Snackjack: A Toy Model of Blackjack

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Abstract

Snackjack is a highly simplified version of blackjack that was proposed by Ethier (2010) and given its name by Epstein (2013). The eight-card deck comprises two aces, two deuces, and four treys, with aces having value either 1 or 4, and deuces and treys having values 2 and 3, respectively. The target total is 7 (vs. 21 in blackjack), and ace-trey is a natural. The dealer stands on 6 and 7, including soft totals, and otherwise hits. The player can stand, hit, double, or split, but split pairs receive only one card per paircard (like split aces in blackjack), and there is no insurance.

We analyze the game, both single and multiple deck, deriving basic strategy and one-parameter card-counting systems. Unlike in blackjack, these derivations can be done by hand, though it may nevertheless be easier and more reliable to use a computer. More importantly, the simplicity of snackjack allows us to do computations that would be prohibitively time-consuming at blackjack. We can thereby enhance our understanding of blackjack by thoroughly exploring snackjack.

Keywords: Blackjack; grayjack; snackjack; basic strategy; card counting; bet variation; strategy variation

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Introduction

According to Marzuoli (2009),

> Toy models in theoretical physics are invented to make simpler the modelling of complex physical systems while preserving at least a few key features of the originals. Sometimes toy models get a life of their own and have the chance of emerging as paradigms.

For example, the simple coin-tossing games of Parrondo form a toy model of the rather complex flashing Brownian ratchet in statistical physics (see, e.g., Ethier & Lee, 2018). Our aim here is to explore a toy model of the game of blackjack, primarily as a way of gaining insight.

One possible toy model of blackjack is the contrived game of *red-and-black* in which one can bet, at even money, that the next card dealt will be red. This game has been studied by Thorp & Walden (1973), Griffin (1976), Ethier & Levin (2005), and others. Its simplicity allows the card counter to play perfectly, and analysis is straightforward. However, because the game is vastly simpler than blackjack and rather unlike blackjack, the insights it offers are limited.

Epstein, in the first edition of *The Theory of Gambling and Statistical Logic* (1967, p. 269), proposed the game of *grayjack*, a simplified version of blackjack, "offering an insight into the structure of the conventional game." Grayjack uses a 13-card deck comprising one ace, two twos, two threes, two fours, two fives, and four sixes, with aces having value 1 or 7, and the other cards having their nominal values. The target total is 13, and ace-six is a natural. The dealer stands on 11, 12, and 13, including soft totals, and otherwise hits. The player can stand, hit, double, or split, just as in blackjack, but there is no resplitting. The problem with grayjack, as a toy model of blackjack, is that its analysis is only marginally simpler than that of blackjack itself. Both require elaborate computer programs. Fifty years after grayjack was introduced, its basic strategy was still unpublished and perhaps even unknown (but see Appendix A of Ethier & Lee, 2019).

Ethier, in *The Doctrine of Chances* (2010, Problem 21.19), proposed an even simpler toy model of blackjack, which was renamed *snackjack* by Epstein in the latest edition of *TGSL* (2013, p. 291). The eight-card deck comprises two aces, two deuces, and four treys, with aces having value either 1 or 4, and deuces and treys having values 2 and 3, respectively. The target total is 7, and ace-trey is a natural. The dealer stands on 6 and 7, including soft totals, and otherwise hits. The player can stand, hit, double, or split, but split pairs receive only one card per paircard (like split aces in blackjack), and there is no insurance. Unlike blackjack or grayjack, snackjack can be analyzed by hand, though it may nevertheless be easier and more reliable to use a computer. We used a *Mathematica* program to confirm our "by hand" analysis.

Our aim here is to thoroughly analyze the game of snackjack, both single and multiple deck. We do not propose snackjack as a new casino game; rather, we believe that it offers insight into the more complex game of blackjack. Specifically, the simplicity of snackjack allows us to do computations that would be prohibitively time-consuming at blackjack. In addition, snackjack has pedagogical value: The very elaborate theory of blackjack becomes a bit more transparent when viewed through the lens of this simple toy model.

In the next section we give a self-contained description of the rules of snackjack. The third section tries to justify our claim that snackjack, but not blackjack or grayjack, can be analyzed by hand. The fourth section describes our methodology for deriving basic strategy, which is consistent with what was used for single-deck blackjack in Ethier (2010, Section 21.2). It can also be adapted to grayjack. The fifth section applies this methodology to snackjack, both single and multiple deck. The sixth section explores some of the consequences for snackjack of the fundamental theorem of card counting (Thorp & Walden, 1973; Ethier & Levin, 2005). The seventh section investigates card counting at snackjack and its application to bet variation. In all discussions of card counting, we emphasize the

Table 1
Salient features of blackjack, grayjack, and snackjack.

	blackjack	grayjack	snackjack
deck size	52	13	8
deck composition	four aces; four each of 2–9; 16 tens	one ace; two each of 2–5; four sixes	two aces; two deuces; four treys
ace value	1 or 11	1 or 7	1 or 4
target total	21	13	7
natural	ace-ten	ace-six	ace-trey
dealer stands	17–21 (incl. soft)	11–13 (incl. soft)	6–7 (incl. soft)

39-deck (312-card) game. The eighth section explores the potential for gain by varying basic strategy. Finally, the last section summarizes what snackjack tells us about blackjack.

Detailed rules of snackjack

Snackjack is played with a single eight-card deck comprising two aces, two deuces, and four treys, or with multiple such decks mixed together. Aces have value either 1 or 4, and deuces and treys have values 2 and 3, respectively. Suits do not play a role. A hand comprising two or more cards has value equal to the total of the values of the cards. The total is called *soft* if it is at most 7 and the hand contains an ace valued as 4, otherwise it is called *hard*.

Each player competes against the dealer. (In the single-deck game there can be only one player.) After making a bet, each player receives two cards, usually face down (but it does not actually matter), and the dealer receives two cards, one face down (the *downcard* or *hole card*) and one face up (the *upcard*). If the player has a two-card total of 7 (a *natural*) and the dealer does not, the player wins and is paid 3 to 2. If the dealer has a natural and the player does not, the player loses his bet. If both player and dealer have naturals, a push is declared. If the dealer's upcard is an ace or a trey, he checks his downcard to determine whether he has a natural before proceeding. There is no insurance bet in snackjack.

If the dealer and at least one player fail to have naturals, play continues. Starting with the player to the dealer's left and moving clockwise, each player completes his hand as follows. He must decide whether to *stand* (take no more cards) or to *hit* (take an additional card). If he chooses the latter and his new total does not exceed 7, he must make the same decision again and continue to do so until he either stands or *busts* (his total exceeds 7). If the player busts, his bet is lost, even if the dealer subsequently busts. The player has one or two other options after seeing his first two cards. He may *double down*, that is, double his bet and take one, and only one, additional card. If he has two cards of the same value, he may *split* his pair, that is, make an additional bet equal to his initial one and play two hands, with each of his first two cards being the initial card for one of the two hands and each of his two bets applying to one of the two hands. Each card of the split pair receives one and only one card. (This last rule was mistakenly omitted from Ethier, 2010, Problem 21.19. It is needed to avoid the possibility of running out of cards.[1]) A two-card 7 after a

[1] While it is not possible to run out of cards in a one-player vs. dealer single-deck game, it *is* possible that all eight cards are needed. For example, consider a pair of treys against a dealer deuce. Player splits, getting a trey on each trey. Dealer's downcard is also a deuce, and he draws an ace, then another ace, exhausting the deck for a total of six and a double push. This is just one of several scenarios in which the full deck is needed.

split is not regarded as a natural and is therefore not entitled to a 3-to-2 payoff; in addition, it pushes a dealer 7 comprising three or more cards.

As we have already assumed, the dealer checks for a natural when his upcard is an ace or a trey. This is sometimes stated by saying that an untied dealer natural wins original bets only — additional bets due to doubling or splitting, if they could be made, would be pushed.

After each player has stood, busted, doubled down, or split pairs, the dealer acts according to a set of mandatory rules. The dealer stands on hands of 6 and 7, including soft totals, and otherwise hits.

If the dealer busts, all remaining players are paid even money. If the dealer stands, his total is compared to that of each remaining player. If the player's total (which does not exceed 7) exceeds the dealer's total, the player is paid even money. If the dealer's total (which does not exceed 7) exceeds the player's total, the player loses his bet. If the player's total and the dealer's total are equal (and do not exceed 7), the hand is declared a push.

Blackjack vs. grayjack vs. snackjack

We pause to compare blackjack, grayjack, and snackjack. First, it is important to clarify the specific blackjack rules that we assume. The assumed set of rules was at one time standard on the Las Vegas Strip, so we consider it the benchmark against which we can measure other sets of rules. In the notation of the blackjack literature, we assume S17 (dealer stands on soft 17), DOA (double down any first two cards), NDAS (no double after splits), SPA1 (split aces once, receiving only one card per ace), SPL3 (split non-ace pairs up to three times [up to four hands]), 3:2 (untied player natural pays 3 to 2), OBO (untied dealer natural wins original bets only), and NS (no surrender).

As for grayjack, we assume similarly S11, DOA, NDAS, SPA1 (in the case of multiple decks), SPL1 (no resplitting), 3:2, OBO, and NS.

Finally, snackjack rules can be summarized by S6, DOA, NDAS, SPP1 (split pairs once, receiving only one card per paircard), 3:2, OBO, NS, and NI (no insurance).

The more restrictive pair-splitting rules in grayjack and snackjack ensure against running out of cards in the single-deck (one player vs. dealer) games.[2] We maintain these rules in the multiple-deck games even if running out of cards is no longer an issue.

Table 2 (single deck) and Table 3 (multiple deck) compare various statistics for blackjack, grayjack, and snackjack. The aim is to justify our claim that blackjack and grayjack analyses require a computer, whereas a comparable analysis of snackjack, while tedious, does not.

Let us briefly explain these statistics. The number of unordered two-card player hands in blackjack is well known to be $\binom{10}{2}+\binom{10}{1}=55$. Similar calculations apply to grayjack and snackjack, except that a pair of aces is impossible in single-deck grayjack. In single-deck blackjack the number of unordered unbusted player hands (of any size) is 2,008. This is simply the sum over $2 \le n \le 21$ of the number of partitions of the integer n into two or more parts with no part greater than 10 and no part having multiplicity greater than 4. It is the number of hands that we must analyze for composition-dependent basic strategy.

The number of composition-dependent basic strategy decision points is the number of unordered unbusted player hands multiplied by the number of possible dealer upcards, excluding those cases that require more cards than are available. For example, in single-deck snackjack, $A,2,2$ (an ace and two deuces) vs. 2 is ruled out because it requires three deuces, more than are in the deck. Epstein (2013, p. 291) reported 33 decision points because he included a spurious one, namely $A,2,2$ vs. A. Indeed, all basic strategy expectations are conditioned on the dealer not having a natural, but in this case, only treys

[2]To see that SPL2 could result in an incomplete game in single-deck grayjack, consider a player $6,6$ vs. a dealer ace. Player splits, draws another 6, and resplits. First hand is $6,2,2,3$, second hand is $6,4,5$, and third hand is $6,4,5$. Dealer's hand is $A,3,6$, exhausting the deck before completing the hand. We note that player violated basic strategy only when splitting 6s.

Table 2
Single-deck comparisons of blackjack and its toy models.

statistic	blackjack	grayjack	snackjack
number of cards	52	13	8
number of unordered two-card player hands	55	20	6
number of unordered unbusted player hands	2,008 [1]	87	14
number of comp.-dep. basic strategy decision points	19,620 [2]	430	32
number of ordered dealer drawing sequences	48,532 [3]	498	17
number of unordered dealer drawing sequences	2,741 [4]	93	11
mimic-the-dealer strategy expectation	−0.0568456 [5]	−0.0584311	+0.0952381
composition-dep. basic strategy expectation	+0.000412516 [6]	+0.0218749	+0.192857

[1] Epstein (2013, p. 275) or Ethier (2010, p. 655). [2] Ethier (2010, p. 655). [3] Epstein (2013, p. 275), Ethier (2010, pp. 9–11, 649), or Griffin (1999, p. 158). [4] Ethier (2010, pp. 646, 648). [5] Ethier (2010, p. 647). [6] Ethier (2010, p. 661).

remain, so the dealer's downcard must be a trey. In effect, we are conditioning on an event of probability 0, so we must exclude this case.

We can readily compute the number of ordered dealer drawing sequences by direct enumeration. For example, sequence number 24,896 (in reverse lexicographical order) of the 48,532 such sequences in single-deck blackjack is $3,2,2,2,2,\mathrm{A},\mathrm{A},\mathrm{A},\mathrm{A},3$. Without regard to order, this sequence would be listed as $(4,4,2,0,0,0,0,0,0,0)$ (i.e., 4 aces, 4 twos, and 2 threes), with total 18 and multiplicity 15 (i.e., 15 permutations of $3,2,2,2,2,\mathrm{A},\mathrm{A},\mathrm{A},\mathrm{A},3$ appear in the ordered list). We use the ordered dealer dealer drawing sequences to compute conditional expectations when standing. We use the unordered dealer drawing sequences to evaluate the player's expectation under the mimic-the-dealer strategy (Ethier, 2010, p. 647), which depends on P(both player and dealer bust). (The double bust is hypothetical; it assumes that the dealer deals out his hand even after the player busts.) In both blackjack and grayjack, the dealer advantage of acting last (because the dealer wins double busts) dominates the player advantage of a 3-to-2 payoff for an untied natural. In snackjack, the opposite is true because double busts are rare (probability 2/105 in single deck) and winning player naturals are quite common (probability 8/35 in single deck).

Finally, we and others have computed the player's expectation under composition-dependent basic strategy in blackjack, grayjack, and snackjack. Of course it is substantially larger than that for the mimic-the-dealer strategy. In single-deck blackjack this expectation is positive, barely, which may explain why the assumed set of rules is obsolete. In single-deck grayjack it is about +2.19%, well below Epstein's (2013, p. 291) estimate of +7.5%, but the positive expectation nevertheless "mitigates its suitability as a casino game," as Epstein noted. Perhaps 24-deck grayjack (−1.89%) would be viable as a casino game.

Table 3
Multiple-deck comparisons of blackjack and its toy models.

statistic	six-deck blackjack	24-deck grayjack	39-deck snackjack
number of cards	312	312	312
number of unordered two-card player hands	55	21	6
number of unordered unbusted player hands	3,072 [1]	291	27
number of comp.-dep. basic strategy decision points	30,720	1,746	81
number of ordered dealer drawing sequences	54,433 [2]	1,121	21
number of unordered dealer drawing sequences	3,357	257	15
mimic-the-dealer strategy expectation	-0.0567565	-0.0628381	$+0.0720903$
composition-dep. basic strategy expectation	-0.00544565 [3]	-0.0189084	$+0.139309$

[1] Griffin (1999, p. 172). [2] Griffin (1999, p. 158). [3] Computed by Marc Estafanous.

