# A Game-Theoretic Analysis of Baccara Chemin de Fer

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# Baccara chemin de fer — review of main contributions

- Baccara was first mentioned in print by Van Tenac in 1847.
- It was analyzed by Dormoy in 1872 and Bertrand in 1889.
- Borel called Bertrand's study "extremely incomplete," but it motivated Borel to develop game theory in the 1920s.
- Von Neumann planned to study baccara after proving the minimax theorem in 1928, but he didn't.
- The first game-theoretic solution was by Kemeny and Snell in 1957.
- In 1964, Foster gave a solution based on a new algorithm, unaware of the Kemeny–Snell solution.
- A solution under more realistic assumptions was found by Downton and Lockwood in 1975 using Foster's algorithm.
- Based on the extensive form of the game, the Kemeny–Snell solution was rederived by Deloche and Oguer in 2007.

THÉORIE MATHÉMATIQUE

DU

# JEU DE BACCARAT

PAR

**M. EMILE DORMOY**

Ingénieur des mines

AVEC UNE PRÉFACE PAR M. FRANCISQUE SARCEY

PARIS

ARMAND ANGER, LIBRAIRE-ÉDITEUR

48, RUE LAFFITTE, 48

1875

# The rules of baccara chemin de fer

- We consider the classical parlor game, not the modern casino game.
- Dealt from a *sabot*, or shoe, containing six 52-card decks.
- Cards A, 2–9, 10, J, Q, K have values 1, 2–9, 0, 0, 0, 0.
- The total of a hand, comprising two or three cards, is the sum of the values, *modulo* 10 (i.e., only the last digit of the sum is used).
- Two cards are dealt face down to Player, two face down to Banker.
- If either Player or Banker has a two-card total of 8 or 9 (a *natural*), the game is over.
- If not, Player has the option of requesting a third card, dealt face up.
- Player must draw to 4 or less, stand on 6 and 7. Player has a choice on 5.
- Then Banker, seeing Player's third card if any, has the option of requesting a third card. (There are no restrictions on Banker's strategy under classical rules.)
- Hands are compared, with the total closer to 9 winning. A tie is a push.

# Other forms of baccarat

- In the modern casino game of baccara chemin de fer, Banker's strategy is highly constrained, and the casino collects a 5% commission on Banker wins. This is a two-person game but but not a zero-sum game, as we will point out below.
- Baccara en banque or baccara à deux tableaux was popularized by the Prince of Wales (in the Royal Baccarat Scandal of 1891) and by Nicolas Zographos (of the Greek Syndicate of the post-WWI, pre-WWII era). Here two Player hands compete against one Banker hand, so this is a three-person game.
- In punto banco (modern casino baccarat), there are no strategy choices for Player or Banker, so this is not a strategic game. However, house-banked bets are available on Player as well as on Banker. A winning Banker bet pays 19 to 20 (equivalent to a 5% commission on Banker wins).

# The concept of a matrix game

An $m \times n$ *matrix game* is a two-person zero-sum strategic game in which Player I has $m$ pure strategies and Player II has $n$ pure strategies. It is specified by the *payoff matrix*

$$\boldsymbol{A} = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ m \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & \cdots & n \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array},$$

the $(i,j)$th entry of which is the expected payoff to player I if player I adopts pure strategy $i$ and player II adopts pure strategy $j$.

Example (Rock-Scissors-Paper)

$$\boldsymbol{A} = \begin{array}{c} \\ R \\ S \\ P \end{array} \begin{array}{c} \begin{array}{ccc} R & S & P \end{array} \\ \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \end{array}$$

# Baccarat: The parlor game vs. the casino game

| parlor game | | |
|---|---|---|
| event | (Player payoff, Banker payoff) | sum of payoffs |
| Player win | $(1,-1)$ | 0 |
| Banker win | $(-1,1)$ | 0 |
| Tie | $(0,0)$ | 0 |

| casino game | | |
|---|---|---|
| event | (Player payoff, Banker payoff) | sum of payoffs |
| Player win | $(1,-1)$ | 0 |
| Banker win | $(-1,19/20)$ | $-1/20$ |
| Tie | $(0,0)$ | 0 |

# Modeling baccara as a matrix game

We classify baccara models in two ways. First, according to how cards are dealt.

- Model A. Cards are dealt with replacement from a single deck.
- Model B. Cards are dealt without replacement from a $d$-deck shoe.

Second, according what Player and Banker see.

- Model 1. Each of Player and Banker sees the total of his own two-card hand but not its composition. $(2 \times 2^{88})$
- Model 2. Banker sees the composition of his own two-card hand while Player sees only his own total. $(2 \times 2^{484})$
- Model 3. Each of Player and Banker sees the composition of his own two-card hand. $(2^5 \times 2^{484})$

Model A1: Kemeny and Snell (1957).
Model B2: Downton and Lockwood (1975).
Model B3: our focus.

