A Fast and Simple Algorithm for Computing M Shortest Paths in Stage Graph

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A FAST AND SIMPLE ALGORITHM FOR COMPUTING M SHORTEST PATHS IN STAGE GRAPH

We consider the problem of computing $m$ shortest paths between a source node $s$ and a target node $t$ in a stage graph. Polynomial time algorithms known to solve this problem use complicated data structures. This paper proposes a very simple algorithm for computing all $m$ shortest paths in a stage graph efficiently. The proposed algorithm does not use any complicated data structure and can be implemented in a straightforward way by using only array data structure. This problem appears as a sub-problem for planning risk reduced multiple k-legged trajectories for aerial vehicles.

1. INTRODUCTION

Computing the shortest collision-free path in the presence of obstacles is a well investigated problem in algorithm analysis and robotics. Rather efficient algorithms are known to solve this problem in two dimensions and the problem is intractable in three dimensions [1-4]. One of the effective approaches for solving the collision-free path problem is to model it as a path problem in weighted graphs. A graph structure called the visibility graph is very effective in capturing shortest paths in two dimensions. In fact, it is known that the shortest collision-free path connecting two points in the presence of two dimensional obstacles is contained in the corresponding visibility graph [7]. However, the concept of visibility graph does not work in three dimensions for capturing the shortest path.

In recent years, researchers have considered the problem of computing $m$ shortest paths connecting two given nodes in a weighted graph. The objective is to compute the shortest path, second shortest path, third shortest path, up to $m^{th}$ shortest path in a weighted graph. Solution for $m$ shortest paths problem has important applications for planning risk reduced trajectories for aerial vehicles [4,5]. In real world applications, several short length collision-free paths are required. In a changing environment, a selected collision-free path may become forbidden due to the emergence of new obstacle. In such situations it would be highly desirable to have alternative short length collision-free paths. All known fast algorithms for computing $m$ shortest paths use sophisticated data structures [3]. For reducing the use of sophisticated data structures we propose an algorithm for computing $m$ shortest paths in a stage graph. The paper proposes a very simple algorithm that can be implemented in a straightforward way by using only array data structure.

2. PRELIMINARIES

Consider a stage graph $G(V,E)$ where the vertices are partitioned into $k$ stages $V_0, V_1, ..., V_{k-1}$ and the target vertex $V_k$. The first $(V_0)$ and the last $(V_k)$ stages of the graph contain source and target vertex $s$ and $t$, respectively. To compute the shortest path in such graphs, we introduce edges between the vertices of each stage of the graph with weights equal to infinity. This forms a complete weighted graph $G(V,E')$ where $E'$ is the set of edges introduced between the vertices of each stage. Figure 1 illustrates a stage graph $G(V,E')$ with $k = 3$ stages.

Let $W(v_i)$ denote the weight of vertex $v_i$ and let $w(v_{i,j}, v_{i+1,j})$ denote the weight of edge $(v_{i,j}, v_{i+1,j})$.
The main contribution of this paper is the development of a fast and simple algorithm for computing all $m$ shortest paths in a stage graph. The proposed algorithm outputs all $m$ shortest paths in $O(km^3 \log m)$ time using only array data structure and can be very easily implemented.

2. PRELIMINARIES

Consider a stage graph $G(V,E)$ where $V$ is the set of vertices and $E$ is the set of edges. In a stage graph the vertices are partitioned into $k+2$ disjoint sets (stages) $V_0, V_1, V_2, \ldots, V_k, V_{k+1}$. The first ($V_0$) and the last ($V_{k+1}$) stage contain one vertices each, the source vertex $s$ and the target vertex $t$, respectively. The vertices in stage $i$ ($1 \leq i \leq k$) are denoted as $V_{i,1}, V_{i,2}, V_{i,3}, \ldots, V_{i,m}$. In a stage graph edges are present only between the vertices of consecutive stages. Figure 1 illustrates a stage graph and the notations.

For clarity of presentation, we assume without loss of generality that the number of vertices in each stage is the same and is equal to $m$. If not, we can introduce extra vertices in stages containing fewer than $m$ vertices to make the number of vertices equal throughout. We then introduce edges between the extra vertices and the vertices of the previous/next stages with weights equal to infinity. This weight assignment ensures that the shortest route will never go through any of the extra vertices. We could apply Dijkstra’s algorithm on the graph to compute the shortest path which would take $O(km^2)$ time. However we can exploit the structure of the graph to obtain a faster algorithm.

