

Gaming Applications of a Forgotten Distribution

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Introduction

The bonus game has become a very important part of the slot machine, as indicated by a large number of patent applications and patents (Bennett, 2000; Slomiany and Grupp, 2000; Vancurra, 2000; Baerlocher et al, 2002; Jaffe et al, 2002; Cannon and Donovan, 2003; Dickerson, 2005). Therefore, probability distributions suitable for developing games are very much in need. In this study, we illustrate how a forgotten discrete probability distribution, viz. the negative hypergeometric distribution, can be used to develop bonus games for slot machines, and also to analyze existing bonus games. The mean and the variance for this distribution were not readily available. However, these measures were recently computed using a new technique (Jones, 2013) and are available at present. The mean of the probability distribution is needed to compute the house advantage or par of a slot game, and the variance is needed for calculating the variance of the slot game payout distribution. The variance of a slot game significantly impacts the ‘time on device’ that a player gets (Lucas et al, 2007; Lucas and Singh, 2008) from the slot game, and therefore is an important parameter of a slot game.

The Negative Hypergeometric Distribution

A finite population consists of N objects of which n are of a particular type. Objects are drawn without replacement from this population until k objects of the particular type are drawn, where $1 \leq k \leq n$. Let X be the number of the draw on which the k^{th} particular object is drawn. Then the distribution of X is the negative hypergeometric distribution with probability function (Jones, pp. 3-4):

$$p(x) = \frac{\binom{N-n}{x-k} \binom{n}{k-1}}{\binom{N}{x-1}} \frac{(n-k+1)}{(N-x+1)} = \frac{\binom{x-1}{k-1} \binom{N-x}{n-k}}{\binom{N}{n}}$$

The mean and the variance of X are given by (Jones, pp. 3-4):

$$E[X] = \mu_x = \frac{k(N+1)}{n+1}$$

$$\text{Var}(X) = \sigma_x^2 = \frac{k(N+1)(N-n)(n-k+1)}{(n+1)^2(n+2)}.$$

Gaming Applications

We present two different types of applications for bonus games.

1. In this type of bonus game, the player is presented with N covered locations (squares, bubbles, eggs, etc.), with each location containing either a prize or a ‘devil’ (joker). There are n locations with the ‘devil’. The player uncovers each location until the k^{th} ‘devil’ is uncovered. The player is then awarded the sum of all the prize money uncovered plus a consolation award for uncovering k ‘devils’. The latter award may be zero, a fixed award, or an award proportional to the remaining covered locations.

(a) The simplest possible game is one in which each of the non-devil locations has a fixed amount of A_1 and each of the ‘devil’ locations has an amount of A_0 . The player is awarded all the prize money uncovered, including the awards for the ‘devils’. For this game, let X be the total number of uncovered locations and let W be the total award for the game. Then W is given by:

$$W = A_0 k + A_1 (X - k)$$

Since A_0 and A_1 are constants,

$$E[W] = A_0 k + A_1 (E[X] - k) = A_1 \mu_X - (A_1 - A_0)k$$

$$E[W] = A_1 \mu_X - (A_1 - A_0)k$$

$$Var(W) = A_1^2 Var(X).$$

Example

In this example, there are 15 locations of which 3 are occupied by ‘devils’. The locations may be given by a matrix or 15 ‘eggs’ or ‘balloons’. Furthermore, let the fixed prize award be 25 and the consolation award for the ‘devil’ be 2. Therefore, we have:

$$N = 15, n = 3, A_0 = 2, \text{ and } A_1 = 25.$$

The mean and the variance of X are given by:

$$\mu_x = \frac{k(N+1)}{n+1} = \frac{k(16)}{4} = 4k$$

$$\begin{aligned}\sigma_x^2 &= \frac{k(N+1)(N-n)(n-k+1)}{(n+1)^2(n+2)} = \frac{k(16)(12)(4-k)}{(4)^2(5)} \\ &= \frac{12k(4-k)}{5}.\end{aligned}$$

The mean and the variance of the bonus game are given by:

$$E[W] = (25)(4k) - (25 - 2)k = 77k$$

$$Var(W) = (25)^2 \frac{12k(4-k)}{5} = 1500k(4-k)$$

The following table provides the mean, the variance, and the standard deviation for all possible values of k :

k	$E[W]$	$Var(W)$	σ_W
1	77	4500	67.1
2	154	6000	77.5
3	231	4500	67.1

(b) In this game, the (non-devil) prize awards are not fixed. Let these be: A_1, A_2, \dots, A_{N-n} . The total award for the game is given by:

$$W = A_0 k + V$$

where V is the sum of all the prizes that have been uncovered at the non-devil locations. The mean and the variance of V are not easily computed. However, it was shown recently (Jones, pp. 11-17) that these quantities have the following form:

$$E[V] = E[X - k] \bar{A}$$

$$Var(V) = \frac{k \sigma_X^2 \left(\overline{A^2} - (\bar{A})^2 \right)}{\mu_X} + (\bar{A})^2 \sigma_X^2$$

where

$$\bar{A} = \left[\sum_{i=1}^{N-n} A_i \right] / (N - n) \qquad \overline{A^2} = \left[\sum_{i=1}^{N-n} A_i^2 \right] / (N - n)$$

