# Casino Games and the Central Limit Theorem[1]

*Ashok Singh, Ph. D.**
*Anthony F. Lucas, Ph.D.*
*Rohan J. Dalpatadu, Ph.D.*
*Dennis J. Murphy, Ph.D.*

**Abstract**

The central limit theorem, in simple terms, states that the probability distribution of the mean of a random sample, for most probability distributions, can be approximated by a normal distribution when the number of observations in the sample is 'sufficiently' large. Most applied statistics books recommend using the normal approximation for the probability distribution of the sample mean when the number of observations exceeds 30. It is commonly known in the discipline of statistics that larger samples will be needed when the underlying probability distribution is heavily skewed. However, the minimum number of samples needed for the CLT to yield a reasonable approximation, when the distribution being sampled is heavily skewed, is not known. The Berry-Esseen theorem does provide an upper bound on the error in approximating the probability distribution of the sample mean by the normal distribution, but this upper bound turns out to be of no value when applied to slot games. The pay-out probability distributions of many casino games such as slots are heavily skewed, yet the CLT is used for calculating 'confidence limits' for total casino win, or rebates on losses, for these games. We will use Monte Carlo experiments to simulate the play of a few slot games and the table game of baccarat to estimate the probability distribution of the mean payout for sample sizes as large as 4,000, and compare it to the normal distribution.

## Introduction

The law of large numbers (LLN) and the central limit theorem (CLT) are the cornerstones of inferential statistics. The LLN ensures that as the number of random samples collected from a probability distribution is increased, the sample mean converges to the true population mean, and the CLT guarantees that the sampling distribution of the mean will be Gaussian, provided there are a sufficient number of independent observations. The law of large numbers was utilized by the Chevalier de Mere (1607 - 1684), an astute gambler (see Maxwell, 1999), and first proven by Jakob Bernoulli in 1713 (Bauer, 1996). In 1733, Abraham de Moivre introduced the concept of the Gaussian distribution as an approximation to the binomial distribution. This result, now called the theorem of de Moivre – Laplace (Feller, 1968), is a special case of the CLT. The CLT is invoked in deriving formulas for confidence intervals in estimation problems and critical regions in hypothesis testing problems.

There are many practical applications of the central limit theorem. The CLT provides justification for many procedures in statistical process control and statistical quality

Ashok Singh, Ph. D.*
Professor
University of Nevada Las Vegas
William F. Harrah College of Hotel Administration
4505 Maryland Parkway
Las Vegas, NV, 89154
Email: aksingh@unlv.nevada.edu

Anthony F. Lucas, Ph.D.
Professor
William F. Harrah College of Hotel Administration
University of Nevada, Las Vegas
Email: AFL2@cox.net

Rohan J. Dalpatadu , Ph.D.
Associate Professor
University of Nevada Las Vegas
Department of Mathematical Sciences
4505 Maryland Parkway
Las Vegas, NV, 89154
Email: dalpatad@unlv.nevada.edu

Dennis J. Murphy, Ph.D.
Senior Statistician
ADI-NV, Inc.
Email: djmurphy0420@yahoo.com

*Corresponding author

1 This research was conducted with funding support from the Caesars Foundation

control. The concept of Six Sigma derives from the central limit theorem. The best-known practical application of the LLN and the CLT is the casino gaming industry. The profitability of all of the casino games is guaranteed by the LLN and the CLT. In other words, these two theorems ensure that, in the long run, any casino game will result in a positive casino win, provided the game carries a positive expected value. For example, some video poker games have offered a positive expected value for the player, if played perfectly.

Applications of the central limit theorem exist in the gaming literature as well. Johnson (2006) used the central limit theorem to calculate optimal keno strategies. Ethier and Levin (2005) have derived a variation of the central limit theorem and provided a simplified proof of Thorp and Walden's theorem of card counting.

The approximate normality of the sample mean for large samples is routinely used for calculating confidence limits on actual casino win from a table game, or actual casino win from a given player (Hannum and Cabot,2005). The PAR (probability accounting report) sheet of every slot game on a casino floor includes, among other things, confidence limits of payback percentage for given numbers of games played (Hannum and Cabot, 2005; Harrigan and Dixon, 2009). The CLT is also used for calculating rebate on actual loss (Hannum and Cabot, 2005).

The probability distribution of the payouts from a typical slot game, however, is heavily skewed, and the number of samples (i.e., the number of pulls or spins) would need to be larger than 30, the magic number recommended by most applied statistics textbooks (Webster, 1998; Ott and Longnecker; 2010; Devore, 2012, to name a few) for the normal approximation to yield accurate results. In this article, we use Monte Carlo simulation experiments from a few slot games and the table game of baccarat to compare the approximate results obtained from the CLT to the results from simulation experiments.

By way of examples, this study seeks to demonstrate the applicability of the LLN and the CLT to common problems facing modern gaming operators. In doing so, the findings will shine a light on the viability of popular heuristics that are routinely employed in the gaming industry. The outcomes of play are simulated and analyzed at sample sizes that are meaningful to gaming operators.

**CLT and the Berry-Esseen Theorem**

The CLT states that the probability distribution of the mean of a random sample from a population with mean μ and finite standard deviation σ approaches the normal distribution as the sample size n is increased:

$$P\left(\frac{\bar{x}-\mu}{\sigma/\sqrt{n}} \le x\right) \to \Phi(x), \text{ where } \Phi(x) \text{ is the standard normal cumulative distribution.}$$

The Berry-Esseen Theorem (see, for example, Gut, 2012) provides an upper bound on the error in the above approximation:

$$\max\left| F_{\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}}(x) - \Phi(x) \right| \le C\frac{\gamma^3}{\sigma^3\sqrt{n}}, \text{ where } \gamma^3 = E(|X-\mu|^3),$$

$\sigma < \infty$ is the standard deviation, and C is a constant.

Much work has been done to accurately determine the value of the constant C; the best value of C in 1972 was 0.7975 which was replaced by 0.7655 in 1986 (Gut, 2012). The current best value is 0.4784, due to Shvetsova (2007). We have calculated the

Berry-Esseen upper bound for the sample size n of 100000 in the error of approximation provided by the CLT for the four slot games considered in this paper (see Appendix A); these results are shown in Table 1 below.

Table 1: Berry-Esseen upper bound in the error of approximation provided by CLT

| Slot game | μ | Standard deviation σ | γ3 | Upper bound |
|---|---|---|---|---|
| 1 | 0.88 | 11.28 | 1263630 | 1.33 |
| 2 | 0.91 | 11.28 | 1263693 | 1.33 |
| 3 | 0.94 | 11.28 | 1236563 | 1.33 |
| 4 | 0.97 | 11.28 | 1263550 | 1.33 |

Clearly the upper bound provided by Berry-Esseen theorem for a sample size of 100000 is of no practical value in the present situation.