Let $W(v_{ij})$ denote the weight of the shortest path from source vertex $s$ to the $j^{th}$ vertex in stage $i$ and let $w(v_{ij}, v_{il,j'})$ denote the weight of the edge connecting vertex $v_{ij}$ to vertex $v_{il,j'}$. All known fast algorithms for computing $m$ shortest paths reported in the literature use very complicated data structures [3]. For reducing asymptotic complexity, researchers are forced to use highly sophisticated data structures which are extremely hard to implement. The main contribution of this paper is the development of a fast and simple algorithm for computing all $m$ shortest paths in a stage graph. The proposed algorithm outputs all $m$ shortest paths in $O(km^3 \log m)$ time using only array data structure and can be very easily implemented.
Suppose we know the shortest path from source vertex \( s \) to all vertices \( v_1, v_2, \ldots, v_m \) in region \( i \). Observe that the weight of the shortest path from \( s \) to \( v_{i+1,j} \) can be written as:

\[
W(v_{i+1,j}) = \min\{W(v_{i,j}) + w(v_{i,j}, v_{i+1,j}), W(v_{i+1,1}) + w(v_{i+1,1}, v_{i+1,j}), \ldots, W(v_{i,m}) + w(v_{i,m}, v_{i+1,j})\}
\]

**Lemma 2:** If we know the weights of the shortest paths from source vertex \( s \) to all vertices in stage \( i \), then we can compute the weights of the shortest paths from \( s \) to all vertices in stage \( V_{i+1} \) in \( O(m^2) \) time where \( m \) is the number of vertices per leg region.

**Proof:** (Omitted)

By repeating the above lemma on stages from left to right, starting from the first stage, the weight of the shortest path from \( s \) to \( t \) can be constructed in a straightforward way. A formal sketch of the algorithm is as listed below.

**Stage-Step Shortest Path Algorithm**

**Input:** Weighted Stage Graph \( G(V,E) \)

**Output:** Weight and description of the shortest \( k \)-legged path connecting \( s \) and \( t \).

**Step 1:** //Compute the path weights from \( s \) to each vertex in stage 1.

\[
\text{for } (j=1; j<=m; j++) \quad W(v_{1,j}) = w(s, v_{1,j});
\]

**Step 2:** Compute the path weight to each vertex in all subsequent stages.

\[
\text{for } (i=1; i<k; i++) \quad \text{for } (j=1; j<=m; j++) \quad W(v_{i+1,j}) = \min\{W(v_{i,j}) + w(v_{i,j}, v_{i+1,j}), W(v_{i+1,1}) + w(v_{i+1,1}, v_{i+1,j}), \ldots, W(v_{i,m}) + w(v_{i,m}, v_{i+1,j})\};
\]

**Step 3:** //Compute the path weight to \( t \).

\[
W(t) = \min\{W(v_{k,j}) + w(v_{k,j}, t), W(v_{k,2}) + w(v_{k,2}, t), W(v_{k,3}, t), \ldots, W(v_{k,m}) + w(v_{k,m}, t)\};
\]

A simple analysis shows that the above Stage Step Shortest Path algorithm takes \( O(km^2) \) time.

### 3. ALGORITHM DEVELOPMENT

A pair of paths connecting \( s \) to \( t \) is obviously the simplest example of multiple paths. It is therefore useful to consider the construction of the second shortest path in the stage graph. The second shortest path and the first shortest path could be completely disjoint in their interior or could share some edges. It is critical to note that the first shortest path and the second shortest path must have at least one edge not common between them; otherwise both paths will be identical. Hence, if we execute the shortest path algorithm on the graph by removing an “appropriate” edge of the shortest path then the resulting path will be the second shortest path. But it is edges of the first shortest path below.

**Edge Elimination Second Shortest Path Algorithm**

**Step 1:** Run the stage-step algorithm on the stage graph \( p_1 \).

**Step 2:** Let \( e_1, e_2, \ldots, e_4 \) denote the edges of the first shortest path.

**Step 3:** Successively delete one edge \( e_i \) from \( p_1 \) and repeat the above algorithm again on the resulting stage graph. Note: replace the previously deleted edge every time.

**Step 4:** The shortest path from \( s \) to \( t \) is the second shortest path \( p_2 \).

A straightforward analysis of the above algorithm shows that the second shortest path can always be constructed.