The mean and the variance of W are given by:

$$E[W] = A_0 k + E[V] = \bar{A} \mu_x - (\bar{A} - A_0) k$$

$$\begin{aligned} Var(W) &= Var(V) = \frac{k \sigma_x^2 \left(\overline{A^2} - (\bar{A})^2 \right)}{\mu_x} + (\bar{A})^2 \sigma_x^2 \\ &= \frac{\sigma_x^2}{\mu_x} \left[k \left(\overline{A^2} - (\bar{A})^2 \right) + \mu_x (\bar{A})^2 \right]. \end{aligned}$$

Example

In this example, there are 20 locations of which 4 are occupied by ‘devils’. The locations may be given by a 4x5 matrix or 20 ‘eggs’ or ‘balloons’. Furthermore let there be 7 prizes of 10, 2 prizes of 15, 4 prizes of 25, 2 prizes of 50, and 1 prize of 100 along with a consolation award of 2 for each of the ‘devils’. For this case, we have:

$$N = 20, n = 4, A_0 = 2, A_1, \dots, A_7 = 10, A_8 = A_9 = 15, A_{10}, \dots, A_{13} = 25, A_{14} = A_{15} = 50, A_{16} = 100.$$

The mean and the variance of X are given by:

$$\mu_X = \frac{k(N+1)}{n+1} = \frac{k(21)}{5} = \frac{21k}{5}$$

$$\begin{aligned}\sigma_X^2 &= \frac{k(N+1)(N-n)(n-k+1)}{(n+1)^2(n+2)} = \frac{k(21)(16)(5-k)}{(5)^2(6)} \\ &= \frac{56k(5-k)}{25}.\end{aligned}$$

The mean and the variance of the bonus game are given by:

$$\begin{aligned} E[W] &= 25 \left(\frac{21k}{5} \right) - (25 - 2)k \\ &= 82k \end{aligned}$$

$$\begin{aligned} Var(W) &= \frac{56k(5-k)}{25(21k/5)} \left[k(1165.6 - 625) + (21k/5)(625) \right] \\ &= 1688k(5-k). \end{aligned}$$

The following table provides the mean, the variance, and the standard deviation for all possible values of k :

k	$E[W]$	$Var(W)$	σ_W
1	82	6752	82.2
2	164	10128	100.6
3	246	10128	100.6
4	328	6752	82.2

(c) In this game, the (non-devil) prize awards are not fixed. Let these be: A_1, A_2, \dots, A_{N-n} . The player is awarded the non-devil prizes that were uncovered along with an award of A_0 for each of the remaining ('devil' plus covered) locations. The total award for the game is given by:

$$W = A_0(N - X) + V$$

where V is the sum of all the prizes that have been uncovered at the non-devil locations. As in (b), the mean and variance of V are given by:

$$E[V] = E[X - k] \bar{A}$$

$$Var(V) = \frac{k\sigma_x^2 \left(\overline{A^2} - (\bar{A})^2 \right)}{\mu_x} + (\bar{A})^2 \sigma_x^2$$

where

$$\bar{A} = \left[\sum_{i=1}^{N-n} A_i \right] / (N - n) \quad \quad \overline{A^2} = \left[\sum_{i=1}^{N-n} A_i^2 \right] / (N - n).$$

The mean and the variance of W are given by:

$$\begin{aligned} E[W] &= A_0 \{N - E[X]\} + E[V] \\ &= (\bar{A} - A_0) \mu_X - (k\bar{A} - NA_0) \end{aligned}$$

$$\begin{aligned} Var(W) &= Var(V - A_0X + A_0N) = Var(V - A_0X) \\ &= Var(V) + A_0^2 Var(X) - 2A_0 Cov(V, X) \\ &= \frac{\sigma_X^2}{\mu_X} \left[k \left(\overline{A^2} - (\bar{A})^2 \right) + \mu_X \left((\bar{A})^2 + A_0^2 \right) \right] - 2A_0 Cov(V, X) \end{aligned}$$

The covariance term in the last expression is rather complicated and in most cases could be omitted due to its relative size when compared to the other terms. The actual formula for it will be given in a later study.