# What was the purpose of this research?

We wanted to answer the following questions:

- What do the exactly optimal composition-dependent strategies, for Player and Banker, look like? [The answer is that each of Player and Banker mixes two of his pure strategies.]
- Would software not available in 1975 (namely, computer-algebra software such as *Mathematica*) allow a more complete solution than that of Downton and Lockwood? [The answer is yes.]
- Most important, is there an algorithm for solving a large game (e.g., $2^5 \times 2^{484}$) of the type that appears in baccara? (Foster's algorithm applies only to certain $2 \times 2^n$ games.) [The answer is yes, not necessarily in general but at least in this case.]

# Banker strategy under Model A1

First step: Eliminate strictly dominated Banker strategies.

| Banker's total | Player's third card ($\varnothing$ if Player stands) 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | $\varnothing$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0, 1, 2 | D | D | D | D | D | D | D | D | D | D | D |
| 3 | D | D | D | D | D | D | D | D | S | $*$ | D |
| 4 | S | $*$ | D | D | D | D | D | D | S | S | D |
| 5 | S | S | S | S | $*$ | D | D | D | S | S | D |
| 6 | S | S | S | S | S | S | D | D | S | S | $*$ |
| 7 | S | S | S | S | S | S | S | S | S | S | S |

D means draw, S means stand

$*$ means uncertain—depends on Player's strategy

This reduces the game from $2 \times 2^{88}$ to $2 \times 2^{4}$.

# Solution under Model A1

Second step: Solve the $2 \times 16$ game.

Player:

5: $(\mathrm{D}, \mathrm{S})$ with probabilities $(9/11, 2/11)$.

Banker: Follow strategy in preceding table, except

$(3, 9)$: D
$(4, 1)$: S
$(5, 4)$: D
$(6, \varnothing)$: $(\mathrm{D}, \mathrm{S})$ with probabilities $(859/2288, 1429/2288)$.

Value of game to Player (pushes included):

$$V = -\frac{679\,568}{53\,094\,899} \approx -0.0127991.$$

# Banker strategy under Model B3, $d = 6$

First step: Eliminate strictly dominated Banker strategies.

| Banker's total | Banker's hand | Player's third card ($\varnothing$ if Player stands) 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | $\varnothing$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $0, 1, 2$ | | D | D | D | D | D | D | D | D | D | D | D |
| 3 | | D | D | D | D | D | D | D | D | S | $*$ | D |
| 4 | $(0,4), (1,3), (5,9)$ | S | S | D | D | D | D | D | D | S | S | D |
| 4 | $(2,2), (6,8), (7,7)$ | S | $*$ | D | D | D | D | D | D | S | S | D |
| 5 | $(0,5), (6,9), (7,8)$ | S | S | S | S | $*$ | D | D | D | S | S | D |
| 5 | $(1,4), (2,3)$ | S | S | S | S | S | D | D | D | S | S | D |
| 6 | $(3,3)$ | S | S | S | S | S | S | $*$ | D | S | S | $*$ |
| 6 | all others | S | S | S | S | S | S | D | D | S | S | $*$ |
| 7 | | S | S | S | S | S | S | S | S | S | S | S |

This reduces the game from $2^5 \times 2^{484}$ to $2^5 \times 2^{18}$.

# Solution under Model B3, $d = 6$ (Baccara solved!)

| Player's two-card hand | optimal move |
|---|---|
| $(0,5), (6,9), (7,8)$ | D |
| $(1,4)$ | (D, S) |
| $(2,3)$ | S |

| Banker's total | Player's third card ($\varnothing$ if Player stands) | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | $\varnothing$ |
| 0, 1, 2 | D | D | D | D | D | D | D | D | D | D | D |
| 3 | D | D | D | D | D | D | D | D | S | D | D |
| 4 | S | S | D | D | D | D | D | D | S | S | D |
| 5 | S | S | S | S | $*$ | D | D | D | S | S | D |
| 6 | S | S | S | S | S | S | D | D | S | S | $\dagger$ |
| 7 | S | S | S | S | S | S | S | S | S | S | S |

# Footnotes and parameters

*Banker's two-card total is 5 and Player's third card is 4: D on $(0,5)$, $(6,9)$, $(7,8)$, S on $(1,4)$, $(2,3)$.

†Banker's two-card total is 6 and Player stands: $(\mathrm{D},\mathrm{S})$ on $(0,6)$, S on $(1,5)$, $(2,4)$, $(3,3)$, $(7,9)$, D on $(8,8)$.

Player draws on $(1,4)$ with probability

$$p = \frac{35\,003}{74\,880} \approx 0.467455,$$

and Banker draws on $(0,6)$, when Player stands, with probability

$$q = \frac{18\,885\,571}{36\,781\,056} \approx 0.513459.$$

The value of the game (to Player) is

$$V = -\frac{73\,356\,216\,203\,119}{5\,712\,649\,844\,821\,920} \approx -0.0128410.$$

# What is missing from this presentation?

- Solution under Model B3 for every positive integer $d$. For $d \geq 9$, the parameters $p$, $q$, and $V$ are given by formulas (rational functions of $d$).
- Solutions under Models B1 and B2 for every positive integer $d$. Here the solutions can be shown to be unique.
- Solutions under Models A2 and A3. Here there are multiple optimal strategies.
- Proofs of optimality in all cases.
- Want more detail? Then download our (28-page) paper at `http://arxiv.org/abs/1305.5468`.