon_1}\right) + v \]  

(4.18)

where 'v' is the stabilization signal. Obviously, v = 0 when the stabilizer is not used.

We know that \( \Delta y = \ddot{y} = y - y_c \)

From (4.1), we have

\[ \Delta \dot{y} = \dot{\ddot{y}} = S - 2\zeta \omega_n \ddot{y} - \omega_n^2 W_s \]  

(4.19)

And from (4.3),

\[ \Delta \ddot{y} = \dddot{y} = \ddot{S} - 2\zeta \omega_n \dot{\ddot{y}} - \omega_n^2 \ddot{y} \]  

(4.20)

4.5.4 State Variable Representation of the System

Using (4.19) in (4.20) and subsequent substitution in (4.17) yields an expression for \( \Delta \ddot{q} \) in terms of the elements of 'ξ'. The resulting expression for \( \Delta \ddot{q} \) along with (4.18), (4.19) and (4.2) results in the following state variable representation of the system.
\[
\begin{bmatrix}
A_{11} & I & 0 & 0 & A_{15} \\
0 & A_{22} & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 \\
F_1 & F_2 & F_3 & 0 & F_4 \\
I & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
\xi \\
\end{bmatrix}
+ \begin{bmatrix}
0 \\
I \\
0 \\
0 \\
0 \\
\end{bmatrix}
v
= \begin{bmatrix}
A \xi + Bv \\
\end{bmatrix}
\]

\[
\triangleq A \xi + Bv
\]

where

\[
F_1 = \tilde{M}_{22}^{-1}[M_{21}(q^*)(b_1^2 - \omega_{ne}^2) - P_{q}(q^*)]
\]

\[
\tilde{M}_{22} = M_{22}^{*2} - M_{21}^{*1}E
\]

\[
b_1 = 2\zeta_\omega \omega_{ne}
\]

\[
F_2 = \tilde{M}_{22}^{-1}M_{21}(q^*)(\frac{1}{\zeta} + b_1)
\]

\[
F_3 = \tilde{M}_{22}^{-1}\tilde{P}_{qq}
\]

\[
\tilde{P}_{qq} = P_{qq}^*E - P_{qq}
\]

\[
F_4 = -\tilde{M}_{22}^{-1}M_{21}(q^*)b_1\omega_{ne}^2
\]

\[
B_4 = -\tilde{M}_{22}^{-1}M_{21}(q^*)
\]

\[
A_{11} = \begin{bmatrix}
-b_1 & 0 \\
0 & -b_1 \\
\end{bmatrix}
\]

\[
A_{15} = \begin{bmatrix}
-\omega_{ne}^2 & 0 \\
0 & -\omega_{ne}^2 \\
\end{bmatrix}
\]

\[
A_{22} = \begin{bmatrix}
-\frac{K}{\zeta} & 0 \\
0 & -\frac{K}{\zeta} \\
\end{bmatrix}
\]

### 4.5.5 Design of the Linear Stabilizer

A stabilizer is then designed using the pole assignment technique. A proper selection of poles of the closed-loop system is essential for obtaining good responses. It is worth noting that once the stabilizing signal is superimposed on the control signal, the tracking ability of the control law is affected. All the same, the stabilizing signal
is required to dampen the elastic oscillations and its magnitude must be small such
that the tracking ability of the inverse controller is not adversely affected. But it
should be of sufficient magnitude so that fast damping of the elastic oscillations can
be achieved.

A set of desirable eigen values of the closed-loop system is taken as $S_{cl} = S_r U S_c$
where

$$S_c = \{ \mu : \mu = -c + \text{Im}(\lambda), \lambda \in S_c, c > 0 \}$$

In the set $S_{cl}$, the eigenvalues associated with the $y$-responses are retained, but
$\lambda_i \in S_c$ are shifted to the left by ‘c’ units in the complex plane. The control law for
the stabilization is of the form

$$v = -L \xi,$$  \hspace{1cm} (4.22)

such that the closed-loop system matrix $[F_{cl}] = [\tilde{A} - \tilde{B} L]$ has the desirable eigenvalues.