## Methodology

Monte Carlo simulation was used to simulate the play of n hands for a slot game and also the table game of baccarat, and the probability distribution of the simulated payback (for slot games) or player win/loss (for baccarat) was estimated by the histograms of the simulated means. The coefficients of skewness and kurtosis (Sharma, J.K., 2007; Rupert, 2004), which equal 0 for any normal distribution, are useful measures for assessing departures from normality and have been included in the descriptive statistics of the simulated results. The R programming language (R Core Team, 2012) was used to code all of the simulation experiments utilized in this study. All graphs in this paper were programmed using the ggplot2 package in R.

In Nevada, for the calendar year ended December 31, 2012, over 63% of the State's total gaming win came from slot machines (Nevada Gaming Control Board, 2012). During the same reporting period, baccarat produced more win than any other table game. Both slots and baccarat serve as perennial revenue juggernauts for Nevada casinos. Because of these contributions, this study examines the effects of the LLN and CLT on these two areas of the casino.

### Slot Games

In an earlier study (Lucas and Singh, 2011), the pay table of an actual slot game was modified to create twelve different versions of the same slot game, with four payback percentages (88%, 91%, 94%, and 97%) and three standard deviations (6, 11, and 15 coins); Monte Carlo simulation was used to show the inability of players to detect differences in the payback percentages. In this study, we have used four of these slot games corresponding to the standard deviation of 11 coins, and investigated the effect of an increase in the number of spins on the actual payback to a player who wagers one credit and plays one line on each spin. The four pay tables examined in this study can be found in Appendix A.A brief description of the simulation follows:

1) Generate n random numbers from the slot game pay table, which represent the actual amounts paid back to a player after n spins.

2) Calculate the mean $\overline{x}$ of the sample of size n obtained in Step 1. This sample mean represents the actual average payback to a player after n spins.

A large number (N) of iterations of the above two steps yields N values of the sample mean, $\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_N$.. The probability distribution or the sampling distribution of the sample mean is estimated from these N sample means. The number of iterations N used in this study was 100,000 and six values of the sample size n were included in this study: 100, 250, 500, 750, 1000, 2000, and 4000. Using 500 as the average number of trials or slot pulls per hour (Hannum and Cabot, 2005), the above sample sizes cover the average slot play time range of 12 minutes (100 spins) to 8 hours (4000 spins). The descriptive statistics, histograms, and normal quantile-quantile (q-q) plots were used to assess the normality of the probability distribution of the sample mean in each case.

As mentioned earlier, the PAR sheet of a slot game includes 95% confidence limits of the payback percentage for n games played, calculated from the following CLT-based approximate formula:

Payback Percentage ± VI/√n,

where VI is the 95% volatility index of the slot, given by VI = 1.96 x standard deviation. It should be noted that in the above formula, the true expected payback and true sd are being used, and therefore the result is not a 'confidence interval' but in fact behaves rather like a set of quality control limits for the actual average payback from n hands.

In this paper, the 95% approximate control limits obtained from the above formula are compared to the equal-tailed non-parametric 95% confidence limits obtained from simulations. Any differences highlight the extent to which the simulated sample sizes produced results deviate from those calculated by the popular formula.

**Baccarat**

In the standard form of baccarat, the dealer deals two cards each to the Bank hand and the Player hand, and a gambler can bet on either of the two hands to win, or that a tie will occur. Of course, when dictated by the rules, either or both of these two hands may receive a third card, before a winning outcome is determined. The payout on the Player hand is 1 to 1, while winning Bank wagers are paid at a rate of 0.95 to 1, assuming the standard 5% commission on winning Bank wagers. Winning tie bets are paid at a rate of 8 to 1. In the event of a tie hand, a bet on either the Bank or the Player will push, i.e., the bet will neither win nor lose. The probabilities of a winning Player bet, a winning Bank bet, and a winning tie bet are known (See Kilby, Fox, and Lucas, 2004):

Probability that the Player hand wins = 0.4462466

Probability that the Bank hand wins = 0.4585974

Probability that the two hands are tied = 0.0951560

The expected value of a single unit wager on the Player hand in baccarat is -0.01235, and the per unit standard deviation is 0.9512.

In order to generate results of a sequence of n hands of baccarat, in which a player wagers 1 unit each time on the Player hand, n random numbers were generated from the above probability distribution and the sample mean was calculated. By repeating the above steps a large number (N) of times, N values of the sample mean,

$\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_N$ , were obtained. The number of iterations N used was 100,000 and the same six values of the sample sizes n (i.e, hands) were included in this study: 100, 250, 500, 750, 1000, 2000, and 4000. The descriptive statistics, histograms, and normal quantile-quantile (q-q) plots were used to assess the normality of the probability distribution of the sample mean in each case. In addition, the CLT based 95%

confidence limits are compared to the non-parametric 95% confidence limits obtained from simulations, to highlight any relevant differences. That is, the limits computed by the CLT-based formula and are compared to those produced by the simulations, for each sample size.

## Results

The results of our simulation experiments with the slot games and baccarat are summarized in this section.

### Slot Games

Tables 1-4 show the descriptive statistics, including the sample skewness and kurtosis, of simulated average paybacks. Figures 1, 3, 5 and 7 show the sample histograms of observed payback values, and Figures 2, 4, 6 and 8 show the corresponding normal q-q plots. It should be kept in mind that the true skewness and kurtosis values for a normal distribution are both equal to 0, and the normal q-q plot of a sample generated from a normal distribution should fall along the $45^o$ line. It can be seen from Tables 1-4 that:

(i) for each of the four slot games, the observed average payback from n slot pulls equals the corresponding true payback percentage for a 1-credit wager; for example, the sample mean of simulated paybacks from Game 2 equals the true payback of 0.91 even for 100 slot pulls, keeping in mind that this 100-pull average was computed from the results of 100,000 trips, each comprised of 100 pulls (see Table 2),

(ii) for each of the four slot games, the observed median payback from n slot pulls is smaller than the corresponding true mean, with the gap between the two values (observed median and true payback) decreasing as the number of slot pulls n is increased,

(iii) both the skewness and the kurtosis values decrease as n is increased, but even for samples of size 4000, both skewness and kurtosis are quite far from 0, indicating a serious departure from normality in the probability distribution of sample mean of n slot pulls from any of the four games considered in our simulation, and

(iv) the range of average observed paybacks shrinks as the number of spins is increased as can be seen from the minimum and maximum average paybacks.

Figure 1 shows that, for slot game 1, the sample histograms of observed paybacks for the six sample sizes considered are highly skewed; the same can be seen from the corresponding normal q-q plots of observed paybacks, shown in Figure 2.

Figures 3-8 show that the same conclusions can be drawn for all of the other three slot games included in this simulation study.