Snackjack (+19.3% for single deck, +13.9% for 39 decks) would certainly not be. But that is not our concern. Instead, we want to gain insight into blackjack by studying snackjack.

Certainly, it would be possible to make a simple rules change that would give the advantage to the house and make snackjack a potential casino game. There are probably many ways to do this, but an especially simple approach would be to impose the rule, "A player natural pays even money [instead of 3 to 2], with the exception that it loses to a dealer natural [instead of pushing]." The result is +3.10% for single deck, −0.0959% for double deck, −0.713% for triple deck, and −1.73% for 39 decks. We do not pursue this, however. Instead, when we want the game to be slightly disadvantageous for the purpose of our card-counting analysis, we impose a suitable commission, specifically 1/7 of the amount initially bet in the 39-deck game, resulting in a net expectation of −0.355%.

Snackjack basic strategy methodology

The term "basic strategy" has several interpretations. See Schlesinger (2018, Appendix A) for a thorough discussion of the issues. We will interpret it as composition-dependent basic strategy, since total-dependent or partially total-dependent basic strategy is an unnecessary compromise in this simple game. Because of our restrictive rules on splitting, we need not concern ourselves with which cards are used for decisions about split hands. We follow the approach originated for blackjack by Manson et al. (1975) and used by Griffin (1999, p. 172) and Ethier (2010, Section 21.2), and we use the notation of the latter source. A completely different approach was taken by Werthamer (2018, Section 7.2.1), who wrote (p. 74), "… no [previous] study describes its methodology in detail …," regrettably overlooking Ethier (2010).[3]

[3] In fairness, the claim first appeared in Werthamer (2009), at which time it was accurate.

A similar approach applies to grayjack, but here we must clarify how splits are treated. Our convention is that the player makes use only of the cards in the hand he is currently playing, and of course the dealer's upcard.

Returning to snackjack, an arbitrary pack is described by $\boldsymbol{n} = (n_1, n_2, n_3)$, meaning that it comprises n_1 aces, n_2 deuces, and n_3 treys, with

$$|\boldsymbol{n}| := n_1 + n_2 + n_3$$

being the number of cards. An unordered player hand is denoted by $\boldsymbol{l} = (l_1, l_2, l_3)$ if it comprises $l_1 \le n_1$ aces, $l_2 \le n_2$ deuces, and $l_3 \le n_3$ treys. The number of cards in the hand is

$$|\boldsymbol{l}| := l_1 + l_2 + l_3,$$

and the hand's total is

$$T(\boldsymbol{l}) := \begin{cases} l_1 + 2l_2 + 3l_3 + 3 & \text{if } l_1 \ge 1 \text{ and } l_1 + 2l_2 + 3l_3 \le 4, \\ l_1 + 2l_2 + 3l_3 & \text{otherwise,} \end{cases}$$

with the two cases corresponding to soft and hard totals. For the hand to be unbusted, $\boldsymbol{l}$ must satisfy $T(\boldsymbol{l}) \le 7$.

Let $\boldsymbol{X}$ denote the player's hand, let Y denote the player's next card, if any, and let U denote the dealer's upcard, D his downcard, and S his final total. Finally, let $G_{\rm std}$, $G_{\rm hit}$, $G_{\rm dbl}$, and $G_{\rm spl}$ denote the player's profit from standing, hitting, doubling, and splitting, assuming an initial one-unit bet.

Here and in what follows, we occasionally denote an ace not by A but by 1.

We denote the events on which we will condition by

$$A(\boldsymbol{l}, u) := \begin{cases} \{\boldsymbol{X} = \boldsymbol{l},\ U = 1,\ D \ne 3\} & \text{if } u = 1, \\ \{\boldsymbol{X} = \boldsymbol{l},\ U = 2\} & \text{if } u = 2, \\ \{\boldsymbol{X} = \boldsymbol{l},\ U = 3,\ D \ne 1\} & \text{if } u = 3, \end{cases}$$

and we define the conditional expectations associated with each player hand, dealer upcard, and strategy:

$$\begin{aligned} E_{\rm std}(\boldsymbol{l}, u) &:= \mathrm{E}[G_{\rm std} \mid A(\boldsymbol{l}, u)], \\ E_{\rm hit}(\boldsymbol{l}, u) &:= \mathrm{E}[G_{\rm hit} \mid A(\boldsymbol{l}, u)], \\ E_{\rm dbl}(\boldsymbol{l}, u) &:= \mathrm{E}[G_{\rm dbl} \mid A(\boldsymbol{l}, u)] \quad (|\boldsymbol{l}| = 2), \\ E_{\rm spl}(\boldsymbol{l}, u) &:= \mathrm{E}[G_{\rm spl} \mid A(\boldsymbol{l}, u)] \quad (\boldsymbol{l} = 2\boldsymbol{e}_i,\ i \in \{1,2,3\}), \end{aligned}$$

where $\boldsymbol{e}_1 := (1,0,0)$, $\boldsymbol{e}_2 := (0,1,0)$, and $\boldsymbol{e}_3 := (0,0,1)$. Temporarily, we define the maximal stand/hit conditional expectation for each player hand and dealer upcard by

$$E^*_{\max}(\boldsymbol{l}, u) := \max\{E_{\rm std}(\boldsymbol{l}, u), E_{\rm hit}(\boldsymbol{l}, u)\}. \tag{1}$$

We specify more precisely the set of player hands and dealer upcards we will consider. We denote the set of all unordered unbusted player hands of two or more cards by

$$\mathscr{L} := \{\boldsymbol{l} \le \boldsymbol{n} : |\boldsymbol{l}| \ge 2,\ T(\boldsymbol{l}) \le 7\}$$

and the set of all pairs of such hands and dealer upcards by

$$\mathscr{M} := \{(\boldsymbol{l}, u) \in \mathscr{L} \times \{1,2,3\} : l_u \le n_u - 1\}.$$

The cardinality of $\mathscr{L}$ is the sum over $2 \le n \le 7$ of the number of partitions of the integer n into two or more parts with no part greater than 3 and 1s having multiplicity at most n_1, 2s having multiplicity at most n_2, and 3s having multiplicity at most n_3.

The basic relations connecting the conditional expectations defined above include, for all $(\boldsymbol{l},u)\in\mathscr{M}$,

$$E_{\rm std}(\boldsymbol{l},u)=\mathrm{P}(S<T(\boldsymbol{l})\text{ or }S>7\mid A(\boldsymbol{l},u))-\mathrm{P}(T(\boldsymbol{l})<S\le 7\mid A(\boldsymbol{l},u)), \tag{2}$$

$$\begin{aligned}E_{\rm hit}(\boldsymbol{l},u)=&\sum_{1\le k\le 3:\ (\boldsymbol{l}+\boldsymbol{e}_k,u)\in\mathscr{M}}p(k\mid\boldsymbol{l},u)E^*_{\max}(\boldsymbol{l}+\boldsymbol{e}_k,u)\\&+\sum_{1\le k\le 3:\ \boldsymbol{l}+\boldsymbol{e}_k\notin\mathscr{L}}p(k\mid\boldsymbol{l},u)(-1),\end{aligned} \tag{3}$$

$$\begin{aligned}E_{\rm dbl}(\boldsymbol{l},u)=&\,2\sum_{1\le k\le 3:\ (\boldsymbol{l}+\boldsymbol{e}_k,u)\in\mathscr{M}}p(k\mid\boldsymbol{l},u)E_{\rm std}(\boldsymbol{l}+\boldsymbol{e}_k,u)\\&+2\sum_{1\le k\le 3:\ \boldsymbol{l}+\boldsymbol{e}_k\notin\mathscr{L}}p(k\mid\boldsymbol{l},u)(-1)\qquad(|\boldsymbol{l}|=2),\end{aligned} \tag{4}$$

$$E_{\rm spl}(2\boldsymbol{e}_i,u)=2\sum_{1\le k\le 3}p(k\mid 2\boldsymbol{e}_i,u)E_{\rm std}(\boldsymbol{e}_i+\boldsymbol{e}_k,u\mid\boldsymbol{e}_i)\qquad(i=1,2,3), \tag{5}$$

where

$$p(k\mid\boldsymbol{l},u):=\mathrm{P}(Y=k\mid A(\boldsymbol{l},u)).$$

The probabilities $p(k\mid\boldsymbol{l},u)$ are derived from Bayes' law for $u=1$ and $u=3$:

$$p(k\mid\boldsymbol{l},1)=\frac{n_k-l_k-\delta_{1,k}}{|\boldsymbol{n}|-|\boldsymbol{l}|-1}\left(\frac{1-(n_3-l_3-\delta_{3,k})/(|\boldsymbol{n}|-|\boldsymbol{l}|-2)}{1-(n_3-l_3)/(|\boldsymbol{n}|-|\boldsymbol{l}|-1)}\right), \tag{6}$$

$$p(k\mid\boldsymbol{l},2)=\frac{n_k-l_k-\delta_{2,k}}{|\boldsymbol{n}|-|\boldsymbol{l}|-1}, \tag{7}$$

$$p(k\mid\boldsymbol{l},3)=\frac{n_k-l_k-\delta_{3,k}}{|\boldsymbol{n}|-|\boldsymbol{l}|-1}\left(\frac{1-(n_1-l_1-\delta_{1,k})/(|\boldsymbol{n}|-|\boldsymbol{l}|-2)}{1-(n_1-l_1)/(|\boldsymbol{n}|-|\boldsymbol{l}|-1)}\right), \tag{8}$$

where $\delta_{u,k}$ is the Kronecker delta. Equation (5) comes from Ethier (2010, Eq. (21.53)) and requires a slight extension of our notation. We define $E_{\rm std}(\boldsymbol{l},u\mid\boldsymbol{m})$ analogously to $E_{\rm std}(\boldsymbol{l},u)$, but with $\boldsymbol{m}=(m_1,m_2,m_3)$ indicating that the initial pack is depleted by removing m_1 aces, m_2 deuces, and m_3 treys (in addition to the cards in the player's hand and the dealer's upcard). Thus, $E_{\rm std}(\boldsymbol{l},u)=E_{\rm std}(\boldsymbol{l},u\mid\boldsymbol{0})$.

The quantities (2) are computed directly, while those in (3) are obtained recursively. They are recursive in the player's hard total

$$T_{\rm hard}(\boldsymbol{l}):=l_1+2l_2+3l_3.$$

The recursion is initialized with

$$E_{\rm hit}(\boldsymbol{l},u)=-1,\qquad(\boldsymbol{l},u)\in\mathscr{M},\ T_{\rm hard}(\boldsymbol{l})=7. \tag{9}$$

There is one exception to (2) because an untied player natural is paid 3 to 2:

$$E_{\rm std}(\boldsymbol{e}_1+\boldsymbol{e}_3,u)=\frac{3}{2},\qquad u=1,2,3. \tag{10}$$

We begin by computing $E_{\rm std}(\boldsymbol{l},u)$ for all $(\boldsymbol{l},u)\in\mathscr{M}$ using (2) (except for (10)). The number of ordered dealer drawing sequences that must be analyzed for each such $\boldsymbol{l}$ is at most 21. Then we go back and compute $E^*_{\max}(\boldsymbol{l},u)$ of (1) for $T_{\rm hard}(\boldsymbol{l})=7,6,5,4,3,2$ (in that order) and all u using (2), (3), (6)–(8), and (9). Finally, we compute $E_{\rm dbl}(\boldsymbol{l},u)$ using (2), (4), and (6)–(8), and $E_{\rm spl}(\boldsymbol{l},u)$ using (2), (5), and (6)–(8). We can finally evaluate

$$E_{\max}(\boldsymbol{l},u):=\begin{cases}\max\{E_{\rm std}(\boldsymbol{l},u),E_{\rm hit}(\boldsymbol{l},u),E_{\rm dbl}(\boldsymbol{l},u),E_{\rm spl}(\boldsymbol{l},u)\} & \text{if } \boldsymbol{l}=2\boldsymbol{e}_i,\\ \max\{E_{\rm std}(\boldsymbol{l},u),E_{\rm hit}(\boldsymbol{l},u),E_{\rm dbl}(\boldsymbol{l},u)\} & \text{if } \boldsymbol{l}=\boldsymbol{e}_i+\boldsymbol{e}_j,\\ \max\{E_{\rm std}(\boldsymbol{l},u),E_{\rm hit}(\boldsymbol{l},u)\} & \text{if } |\boldsymbol{l}|\ge 3,\end{cases}$$

where $i \in \{1,2,3\}$ in the first line and $i, j \in \{1,2,3\}$ with $i < j$ in the second.

We can also evaluate the player's overall expectation E using the optimal strategy thus derived. It is simply a matter of conditioning on the player's initial two-card hand and the dealer's upcard. Now the 18 events $A(\boldsymbol{l}, u)$ for $|\boldsymbol{l}| = 2$ and $u = 1,2,3$ do not partition the sample space, but if we include the 12 events

$$B(\boldsymbol{e}_i + \boldsymbol{e}_j, 1) := \{\boldsymbol{X} = \boldsymbol{e}_i + \boldsymbol{e}_j,\ U = 1,\ D = 3\}$$
$$B(\boldsymbol{e}_i + \boldsymbol{e}_j, 3) := \{\boldsymbol{X} = \boldsymbol{e}_i + \boldsymbol{e}_j,\ U = 3,\ D = 1\}$$

as well, where $1 \le i \le j \le 3$, then we do have a partition, and conditioning gives the desired result, namely

$$\begin{aligned} E &= \sum_{u=1}^{3} \sum\sum_{1\le i\le j\le 3} \mathrm{P}(A(\boldsymbol{e}_i + \boldsymbol{e}_j, u)) E_{\max}(\boldsymbol{e}_i + \boldsymbol{e}_j, u) + \sum\sum_{1\le i\le j\le 3:\ (i,j)\ne(1,3)} \mathrm{P}(B(\boldsymbol{e}_i + \boldsymbol{e}_j, 1))(-1) \\ &\quad + \sum\sum_{1\le i\le j\le 3:\ (i,j)\ne(1,3)} \mathrm{P}(B(\boldsymbol{e}_i + \boldsymbol{e}_j, 3))(-1) + \mathrm{P}(B(\boldsymbol{e}_1 + \boldsymbol{e}_3, 1))(0) + \mathrm{P}(B(\boldsymbol{e}_1 + \boldsymbol{e}_3, 3))(0) \\ &= \sum_{u=1}^{3} \sum\sum_{1\le i\le j\le 3} \mathrm{P}(A(\boldsymbol{e}_i + \boldsymbol{e}_j, u)) E_{\max}(\boldsymbol{e}_i + \boldsymbol{e}_j, u) - \frac{\binom{n_1}{1}\binom{n_3}{1}}{\binom{|\boldsymbol{n}|}{2}} \left(1 - \frac{\binom{n_1-1}{1}\binom{n_3-1}{1}}{\binom{|\boldsymbol{n}|-2}{2}}\right). \end{aligned}$$