Example

In this example, there are 20 locations of which 4 are occupied by ‘devils’. The locations may be given by a 4x5 matrix or 20 ‘eggs’ or ‘balloons’. Furthermore let there be 7 prizes of 10, 2 prizes of 15, 4 prizes of 25, 2 prizes of 50, and 1 prize of 100 along with a consolation award of 2 for each of the ‘devils’. For this case, we have:

$$N = 20, n = 4, A_0 = 2, A_1, \dots, A_7 = 10, A_8 = A_9 = 15, A_{10}, \dots, A_{13} = 25, A_{14} = A_{15} = 50, A_{16} = 100.$$

The mean and the variance of X are given by:

$$\mu_x = \frac{k(N+1)}{n+1} = \frac{k(21)}{5} = \frac{21k}{5}$$

$$\begin{aligned}\sigma_x^2 &= \frac{k(N+1)(N-n)(n-k+1)}{(n+1)^2(n+2)} = \frac{k(21)(16)(5-k)}{(5)^2(6)} \\ &= \frac{56k(5-k)}{25}.\end{aligned}$$

The mean and the variance of the bonus game are given by:

$$\begin{aligned} E[W] &= (25 - 2) \left(\frac{21k}{5} \right) - (25k - 40) \\ &= 71.6k + 40 \end{aligned}$$

$$\begin{aligned} Var(W) &\approx \frac{56k(5-k)}{25(21k/5)} \left[k(1165.6 - 625) + (21k/5)(625 + 4) \right] \\ &= 1697k(5-k). \end{aligned}$$

The following table provides the mean, the variance, and the standard deviation for all possible values of k :

k	$E[W]$	$Var(W)$	σ_W
1	111.6	6788	82.4
2	183.2	10182	100.9
3	254.8	10182	100.9
4	326.4	6788	82.4

2. This type of bonus game is a simple match game: The player first selects an object from a population of N objects of which there are n_1 objects of type 1, n_2 objects of type 2, ..., n_r objects of type r , where $n_1 + n_2 + \dots + n_r = N$. The selected object is removed from the population and the player selects objects, one at a time, without replacement, from the remaining set of objects, until a match is found. The player is then awarded a prize that is dependent on the number of the matching draw. An alternative approach would be to have two identical populations and for the player to select an object from the first population and try to match it with an object from the second population.

For the first approach, if object i were selected, then there would be $N - 1$ objects left in the population out of which $n_i - 1$ objects would be of the selected type. Therefore, if the match, i.e., $k = 1$, is made on the the X^{th} draw, the mean and variance of X would be given by:

$$\mu_X = \frac{(1)\{(N-1)+1\}}{(n_i-1)+1}$$

$$\sigma_X^2 = \frac{(1)\{(N-1)+1\}\{(N-1)-(n_i-1)\}\{(n_i-1)+(1)-1\}}{\{(n_i-1)+1\}^2 \{(n_i-1)+2\}}$$

These expressions reduce to:

$$\mu_X = \frac{N}{n_i}, \quad i = 1, \dots, r$$

$$\sigma_X^2 = \frac{N(N - n_i)(n_i - 1)}{n_i^2(n_i + 1)}, \quad i = 1, \dots, r$$

For the second approach, if object i were selected, then there would be N objects in the second population out of which n_i objects would be of the selected type. Therefore, if the match, i.e., $k = 1$, is made on the the X^{th} draw, the mean and variance of X would be given by:

$$\mu_X = \frac{N + 1}{n_i + 1}, \quad i = 1, \dots, r$$

$$\sigma_X^2 = \frac{(N + 1)(N - n_i)n_i}{(n_i + 1)^2(n_i + 2)}, \quad i = 1, \dots, r$$

The probability functions for both approaches are easily computed using the values of the parameters: N , n , and $k = 1$.

Example

In this example, the population is a standard deck of cards and the first approach is used. The player is awarded the prizes that depend on the placement of the match. The distinct objects, in the population are denoted by the ranks of the cards: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A. Therefore, $N = 52$, and $n_i = 4$ for all i , and we present the following table for a bonus game with an expected value of approximately 25:

x	$\Pr(X = x)$	Award	$E[\text{Award}]$
1-10	0.488115	10	4.881150
11-20	0.296039	15	4.440585
21-30	0.151980	30	4.559400
31-40	0.055943	100	5.559430
41-49	0.007923	700	5.546100
Total	1.000000	-	24.986665

The variance of the bonus game could be computed using the ideas in the first part of the presentation, or by simulation.

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