Table 5 shows the 95% confidence limits for payback percentage calculated from the CLT-based approximate formula (CLT column) and the same obtained from simulated payback averages (SIM column) by calculating their lower and upper 2.5% quantiles. It can be seen from Table 5 that, in each case, the 95% confidence intervals obtained from simulations are narrower (i.e., more accurate) than the ones obtained from the CLT-based approximate formula.

*Table 2:* Descriptive statistics of average payback from n slot pulls from slot game 1

| n | Minimum | Maximum | Mean | Median | sd | Skewness | Kurtosis |
|---|---|---|---|---|---|---|---|
| 100 | 0.10 | 111.14 | 0.88 | 0.8 | 1.21 | 83.46 | 7604.75 |
| 250 | 0.28 | 45.24 | 0.88 | 0.84 | 0.78 | 51.69 | 2904.15 |
| 500 | 0.37 | 23.16 | 0.88 | 0.85 | 0.46 | 41.9 | 1987.55 |
| 1000 | 0.53 | 12.14 | 0.88 | 0.86 | 0.37 | 27.16 | 810.42 |
| 2000 | 0.6 | 6.62 | 0.88 | 0.86 | 0.27 | 18.62 | 377.93 |
| 4000 | 0.67 | 6.39 | 0.88 | 0.87 | 0.18 | 13.84 | 212.74 |

Table 3: Descriptive statistics of average payback from n slot pulls from slot game 2

| n | Minimum | Maximum | Mean | Median | sd | Skewness | Kurtosis |
|---|---|---|---|---|---|---|---|
| 100 | 0.10 | 112.37 | 0.91 | 0.84 | 1.21 | 883.26 | 7578.93 |
| 250 | 0.32 | 45.53 | 0.91 | 0.87 | 0.70 | 56.11 | 3499.88 |
| 500 | 0.41 | 23.40 | 0.91 | 0.88 | 0.55 | 36.46 | 1445.98 |
| 1000 | 0.53 | 12.21 | 0.91 | 0.89 | 0.3 | 27.72 | 850.14 |
| 2000 | 0.62 | 6.58 | 0.91 | 0.89 | 0.25 | 19.65 | 425.96 |
| 4000 | 0.69 | 3.79 | 0.91 | 0.90 | 0.17 | 14.16 | 222.11 |

*Table 4:* Descriptive statistics of average payback from n slot pulls from slot game 3

| n | Minimum | Maximum | Mean | Median | sd | Skewness | Kurtosis |
|---|---|---|---|---|---|---|---|
| 100 | 0.10 | 112.30 | 0.94 | 0.87 | 1.25 | 80.76 | 7070.17 |
| 250 | 0.34 | 45.49 | 0.94 | 0.90 | 0.74 | 53.85 | 3180.98 |
| 500 | 0.47 | 23.48 | 0.94 | 0.91 | 0.56 | 36.32 | 1430.57 |
| 1000 | 0.57 | 12.49 | 0.94 | 0.92 | 0.34 | 28.67 | 915.89 |
| 2000 | 0.65 | 6.66 | 0.94 | 0.93 | 0.24 | 20.25 | 455.95 |
| 4000 | 0.70 | 6.49 | 0.94 | 0.93 | 0.19 | 13.74 | 212.47 |

Table 5: Descriptive statistics of average payback from n slot pulls from slot game 4

| n | Minimum | Maximum | Mean | Median | sd | Skewness | Kurtosis |
|---|---|---|---|---|---|---|---|
| 100 | 0.13 | 111.26 | 0.97 | 0.90 | 1.25 | 81.02 | 7100.48 |
| 250 | 0.34 | 46.10 | 0.97 | 0.93 | 0.73 | 54.63 | 3286.07 |
| 500 | 0.50 | 23.32 | 0.97 | 0.94 | 0.50 | 39.80 | 1756.69 |
| 1000 | 0.57 | 12.22 | 0.97 | 0.95 | 0.34 | 28.55 | 907.30 |
| 2000 | 0.70 | 6.67 | 0.97 | 0.96 | 0.26 | 19.42 | 414.37 |
| 4000 | 0.76 | 6.57 | 0.97 | 0.96 | 0.18 | 13.74 | 213.49 |

Table 6: Comparison of 95% confidence limits for payback% from n slot pulls calculated from the CLT-based formula (CLT column) and the same obtained from 100,000 simulations (SIM column)

| | | Game 1 | | Game 2 | | Game 3 | | Game 4 | |
|---|---|---|---|---|---|---|---|---|---|
| n | | CLT | SIM | CLT | SIM | CLT | SIM | CLT | SIM |
| 100 | L95% | -1.33 | 0.39 | -1.31 | 0.42 | -1.27 | 0.44 | -1.24 | 0.47 |
| | U95% | 3.09 | 1.71 | 3.11 | 1.73 | 3.15 | 1.76 | 3.18 | 1.78 |
| 250 | L95% | -0.52 | 0.53 | -0.50 | 0.56 | -0.46 | 0.58 | -0.43 | 0.61 |
| | U95% | 2.28 | 1.38 | 2.30 | 1.41 | 2.34 | 1.44 | 2.37 | 1.48 |
| 500 | L95% | -0.11 | 0.61 | -0.09 | 0.64 | -0.05 | 0.67 | -0.02 | 0.7 |
| | U95% | 1.87 | 1.24 | 1.89 | 1.27 | 1.93 | 1.29 | 1.96 | 1.32 |
| 1000 | L95% | 0.18 | 0.68 | 0.20 | 0.71 | 0.24 | 0.74 | 0.27 | 0.77 |
| | U95% | 1.58 | 1.12 | 1.60 | 1.15 | 1.64 | 1.18 | 1.67 | 1.21 |
| 2000 | L95% | 0.39 | 0.73 | 0.41 | 0.76 | 0.45 | 0.79 | 0.48 | 0.82 |
| | U95% | 1.37 | 1.04 | 1.39 | 1.07 | 1.43 | 1.1 | 1.46 | 1.13 |
| 4000 | L95% | 0.53 | 0.77 | 0.55 | 0.8 | 0.59 | 0.83 | 0.62 | 0.86 |
| | U95% | 1.23 | 0.99 | 1.25 | 1.02 | 1.29 | 1.05 | 1.32 | 1.08 |

**Baccarat**

The simulation results for baccarat are very different from the ones previously reported for the slot games. Table 6 shows the descriptive statistics of average payback for n wagers of 1 unit each on the player hand. Per Lucas & Kilby (2012), the true house advantage of the player hand bet in baccarat is 0.0124, and the average simulated player win of n wagers is very close to -0.0124 for each n; moreover, the simulated skewness and kurtosis values are very close to 0. The histogram plots (Figure 9) and the normal q-q plots (Figure 10) of simulated average player win also indicate that the probability distributions of the average player win from simulation are quite close to the normal distribution. Table 7 shows that the 95% confidence limits for the true payback percentage computed using the CLT based formula are quite close to the corresponding values obtained from simulations.