The second equality uses the fact that the union of the events $B(\boldsymbol{e}_i + \boldsymbol{e}_j, u)$ ($1 \le i \le j \le 3$, $(i,j) \ne (1,3)$, $u \in \{1,3\}$) is the event that the dealer has a natural and the player does not. Finally, we observe that, for $1 \le i < j \le 3$ or $1 \le i \le 3$,

$$\begin{aligned} \mathrm{P}(A(\boldsymbol{e}_i + \boldsymbol{e}_j, 1)) &= \frac{\binom{n_i}{1}\binom{n_j}{1}}{\binom{|\boldsymbol{n}|}{2}} \frac{n_1 - \delta_{1,i} - \delta_{1,j}}{|\boldsymbol{n}| - 2} \left(1 - \frac{n_3 - \delta_{3,i} - \delta_{3,j}}{|\boldsymbol{n}| - 3}\right), \\ \mathrm{P}(A(2\boldsymbol{e}_i, 1)) &= \frac{\binom{n_i}{2}}{\binom{|\boldsymbol{n}|}{2}} \frac{n_1 - 2\delta_{1,i}}{|\boldsymbol{n}| - 2} \left(1 - \frac{n_3 - 2\delta_{3,i}}{|\boldsymbol{n}| - 3}\right), \\ \mathrm{P}(A(\boldsymbol{e}_i + \boldsymbol{e}_j, 2)) &= \frac{\binom{n_i}{1}\binom{n_j}{1}}{\binom{|\boldsymbol{n}|}{2}} \frac{n_2 - \delta_{2,i} - \delta_{2,j}}{|\boldsymbol{n}| - 2}, \\ \mathrm{P}(A(2\boldsymbol{e}_i, 2)) &= \frac{\binom{n_i}{2}}{\binom{|\boldsymbol{n}|}{2}} \frac{n_2 - 2\delta_{2,i}}{|\boldsymbol{n}| - 2}, \\ \mathrm{P}(A(\boldsymbol{e}_i + \boldsymbol{e}_j, 3)) &= \frac{\binom{n_i}{1}\binom{n_j}{1}}{\binom{|\boldsymbol{n}|}{2}} \frac{n_3 - \delta_{3,i} - \delta_{3,j}}{|\boldsymbol{n}| - 2} \left(1 - \frac{n_1 - \delta_{1,i} - \delta_{1,j}}{|\boldsymbol{n}| - 3}\right), \\ \mathrm{P}(A(2\boldsymbol{e}_i, 3)) &= \frac{\binom{n_i}{2}}{\binom{|\boldsymbol{n}|}{2}} \frac{n_3 - 2\delta_{3,i}}{|\boldsymbol{n}| - 2} \left(1 - \frac{n_1 - 2\delta_{1,i}}{|\boldsymbol{n}| - 3}\right), \end{aligned}$$

and the derivation is complete.

Snackjack basic strategy results

In the case of a single deck, $(n_1, n_2, n_3) = (2,2,4)$, so by direct enumeration, $|\mathscr{L}| = 14$ and $|\mathscr{M}| = 32$ after we exclude $((1,2,0),1)$ from $\mathscr{M}$, as explained earlier. The 87 conditional expectations (32 stand, 32 hit, 16 double, 7 split) needed for composition-dependent basic strategy are shown in Table 4. The inner product of the last two columns, divided by 420, is the player's expectation under basic strategy, $27/140 \approx 0.192857$.

To clarify our method for determining composition-dependent basic strategy, we provide several examples of how the conditional expectations in Table 4 were computed. Let us begin with an example, suggested by a reviewer, that does not require much computation. Consider A, 2 vs. A. The dealer's downcard cannot be an ace because both aces have been dealt. It cannot be a trey because that would give the dealer a natural and an automatic win. So the dealer also has A, 2 and will stand with his soft 6. Furthermore, the remainder of the deck has only treys. So whether the player stands, hits, or doubles, he will end up with a soft or hard 6 and will push the dealer's soft 6. The basic strategist therefore is indifferent to standing, hitting, or doubling.

Table 4
Derivation of composition-dependent basic strategy (bs) for single-deck snackjack. For computational convenience, rows are arranged in descending order of the hard total (htot). For a more concise description of basic strategy, refer to Table 5 below.

											two-card hands
no	nos of As, 2s, 3s	htot	tot	up	E_{std}	E_{hit}	E_{dbl}	E_{spl}	bs	$E_{\max}$	$420 \times$ probab
1	(0,2,1)	7	h7	A	1	−1	na	na	S		
2	(0,2,1)	7	h7	3	1	−1	na	na	S		
3	(1,0,2)	7	h7	A	1	−1	na	na	S		
4	(1,0,2)	7	h7	2	2/3	−1	na	na	S		
5	(1,0,2)	7	h7	3	7/9	−1	na	na	S		
6	(2,1,1)	7	h7	2	1	−1	na	na	S		
7	(2,1,1)	7	h7	3	1	−1	na	na	S		
8	(0,0,2)	6	h6	A	−2/9	−2/3	−4/3	−4/9	S	−2/9	18
9	(0,0,2)	6	h6	2	−1/30	−1/3	−2/3	1/5	Spl	1/5	30
10	(0,0,2)	6	h6	3	0	−1/9	−2/9	2/9	Spl	2/9	18
11	(1,1,1)	6	h6	A	0	−1	na	na	S		
12	(1,1,1)	6	h6	2	1/2	−1/2	na	na	S		
13	(1,1,1)	6	h6	3	2/9	−1/3	na	na	S		
14	(2,2,0)	6	h6	3	0	−1	na	na	S		
15	(0,1,1)	5	h5	A	−1/2	−5/8	−5/4	na	S	−1/2	16
16	(0,1,1)	5	h5	2	−2/5	−2/5	−4/5	na	S/H	−2/5	20
17	(0,1,1)	5	h5	3	−2/3	−1/18	−1/9	na	H	−1/18	36
18	(1,2,0)	5	h5	3	−1	−2/3	na	na	H		
19	(2,0,1)	5	h5	2	0	−1/2	na	na	S		
20	(2,0,1)	5	h5	3	−1/3	0	na	na	H		
21	(0,2,0)	4	h4	A	1	1	2	2	D/Spl	2	1
22	(0,2,0)	4	h4	3	−1	1/6	0	−1	H	1/6	6
23	(1,0,1)	4	s7	A	3/2	3/4	3/2	na	S/D	3/2	8
24	(1,0,1)	4	s7	2	3/2	1/2	1	na	S	3/2	40
25	(1,0,1)	4	s7	3	3/2	3/8	7/12	na	S	3/2	48
26	(2,1,0)	4	s7	2	1	1	na	na	S/H		
27	(2,1,0)	4	s7	3	1	3/4	na	na	S		
28	(1,1,0)	3	s6	A	0	0	0	na	S/H/D	0	2
29	(1,1,0)	3	s6	2	3/5	3/5	6/5	na	D	6/5	10
30	(1,1,0)	3	s6	3	3/16	1/4	3/8	na	D	3/8	32
31	(2,0,0)	2	s5	2	1/5	1/5	2/5	6/5	Spl	6/5	5
32	(2,0,0)	2	s5	3	−2/5	2/5	2/5	6/5	Spl	6/5	10
dealer has natural, player does not										−1	96
both player and dealer have naturals										0	24
total											420

Other examples are a little more involved and require some notation. Let us denote by Y the player's next card and by $\mathbf{Z} = (Z_1, Z_2, \dots)$ the dealer's hand beginning with $Z_1 = U$ and $Z_2 = D$. Z_3 and Z_4 would be the third and fourth cards in the dealer's hand if needed.

No more than four cards are ever needed.

First, the conditional expectation when standing with 3, 3 vs. A can be evaluated with a tree diagram. See Figure 1. More formally,

$$\begin{aligned}
E_{\rm std}(2\boldsymbol{e}_3,1) &= {\rm E}[G_{\rm std}\mid \boldsymbol{X}=2\boldsymbol{e}_3,\, U=1,\, D\neq 3)\\
&= {\rm P}(S=6\mid \boldsymbol{X}=2\boldsymbol{e}_3,\, U=1,\, D\neq 3)(0)\\
&\quad + {\rm P}(S=7\mid \boldsymbol{X}=2\boldsymbol{e}_3,\, U=1,\, D\neq 3)(-1)\\
&\quad + {\rm P}(S=8\mid \boldsymbol{X}=2\boldsymbol{e}_3,\, U=1,\, D\neq 3)(1)\\
&= {\rm P}(\boldsymbol{Z}=(1,2)\mid \boldsymbol{X}=2\boldsymbol{e}_3,\, U=1,\, D\neq 3)(0)\\
&\quad + {\rm P}(\boldsymbol{Z}=(1,1,2)\text{ or }(1,1,3,2)\mid \boldsymbol{X}=2\boldsymbol{e}_3,\, U=1,\, D\neq 3)(-1)\\
&\quad + {\rm P}(\boldsymbol{Z}=(1,1,3,3)\mid \boldsymbol{X}=2\boldsymbol{e}_3,\, U=1,\, D\neq 3)(1)\\
&= \frac{2}{3}(0)+\left(\frac{1}{3}\frac{2}{4}+\frac{1}{3}\frac{2}{4}\frac{2}{3}\right)(-1)+\left(\frac{1}{3}\frac{2}{4}\frac{1}{3}\right)(1) = -\frac{2}{9}.
\end{aligned}$$

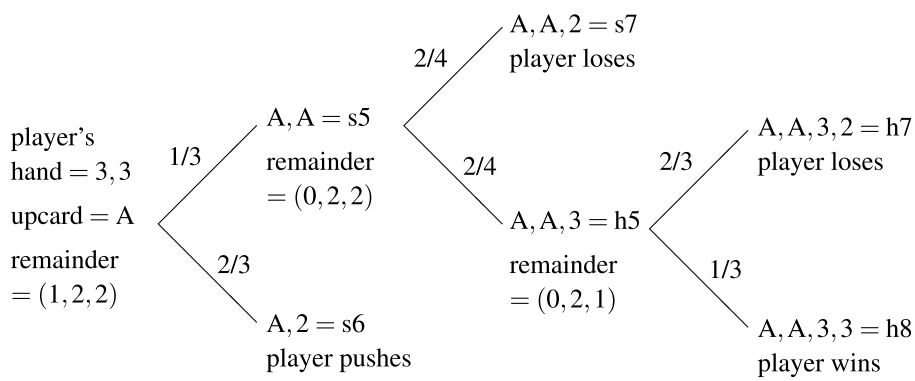

Figure 1
The tree diagram used to evaluate $E_{\rm std}(2\boldsymbol{e}_3,1)$. *(A remainder of* $(1,2,2)$ *means that* 1 *ace,* 2 *deuces, and* 2 *treys remain;* s5 *and* h5 *stand for soft* 5 *and hard* 5.*) Notice that we are conditioning on the dealer not having a natural (i.e., the dealer's downcard is not a trey).*

Second, the conditional expectation when hitting with A, 2 vs. 3 is

$$\begin{aligned}
E_{\rm hit}(\boldsymbol{e}_1+\boldsymbol{e}_2,3) &= {\rm E}[G_{\rm hit}\mid \boldsymbol{X}=\boldsymbol{e}_1+\boldsymbol{e}_2,\, U=3,\, D\neq 1)\\
&= {\rm P}(Y=1\mid \boldsymbol{X}=\boldsymbol{e}_1+\boldsymbol{e}_2,\, U=3,\, D\neq 1)E_{\rm std}((2,1,0),3)\\
&\quad + {\rm P}(Y=2\mid \boldsymbol{X}=\boldsymbol{e}_1+\boldsymbol{e}_2,\, U=3,\, D\neq 1)E_{\rm hit}((1,2,0),3)\\
&\quad + {\rm P}(Y=3\mid \boldsymbol{X}=\boldsymbol{e}_1+\boldsymbol{e}_2,\, U=3,\, D\neq 1)E_{\rm std}((1,1,1),3)\\
&= \frac{1}{5}\,\frac{1-0}{1-\frac{1}{5}}(1)+\frac{1}{5}\,\frac{1-\frac{1}{4}}{1-\frac{1}{5}}\left(-\frac{2}{3}\right)+\frac{3}{5}\,\frac{1-\frac{1}{4}}{1-\frac{1}{5}}\left(\frac{2}{9}\right)=\frac{1}{4},
\end{aligned}$$

where (8) was used to evaluate the conditional probabilities and we have used the facts that $E_{\rm std}((2,1,0),3)$, $E_{\rm hit}((1,2,0),3)$, and $E_{\rm std}((1,1,1),3)$ have already been computed, and are larger than $E_{\rm hit}((2,1,0),3)$, $E_{\rm std}((1,2,0),3)$, and $E_{\rm hit}((1,1,1),3)$, respectively.

Third, the conditional expectation when doubling with A, 3 vs. A is

$$\begin{aligned}
E_{\rm dbl}(\boldsymbol{e}_1+\boldsymbol{e}_3,1) &= {\rm E}[G_{\rm dbl}\mid \boldsymbol{X}=\boldsymbol{e}_1+\boldsymbol{e}_3,\, U=1,\, D\neq 3)\\
&= 2\{{\rm P}(Y=2\mid \boldsymbol{X}=\boldsymbol{e}_1+\boldsymbol{e}_3,\, U=1,\, D\neq 3)E_{\rm std}((1,1,1),1)\\
&\quad + {\rm P}(Y=3\mid \boldsymbol{X}=\boldsymbol{e}_1+\boldsymbol{e}_3,\, U=1,\, D\neq 3)E_{\rm std}((1,0,2),1)\}\\
&= 2\left[\frac{2}{5}\,\frac{1-\frac{3}{4}}{1-\frac{3}{5}}(0)+\frac{3}{5}\,\frac{1-\frac{2}{4}}{1-\frac{3}{5}}(1)\right]=\frac{3}{2},
\end{aligned}$$

using (6).

