Self-correcting Kelly strategies for skeptical Bayesians

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• Consider a gamble that returns +20% or -18% with equal probability
• Standard utility theory says you should accept or reject based on your risk aversion
• Expected return +1%
• Standard deviation of return +19%
• Law of large numbers is your friend, if gamble is repeated many times ratio of expected return to standard deviation increases without bound
• Let’s see what really happens...
Simulation of 20 paths of 250 bets

- 50% chance of +20%, 50% chance of -18%
- Vertical axis is wealth starting at 1
- Horizontal axis is number of bets made
- Black line is growth in expected value
- Colored lines are 20 simulated paths of wealth
What happened?

• Most paths go quickly to near zero
• A few paths shoot up far beyond expected value, but eventually crash
• If we ran this longer
  • all paths would go to zero wealth
  • expected value would continue exponential increase
What is the problem?

• Expected return of gamble is +1%
• Median return is -0.8%
  • A win plus a loss leaves you with $1.2 \times 0.82 = 0.984$
  • A loss plus a win is the same
  • Median result is $0.984^{0.5} = 0.992$ or a 0.8% loss
• Law of large numbers is your enemy
  • Expected wealth increases 1% every gamble
  • Median wealth declines 0.8% every gamble
  • Long-term distribution is microscopic chance of astronomical gain, virtual certainty of ruin
  • You need over 52% win rate to break even, as repetitions increase that win rate becomes nearly impossible
It gets worse. . .

- Imagine that the 20 colored paths represent:
  - 20 new traders starting at a prop desk
  - 20 new hedge funds
  - 20 new business initiatives/politicians/military adventures
- Some of them achieve extraordinary success
- Big drawdowns are followed by even bigger recoveries, teaching people to keep the faith and double up after losses
- Successful realizations will attract investment, imitation, larger limits, more freedom
- All will inevitably crash, and crash with far higher exposures than they carried on their upswings
- How does the financial system (or anything) survive?
Go back to our simulation

• Bet wins +20% half the time and loses 18% the other half
• Reduce the size to 5/18 so the bet wins 5.56% half the time and loses 5% the other half
• Now expected return is only 0.28% instead of 1%
• But median return is +0.14% instead of -0.8%
• We make a profit as long as we win 49% of our bets, instead of 52%
• Law of large numbers is our friend again
• Reduced bet simulation using same win/loss results as previous simulation
• Black line is exponential growth at the expected value rate, 0.28% per bet
• Long run results will cluster around growth at half that rate, 0.14%
• Milder ups and downs, no tendency to soar or blow up
Kelly myth #1

Overbetting is the error that leads to predictable crashes
What really changes

• Suppose we gave the original risk takers who were betting +20%/-18% initial capital of 3.6 instead of 1 and let them make the same dollar size bets as before
• Now their gains and losses are +5.56%/-5%
• What if we lied and just told them they had extra capital?
• Obviously it’s not the extra capital that makes a difference, nor the dollar size of the bets
• What matters is how much the risk takers increase their dollar bets after wins and reduce them after losses
Another example

• Suppose someone raised bets 20% after losses and cut them 18% after wins?
• Same average bet size as original bettors since wins and losses are equally likely
• 40% of paths go to 0, 60% go to 2
Kelly insight #1

• Don’t focus on absolute bet size
• Focus on how bets are increased or decreased in response to events
• Dramatic bet increases after success, and decreases after failure, exploit opportunities while limiting downside, but cause volatility drag
• The reverse strategy limits upside and can go to zero, but profits from volatility drag
• Intermediate strategies trade off volatility drag for convexity
Kelly myth #2

There is a single optimal risk strategy
Kelly insight #2

• Risk strategy depends on situation
• Project with fixed upside and resources already spent
  • Decrease risk after success, increase after failure
  • Fail fast or succeed
• Venture with small costs and improbable gains
  • Increase risk faster than Kelly after success, and decrease faster after failure
  • The right tail matters, the median is little better than failure
• Typical intermediate case
  • Right tail is illusory
  • Extended failure to grow or opportunity changes, not losing everything, ends betting
  • Exponential opportunity is key, rate of growth is secondary
Kelly myth #3