## Conclusions and Limitations

It is generally known that statistics in general and statistical simulation in particular do not provide proof of a hypothesis or a rule of thumb (see, for example, Ehninger and Brockriede, 2008); simulation can only be used to provide evidence in favor of a rule of thumb, or to demonstrate that the rule of thumb does not always work. The simulations used in this paper clearly show that the thumb rule of '$n \geq 30$' for the CLT to provide a good normal approximation to the probability distribution of the sample mean does not work in the case of a heavily skewed probability distribution such as a typical slot game, and that the normal approximation is valid even for moderate sample sizes when the game has a symmetric probability distribution of payouts (e.g., the table game of baccarat).

Table 7: Descriptive statistics of average player win from n 1-unit wagers on the player hand in baccarat

| n | Minimum | Maximum | Mean | Median | Skewness | Kurtosis |
|---|---|---|---|---|---|---|
| 100 | -0.4300 | 0.4500 | -0.0120 | -0.0100 | 0.0010 | 0.0100 |
| 250 | -0.2700 | 0.2500 | -0.0120 | -0.0120 | -0.0030 | 0.0200 |
| 500 | -0.1900 | 0.1700 | -0.0120 | -0.0120 | -0.0040 | -0.0100 |
| 1000 | -0.1400 | 0.1100 | -0.0130 | -0.0130 | -0.0070 | -0.0100 |
| 2000 | -0.1000 | 0.0800 | -0.0130 | -0.0130 | -0.0070 | -0.0100 |
| 4000 | -0.1000 | 0.0500 | -0.0120 | -0.0130 | -0.0030 | 0.0100 |

Table 8: Comparison of 95% confidence limits for payback percentage from n 1-unit wagers on player hand in baccarat calculated from the CLT-based formula (CLT column) and the same obtained from 100,000 simulations (SIM column)

| n | CLT | | SIM | |
|---|---|---|---|---|
| 100 | -0.1740 | 0.1986 | -0.2000 | 0.1700 |
| 250 | -0.1056 | 0.1302 | -0.1300 | 0.1000 |
| 500 | -0.0710 | 0.0956 | -0.1000 | 0.0700 |
| 1000 | -0.0466 | 0.0712 | -0.0700 | 0.0500 |
| 2000 | -0.0294 | 0.0540 | -0.0500 | 0.0300 |
| 4000 | -0.0172 | 0.0418 | -0.0400 | 0.0200 |

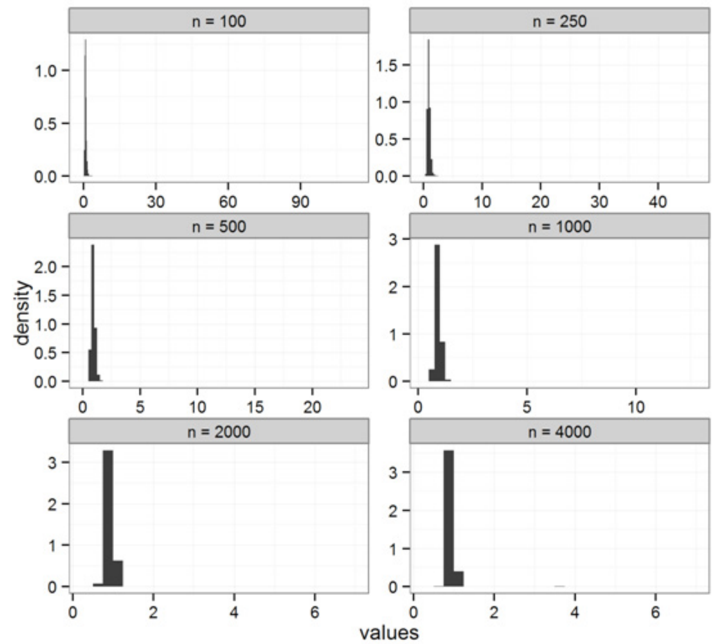


Figure 1: Histograms of average paybacks from Slot Game 1 for n slot pulls (n = 100, 250, 500, 1000, 2000, and 4000); results of 100,000 simulations.

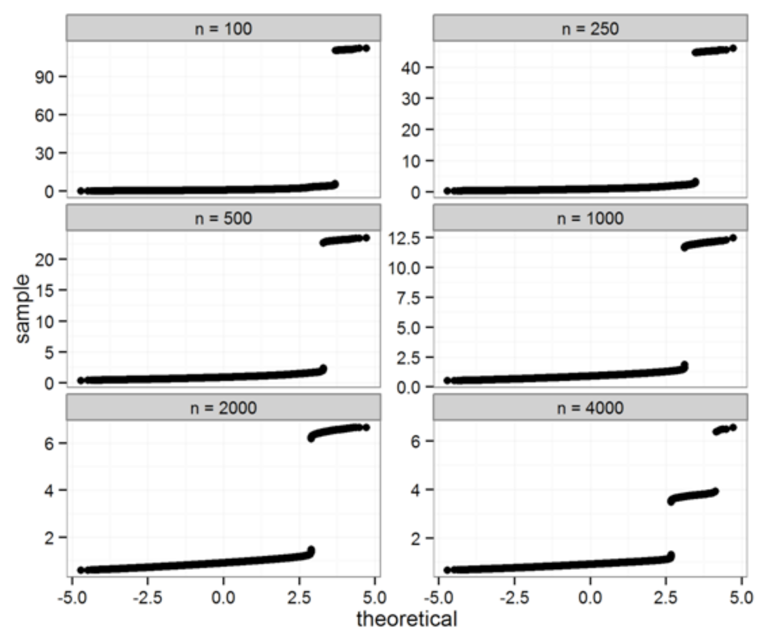


Figure 2: Normal q-q plots of average paybacks from Slot Game 1 for n slot pulls (n = 100, 250, 500, 1000, 2000, and 4000); results of 100,000 simulations.