Finally, the conditional expectation when splitting with 3, 3 vs. 2 is

$$\begin{aligned}
E_{\mathrm{spl}}(2\boldsymbol{e}_3,2) &= 2\sum_{j=1}^{3} \mathrm{P}(Y=j \mid \boldsymbol{X}=2\boldsymbol{e}_3, U=2)E_{\mathrm{std}}(\boldsymbol{e}_3+\boldsymbol{e}_j,2\mid \boldsymbol{e}_3)\\
&= 2\left[\frac{2}{5}E_{\mathrm{std}}(\boldsymbol{e}_3+\boldsymbol{e}_1,2\mid \boldsymbol{e}_3)+\frac{1}{5}E_{\mathrm{std}}(\boldsymbol{e}_3+\boldsymbol{e}_2,2\mid \boldsymbol{e}_3)+\frac{2}{5}E_{\mathrm{std}}(2\boldsymbol{e}_3,2\mid \boldsymbol{e}_3)\right]\\
&= 2\,\frac{2}{5}[\mathrm{P}_{-\boldsymbol{e}_3}(\boldsymbol{Z}=(2,2,3)\text{ or }(2,3,2)\mid \boldsymbol{X}=\boldsymbol{e}_3+\boldsymbol{e}_1, U=2)(0)\\
&\qquad +\mathrm{P}_{-\boldsymbol{e}_3}(\boldsymbol{Z}=(2,1),(2,2,1,3),(2,3,1),\text{ or }(2,3,3)\mid\\
&\qquad\qquad \boldsymbol{X}=\boldsymbol{e}_3+\boldsymbol{e}_1, U=2)(1)]\\
&\quad +2\,\frac{1}{5}[\mathrm{P}_{-\boldsymbol{e}_3}(\boldsymbol{Z}=(2,1)\text{ or }(2,3,1)\mid \boldsymbol{X}=\boldsymbol{e}_3+\boldsymbol{e}_2, U=2)(-1)\\
&\qquad +\mathrm{P}_{-\boldsymbol{e}_3}(\boldsymbol{Z}=(2,3,3)\mid \boldsymbol{X}=\boldsymbol{e}_3+\boldsymbol{e}_2, U=2)(1)]\\
&\quad +2\,\frac{2}{5}[\mathrm{P}_{-\boldsymbol{e}_3}(\boldsymbol{Z}=(2,1),(2,2,1,1),\text{ or }(2,3,1)\mid \boldsymbol{X}=2\boldsymbol{e}_3, U=2)(0)\\
&\qquad +\mathrm{P}_{-\boldsymbol{e}_3}(\boldsymbol{Z}=(2,2,3)\text{ or }(2,3,2)\mid \boldsymbol{X}=2\boldsymbol{e}_3, U=2)(-1)\\
&\qquad +\mathrm{P}_{-\boldsymbol{e}_3}(\boldsymbol{Z}=(2,2,1,3)\mid \boldsymbol{X}=2\boldsymbol{e}_3, U=2)(1)]\\
&= 2\,\frac{2}{5}\left[\left(\frac{1}{4}\frac{2}{3}+\frac{2}{4}\frac{1}{3}\right)(0)+\left(\frac{1}{4}+\frac{1}{4}\frac{1}{3}\frac{2}{2}+\frac{2}{4}\frac{1}{3}+\frac{2}{4}\frac{1}{3}\right)(1)\right]\\
&\quad +2\,\frac{1}{5}\left[\left(\frac{2}{4}+\frac{2}{4}\frac{2}{3}\right)(-1)+\left(\frac{2}{4}\frac{1}{3}\right)(1)\right]\\
&\quad +2\,\frac{2}{5}\left[\left(\frac{2}{4}+\frac{1}{4}\frac{2}{3}\frac{1}{2}+\frac{1}{4}\frac{2}{3}\right)(0)+\left(\frac{1}{4}\frac{1}{3}+\frac{1}{4}\frac{1}{3}\right)(-1)+\left(\frac{1}{4}\frac{2}{3}\frac{1}{2}\right)(1)\right]\\
&= 2\left[\frac{2}{5}\frac{2}{3}+\frac{1}{5}\left(-\frac{2}{3}\right)+\frac{2}{5}\left(-\frac{1}{12}\right)\right]=\frac{1}{5},
\end{aligned}$$

where the subscript $-\boldsymbol{e}_3$ means that the deck has been depleted by one trey.

In the case of two decks, $(n_1,n_2,n_3)=(4,4,8)$, so $|\mathscr{L}|=23$ and $|\mathscr{M}|=66$.

In the case of three decks, $(n_1,n_2,n_3)=(6,6,12)$, so $|\mathscr{L}|=26$ and $|\mathscr{M}|=77$.

In the case of d decks with $d\ge 4$, $(n_1,n_2,n_3)=(2d,2d,4d)$, so $|\mathscr{L}|=27$ and $|\mathscr{M}|=81$. Of course, some of these 27 hands are never seen by the basic strategist. For example, $(3,0,0)$, $(4,0,0)$, $(5,0,0)$, $(6,0,0)$, and $(7,0,0)$ are never encountered because the basic strategist splits $(2,0,0)$.

In Table 5 we present basic strategy for d decks, where d is a positive integer. For $d\ge 9$, composition-dependent basic strategy does not depend on d. It may be surprising that basic strategy has very little dependence on the dealer's upcard, but that is the nature of snackjack. Overall player expectation $E(d)$, as a function of the number of decks d, is shown in Table 6. For $d\ge 9$, it is given by the formula (15) below and therefore satisfies, approximately,

$$E(d)\approx E(\infty)+C/d, \tag{11}$$

where $E(\infty)=283/2^{11}$ and $C=1441/2^{15}$. In fact, this approximation is reasonably good for $1\le d\le 8$ as well. For the blackjack version of (11), see the discussions by Griffin (1999, p. 115 and Appendix 8B) and Werthamer (2018, p. 70).

Potential gain from bet variation

The fundamental theorem of card counting (Thorp & Walden, 1973; Ethier & Levin, 2005) (see Ethier, 2010, Section 11.3, for a textbook treatment) tells us that the player's conditional expectation under basic strategy, given the n cards seen so far, is a random

Table 5
Composition-dependent basic strategy for d-deck snackjack, d a positive integer. For $d = 1$ there are five decision points where composition-dependent basic strategy is nonunique; for $d = 2$ there is one. We have excluded exceptions that do not occur to the basic strategist. For example, with A,2,2 *(hard 5) vs.* 2 *it is correct to stand if $2 \le d \le 6$, but the basic strategist doubles* A,2 *vs.* 2 *and* 2,2 *vs.* 2*, so this exception never arises.*

player	dealer upcard		
total	A	2	3
hard 7	S	S	S
hard 6	S	S	S
hard 5	H[1]	H	H
soft 7	S	S	S
soft 6	H	D	D
(3,3)	Spl[2]	Spl	Spl
(2,2)	D	D	D[3]
(A,A)	Spl	Spl	Spl

[1] S if $d = 1$. [2] S if $d \le 8$.
[3] H if $d = 1$.

Table 6
Player expectation at d-deck snackjack under composition-dependent basic strategy, as a function of d.

d	expectation	d	expectation	d	expectation
1	0.192857	7	0.144558	13	0.141548
2	0.163144	8	0.143639	26	0.139871
3	0.154360	9	0.143031	39	0.139309
4	0.150073	10	0.142550	52	0.139028
5	0.147500	11	0.142156		
6	0.145784	12	0.141827	∞	0.138184

variable with mean that is constant in n, mean positive part that is nondecreasing in n, and standard deviation that is increasing in n.

To illustrate in the simplest possible situation, we consider the toy game of red-and-black mentioned earlier, which could just as well be *odd-and-even*. An advantage of the latter formulation is that the cards can be numbered from 1 to N (N is the size of the deck, assumed even), and then

$$Z_n := \frac{1}{N-n}\sum_{i=1}^{n}(-1)^{X_i}$$

gives the exact player conditional expectation of a one-unit even-money bet that the next card dealt is odd, given that the first n cards, $X_1, X_2, \ldots, X_n$, have been seen. It is easy to verify that $\mathrm{E}[Z_n] = 0$,

$$\mathrm{E}[(Z_n)^+] = \frac{1}{N-n}\sum_{k=0}^{\lfloor n/2 \rfloor}(n-2k)\frac{\binom{N/2}{k}\binom{N/2}{n-k}}{\binom{N}{n}}, \qquad (12)$$

and

$$\mathrm{SD}(Z_n) = \sqrt{\frac{n}{(N-n)(N-1)}}, \tag{13}$$

for $n = 1, 2, \ldots, N-1$, where $a^+ := \max(a, 0)$. The expectation (12) is nondecreasing in n, and the standard deviation (13) is increasing in n, both as a result of the FTCC (Ethier & Levin, 2005). (We cannot express (12) in closed form, but we can show analytically that $\mathrm{E}[(Z_n)^+] = \frac{1}{2}\mathrm{E}[|Z_n|] \le \frac{1}{2}\,\mathrm{SD}(Z_n)$.)

Let us consider a shoe comprising 39 decks, or 312 cards, at snackjack. Basic strategy is the strategy of Table 5 without the footnotes. To summarize it, the player mimics the dealer except when he has a soft 6 or a pair. He hits a soft 6 against an ace and otherwise doubles. He splits a pair of aces and a pair of treys and doubles a pair of deuces. A single round with one player can be completed with certainty if at least eight cards remain. The mean profit, given that n_1 aces, n_2 deuces, and n_3 treys remain, is

$$E(n_1, n_2, n_3) = \frac{P(n_1, n_2, n_3)}{(n_1 + n_2 + n_3)_8}, \tag{14}$$

where $P(n_1, n_2, n_3)$ is a polynomial of degree 8 in n_1, n_2, and n_3 with 147 terms, and $(N)_8 := N(N-1)\cdots(N-7)$. (See Ethier & Lee, 2019, Appendix C, for the explicit formula for $P(n_1, n_2, n_3)$.) For example,

$$\begin{aligned} &E(2d, 2d, 4d) \\ &\quad = \frac{-630 + 4{,}017d - 2{,}673d^2 - 32{,}132d^3 + 92{,}560d^4 - 97{,}144d^5 + 36{,}224d^6}{(8d-1)_3(8d-5)_3}, \end{aligned} \tag{15}$$

which yields the entries in Table 6 for $d \ge 9$ because basic strategy optimized for d decks coincides with 39-deck basic strategy provided $d \ge 9$. As a check of (14), we can confirm that $E(n_1, 0, 0) = -2$ (player splits, gets two soft 5s, dealer wins both with soft 6), $E(0, n_2, 0) = 0$ (player doubles, gets hard 6, dealer pushes with hard 6), and $E(0, 0, n_3) = 0$ (player splits, gets two hard 6s, dealer pushes both with hard 6).

Because of the simplicity of snackjack, we can compute the means and variances arising in the fundamental theorem of card counting. The analogous computations at blackjack would be prohibitively time-consuming. To justify this claim, we need to do some counting.

In 39-deck snackjack, if n cards have been seen, the numbers M_1, M_2, and M_3 of aces, deuces, and treys among them are such that (M_1, M_2, M_3) has the multivariate hypergeometric distribution

$$\mathrm{P}(M_1 = m_1, M_2 = m_2, M_3 = m_3) = \frac{\binom{78}{m_1}\binom{78}{m_2}\binom{156}{m_3}}{\binom{312}{n}},$$

where $0 \le m_1 \le 78$, $0 \le m_2 \le 78$, $0 \le m_3 \le 156$, and $m_1 + m_2 + m_3 = n$. The number $s(n)$ of distinct values of (M_1, M_2, M_3) such that $M_1 + M_2 + M_3 = n$ satisfies $s(n) = s(312-n)$ and is given by

$$\begin{aligned} s(n) = \sum_{k=0}^{2} (-1)^k \binom{2}{k} \bigg[&\binom{n - 79k + 2}{2} \mathbf{1}\{n \ge 79k\} \\ &- \binom{n - 79k - 157 + 2}{2} \mathbf{1}\{n \ge 79k + 157\} \bigg]; \end{aligned} \tag{16}$$

see below for details. In particular, $\max_n s(n) = s(156) = 6{,}241$ and $\sum_n s(n) = (79)^2 157 = 979{,}837$. That is, there are fewer than one million distinguishable subsets of the 39-deck snackjack shoe.

In 24-deck grayjack, if n cards have been seen, the numbers M_1, M_2, ..., M_6 of aces, 2s, ..., 6s among them are such that $(M_1, M_2, \ldots, M_6)$ has the multivariate hypergeometric distribution

$$\mathrm{P}(M_1 = m_1, M_2 = m_2, \ldots, M_6 = m_6) = \frac{\binom{24}{m_1}\left[\prod_{i=2}^{5}\binom{48}{m_i}\right]\binom{96}{m_6}}{\binom{312}{n}},$$

where $0 \le m_1 \le 24$, $0 \le m_i \le 48$ for $2 \le i \le 5$, $0 \le m_6 \le 96$, and $m_1 + m_2 + \cdots + m_6 = n$. The number $g(n)$ of distinct values of $(M_1, M_2, \ldots, M_6)$ such that $M_1 + M_2 + \cdots + M_6 = n$ satisfies $g(n) = g(312 - n)$ and is given by

$$\begin{aligned} g(n) = \sum_{k=0}^{4} (-1)^k \binom{4}{k} \bigg[&\binom{n - 49k + 5}{5} \mathbf{1}\{n \ge 49k\} \\ &- \binom{n - 25 - 49k + 5}{5} \mathbf{1}\{n \ge 25 + 49k\} \\ &- \binom{n - 49k - 97 + 5}{5} \mathbf{1}\{n \ge 49k + 97\} \\ &+ \binom{n - 25 - 49k - 97 + 5}{5} \mathbf{1}\{n \ge 25 + 49k + 97\} \bigg]. \end{aligned} \tag{17}$$

Thus, $\max_n g(n) = g(156) = 130{,}046{,}539$ and $\sum_n g(n) = 25(49)^4 97 = 13{,}979{,}642{,}425$. That is, there are nearly 14 billion distinguishable subsets of the 24-deck grayjack shoe.

In six-deck blackjack, if n cards have been seen, the numbers M_1, M_2, ..., M_{10} of aces, 2s, ..., tens among them are such that $(M_1, M_2, \ldots, M_{10})$ has the multivariate hypergeometric distribution

$$\mathrm{P}(M_1 = m_1, M_2 = m_2, \ldots, M_{10} = m_{10}) = \frac{\left[\prod_{i=1}^{9}\binom{24}{m_i}\right]\binom{96}{m_{10}}}{\binom{312}{n}},$$

where $0 \le m_i \le 24$ for $1 \le i \le 9$, $0 \le m_{10} \le 96$, and $m_1 + m_2 + \cdots + m_{10} = n$. The number $b(n)$ of distinct values of $(M_1, M_2, \ldots, M_{10})$ such that $M_1 + M_2 + \cdots + M_{10} = n$ satisfies $b(n) = b(312 - n)$ and is given by

$$\begin{aligned} b(n) = \sum_{k=0}^{9} (-1)^k \binom{9}{k} \bigg[&\binom{n - 25k + 9}{9} \mathbf{1}\{n \ge 25k\} \\ &- \binom{n - 25k - 97 + 9}{9} \mathbf{1}\{n \ge 25k + 97\} \bigg]; \end{aligned} \tag{18}$$

$\max_n b(n) = b(156) = 3{,}726{,}284{,}230{,}655$ and $\sum_n b(n) = (25)^9 97 = 370{,}025{,}634{,}765{,}625$. That is, there are more than 370 trillion distinguishable subsets of the six-deck blackjack shoe.

Denoting by $b_1(n)$ the analogous quantity in single-deck blackjack, we have $b_1(n) = b_1(52 - n)$, $\max_n b_1(n) = b_1(26) = 1{,}868{,}755$, and $\sum_n b_1(n) = 5^9 17 = 33{,}203{,}125$. See Griffin (1999, p. 159) and Thorp (2000, p. 126).