**Lemma 3:** The edge elimination algorithm correctly computes the second shortest path.

**Proof:** (Omitted)

### 3.1: COMPUTING ALL SHORTEST PATHS

As seen above, computing the shortest paths of the stage graph is difficult because we need to somehow modify our data structure for each stage. Let \( W'(v_{i,j}) \) denote the weight of the new edges connecting \( v_{i,j} \) to its neighboring vertices. Let the shortest path from \( s \) to \( t \) correspond to the second shortest path in the stage graph results in \( 2m \) possible choices. We can observe that the length of the shortest path weights to \( t \) is the number of connecting edges. Imagine for a particular \( W'(v_{i,j}) \) denote the weight of the new edges connecting \( v_{i,j} \) to its neighboring vertices. Let the shortest path from \( s \) to \( t \) correspond to the second shortest path in the stage graph results in \( 2m \) possible choices. We can observe that the length of the shortest path weights to \( t \) is the number of connecting edges. Imagine for a particular
second shortest path. But it is not clear how to identify the appropriate edge. So we try all edges of the first shortest path one by one. The algorithm based on this approach is sketched below.

Edge Elimination Second Shortest Path Algorithm

Step 1: Run the stage-step algorithm on the stage graph to determine the first shortest path $p_1$.

Step 2: Let $e_0, e_1, ..., e_k$ denote the edges of $p_1$.

Step 3: Successively delete one edge at a time ($e_0, e_1, ..., e_k$) from $p_1$ and run Stage Step Algorithm again on the resulting graph to create a pool of potential second shortest paths. Note: replace the previously deleted edge before deleting a new edge.

Step 4: The shortest path from the pool of potential second shortest paths is the second shortest path $p_2$.

A straightforward analysis of the edge elimination second shortest path algorithm reveals that the second shortest path can be computed in $O(k^2m^2)$ time, if the weighted graph is available.

Lemma 3: The edge elimination second shortest path algorithm constructs the second shortest path correctly.

Proof: (Omitted)

3.1: COMPUTING ALL SHORTEST PATHS

As seen above, computing the second and subsequent shortest paths in a k-legged stage graph is difficult because we only capture the shortest path to any particular vertex. We need to somehow modify our data structure to retain the shortest paths into a vertex. Let $W_q(v_i)$ denote the weight of the $q^{th}$ shortest path from source vertex $s$ to $i^{th}$ vertex in stage $i$.

We can observe that the length of the shortest path to a vertex is the minimum of the sums of the shortest path weights to the prior vertices plus the weights of the corresponding connecting edges. Imagine for a moment that each vertex in $i^{th}$ stage is really two co-positioned vertices. Let the shortest path weights to the new vertices vary while the weights of the new edges connecting to vertex $v_{i+1,1}$ remain the same. The weights to the new vertices correspond to the second shortest path weights to the original vertex. Our new graph results in $2m$ possible choices for the shortest path to vertex $v_{i+1,1}$. It is of interest to note that while in the original graph, the second shortest path to vertex $v_{i+1,1}$ might not have been one of the $m$ possible choices, in the new graph, the second shortest path must be
contained in the $2m$ possible choices. Adding an additional vertex representing the third shortest path to each vertex in stage $i$ would result in $3m$ possible choices for the shortest path to vertex $v_{i+1,l}$ and these $3m$ possible choices must include the three shortest paths to vertex $v_{i+1,l}$. We can continue adding vertices until there are $m$ vertices per vertex in stage $i$ for a total of $m^2$ possible shortest paths to vertex $v_{i+1,l}$ which must contain the $m$ shortest paths to vertex $v_{i+1,l}$.

Suppose we know the $m$ shortest paths from source vertex $s$ to all vertices $v_{i,1}, v_{i,2}, ..., v_{i,m}$ in $i$th stage. Observe that the weight of the $q$th shortest path from $s$ to $v_{i,j}$ can be written as:

$$W_q(v_{i,j}) = \min_q \{ W_1(v_{i,1}) + w(v_{i,1}, v_{i+1,j}), \ldots, W_m(v_{i,1}) + w(v_{i,1}, v_{i+1,j}), \ldots, W_1(v_{i,m}) + w(v_{i,m}, v_{i+1,j}), \ldots, W_m(v_{i,m}) + w(v_{i,m}, v_{i+1,j}) \}$$

where, $\min_q \{a_1, a_2, \ldots, a_m\}$ returns the $q$th smallest of the arguments.