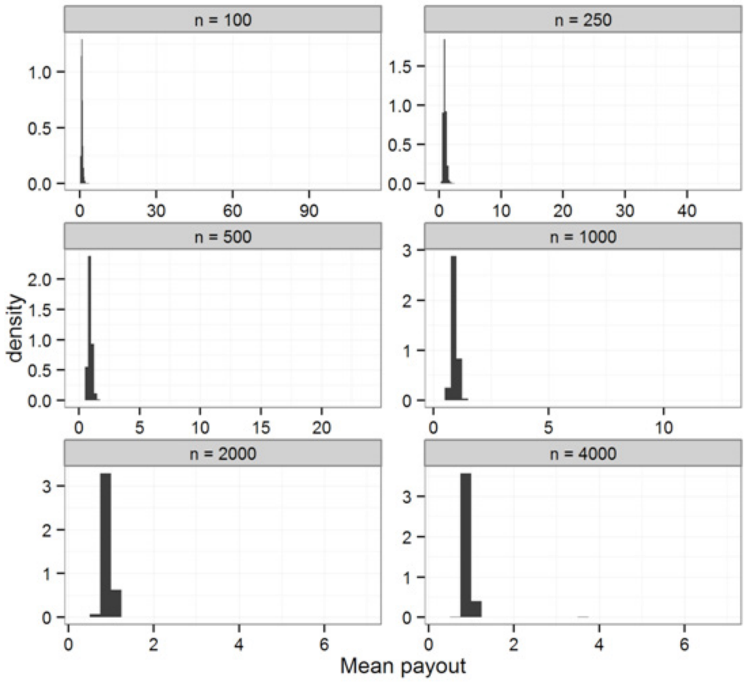


Figure 3: Histograms of average paybacks from Slot Game 2 for n slot pulls (n = 100, 250, 500, 1000, 2000, and 4000); results of 100,000 simulations.

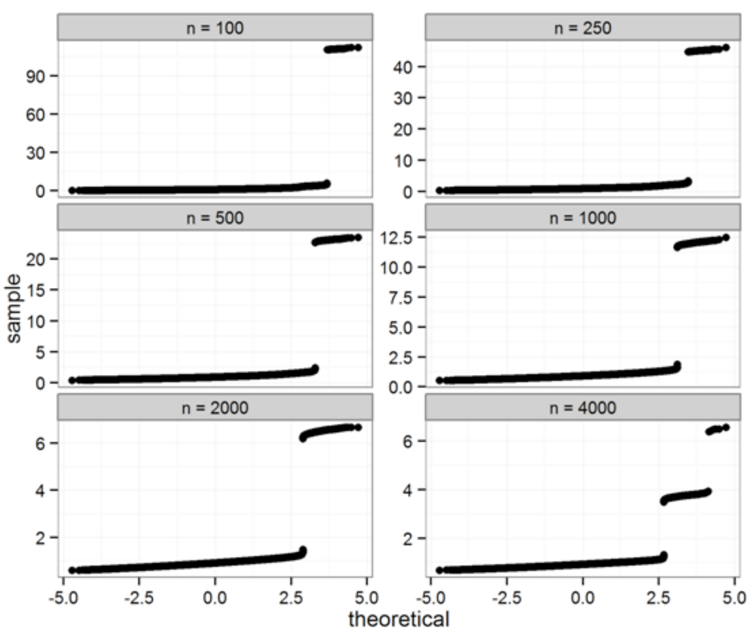


Figure 4: Normal q-q plots of average paybacks from Slot Game 2 for n slot pulls (n = 100, 250, 500, 1000, 2000, and 4000); results of 100,000 simulations.

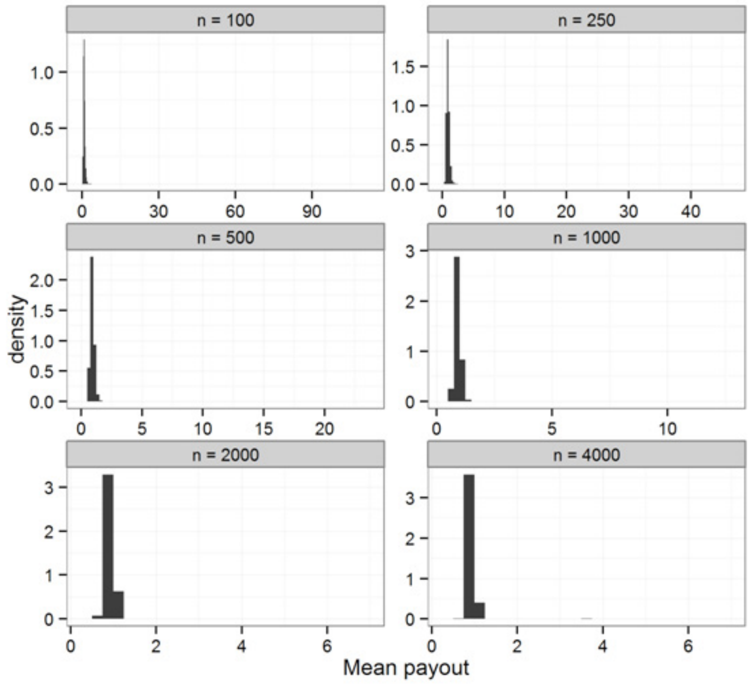


Figure 5: Histograms of average paybacks from Slot Game 3 for n slot pulls (n = 100, 250, 500, 1000, 2000, and 4000); results of 100,000 simulations.

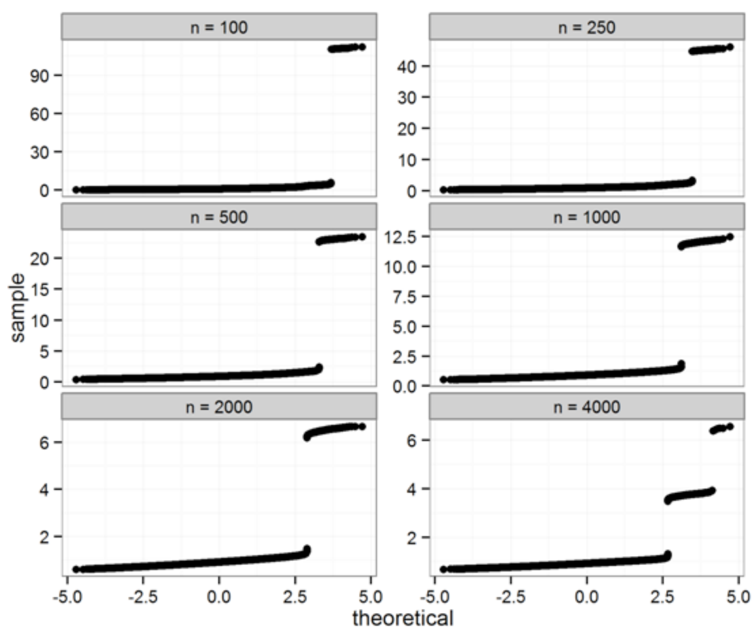


Figure 6: Normal q-q plots of average paybacks from Slot Game 3 for n slot pulls (n = 100, 250, 500, 1000, 2000, and 4000); results of 100,000 simulations.