To clarify how (16)–(18) were derived, we elaborate on (16). Let

$$\begin{aligned} A &:= \{(m_1, m_2, m_3) : m_1 \ge 0, m_2 \ge 0, m_3 \ge 0, m_1 + m_2 + m_3 = n\}, \\ B_1 &:= \{(m_1, m_2, m_3) : m_1 \ge 79, m_2 \ge 0, m_3 \ge 0, m_1 + m_2 + m_3 = n\}, \\ B_2 &:= \{(m_1, m_2, m_3) : m_1 \ge 0, m_2 \ge 79, m_3 \ge 0, m_1 + m_2 + m_3 = n\}, \\ B_3 &:= \{(m_1, m_2, m_3) : m_1 \ge 0, m_2 \ge 0, m_3 \ge 157, m_1 + m_2 + m_3 = n\}. \end{aligned}$$

Then $|A| = \binom{n+2}{2}$, $|B_1| = \binom{n-79+2}{2}\mathbf{1}\{n \ge 79\}$, $|B_1 \cap B_3| = \binom{n-79-157+2}{2}\mathbf{1}\{n \ge 79 + 157\}$, and so on. By inclusion-exclusion,

$$|A - (B_1 \cup B_2 \cup B_3)| = |A| - |B_1| - |B_2| - |B_3| + |B_1 \cap B_2| + |B_1 \cap B_3| + |B_2 \cap B_3|$$

$$= |A| - |B_3| - 2(|B_1| - |B_1 \cap B_3|) + |B_1 \cap B_2|,$$

where we have used $B_1 \cap B_2 \cap B_3 = \varnothing$, $|B_1| = |B_2|$, and $|B_1 \cap B_3| = |B_2 \cap B_3|$, and the result follows.

Returning to snackjack, let Z_n denote the player's conditional expectation, given that n cards have been seen. Then

$$\begin{aligned}\mathrm{E}[Z_n] &= \sum_{m_1+m_2+m_3=n} \frac{\binom{78}{m_1}\binom{78}{m_2}\binom{156}{m_3}}{\binom{312}{n}} E(78-m_1, 78-m_2, 156-m_3) \\ &= \mathrm{E}[Z_0] = E(78,78,156) = \frac{220{,}204{,}549{,}189}{1{,}580{,}689{,}046{,}285} =: \mu, \end{aligned} \tag{19}$$

where $0 \le m_1 \le 78$, $0 \le m_2 \le 78$, and $0 \le m_3 \le 156$ in the sum. The second equality is a consequence of the martingale property of $\{Z_n\}$, for which see Ethier & Levin (2005).

The expected positive part of the difference between the player's conditional expectation and a positive number ν is

$$\mathrm{E}[(Z_n - \nu)^+] = \sum_{m_1+m_2+m_3=n} \frac{\binom{78}{m_1}\binom{78}{m_2}\binom{156}{m_3}}{\binom{312}{n}} [E(78-m_1, 78-m_2, 156-m_3) - \nu]^+, \tag{20}$$

where the sum is constrained as in (19). To interpret this, suppose that players are required to pay a commission ν per unit bet initially on each hand. (Doubling and splitting do not require any additional commission.) Then this is the player's expected profit, assuming n cards have been seen and assuming he bets one unit if and only if his net conditional expectation (taking the commission into account) is nonnegative.

Finally, the variance of the player's conditional expectation is

$$\mathrm{Var}(Z_n) = \sum_{m_1+m_2+m_3=n} \frac{\binom{78}{m_1}\binom{78}{m_2}\binom{156}{m_3}}{\binom{312}{n}} [E(78-m_1, 78-m_2, 156-m_3) - \mu]^2, \tag{21}$$

and again the sum is constrained as in (19).

The quantities (19)–(21) can be computed for $n = 1, 2, 3, \ldots, 304$ $(= 312 - 8)$, and Figure 2 displays the graph of $f(n) := \mathrm{E}[(Z_n - \nu)^+]$ with $\nu = 1/7$, as well as the graph of the standard deviations $g(n) := \mathrm{SD}(Z_n)$. The two curves have similar shapes, and are also very similar to the graphs of (12) and (13) with $N = 312$. They increase gradually over the first 2/3 of the shoe and more rapidly over the final 1/6. The increase in the slope is gradual throughout, unlike with the famous "hockey stick graph" of climate science.

Notice that, for $n = 1, 2, 3, \ldots, 304$, each of the quantities in (19)–(21) requires up to 6,241 evaluations of the rational function $E(n_1, n_2, n_3)$, which is computationally routine. The corresponding quantities in blackjack would require up to 3.7 trillion evaluations of the basic strategy expectation (which itself is too complicated to be usefully expressed as a rational function; see Table 3), and would be computationally prohibitive.

Card counting and bet variation

It is well known in blackjack (Griffin, 1999, Chap. 4) that when the point values of a card-counting system are highly correlated with the effects of removal, a high betting efficiency is achieved but not necessarily a high strategic efficiency. In snackjack, we continue to treat the case of a 39-deck, 312-card, shoe.

Let us denote by $\mu(\boldsymbol{m})$, where $\boldsymbol{m} = (m_1, m_2, m_3)$, the expected profit from an initial one-unit snackjack wager, assuming composition-dependent basic strategy (optimized for the 39-deck shoe), when the 39-deck shoe is depleted by m_1 aces, m_2 deuces, and m_3 treys. Using (14),

$$\mu(\boldsymbol{m}) = E(78 - m_1, 78 - m_2, 156 - m_3).$$

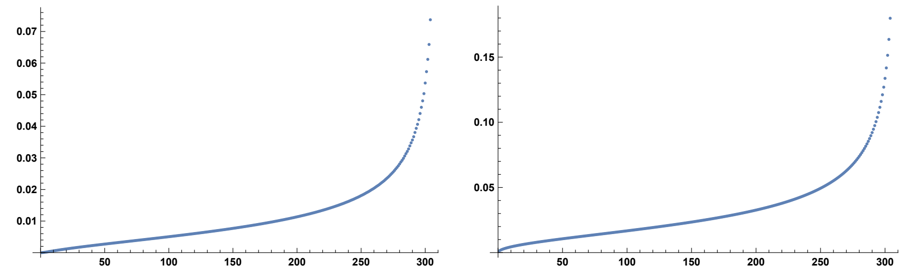

Figure 2
On the left is the graph of $f(n) := \mathrm{E}[(Z_n - \nu)^+]$ *with* $\nu = 1/7$, $1 \le n \le 304$, *for* 39*-deck snackjack. On the right is the graph of* $g(n) := \mathrm{SD}(Z_n)$, $1 \le n \le 304$, *for the same game.*

We can then evaluate the *effects of removal* on the expected profit from an initial one-unit snackjack wager, assuming composition-dependent basic strategy:

$$\mathrm{EoR}(i) := \mu(\boldsymbol{e}_i) - \mu(\mathbf{0}), \qquad i = 1,2,3. \tag{22}$$

The numbers (22), multiplied by 311, are

$$E_1 = -\frac{849{,}581{,}527}{1{,}793{,}859{,}330}, \quad E_2 = \frac{3{,}539{,}587{,}453}{5{,}082{,}601{,}435}, \quad E_3 = -\frac{6{,}794{,}638{,}759}{60{,}991{,}217{,}220},$$

with decimal equivalents listed in Table 7. A simple probabilistic argument shows that

$$E_1 + E_2 + 2E_3 = 0. \tag{23}$$

We provide the exact fractions above to allow confirmation that (23) holds exactly, not just to a certain number of decimal places.

Table 7
Effects of removal, multiplied by 311, *for an initial one-unit bet in* 39*-deck snackjack, assuming composition-dependent basic strategy for the* 39*-deck shoe. Results are rounded to six decimal places. Also included are two card-counting systems, one of level one, the other of level six.*

card value i	$E_i :=$ 311 EoR(i)	level one system	level six system
1	−0.473605	−1	−4
2	0.696413	1	6
3	−0.111404	0	−1
correlation ρ		0.965597	0.999921
regression coefficient γ		0.585009	0.116587

In the blackjack literature (e.g., Schlesinger, 2018, pp. 503–504, 522), it is conventional to evaluate the effects of removal for the single-deck game and then use a conversion factor to handle the multiple-deck games. We could follow this precedent with $6\frac{1}{2}$-deck, 52-card, snackjack playing the role of single-deck blackjack, but we prefer to work directly with the 39-deck, 312-card, game.

Recall that, in a *balanced* card-counting system, the sum of the point values over the entire pack is 0. For the system (J_1, J_2, J_3), this means that

$$J_1 + J_2 + 2J_3 = 0.$$

Table 7 lists two balanced card-counting systems, the best level-one system and the best level-six system, the level being defined by $\max(|J_1|,|J_2|,|J_3|)$. In each case we indicate the correlation ρ with the effects of removal, and the relevant regression coefficient γ defined below. Based on Ethier (2010, Eqs. (11.76), (11.95), (21.69), and (21.70)), an estimate of Z_n (the player's conditional expectation under basic strategy, given that n cards have been seen) is

$$\widehat{Z}_n := \mu + \frac{1}{312-n}\sum_{j=1}^{n} E_{X_j}, \tag{24}$$

where $E_i := 311\,\mathrm{EoR}(i)$ and $X_1, X_2, \dots, X_{312}$ is the sequence of card values in the order in which they are exposed, which in turn is approximated by

$$Z_n^* := \mu + \frac{\gamma}{52}\left(\frac{52}{312-n}\sum_{j=1}^{n} J_{X_j}\right) = \mu + \frac{\gamma}{52}\mathrm{TC}_n, \tag{25}$$

where (J_1, J_2, J_3) is one of the two card-counting systems listed in Table 7, and

$$\gamma := \frac{E_1J_1 + E_2J_2 + 2E_3J_3}{J_1^2 + J_2^2 + 2J_3^2}$$

is the regression coefficient that minimizes the sum of squares $(E_1-\gamma J_1)^2 + (E_2-\gamma J_2)^2 + 2(E_3-\gamma J_3)^2$. We find that

$$\rho \approx 0.965597, \quad \gamma = \frac{35{,}680{,}410{,}677}{60{,}991{,}217{,}220}, \qquad \text{if } (J_1,J_2,J_3) = (-1,1,0),$$
$$\rho \approx 0.999921, \quad \gamma = \frac{63{,}997{,}110{,}301}{548{,}920{,}954{,}980}, \qquad \text{if } (J_1,J_2,J_3) = (-4,6,-1).$$

Finally, TC_n is the *true count*, which is the *running count* (the sum of the point values of the cards seen so far) divided by the number of unseen 52-card packs, namely $(312-n)/52$. We use 52 instead of 8 here because it may be easier to estimate the number of unseen 52-card packs than the number of unseen 8-card decks.

The first question we would like to address is, how accurate is card counting? There are several ways to answer this question, but a first step would be to compare Z_n with its approximations $\widehat{Z}_n$ and Z_n^*. More specifically, we compare the L^1 distances between Z_n and its approximations. So we evaluate

$$\|Z_n - \widehat{Z}_n\|_1 = \sum_{m_1+m_2+m_3=n} \frac{\binom{78}{m_1}\binom{78}{m_2}\binom{156}{m_3}}{\binom{312}{n}} \bigg| E(78-m_1, 78-m_2, 156-m_3) - \left(\mu + \frac{1}{312-n}(m_1E_1 + m_2E_2 + m_3E_3)\right)\bigg|,$$

where $0 \le m_1 \le 78$, $0 \le m_2 \le 78$, and $0 \le m_3 \le 156$ in the sum, and

$$\|Z_n - Z_n^*\|_1 = \sum_{m_1+m_2+m_3=n} \frac{\binom{78}{m_1}\binom{78}{m_2}\binom{156}{m_3}}{\binom{312}{n}} \bigg| E(78-m_1, 78-m_2, 156-m_3) - \left(\mu + \frac{\gamma}{312-n}(m_1J_1 + m_2J_2 + m_3J_3)\right)\bigg|,$$

where the sum is constrained in the same way, with partial results appearing in Table 8. By definition, $\widehat{Z}_1 = Z_1$. We can regard $\|Z_n - \widehat{Z}_n\|_1$ as a measurement of the lack of linearity of the player's conditional expectation under basic strategy when n cards have been seen. It increases gradually as cards are dealt and then more sharply near the end of the shoe. Replacing the EoRs of $\widehat{Z}_n$ by the level-six point count has only a small effect, whereas the use of the rather crude level-one point count has a rather substantial effect.

Table 8
L^1 distances between Z_n (exact player conditional expectation under 39-deck composition-dependent basic strategy), $\widehat{Z}_n$ (approximate player conditional expectation based on effects of removal), and Z_n^* (approximate player conditional expectation based on a card-counting system).

n seen	(a) $\|Z_n - \widehat{Z}_n\|_1$	(b) $\|Z_n - Z_n^*\|_1$ for $(-4,6,-1)$	% incr. of (b) over (a)	(c) $\|Z_n - Z_n^*\|_1$ for $(-1,1,0)$	% incr. of (c) over (a)
1	0	0.00001667	–	0.0003582	–
2	0.000006083	0.00001886	210.1	0.0003614	5842.
3	0.00001110	0.00002510	126.1	0.0005390	4755.
4	0.00001631	0.00002843	74.27	0.0005442	3236.
26	0.0001120	0.0001243	11.05	0.001515	1253.
52	0.0002480	0.0002607	5.121	0.002257	810.1
78	0.0004141	0.0004275	3.245	0.002919	605.0
104	0.0006225	0.0006364	2.233	0.003579	474.9
130	0.0008911	0.0009063	1.709	0.004282	380.5
156	0.001249	0.001265	1.332	0.005072	306.1
182	0.001754	0.001771	0.9314	0.006013	242.8
208	0.002514	0.002536	0.8654	0.007220	187.1
234	0.003788	0.003821	0.8801	0.008960	136.5
260	0.006394	0.006433	0.6069	0.01197	87.14
286	0.01457	0.01466	0.6154	0.02013	38.22
301	0.04268	0.04286	0.4175	0.04682	9.704
302	0.04954	0.04973	0.3965	0.05239	5.753
303	0.05917	0.05912	–	0.06177	4.409
304	0.06990	0.07004	0.1882	0.07335	4.929

Next, we return to a previously computed quantity. We supposed that players are required to pay a commission on each hand equal to $\nu = 1/7$ of the initial amount bet, which would make snackjack a subfair game for the basic strategist. Then $\mathrm{E}[(Z_n - \nu)^+]$ is the player's expected profit, assuming n cards have been seen and assuming he bets one unit if and only if his net conditional expectation (taking the commission into account) is nonnegative. The only problem is how does the player know whether his net conditional expectation is nonnegative? Unless he has an electronic device programmed to evaluate $E(78 - M_1, 78 - M_2, 156 - M_3)$ (which would be illegal in Nevada), he does not. The best he can do is estimate his conditional expectation using card counting. The *betting efficiency* of a card-counting system could then be defined in terms of how close to the ideal $\mathrm{E}[(Z_n - \nu)^+]$ one could come in practice. This would be

$$\begin{aligned}
&\mathrm{E}[(Z_n - \nu)\mathbf{1}\{Z_n^* - \nu \geq 0\}] \\
&\quad = \mathrm{E}[(Z_n - \nu)\mathbf{1}\{\mu + (\gamma/52)\mathrm{TC}_n \geq \nu\}] \\
&\quad = \sum_{m_1+m_2+m_3=n} \frac{\binom{78}{m_1}\binom{78}{m_2}\binom{156}{m_3}}{\binom{312}{n}} [E(78 - m_1, 78 - m_2, 156 - m_3) - \nu] \\
&\qquad\qquad \cdot \mathbf{1}\{\mathrm{TC}_n \geq 52(\nu - \mu)/\gamma\},
\end{aligned}$$

where $0 \leq m_1 \leq 78$, $0 \leq m_2 \leq 78$, and $0 \leq m_3 \leq 156$ in the sum. Thus, the ratio

$$\mathrm{BE}_n = \frac{\mathrm{E}[(Z_n - \nu)\mathbf{1}\{Z_n^* - \nu \geq 0\}]}{\mathrm{E}[(Z_n - \nu)^+]}$$

is the *betting efficiency* when n cards have been seen. We note that the threshold for the true count to suggest a positive expectation is $52(\nu-\mu)/\gamma \approx 0.315367$ in the level-one system. Partial results are shown in Table 9. BE_n is undefined if $n=1$ and is 1 (i.e., 100%) for $n=2,3,4$ for the level-one system and for $n=2,3,\ldots,17$ except $n=10$ and $n=15$ for the level-six system.