The synthesis of the controller is as follows. First, the variable structure controller
is used for large maneuvers of the arm. When the trajectory reaches the neighbour-
hood of the terminal value, the linear stabilizer loop is closed for the capture of the
terminal state and to dampen the vibrations. The stabilizer is effective if the
$\theta$ and
$q$ - responses remain in a small neighbourhood of the equilibrium state at the instant
when the stabilizer is switched. Apparently, $\alpha > \alpha^*$ can also be chosen as long as the
zero dynamics is lightly unstable and the trajectory remains close to the equilibrium
point at the instant of switching of the stabilizer.
4.6 Simulation Results

4.6.1 Introduction to the Digital Simulations

This section explores the results of the digital simulations. The parameters assume the nominal values that are listed in the appendix. The mode shapes \( \phi_{ij} \) are selected as clamped-free modes [1]. Also, the following initial conditions are assumed.

\[
y_c(0) = \ddot{y}_c(0) = \dddot{y}_c(0) = 0, \quad x(0) = 0 \text{ and } w(0) = 0.
\]

A command trajectory \( y_c(t) \) was generated to control \( y(0) = 0 \) to \( y^* \). It was assumed that the given tip position corresponds to \( \theta^* = (90^\circ, 60^\circ)^T \). The terminal value \( y^{**} \) was set to \( y^* = \theta^* + \alpha q^* \) with \( \alpha = 0.34 \).

Let \( y_t = (y_{t1}, y_{t2})^T \) denote the angular positions of the tips of the two links and \( \theta = (\theta_1, \theta_2)^T \), then we have

\[
y_t = \theta + \left( \frac{E q(t)}{\alpha} \right)
\]

\[
= \theta + \left( \frac{D_{1e}}{l_1} \frac{D_{2e}}{l_2} \right)^T.
\]

where \( 'D_{1e}' \) and \( 'D_{2e}' \) are the tip elastic deflections for link 1 and link 2 respectively.

The matrices \( P_i \) of the command generator are taken as \( P_i = p_i I_{2x2}, \ i=0,1,2 \) and are selected such that the poles associated with \( y_{ci}(t) \), the \( i^{th} \) component of \( y_c(t) \), are at \( \{-2, -2 \pm i2\} \). The feedback matrices \( G_i \) are selected and are set to yield poles associate with \( \tilde{y}_i \) in (2.4), of values \( -333.33, -2.47 \pm i 2.48 \} \), where

\[
\tilde{y} = (\tilde{y}_1, \tilde{y}_2)^T.
\]

These poles are chosen to obtain fast tracking error responses. For the chosen
feedback gains, the sets $S_r$ and $S_e$ are

$$S_r = [-333.33, -333.33, -2.47 \pm 2.48i, -2.47 \pm 2.48i]$$

$$S_e = [\pm21.15i, \pm23.77i, \pm108.22i, \pm228.89i]$$

The feedback matrix ‘$L$’ of the stabilizer was chosen such that the set of eigenvalues of the matrix ‘$S_{cl}$’ is

$$S_{cl} = S_r \cup S_e$$

where $S_e = \{-2.5 + Im(\lambda), \lambda \in S_e\}$ and $Im(\lambda)$ denote the imaginary part of $\lambda$. Notice that in the closed-loop system the set of eigenvalues of $S_r$ is retained, and the elements of $S_e$ are simply moved to the left by nearly 2.5 units in the complex plane.

In the simulations that follow, $y_1$ and $y_2$ denote the chosen outputs; $e_1 = \dot{y}_1$ and $e_2 = \dot{y}_2$ are the trajectory errors; $\theta_1$ and $\theta_2$ are the joint angles; $u_1$ and $u_2$ denote the control inputs; $D_{1e}$ and $D_{2e}$ are the tip deflections for the two links and $y_{1t}$ and $y_{2t}$ give actual end effector positions. It should be noted here that the outputs $y_1$ and $y_2$ are those that actually correspond to the point on the beam we have chosen to control (a point close to the actual tip). The simulation results show that these outputs follow the tip positions $y_{1t}$ and $y_{2t}$, thus validating our choice.

4.6.2 Trajectory tracking: Control without stabilization

In this case, simulation was carried out for nominal payload and the stabilizer loop was left open. Digital simulations were performed using (4.2), (4.6) and (4.11) with the stabilizer loop left open. The value of the constant factor ‘$a$’ is chosen to be 0.34.
Figure 4.1: Trajectory tracking without stabilization; Chosen outputs

The trajectory errors were found to be equal to zero as predicted and efficient tracking is evident from Fig.4.1 and Fig.4.2. The absence of the stabilizer leads to persistent elastic mode oscillations of the tips of the two links (Fig.4.3). The $\theta$ response (Fig.4.2) depict oscillations of extremely small magnitudes.