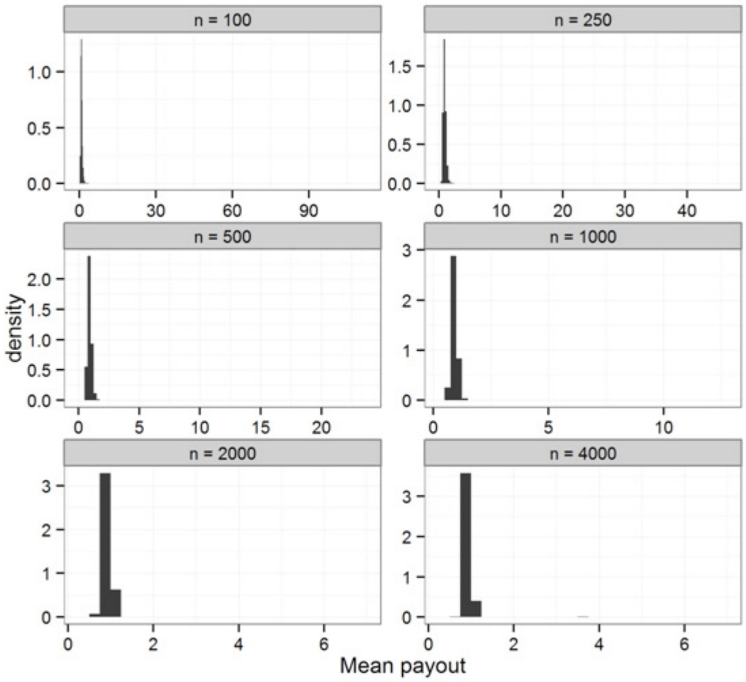


Figure 7: Histograms of average paybacks from Slot Game 4 for n slot pulls (n = 100, 250, 500, 1000, 2000, and 4000); results of 100,000 simulations.

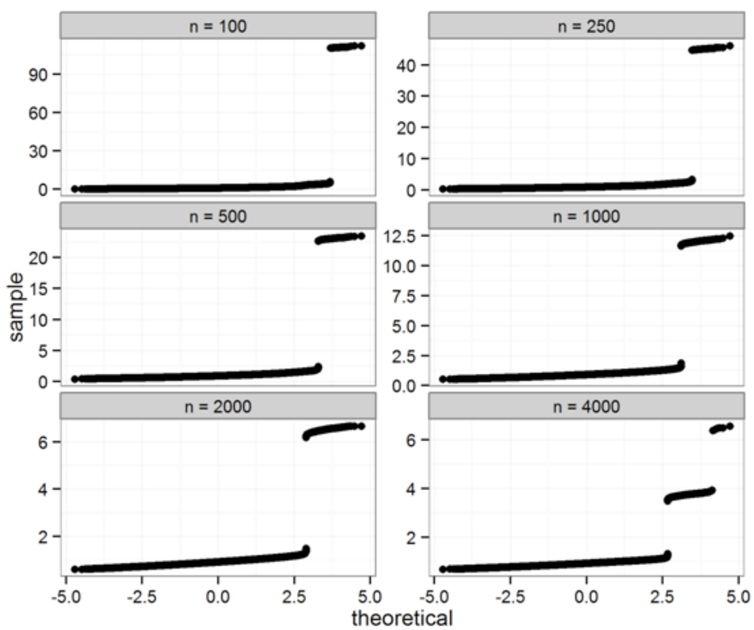


Figure 8: Normal q-q plots of average paybacks from Slot Game 4 for n slot pulls (n = 100, 250, 500, 1000, 2000, and 4000); results of 100,000 simulations.

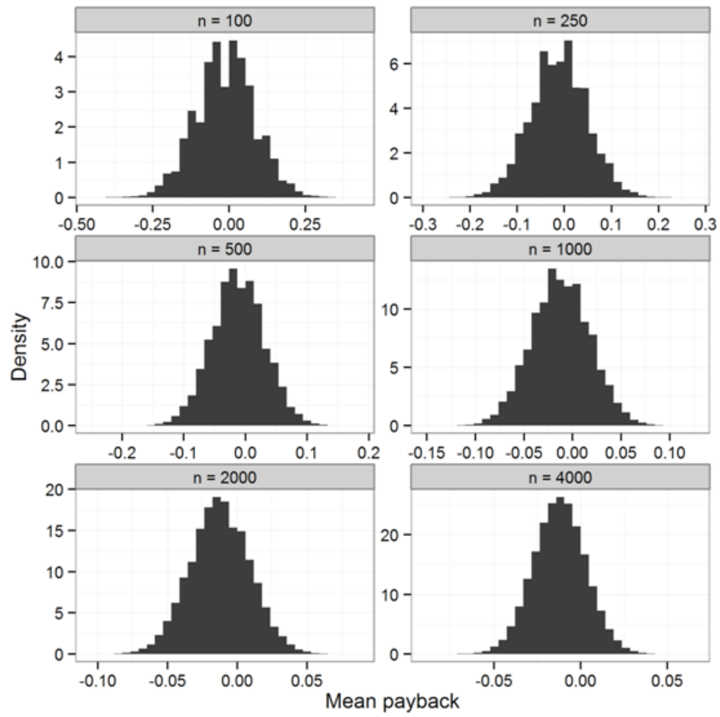


Figure 9: Histograms of average player win for n wagers of 1 unit on the player hand in baccarat. (n = 100, 250, 500, 1000, 2000, and 4000); results of 100,000 simulations.

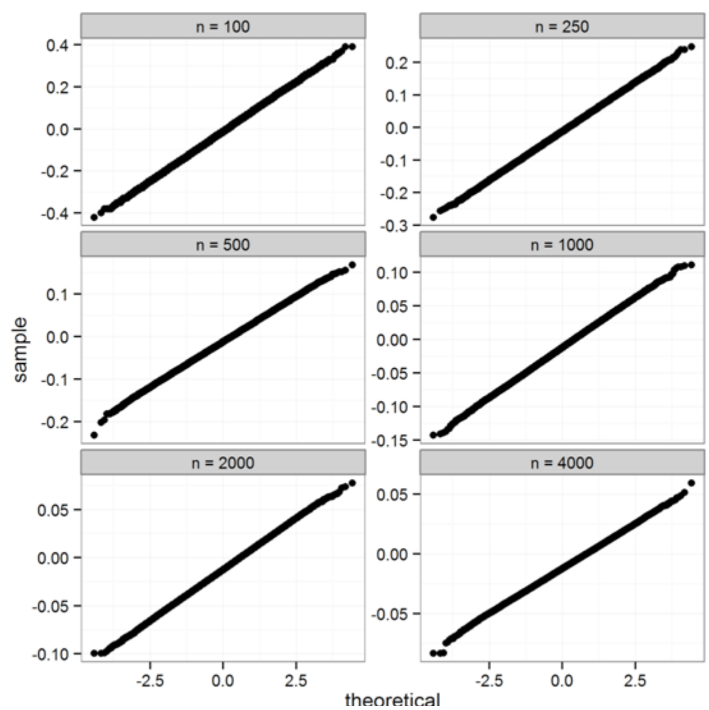


Figure 10: Normal q-q plots of average player win for n wagers of 1 unit on the player hand in baccarat. (n = 100, 250, 500, 1000, 2000, and 4000); results of 100,000 simulations.