Table 9
The betting efficiency of two card-counting systems at 39*-deck snackjack as a function of the number of cards seen.*

n seen	BE_n of $(-4,6,-1)$	BE_n of $(-1,1,0)$	n seen	BE_n of $(-4,6,-1)$	BE_n of $(-1,1,0)$
26	0.999989	0.939475	234	0.992589	0.957591
52	0.998986	0.948355	260	0.991316	0.956100
78	0.999497	0.950998	286	0.986838	0.942744
104	0.998716	0.951662			
130	0.998168	0.951270	301	0.950243	0.895330
156	0.998069	0.954986	302	0.953991	0.892247
182	0.997312	0.956942	303	0.954517	0.889094
208	0.995350	0.957795	304	0.892340	0.862632

It is useful to have a single number that can be called the *betting efficiency* of a card-counting system. For this we use an average of the quantities BE_n. Since it is likely that the last one-quarter of the shoe is not dealt, we exclude decisions based on 234 or more cards. This leads to

$$\mathrm{BE} := \frac{1}{232}\sum_{n=2}^{233}\mathrm{BE}_n.$$

We find that $\mathrm{BE} \approx 0.9982$ for the level-six system $(-4,6,-1)$, and $\mathrm{BE} \approx 0.9508$ for the level-one system $(-1,1,0)$. These numbers are not far from the correlations, 0.9999 and 0.9656, between the EoRs and the numbers of the point count. Griffin (1999, Chapter 4) used this correlation as a proxy for betting efficiency, unable to compute for blackjack numbers analogous to those in Table 9 other than by computer simulation.

Now let us examine the level-one counting system, which we call the *deuces-minus-aces* system, in more detail. It is snackjack's analogue of the Hi-Lo system at blackjack. The true count, when n cards have been seen, including M_1 aces, M_2 deuces, and M_3 treys, is given by

$$\mathrm{TC}_n := \frac{52(M_2-M_1)}{312-n},$$

and the *rounded true count* is TC_n rounded to the nearest integer, denoted by $[\mathrm{TC}_n]$. More precisely, if $k-1/2 < \mathrm{TC}_n < k+1/2$, we define $[\mathrm{TC}_n] := k$ and, if $\mathrm{TC}_n = k+1/2$, then $[\mathrm{TC}_n] = k$ with probability 1/2 and $[\mathrm{TC}_n] = k+1$ with probability 1/2. This symmetric rounding ensures that the distribution of $[\mathrm{TC}_n]$ is symmetric about 0. Indeed,

$$\mathrm{P}([\mathrm{TC}_n]=k) = \sum_{m_1+m_2+m_3=n} \frac{\binom{78}{m_1}\binom{78}{m_2}\binom{156}{m_3}}{\binom{312}{n}}\bigg[\mathbf{1}\bigg\{k-\frac{1}{2} < \frac{52(m_2-m_1)}{312-n} < k+\frac{1}{2}\bigg\} + \frac{1}{2}\,\mathbf{1}\bigg\{\frac{52(m_2-m_1)}{312-n} = k-\frac{1}{2} \text{ or } k+\frac{1}{2}\bigg\}\bigg],$$

where the sum is constrained by $0 \le m_1 \le 78$, $0 \le m_2 \le 78$, and $0 \le m_3 \le 156$. Figure 3 plots the graph of $[\mathrm{TC}_n]$ for $n = 26m$, $m = 4,5,\ldots,11$.

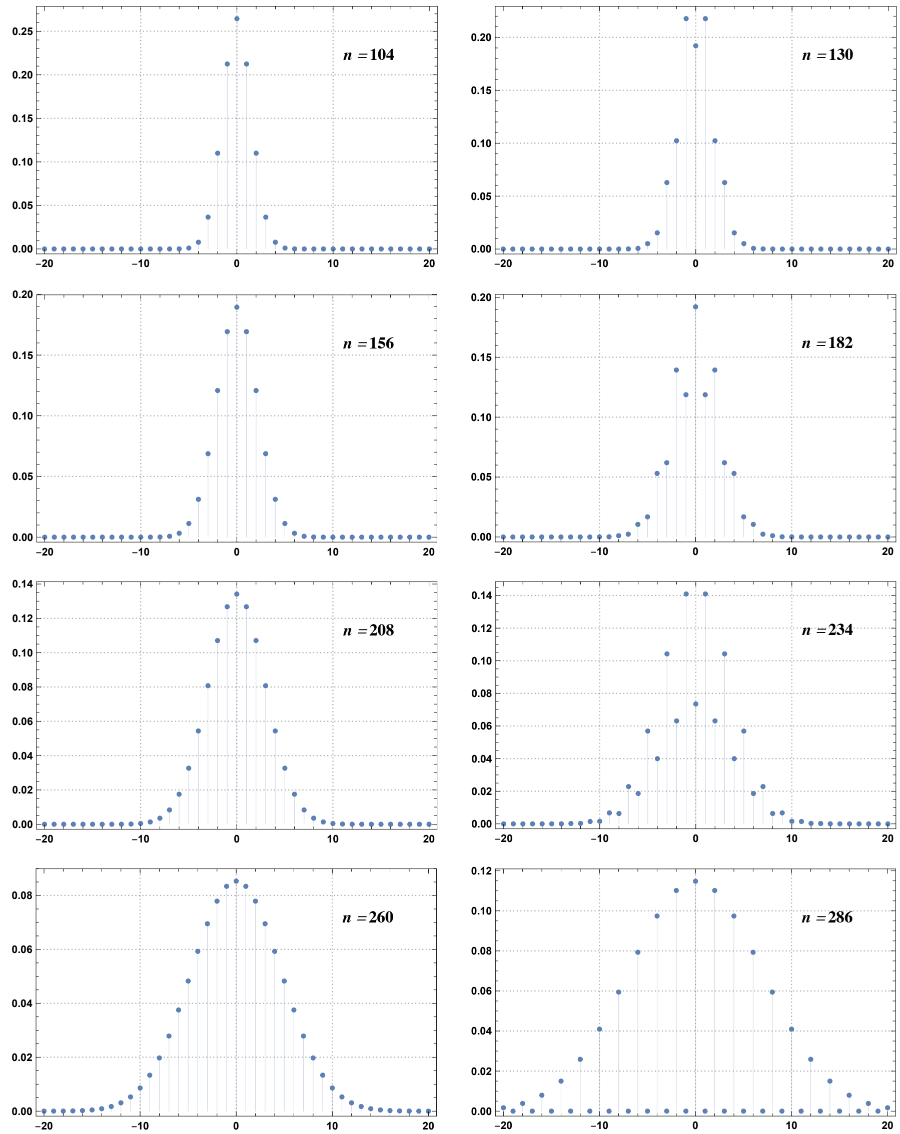

Figure 3
At 39-deck snackjack, the distribution of TC_n *rounded to the nearest integer (assuming the deuces-minus-aces count), with n being the number of cards seen. Notice that the distribution is normal-like for n an even multiple of* 26 *(left column) but not for n an odd multiple of* 26 *(right column).*

Next, we evaluate the conditional expectation at snackjack (assuming the commission of $\nu = 1/7$), given the rounded true count. This is

$$\begin{aligned}
&\mathrm{E}[Z_n - \nu \mid [\mathrm{TC}_n] = k] \\
&\quad = \sum_{m_1+m_2+m_3=n} \frac{\binom{78}{m_1}\binom{78}{m_2}\binom{156}{m_3}}{\binom{312}{n}} [E(78-m_1, 78-m_2, 156-m_3) - \nu] \\
&\qquad\qquad \cdot \left[\mathbf{1}\left\{ k - \frac{1}{2} < \frac{52(m_2-m_1)}{312-n} < k + \frac{1}{2} \right\} \right. \\
&\qquad\qquad\quad \left. + \frac{1}{2}\mathbf{1}\left\{ \frac{52(m_2-m_1)}{312-n} = k - \frac{1}{2} \text{ or } k + \frac{1}{2} \right\} \right] \Big/ \mathrm{P}([\mathrm{TC}_n] = k),
\end{aligned}$$

with the same constraints on the sum, and results are tabulated in Table 10 for $n = 78$, 156, and 234. We find that the rounded true count is a good estimate of the player's expectation in percentage terms.

We now consider a betting strategy similar to the one assumed by Schlesinger (2018, Chap. 5), but a little simpler. We assume that the player bets

$$\max(1, \min([\mathrm{TC}_n], 6)).$$

That is, the player bets the rounded true count, but never less than one unit or more than six units. Thus, this betting strategy has a 6 to 1 spread. It could be argued that the bettor should walk away if the true count falls below some threshold, but we assume that he continues to play and bet one unit, perhaps to hold his place at the table or to disguise his status as a card counter.

Table 10
The conditional expectation at snackjack (assuming a commission of $\nu = 1/7$ per unit initially bet), given the (rounded) true count.

	$n = 78$		$n = 156$		$n = 234$	
$[\mathrm{TC}_n]$	cond'l ex	probab	cond'l ex	probab	cond'l ex	probab
−6	−0.0688	0.00000221	−0.0724	0.00320	−0.0742	0.0186
−5	−0.0585	0.0000640	−0.0606	0.0112	−0.0614	0.0569
−4	−0.0463	0.00195	−0.0489	0.0312	−0.0497	0.0400
−3	−0.0361	0.0146	−0.0373	0.0688	−0.0372	0.104
−2	−0.0246	0.0984	−0.0257	0.121	−0.0254	0.0631
−1	−0.0143	0.207	−0.0143	0.169	−0.0134	0.141
0	−0.00352	0.355	−0.00311	0.189	−0.00183	0.0735
1	0.00723	0.207	0.00791	0.169	0.00950	0.141
2	0.0175	0.0984	0.0187	0.121	0.0207	0.0631
3	0.0288	0.0146	0.0292	0.0688	0.0312	0.104
4	0.0387	0.00195	0.0395	0.0312	0.0418	0.0400
5	0.0506	0.0000640	0.0494	0.0112	0.0513	0.0569
6	0.0606	0.00000221	0.0590	0.00320	0.0611	0.0186

We can then evaluate the player's expected profit at each level of penetration. The formula is

$$\begin{aligned}
&\mathrm{E}[\max(1, \min([\mathrm{TC}_n], 6))(Z_n - \nu)] \\
&\quad = \sum_{m_1+m_2+m_3=n} \frac{\binom{78}{m_1}\binom{78}{m_2}\binom{156}{m_3}}{\binom{312}{n}} [E(78 - m_1, 78 - m_2, 156 - m_3) - \nu] \\
&\qquad\qquad \cdot \sum_k \max(1, \min(k, 6)) \bigg[\mathbf{1}\bigg\{ k - \frac{1}{2} < \frac{52(m_2 - m_1)}{312 - n} < k + \frac{1}{2} \bigg\} \\
&\qquad\qquad\qquad + \frac{1}{2} \mathbf{1}\bigg\{ \frac{52(m_2 - m_1)}{312 - n} = k - \frac{1}{2} \text{ or } k + \frac{1}{2} \bigg\} \bigg],
\end{aligned}$$

with the same constraints on the outer sum, and the results are plotted in Figure 4.

The average expected value over the first 3/4 of the shoe ($0 \le n \le 233$) is 0.00779463. The average over the first 5/6 of the shoe ($0 \le n \le 259$) is 0.0123218.

Card counting and strategy variation

In this section we show that card counting can be used to determine when a departure from basic strategy is called for. A table analogous to Table 4 for 39-deck snackjack could

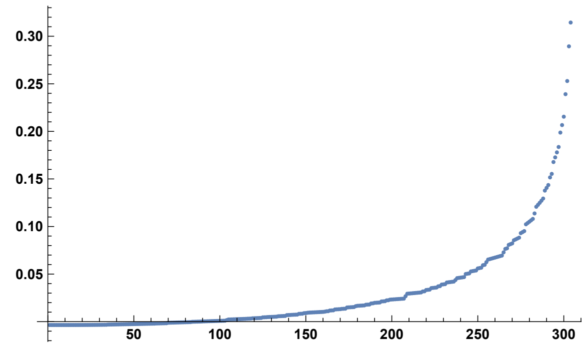

Figure 4
Snackjack expectation as a function of the number n of cards seen, assuming a commission of $\nu = 1/7$ per unit initially bet and bets equal to the rounded true count, but always at least one unit and at most six units.

be generated, but it would have 81 rows. Restricting attention to two-card hands would result in an 18-row table. We omit the details, but computations show that (0,0,2), (0,1,1), and (1,1,0) have the greatest potential for profitable strategy variation. In this section we attempt to quantify this claim.

To clarify the methodology, we treat the case of $(0,0,2)$ (i.e., a pair of treys), which the 39-deck basic strategist splits against any dealer upcard, but with which standing may be preferable in some situations.

We denote $E_{\rm std}((0,0,2),u)$ from the discussion of basic strategy by $E_{{\rm std},(n_1,n_2,n_3)}((0,0,2),u)$ with (n_1,n_2,n_3) indicating the post-deal shoe composition (i.e., the hand's two 3s and dealer upcard u are excluded from the unseen shoe). A similar interpretation applies to $E_{{\rm spl},(n_1,n_2,n_3)}((0,0,2),u)$.