Simulation was also done to show that severe instability of the zero dynamics when the outputs are chosen close to the actual tip angular positions i.e. $\alpha^* \leq \alpha \leq 1$. Divergent responses were obtained and Fig.4.4 illustrates the divergence of the elastic oscillations for each link with the stabilizer loop left open for $\alpha=0.39$ ($\alpha > \alpha^*$).
Figure 4.2: Trajectory tracking without stabilization; Joint angles

Figure 4.3: Trajectory tracking without stabilization; Elastic deflections
4.6.3 Trajectory tracking: Control with Stabilization

Nominal payload

The closed-loop system defined by (4.2), (4.6), (4.11) and (4.22) were digitally simulated (with $\alpha = 0.34$) and the switching logic closes the stabilizer loop when the trajectory enters the neighbourhood of the terminal value (in about 3 seconds). It can be observed from Fig.4.6 that though the tracking error was zero prior to switching of the stabilizer, the error tends to increase at and after the instant of switching but dies down to zero in about 4 seconds. The stabilizer dampens out the elastic mode oscillations (Fig.4.9) and forces the control inputs to a constant value as illustrated in Fig.4.8. The system quickly settles down to its steady state values. It can also be noted that the chosen outputs $y_1$ and $y_2$ (Fig.4.5) very closely follow the actual end
Figure 4.5: Nominal payload with stabilization; Chosen outputs effector outputs $y_{11}$ and $y_{12}$ (Fig.4.10).

Higher payload uncertainty

Simulations were carried for a higher payload uncertainty of 125% ($\alpha = 0.34$). The controller used here is the one that was designed with nominal parameters and the initial conditions were assumed to be zero. In comparison to the nominal payload case, it can be observed that the stabilizer takes a longer time to dampen the oscillations (Fig.4.9 and Fig.4.13) and therefore the system takes a longer time to settle down to its steady state values. It can also be noted that the control torque required in this case (Fig.4.12) is larger than the control torque required for the nominal case (Fig.4.8) and the total elastic mode deflection in this case (Fig.4.13) is more when compared to the nominal payload case (Fig.4.9). Moreover, the chosen outputs $y_1$ and
Figure 4.6: Nominal payload with stabilization; Tracking error

Figure 4.7: Nominal payload with stabilization; Joint angles
Figure 4.8: Nominal payload with stabilization; Control inputs

Figure 4.9: Nominal payload with stabilization; Elastic deflections
Figure 4.10: Nominal payload with stabilization; Actual end effector outputs $y_1$ (Fig.4.11) very closely follow the actual end effector outputs $y_{t1}$ and $y_{t2}$ (Fig.4.14).

The controller was found to tolerate a higher payload uncertainty when compared to the inverse controller of the first scheme. Simulations showed that the controller tolerated an uncertainty of over 150%. With $\alpha = 0$, the controller is similar to the one presented in [19], which resulted in an unstable closed-loop system for this payload uncertainty.

Lower payload uncertainty

Simulation results for a lower payload uncertainty of 100% ($\alpha = 0.34$) are presented in this section. When compared to the higher payload uncertainty case, it can be noted that though the time taken by the system to attain steady state conditions is more or less the same, the frequency of the oscillations in the lower payload case
Figure 4.11: Higher payload uncertainty; Chosen outputs

Figure 4.12: Higher payload uncertainty; Control inputs
Figure 4.13: Higher payload uncertainty; Elastic deflections

Figure 4.14: Higher payload uncertainty; Actual end effector outputs
is much higher (Fig.4.13 and Fig.4.17), which is also a physical reality. Moreover, the values of the control input are lower than the higher payload case (Fig.4.12 and Fig.4.16) and so is the elastic deflections (Fig.4.13 and Fig.4.17). Also, the chosen outputs $y_1$ and $y_2$ (Fig.4.15) very closely follow the actual end effector outputs $y_{11}$ and $y_{12}$ (Fig.4.18).

When compared to the previous scheme, this controller was found to tolerate a wider lower payload uncertainty. For $\alpha = 0$, the controller is similar to the one presented in [19] which could not tolerate this payload uncertainty.