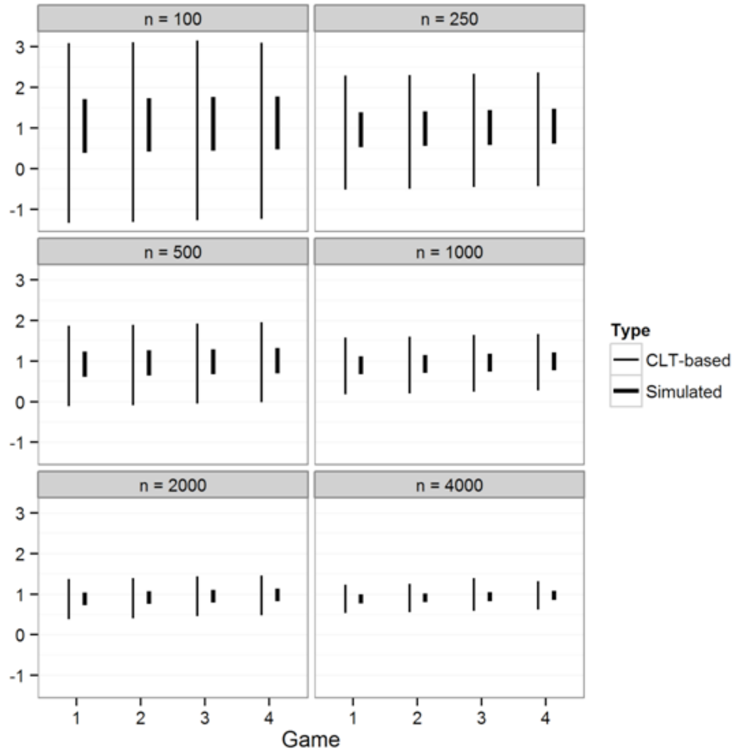


Figure 11: Plots of the CLT-based formula and simulated control limits for the four slot games and number of slot pulls = 100, 250, 500, 1000, 2000, and 4000.

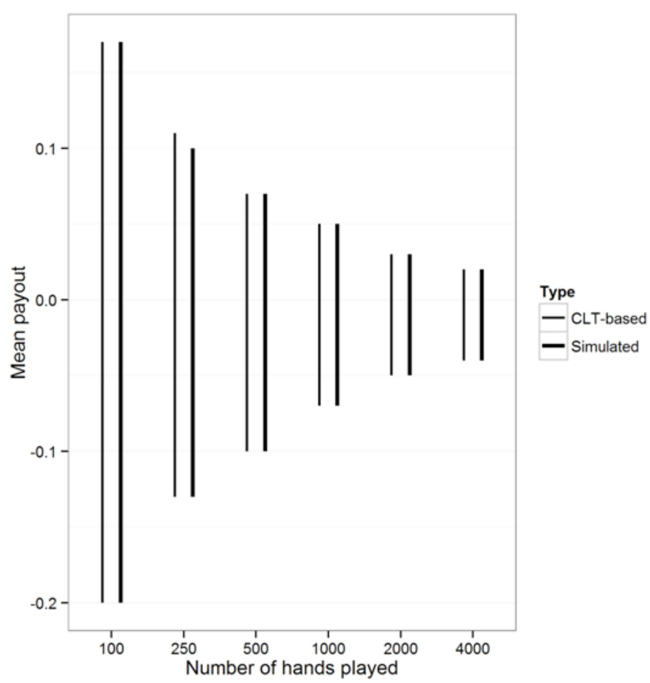


Figure 12: Plots of the CLT-based formula and simulated control limits for the player hand bet in baccarat, and number of hands played = 100, 250, 500, 1000, 2000, and 4000.

| Appendix A | | | | | |
|---|---|---|---|---|---|
| Probability distributions of the four slot games | | | | | |
| PAYOUT FOR 1 CREDIT WAGER | | | | | |
| EVENT | Probability | 88% Game | 91% Game | 94% Game | 97% Game |
| 1 | 0.000259 | 1 | 1 | 1 | 2 |
| 2 | 0.020926 | 1 | 2 | 2 | 3 |
| 3 | 0.018136 | 1 | 2 | 3 | 3 |
| 4 | 0.015545 | 3 | 3 | 4 | 5 |
| 5 | 0.001395 | 7 | 4 | 5 | 4 |
| 6 | 0.001196 | 7 | 5 | 5 | 4 |
| 7 | 0.001036 | 6 | 5 | 5 | 4 |
| 8 | 0.000199 | 6 | 5 | 5 | 4 |
| 9 | 0.000173 | 6 | 5 | 5 | 4 |
| 10 | 0.000013 | 15 | 5 | 5 | 9 |
| 11 | 0.000080 | 15 | 11 | 15 | 9 |
| 12 | 0.009347 | 10 | 12 | 11 | 10 |
| 13 | 0.000257 | 10 | 10 | 10 | 10 |
| 14 | 0.001315 | 20 | 20 | 20 | 20 |
| 15 | 0.000110 | 20 | 20 | 20 | 20 |
| 16 | 0.000427 | 40 | 40 | 40 | 40 |
| 17 | 0.000053 | 40 | 40 | 40 | 40 |
| 18 | 0.000051 | 100 | 60 | 60 | 60 |
| 19 | 0.000009 | 100 | 60 | 60 | 60 |
| 20 | 0.000140 | 75 | 75 | 75 | 75 |
| 21 | 0.000004 | 75 | 75 | 75 | 75 |
| 22 | 0.000046 | 100 | 100 | 100 | 100 |
| 23 | 0.000004 | 100 | 100 | 100 | 100 |
| 24 | 0.000010 | 250 | 300 | 250 | 250 |
| 25 | 0.000001 | 11000 | 11000 | 11000 | 11000 |
| 26 | 0.002591 | 2 | 2 | 2 | 3 |
| 27 | 0.021126 | 2 | 2 | 2 | 2 |
| 28 | 0.018309 | 2 | 2 | 2 | 2 |
| 29 | 0.015545 | 2 | 2 | 2 | 2 |
| 30 | 0.001408 | 5 | 5 | 5 | 5 |
| 31 | 0.001196 | 4 | 4 | 4 | 4 |
| 32 | 0.001036 | 4 | 4 | 4 | 4 |
| 33 | 0.000173 | 4 | 4 | 4 | 4 |
| 34 | 0.000173 | 4 | 4 | 4 | 4 |
| 35 | 0.000013 | 4 | 4 | 4 | 4 |
| 36 | 0.000080 | 4 | 4 | 4 | 4 |
| 37 | 0.009578 | 9 | 9 | 9 | 9 |
| 38 | 0.000263 | 10 | 10 | 10 | 10 |
| 39 | 0.001315 | 20 | 20 | 20 | 20 |
| 40 | 0.000110 | 40 | 40 | 40 | 40 |
| 41 | 0.000456 | 40 | 40 | 40 | 40 |
| 42 | 0.000057 | 40 | 40 | 40 | 40 |
| 43 | 0.000051 | 40 | 40 | 40 | 40 |
| 44 | 0.000009 | 40 | 40 | 40 | 40 |
| 45 | 0.000119 | 40 | 40 | 40 | 40 |