Bet size is what matters
Kelly insight #3

• Smaller than Kelly (take chips off the table)
  • You have more wealth left if the venture fails
  • You get the Kelly exponential growth rate, but applied to a lower base
    • For permanent opportunities, not a big loss
    • For transitory opportunities, it can matter a lot

• Larger than Kelly (pretend capital)
  • Wealth can go to zero (or below)
  • You get more out of transitory opportunities

• Common sense rule: Set absolute bet size so you run out of capital at the point you’d give up or get stopped out anyway
Masters of the universe

• Consider the highest peak in the first simulation, the light blue line with a wealth of 29 after the 198th bet

• This risk taker had won 112 of 198 bets, 57%
Skeptical Bayesian

• You can show her that she paid more in volatility drag than she gained from betting aggressively
• She would have done better with smaller bets
• At an assumed 57% win rate, even her optimal bet size would lead to blow up given the actual 50% win rate
• Key is to persuade her to change bet size based on Bayesian principles
Some quick math

• Conjugate prior for Bernoulli probability is Beta distribution
• Has convenient properties
  • Parameters $w$ and $a$ are equivalent to observing $w$ successes in $a$ attempts before seeing evidence
  • Observing $W$ wins in $A$ attempts leads to a posterior expectation of $p = (W + w) / (A + a)$
  • Optimal Kelly bet is just the bet at the expected $p$
  • Wealth after a series of Kelly bets based on this prior does not depend on the order of the wins and losses
• Example, bet $+20\%$ or $-20\%$, estimate $p = 0.6$
Back to master of the universe

- Won 112 of 198 bets, 57%
- Kelly bet at $p = 0.57$ is 97% of wealth
- Too high to avoid blow up if actual $p = 0.50$
- If she accepted $p = 5/9 \ (0.55)$, median profit is zero
- Need smaller $p$ to be profitable
- If she accepted $p = 0.50$, profit is maximized
Skeptical

• Observed her cohort of 20 risk-takers averaged 50%
• Admits that best of 20 risk-takers at peak wealth is more likely to have been lucky than unlucky
• Accepts that long-term win rate is probably less than 57% historical rate
• Suppose she is skeptical of your arguments, and will only accept that her true long-term win rate is probably 56% (still too high to survive)
Bayesian

- 56% is consistent with a prior belief of 0, 2
- \( p = \frac{112 + 0}{198 + 2} \)
- There is only a small decrease in current bets compared to \( p = \frac{112}{108} \)
- There is a dramatic difference in long-term returns
- 14, 27 implies a prior expectation of 52% success rate
- \( \frac{112 + 14}{198 + 27} = 56\% \)
Why solve for optimal solutions when you know your model is oversimplified and your parameters are just guesses?

• I’m a quant. That’s what we do.
• Most solutions cannot work. At least a solution that is optimal under some conditions might work.
• Optimal solutions decided under calm conditions and applied consistently avoid the many behavioral biases that sabotage ad hoc solutions.
• Some individuals probably can use intuition to do better than simple optimal quant solutions, but...
  • You don’t know for sure who they are
  • Their abilities can wax and wane, things change
  • You cannot systematically test them against a wide range of situations over a long period of time
  • You cannot continuously improve them
Bayesian Kelly approach

• Parametrize results (e.g. lognormal random walk)
• Set prior distribution over parameters
• Set optimal Kelly bet based on posterior distribution
• Update posterior as results come in
• Assumes
  • Wealth is the constraint
  • Wealth is the goal
  • Probability of zero wealth must be zero
  • Time horizon is infinite
Realistic example

• Hiring a trader, investing in a hedge fund, trying out a computer trading algorithm
• How big to start out?
• How to vary size with success?
• When to give up?
Realistic Bayesian Kelly

• Define terminal events
  • Success: you are confident there is a positive edge
  • Failure: you give up hope of a positive edge
• Given parametrization and prior, simulate results for a range of nominal wealth levels
• Estimate probability of success versus failure
• Estimate gain before success, loss before failure
• Add in future value of success
• Select optimal nominal wealth level