By using the above relation, the weights of the $m$ shortest paths from $s$ to all vertices in stage $i+1$ can be found by scanning the weights of edges between stages $i$ and $i+1$ and adding them to the weights of the $m$ shortest paths from $s$ to stages $i$. The time to compute weights $(W_1(v_{i+1,1}), W_2(v_{i+1,1}), \ldots, W_{m^2}(v_{i+1,1}), (W_1(v_{i+1,2}), W_2(v_{i+1,2}), \ldots, W_{m^2}(v_{i+1,2}), \ldots, (W_1(v_{i+1,m}), W_2(v_{i+1,m}), \ldots, W_{m^2}(v_{i+1,m}))$ is bounded by the time to sort the number of edges connecting vertices in $V_i$ to vertices in $V_{i+1}$ multiplied by the number of weights per vertex. Each vertex is taken as an array of size $m$ to record entries for $m$ shortest paths. Each element in the this array contains the weight and index of the $q$th shortest path from source vertex $s$ to vertex $v_{i,j}$. The following algorithm is a modified version of the stage-step algorithm and computes all $m$ shortest paths. In the algorithm the function $\text{ordered}(a_1, a_2, a_3, \ldots, a_j)$ returns the $q$th smallest of the arguments. The array $\text{paths}[1 \ldots m^2]$ stores the sorted weights of $m^2$ shortest paths coming from the previous stage.

3.2 STAGE-STEP ALGORITHM FOR $M$ SHORTEST PATHS

Input: Weighted Stage Graph $G(V,E)$
Output: Weight and description of the shortest $k$-legged path connecting $s$ and $t$.

Step 1: //Compute the path weights from $s$ to each vertex in stage $V_i$.
for ($j=1; j<=m; j+1$) $W_i(v_{i,j}) = w(s, v_{i,j});$

Step 2: //Compute weights of $m$ shortest paths weights to subsequent stages.
for ($i=1; i<k; i+1$) {
    for ($q=1; q<=m; q+1$) $W_i(v_{i,q}) = \text{ordered}(W_i(v_{i,1}) + w(v_{i,1}, v_{i+1,q}), \ldots, W_i(v_{i,m}) + w(v_{i,m}, v_{i+1,q})$)
}

Theorem 2: Given a stage-graph $G(V,E)$, computing all shortest paths is $O(m \log m)$ time.
Proof: (Omitted)

4. CONCLUSION

We presented a fast and easy-to-program algorithm for computing all shortest paths in a stage graph. The scheme can be adapted to programming language. It computing all shortest paths somehow as a $k$ stage graph, generate approximate solution to how the proposed technique triangulation.

REFERENCES

vertex representing the third choice for the shortest:
the three shortest paths to vertices per vertex in stage i must contain the shortest
vertices \(v_{1,1}, v_{2,1}, ..., v_{i,m}\) in \(V_{i+1}\) can be written as:

\[
\begin{aligned}
\text{Step 3: Compute the } m\text{ smallest path weights to } t. \\
\text{paths}[1...m^2] = \\
\text{ordered}(W_1(v_{1,1}) + w(v_{1,1}, v_{1,1}), W_2(v_{1,1}) + w(v_{1,2}, v_{1,1}), ..., \\
W_m(v_{1,1}) + w(v_{1,m}, v_{1,1}), \\
W_1(v_{1,2}) + w(v_{1,2}, v_{1,2}), W_2(v_{1,2}) + w(v_{1,2}, v_{1,1}), ..., \\
W_m(v_{1,2}) + w(v_{1,m}, v_{1,1}), \\
... \\
W_1(v_{i,m}) + w(v_{i,m}, v_{i+1,1}), W_2(v_{i,m}) + w(v_{i,m}, v_{i+1,1}), ..., \\
W_m(v_{i,m}) + w(v_{i,m}, v_{i+1,1}) \\
\end{aligned}
\]

for \((q=1; q\leq m; q+1)\) \(w_q(t) = \text{paths}[q]\);

Theorem 2: Given a stage graph, all shortest k-legged paths can be computed in \(O(km^3 \log m)\) time.

Proof: (Omitted)

4. CONCLUSION

We presented a fast and easy to implement algorithm for computing all shortest paths in a stage graph. The sketched algorithm can be directly coded in C/C++ or any other high level programming language. It would be interesting to further investigate this approach for computing all shortest paths on general graphs. If a general graph can be approximated somehow as a stage graph then the application of the proposed algorithm can be used to generate approximate solution for shortest paths problem. It would also be interesting how the proposed technique works for some planar graphs such as the Delaunay triangulation.

REFERENCES