The difference $E_{{\rm std},(n_1,n_2,n_3)}((0,0,2),u) - E_{{\rm spl},(n_1,n_2,n_3)}((0,0,2),u)$, which represents the expected gain by departing from basic strategy, is well defined under the following conditions: $n_1,n_2,n_3 \ge 0$ and

- if $u=1$, then $n_1+n_2+n_3 \ge 4$, $n_1+n_2 \ge 1$;
- if $u=2$, then $n_1+n_2+n_3 \ge 4$;
- if $u=3$, then $n_1+n_2+n_3 \ge 3$, $n_2+n_3 \ge 1$.

(See explicit formulas for these differences in Ethier & Lee, 2019, Appendix D.) We assume in fact that $n_1+n_2+n_3 \ge 5$ for all u because a new hand should never be dealt with fewer than eight cards remaining.

With $(0,0,2)$ vs. A, the proportion of shoe compositions that call for a departure from basic strategy is $439{,}742/954{,}925 \approx 0.460499$. With $(0,0,2)$ vs. 2, the proportion is $271{,}854/955{,}075 \approx 0.284642$. With $(0,0,2)$ vs. 3, the proportion is $358{,}973/961{,}005 \approx 0.373539$.

With $(0,0,2)$ vs. A, the probability that a departure from basic strategy is called for when n cards have been seen (before the hand is dealt) is

$$\frac{\sum_{(m_1,m_2,m_3)\in\Gamma_n((0,0,2),1)} \binom{78}{m_1}\binom{78}{m_2}\binom{156}{m_3}\alpha_1(m_1,m_2,m_3)}{\sum_{(m_1,m_2,m_3)\in\Gamma_n((0,0,2),1)} \binom{78}{m_1}\binom{78}{m_2}\binom{156}{m_3}} \tag{26}$$

for $n = 1,2,\ldots,304$, where

$$\Gamma_n((0,0,2),1) := \{(m_1,m_2,m_3) : 0 \le m_1 \le 77,\, 0 \le m_2 \le 78,\, 0 \le m_3 \le 154,\\ m_1+m_2 \le 154,\, m_1+m_2+m_3 = n\}$$

and

$$\alpha_1(m_1,m_2,m_3) := \begin{cases} 1 & \text{if } E_{\text{std},(77-m_1,78-m_2,154-m_3)}((0,0,2),1) \\ & \quad > E_{\text{spl},(77-m_1,78-m_2,154-m_3)}((0,0,2),1), \\ 0 & \text{otherwise.} \end{cases} \tag{27}$$

(The condition $m_1 + m_2 \leq 154$ ensures that there are enough aces and deuces remaining to allow the dealer's downcard to be other than a trey.)

The corresponding probability for $(0,0,2)$ vs. 2 is

$$\frac{\sum_{(m_1,m_2,m_3)\in\Gamma_n((0,0,2),2)} \binom{78}{m_1}\binom{78}{m_2}\binom{156}{m_3}\alpha_2(m_1,m_2,m_3)}{\sum_{(m_1,m_2,m_3)\in\Gamma_n((0,0,2),2)} \binom{78}{m_1}\binom{78}{m_2}\binom{156}{m_3}} \tag{28}$$

for $n = 1,2,\ldots,304$, where

$$\Gamma_n((0,0,2),2) := \{(m_1,m_2,m_3) : 0 \leq m_1 \leq 78,\, 0 \leq m_2 \leq 77,\, 0 \leq m_3 \leq 154,\, m_1+m_2+m_3 = n\}$$

and

$$\alpha_2(m_1,m_2,m_3) := \begin{cases} 1 & \text{if } E_{\text{std},(78-m_1,77-m_2,154-m_3)}((0,0,2),2) \\ & \quad > E_{\text{spl},(78-m_1,77-m_2,154-m_3)}((0,0,2),2), \\ 0 & \text{otherwise.} \end{cases} \tag{29}$$

The corresponding probability for $(0,0,2)$ vs. 3 is

$$\frac{\sum_{(m_1,m_2,m_3)\in\Gamma_n((0,0,2),3)} \binom{78}{m_1}\binom{78}{m_2}\binom{156}{m_3}\alpha_3(m_1,m_2,m_3)}{\sum_{(m_1,m_2,m_3)\in\Gamma_n((0,0,2),3)} \binom{78}{m_1}\binom{78}{m_2}\binom{156}{m_3}} \tag{30}$$

for $n = 1,2,\ldots,304$, where

$$\Gamma_n((0,0,2),3) := \{(m_1,m_2,m_3) : 0 \leq m_1 \leq 78,\, 0 \leq m_2 \leq 78,\, 0 \leq m_3 \leq 153,\, m_2+m_3 \leq 230,\, m_1+m_2+m_3 = n\}$$

and

$$\alpha_3(m_1,m_2,m_3) := \begin{cases} 1 & \text{if } E_{\text{std},(78-m_1,78-m_2,153-m_3)}((0,0,2),3) \\ & \quad > E_{\text{spl},(78-m_1,78-m_2,153-m_3)}((0,0,2),3), \\ 0 & \text{otherwise.} \end{cases} \tag{31}$$

The expressions (26), (28), and (30) are graphed in Figure 5.

More important than the probability that a departure from basic strategy is called for is the additional expectation that such a departure provides. With $(0,0,2)$ vs. A, 2, or 3 this is given by (26), (28), or (30) but with

$$\alpha_1(m_1,m_2,m_3) := [E_{\text{std},(77-m_1,78-m_2,154-m_3)}((0,0,2),1) - E_{\text{spl},(77-m_1,78-m_2,154-m_3)}((0,0,2),1)]^+, \tag{32}$$

$$\alpha_2(m_1,m_2,m_3) := [E_{\text{std},(78-m_1,77-m_2,154-m_3)}((0,0,2),2) - E_{\text{spl},(78-m_1,77-m_2,154-m_3)}((0,0,2),2)]^+, \tag{33}$$

or

$$\alpha_3(m_1,m_2,m_3) := [E_{\text{std},(78-m_1,78-m_2,153-m_3)}((0,0,2),3) - E_{\text{spl},(78-m_1,78-m_2,153-m_3)}((0,0,2),3)]^+ \tag{34}$$

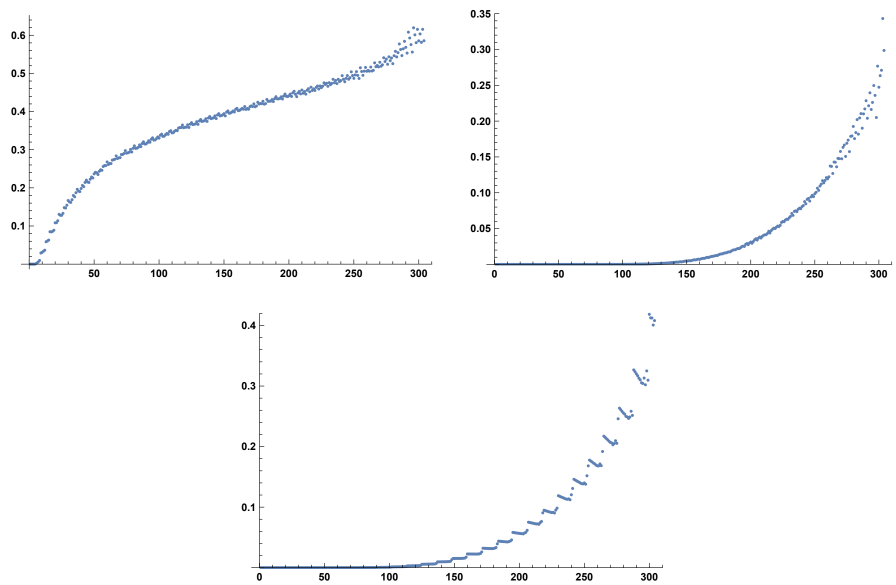

Figure 5
With $(0,0,2)$ *vs.* A *(top left),* $(0,0,2)$ *vs.* 2 *(top right), and* $(0,0,2)$ *vs.* 3 *(bottom), the probability that a departure from basic strategy is called for when n cards have been seen (before the hand is dealt), as a function of n,* $1 \le n \le 304$.

in place of (27), (29), or (31). The expressions (26), (28), and (30), using (32), (33), and (34), are graphed in Figure 6.

As with bet variation, the only way to recognize potentially profitable departures from basic strategy is with card counting. First, we analyze the case $(0,0,2)$ vs. A, which is complicated by the assumption that the dealer does not have a natural. The effects of removal are

$$\begin{aligned}\mathrm{EoR}(i) := {} & E_{\mathrm{std},(77,78,154)-\boldsymbol{e}_i}((0,0,2),1) - E_{\mathrm{spl},(77,78,154)-\boldsymbol{e}_i}((0,0,2),1) \\ & - [E_{\mathrm{std},(77,78,154)}((0,0,2),1) - E_{\mathrm{spl},(77,78,154)}((0,0,2),1)]\end{aligned} \tag{35}$$

for $i = 1,2,3$. The numbers (35), multiplied by 308, are

$$E_1 = \frac{78{,}498{,}676}{49{,}345{,}645}, \quad E_2 = -\frac{895{,}474{,}426}{444{,}110{,}805}, \quad E_3 = \frac{33{,}220{,}264}{148{,}036{,}935},$$

with decimal equivalents 1.59079, −2.01633, and 0.224405. The analogue of (23) is

$$w_1 E_1 + w_2 E_2 + w_3 E_3 = 0 \tag{36}$$

with weights

$$w_1 = \frac{77}{308}\frac{154}{155}, \quad w_2 = \frac{78}{308}\frac{154}{155}, \quad w_3 = \frac{154}{308}; \tag{37}$$

see Epstein (1967, p. 244). The correlation between the effects of removal and the deuces-minus-aces counting system $(J_1, J_2, J_3) = (-1, 1, 0)$ is

$$\rho = \frac{w_1 E_1 J_1 + w_2 E_2 J_2 + w_3 E_3 J_3}{\sigma_E \, \sigma_J} \approx -0.985649,$$

where $\sigma_E^2 := w_1 E_1^2 + w_2 E_2^2 + w_3 E_3^2$ and $\sigma_J^2 := w_1 J_1^2 + w_2 J_2^2 + w_3 J_3^2 - (w_1 J_1 + w_2 J_2 + w_3 J_3)^2$, and the regression coefficient is

$$\gamma = \frac{w_1 E_1 J_1 + w_2 E_2 J_2 + w_3 E_3 J_3}{w_1 J_1^2 + w_2 J_2^2 + w_3 J_3^2} = -\frac{41{,}415{,}529{,}232}{22{,}945{,}724{,}925} \approx -1.80493,$$

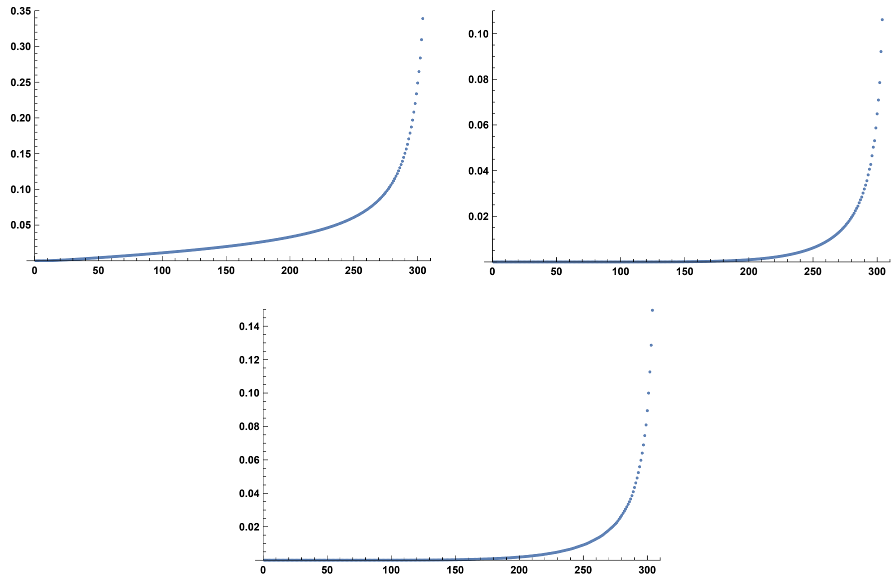

Figure 6
With $(0,0,2)$ *vs.* A *(left top),* $(0,0,2)$ *vs.* 2 *(right top), and* $(0,0,2)$ *vs.* 3 *(bottom), the additional expectation that a departure from basic strategy provides when n cards have been seen (before the hand is dealt), as a function of n,* $1 \leq n \leq 304$*. Notice that the vertical scales differ considerably.*

which is the γ that minimizes the sum of squares $w_1(E_1 - \gamma J_1)^2 + w_2(E_2 - \gamma J_2)^2 + w_3(E_3 - \gamma J_3)^2$.

The analogues of (24) and (25) can be found by observing from (36) and (37) that

$$77\,E_1 + 78\,E_2 + 154\,E_3 = -E_3.$$

Hence

$$\widehat{Z}_n = \frac{1}{309-n}\sum_{j=n+1}^{309}(\mu - E_{X_j}) = \mu + \frac{1}{309-n}\left(E_3 + \sum_{j=1}^{n} E_{X_j}\right)$$

and

$$\begin{aligned} Z_n^* &= \mu + \frac{\gamma}{52}\left(\frac{52}{309-n}\left[J_3 + \sum_{j=1}^{n} J_{X_j}\right]\right) \\ &= \mu + \frac{\gamma}{52}\left(\frac{52}{309-n}\sum_{j=1}^{n} J_{X_j}\right) = \mu + \frac{\gamma}{52}\mathrm{TC}_n^*, \end{aligned} \tag{38}$$

where

$$\mu = E_{\mathrm{std},(77,78,154)}((0,0,2),1) - E_{\mathrm{spl},(77,78,154)}((0,0,2),1) = -\frac{60{,}451}{2{,}426{,}835} \approx -0.0249094.$$

This allows the card counter to know (approximately) when it is advantageous to depart from basic strategy when holding $(0,0,2)$ vs. A. Indeed, $Z_n^* > 0$ is equivalent to

$$\mathrm{TC}_n^* := \frac{52(m_2 - m_1)}{309-n} < -\frac{52\mu}{\gamma}$$

(the inequality is reversed because $\gamma < 0$ or, equivalently, $\rho < 0$). The fraction

$$-\frac{52\mu}{\gamma} = -\frac{7{,}430{,}334{,}665}{10{,}353{,}882{,}308} \approx -0.717638$$

is the *index number* for this departure. In this case, if the *adjusted true count* TC_n^* is less than this index number, standing on $(0,0,2)$ vs. A is called for instead of splitting. We say "adjusted" because the player's treys and the dealer's ace are excluded from the count.