Nominal payload with stabilization; $\alpha = -1$

Here the value of $\alpha$ is taken as -1 (This has been suggested in [10]). Though the system is stable, the magnitude of the oscillations (Fig.4.20) is larger when compared
Figure 4.16: Lower payload uncertainty ; Control inputs

Figure 4.17: Lower payload uncertainty ; Elastic deflections
Figure 4.18: Lower payload uncertainty; Actual end effector outputs to the other nominal cases already discussed and is therefore less efficient. But it can be seen that the chosen outputs $y_1$ and $y_2$ (Fig. 4.19) very closely follow the actual end effector outputs $y_{t1}$ and $y_{t2}$ (Fig. 4.21). This case is similar to the case presented in [10].
Figure 4.19: Nominal payload with $\alpha = -1$; Chosen outputs

Figure 4.20: Nominal payload with $\alpha = -1$; Elastic deflections
Figure 4.21: Nominal payload with $\alpha = -1$; Actual end effector outputs

### 4.7 Conclusion

This Chapter explored the control of a flexible robotic arm using Variable Structure System theory. An appropriate switching surface was chosen and the reaching and sliding modes were rigorously analysed. A suitable control law was formulated and the chosen system trajectories were effectively controlled.

A linear stabilizer was designed using the pole assignment technique. The stabilizer loop is closed as soon as the system trajectories enter a specified neighbourhood of the final state. The stabilizer dampens the rigid and elastic mode vibrations and captures the specified terminal state. Simulations were performed and the controller was found to tolerate a wide range of payload uncertainty.
Chapter 5

SUMMARY

This thesis presented two control schemes for the trajectory control of a two-link flexible robotic manipulator. The first scheme incorporated an Inverse controller while the controller of the second scheme was designed using Variable Structure System theory, which included robustness in its design. The primary aim of these control schemes was to control the actual tip positions of the links. This was however found to be impractical due to the unstability of the zero dynamics of the system. Therefore a constant parameter \( \alpha \) was defined which actually corresponded to particular points on the links. An output was chosen that depended directly on the multiplying factor \( \alpha \). A value of \( \alpha \) was chosen such that the zero dynamics of the system was stable. This actually lead to the control of a point on the link that was close to the actual tip position. Simulation results showed that the actual end effector trajectories closely followed the chosen outputs; thus validating our choice. Extensive simulations were performed in both the schemes. Though both controllers were found to tolerate a wide range of payload uncertainty, the Variable Structure controller was found to tolerate a wider range of payload uncertainty when compared to the Inverse controller.
Appendix A

Parameters of the Robotic Arm

Listed below are the assumed parameter values for the two-link elastic robotic arm for which the digital simulations were carried out: \( \alpha = 0.34 \)

- Mass of each Link = 5.0 \text{ kg}
- Stiffness of each link = 1000
- Length of each link = 1.0 \text{ m}
- Joint mass at Joint 2 = 1.0 \text{ kg}
- Nominal payload = 4.0 \text{ kg}
- Inertia of payload = 1.0 \text{ kgm}^2
- Inertia of mass at Joint 1 = 1.0 \text{ kgm}^2
- Inertia of mass at Joint 2 = 0.8 \text{ kgm}^2
Appendix B

Table 1

The following table shows only the most significant unstable roots from ‘$S_\varepsilon$’ for different values of the constant parameter ‘$\alpha$’:

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$0.415 \pm 23.038 \text{ i}$</th>
<th>$0.000 \pm 108.58 \text{ i}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha = 0.38$</td>
<td>$1.455 \pm 23.184 \text{ i}$</td>
<td>$3.141 \pm 25.169 \text{ i}$</td>
</tr>
<tr>
<td>$\alpha = 0.50$</td>
<td>$4.808 \pm 28.445 \text{ i}$</td>
<td>$4.306 \pm 53.819 \text{ i}$</td>
</tr>
<tr>
<td>$\alpha = 0.80$</td>
<td>$144.1 \pm 0.0000 \text{ i}$</td>
<td>$46.48 \pm 0.0000 \text{ i}$</td>
</tr>
<tr>
<td>$\alpha = 1.00$</td>
<td>$29.41 \pm 0.0000 \text{ i}$</td>
<td></td>
</tr>
</tbody>
</table>
Appendix C

Bibliography