| Appendix A | | | | | |
|---|---|---|---|---|---|
| Probability distributions of the four slot games | | | | | |
| PAYOUT FOR 1 CREDIT WAGER | | | | | |
| EVENT | Probability | 88% Game | 91% Game | 94% Game | 97% Game |
| 46 | 0.000003 | 40 | 40 | 40 | 40 |
| 47 | 0.000034 | 50 | 50 | 50 | 50 |
| 48 | 0.000003 | 50 | 50 | 50 | 50 |
| 49 | 0.000010 | 150 | 150 | 150 | 150 |
| 50 | 0.000001 | 250 | 250 | 250 | 250 |
| 51 | 0.002616 | 2 | 2 | 2 | 1 |
| 52 | 0.018485 | 2 | 2 | 2 | 2 |
| 53 | 0.018485 | 2 | 2 | 2 | 2 |
| 54 | 0.015695 | 2 | 2 | 2 | 2 |
| 55 | 0.001232 | 6 | 4 | 2 | 4 |
| 56 | 0.001046 | 6 | 4 | 3 | 5 |
| 57 | 0.001046 | 6 | 4 | 3 | 5 |
| 58 | 0.000174 | 6 | 4 | 3 | 4 |
| 59 | 0.000174 | 6 | 4 | 3 | 4 |
| 60 | 0.000012 | 15 | 8 | 8 | 8 |
| 61 | 0.000070 | 15 | 8 | 8 | 8 |
| 62 | 0.009787 | 10 | 9 | 10 | 10 |
| 63 | 0.000269 | 10 | 10 | 10 | 10 |
| 64 | 0.001378 | 20 | 20 | 20 | 20 |
| 65 | 0.000115 | 20 | 20 | 20 | 20 |
| 66 | 0.000456 | 40 | 40 | 40 | 40 |
| 67 | 0.000057 | 40 | 40 | 40 | 40 |
| 68 | 0.000051 | 60 | 60 | 60 | 60 |
| 69 | 0.000009 | 60 | 60 | 60 | 60 |
| 70 | 0.000119 | 75 | 75 | 75 | 75 |
| 71 | 0.000003 | 75 | 75 | 75 | 75 |
| 72 | 0.000034 | 100 | 100 | 100 | 100 |
| 73 | 0.000003 | 100 | 100 | 100 | 100 |
| 74 | 0.000010 | 250 | 250 | 250 | 250 |
| 75 | 0.000001 | 250 | 250 | 250 | 250 |
| 76 | 0.784281 | 0 | 0 | 0 | 0 |

## References


Bauer, Heinz (1996). Probability Theory. Volume 23 of De Gruyter Studies in Mathematics, Walter de Gruyter Publishing.

Čisar, Petar and Čisar, Sanja Maravić (2010). Skewness and Kurtosis in Function of Selection of Network Traffic Distribution.ActaPolytechnicaHungarica Vol. 7, No. 2, 2010, 95-106.

Davidson, I., and Aminian, M. (2004). Using the Central Limit Theorem for Belief Network Learning. In The 8th Artificial Intelligence and Mathematics Symposium.

Devore, Jay L. (2012). Probability And Statistics For Engineering And The Sciences, Cengage Learning, 226.

Ethier, S.N. and Levin, David A. (2005). On the fundamental theorem of card counting, with application to the game of trente et quarante. Vol. 37, 90 - 107.

Ehninger, Douglas and Brockriede, Wayne (2008). Decision By Debate. IDEA Press, 148.

Feller, William (1968).An Introduction to Probability Theory and Its Applications, Volume I, 3rd Edition, Wiley.

Hannum, Robert C. and Cabot, Anthony N. (2005). Practical Casino Math, 2nd Edition. Trace Publications.

Harrigan, Kevin A. and Dixon, Mike (2009). PAR Sheets, probabilities, and slot machine play: Implications for problem and non-problem gambling. Journal of Gambling Issues, Vol. 23, 81 - 110.

Johnson, Roger W. (2006). Optimal Keno Strategies and the Central Limit Theorem. Teaching Statistics, Vol. 28, 26 - 29.

Kilby, J., Fox, J., & Lucas, A.F. (2004). Casino Operations Management, 2nd Edition, New York: Wiley, 216.

Lucas, A.F. & Kilby, J. (2012). Introduction to Casino Management. San Diego: Gamma, 271.

Lucas, A.F. & Singh, A.K. (2011). Estimating the ability of gamblers to detect differences in the payback percentages of reel slot machines: A closer look at the slot player experience. UNLV Gaming Research & Review Journal, 15(1), 17-36.

Maxwell, Molly (1999). A Mathematical Perspective on Gambling. The MIT Undergraduate Journal of Mathematics. Retrieved on February 28, 2013 from http//www-math.mit.edu/phase2/UJM/vol1/index.html

Nevada Gaming Control Board (2012). Gaming Revenue Report, December 2012. Retrieved on March 2, 2013 from http://gaming.nv.gov/documents/pdf/1g_10dec.pdf

Ott, Lyman R., and Longnecker, Michael T. (2010). An Introduction to Statistical Methods And Data Analysis. Cengage Learning, 228.

R Core Team (2013). R: A language and environment for statistical computing. R Foundation for Statistical Computing, Vienna, Austria. ISBN 3-900051-07-0, URL http://www.R-project.org/.

Ruppert, David (2004). Statistics and Finance: An Introduction, Springer, 27.

Sharma, J.K. (2007). Business Statistics, 2nd Edition. Pearson Education, India, 189.

Shvetsova, I.G. (2007). Sharpening of the Upper Bound of the Absolute Constant in the Berry-Esseen Inequality. Theory of Probability & Its Applications. Vol. 51, 549-553.

Veluchamy, Suresh K. (2005). A Graphical Approach for Verification of the Central Limit Theorem, M.S. Thesis, University of Nevada, Las Vegas.

Webster, Allen L. (1978). Applied Statistics for Business and Economics, An Essentials Version, Third Edition, Irwin McGraw-Hill, 149.

Wickham, Hadley (2009). ggplot2: elegant graphics for data analysis. Springer, New York.