In practice, we would round the index number to -1, and this play would occur relatively often (the rounded adjusted true count would have to be at -1 or less). Of course we would be betting only one unit, and we can infer an upper bound on the profit potential from the first panel in Figure 6. In fact, we can compute it precisely using (26) with

$$\begin{aligned}\alpha_1(m_1,m_2,m_3) := [&E_{\mathrm{std},(77-m_1,78-m_2,154-m_3)}((0,0,2),1)\\ &- E_{\mathrm{spl},(77-m_1,78-m_2,154-m_3)}((0,0,2),1)]\\ &\cdot \mathbf{1}\left\{\left[\frac{52(m_2-m_1)}{309-n}\right] \le -1\right\}.\end{aligned}$$

The average of these expectations over $1 \le n \le 233$ is approximately 0.0162143.

Next, we analyze the simpler case of $(0,0,2)$ vs. 2. The effects of removal are

$$\begin{aligned}\mathrm{EoR}(i) := &E_{\mathrm{std},(78,77,154)-\boldsymbol{e}_i}((0,0,2),2) - E_{\mathrm{spl},(78,77,154)-\boldsymbol{e}_i}((0,0,2),2)\\ &- [E_{\mathrm{std},(78,77,154)}((0,0,2),2) - E_{\mathrm{spl},(78,77,154)}((0,0,2),2)]\end{aligned} \tag{39}$$

for $i = 1,2,3$. The numbers (39), multiplied by 308, are

$$E_1 = \frac{318{,}420{,}487}{295{,}118{,}793}, \quad E_2 = -\frac{2{,}651{,}203{,}088}{1{,}475{,}593{,}965}, \quad E_3 = \frac{519{,}211{,}999}{1{,}475{,}593{,}965},$$

with decimal equivalents 1.07896, -1.79670, and 0.351866. The analogue of (23) is (36) with weights

$$w_1 = \frac{78}{309}, \quad w_2 = \frac{77}{309}, \quad w_3 = \frac{154}{309}.$$

The correlation between the effects of removal and the deuces-minus-aces counting system $(J_1,J_2,J_3) = (-1,1,0)$ is $\rho \approx -0.943999$, and the regression coefficient is

$$\gamma = -\frac{328{,}326{,}627{,}706}{228{,}717{,}064{,}575} \approx -1.43551.$$

The analogues of (24) and (25) are

$$\widehat{Z}_n = \mu + \frac{1}{309-n}\sum_{j=1}^{n} E_{X_j}$$

and

$$Z_n^* = \mu + \frac{\gamma}{52}\left(\frac{52}{309-n}\sum_{j=1}^{n} J_{X_j}\right) = \mu + \frac{\gamma}{52}\mathrm{TC}_n^*,$$

where

$$\mu = E_{\mathrm{std},(78,77,154)}((0,0,2),2) - E_{\mathrm{spl},(78,77,154)}((0,0,2),2) = -\frac{1{,}452{,}413}{9{,}676{,}026} \approx -0.150104.$$

This analysis is similar to Ethier (2010, pp. 668–670) for 6, 10 vs. 9 in blackjack.

This allows the card counter to know (approximately) when it is advantageous to depart from basic strategy when holding $(0,0,2)$ vs. 2. Indeed, $Z_n^* > 0$ is equivalent to

$$\mathrm{TC}_n^* := \frac{52(m_2-m_1)}{309-n} < -\frac{52\mu}{\gamma}.$$

The fraction

$$-\frac{52\mu}{\gamma} = -\frac{892{,}616{,}719{,}475}{164{,}163{,}313{,}853} \approx -5.43737$$

is the index number for this departure from basic strategy. In this case, if the adjusted true count TC_n^* is less than this index number, standing on $(0,0,2)$ vs. 2 is called for instead of splitting.

In practice, the index number would be rounded to -6. This play would seldom occur, for the rounded adjusted true count would have to be at -6 or less (see Table 10), and when it did occur, the bet size would be one unit. So an upper bound on the value of this departure from basic strategy can be inferred from the second panel in Figure 6.

Finally, we analyze the case $(0,0,2)$ vs. 3, which also involves the assumption that the dealer does not have a natural. The effects of removal are

$$\begin{aligned}\mathrm{EoR}(i) &:= E_{\mathrm{std},(78,78,153)-\boldsymbol{e}_i}((0,0,2),3) - E_{\mathrm{spl},(78,78,153)-\boldsymbol{e}_i}((0,0,2),3)\\ &\quad - [E_{\mathrm{std},(78,78,153)}((0,0,2),3) - E_{\mathrm{spl},(78,78,153)}((0,0,2),3)] \end{aligned} \tag{40}$$

for $i = 1,2,3$. The numbers (40), multiplied by 308, are

$$E_1 = \frac{5{,}425{,}240}{3{,}616{,}767}, \quad E_2 = -\frac{209{,}017{,}702}{138{,}642{,}735}, \quad E_3 = \frac{24{,}804}{46{,}214{,}245},$$

with decimal equivalents 1.50002, -1.50760, and 0.000536718. The analogue of (23) is (36) with weights

$$w_1 = \frac{78}{308}, \quad w_2 = \frac{78}{308}\frac{230}{231}, \quad w_3 = \frac{153}{308}\frac{230}{231}. \tag{41}$$

The correlation between the effects of removal and the deuces-minus-aces counting system $(J_1, J_2, J_3) = (-1,1,0)$ is $\rho \approx -0.999998$, and the regression coefficient is

$$\gamma = -\frac{835{,}778{,}884}{555{,}776{,}529} \approx -1.50380.$$

The analogues of (24) and (25) can be found by observing from (36) and (41) that

$$78E_1 + 78E_2 + 153E_3 = -\frac{78}{230}E_1.$$

Hence

$$\widehat{Z}_n = \frac{1}{309-n}\sum_{j=n+1}^{309}(\mu - E_{X_j}) = \mu + \frac{1}{309-n}\left(\frac{78}{230}E_1 + \sum_{j=1}^{n} E_{X_j}\right)$$

and

$$\begin{aligned} Z_n^* = \mu + \frac{\gamma}{52}\left(\frac{52}{309-n}\left[\frac{78}{230}J_1 + \sum_{j=1}^{n} J_{X_j}\right]\right) &= \mu + \frac{\gamma}{52}\left(\frac{52}{309-n}\left[-\frac{78}{230} + \sum_{j=1}^{n} J_{X_j}\right]\right)\\ &= \mu + \frac{\gamma}{52}\left[-\frac{52}{309-n}\frac{78}{230} + \mathrm{TC}_n^*\right], \end{aligned}$$

where

$$\mu = E_{\mathrm{std},(78,78,153)}((0,0,2),3) - E_{\mathrm{spl},(78,78,153)}((0,0,2),3) = -\frac{229{,}736}{1{,}820{,}203} \approx -0.126214.$$

This allows the card counter to know (approximately) when it is advantageous to depart from basic strategy when holding $(0,0,2)$ vs. 3. Indeed, $Z_n^* > 0$ is equivalent to

$$\mathrm{TC}_n^* := \frac{52(m_2 - m_1)}{309-n} < -\frac{52\mu}{\gamma} + \frac{52}{309-n}\frac{78}{230}.$$

The fraction

$$-\frac{52\mu}{\gamma} = -\frac{70{,}217{,}200{,}248}{16{,}088{,}743{,}517} \approx -4.36437$$

plus the fraction $(52)(39)/[(309-n)115]$ is the variable index number for this departure. As Griffin (1999, p. 181) noted for blackjack, "different change of strategy parameters will be required at different levels of the deck when the dealer's up card is an ace." If the adjusted true count TC_n^* is less than this index number, standing on $(0,0,2)$ vs. 3 is called for instead of splitting.

Even with the extra n-dependent term, the index would be rounded to -5 (if $n \le 260$), so this departure would seldom occur, and the bet size would be one unit when it did occur. Therefore, the profit potential is rather limited, and an upper bound can be inferred from the third panel in Figure 6.

It should be pointed out that the methodology we have used to analyze strategy variation differs slightly from that of the blackjack literature and in particular from that of Griffin (1999, pp. 72–90) and Schlesinger (2018, Appendix D). The distinction was described by Griffin (1999, Appendix to Chap. 6) as follows:

> The strategy tables presented here are not the very best we could come up with in a particular situation. As mentioned in this chapter more accuracy can be obtained with the normal approximation if we work with a 51 rather than a 52 card deck. One could even have separate tables of effects for different two card player hands such as $(\mathrm{T},6)$ v T. Obviously a compromise must be reached, and my motivation has been in the direction of simplicity of exposition and ready applicability to multiple deck play.

There are two issues here. First, in analyzing a particular strategic situation, it is conventional to assume that the player has an "abstract" total, and to even regard the dealer's upcard as "abstract." We regard this convention as an unnecessary simplification, especially in snackjack. It also explains why the player's two cards and dealer's upcard are not included in our "adjusted" true count; indeed, those three cards are not part of the 309-card shoe on which the analysis is based. An advantage of the conventional approach is that the sum of the effects of removal is 0, rather than some weighted average, with the exception of the cases in which the upcard is an ace or a ten in blackjack (an ace or a trey in snackjack). Here Griffin (1999, p. 197) achieved an EoR sum of 0 with an ace up by multiplying the EoR for ten by 36/35. With a ten up, the EoR for ace is multiplied by 48/47. This also seems to be the approach of Schlesinger (2018, Appendix D), but it sacrifices accuracy for simplicity. The second issue is that the strategic EoRs (as well as the betting EoRs) are typically computed for the single-deck game and then converted to the multiple-deck games with the aid of a conversion factor (Griffin, 1999, Chap. 6; Schlesinger, 2018, Appendix D). Here, to maximize accuracy, we compute the effects of removal directly for the game we are interested in, 39-deck snackjack.

The computations that generated the first panel in Figure 6 were exact, but similar computations cannot be done for six-deck blackjack (recall the 370 trillion distinguishable shoe compositions). Instead, approximate methods, developed by Griffin (1976), are available, and it may be of some interest to see how accurate they are in the case of 39-deck snackjack. With Z_n^* defined by (38) and $\sigma_J^2 := w_1 + w_2 - (w_2 - w_1)^2$, a simple computation shows that $\mathrm{E}[Z_n^*] = \mu$ and

$$\mathrm{SD}(Z_n^*) = |\gamma|\sigma_J \sqrt{\frac{n}{(312-n)311}}. \tag{42}$$

(The coefficient of the square root, $|\gamma|\sigma_J$, is sometimes written as $|\rho|\sigma_E$, but this is not quite the same thing.) Notice that (42) is proportional to (13) with $N = 312$. By the normal approximation, $(Z_n^* - \mu)/\mathrm{SD}(Z_n^*)$ is approximately $N(0,1)$. Now in general, if Z is $N(0,1)$, μ is real, and $\sigma > 0$, then

$$\mathrm{E}[(\mu + \sigma Z)^+] = \sigma\,\mathrm{E}[(Z + \mu/\sigma)^+] = \sigma\,\mathrm{UNLLI}(-\mu/\sigma),$$

where UNLLI stands for *unit normal linear loss integral* (Griffin, 1999, p. 87), defined for real x by

$$\mathrm{UNLLI}(x) := \mathrm{E}[(Z-x)^+] = \int_x^\infty (z-x)\phi(z)\,dz = \phi(x) - x(1-\Phi(x)),$$

where ϕ and Φ denote the standard normal probability density function and cumulative distribution function. We conclude from the normal approximation that, with $\sigma_n := \mathrm{SD}(Z_n^*)$,

$$\mathrm{E}[(Z_n^*)^+] \approx \mathrm{E}[(\mu+\sigma_n Z)^+] = \sigma_n\,\mathrm{UNLLI}(-\mu/\sigma_n). \tag{43}$$

Figure 7 shows that the quality of the approximation deteriorates over the course of the shoe. If we average the approximate quantities over $1 \le n \le 233$, we obtain 0.0139785, which underestimates the exact value found above by about 13.8%.

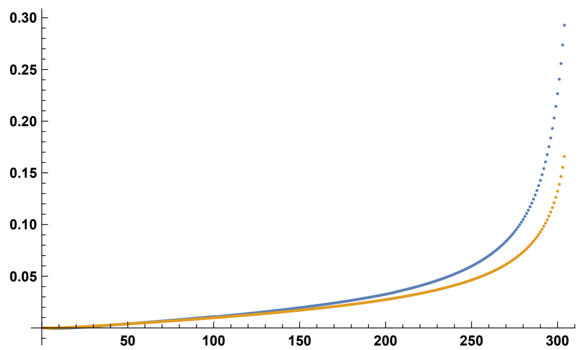

Figure 7
With $(0,0,2)$ *vs.* A*, the exact (blue) and approximate (orange) expected gain by departing from basic strategy when the rounded adjusted true count is at* -1 *or less, as a function of the number n of cards seen (before the hand is dealt),* $1 \le n \le 304$.

We have also analyzed the hands $(0,1,1)$ (hard 5) and $(1,1,0)$ (soft 6), and results are shown in Table 11. In blackjack, the decision points with the greatest profit potential for varying basic strategy are the *Illustrious 18* of Schlesinger (2018, Chap. 5). The results of Table 11 show that only three departures from basic strategy at snackjack are "illustrious." $(0,0,2)$ vs. A (stand instead of split) offers the greatest profit potential, then $(1,1,0)$ vs. 3 (hit instead of double), and finally $(1,1,0)$ vs. A (double instead of hit). None of the others is close to these three.

This table also emphasizes another distinction between our approach and the blackjack literature. Our strategy variations are formulated as departures from basic strategy. Instead of saying "stand instead of split if the rounded adjusted true count is -1 or less," it would be more conventional to say, "split instead of stand if the rounded adjusted true count is 0 or more." This way, all inequalities point in the same direction ($\ge$).

What does snackjack tell us about blackjack?

The simpler a toy model is, the fewer features it shares with the original. Grayjack is closer to blackjack than is snackjack. For example, the proportions of aces and tens in blackjack are 1/13 and 4/13. These proportions are maintained in grayjack for aces and sixes, thereby making naturals about as frequent and allowing insurance. In snackjack, the proportions of aces and treys are unavoidably rather different. In blackjack the numbers of pat totals (17–21) and stiff totals (12–16) are the same, five each. In grayjack the numbers (8–10 stiff and 11–13 pat) are also the same, three each. But in snackjack the numbers (5 stiff and 6–7 pat) are different, again unavoidably. This makes it more difficult to bust, mitigating the dealer's principal advantage, the double bust. Ultimately, we felt that the

Table 11
Analysis of several departures from basic strategy. ρ is the correlation between the effects of removal for the strategic situation and the deuces-minus-aces count $(-1,1,0)$. The inequality listed under "departure criterion" is what $[\mathrm{TC}_n^*]$ must satisfy for a departure from basic strategy to be called for. The quantity ave EV *is the player's expectation averaged over $1 \le n \le 233$ when betting $\max(1,\min([\mathrm{TC}_n^*],6))$. An asterisk in the "index" column signifies a missing n-dependent term. We omit rows for $(0,2,0)$ and $(2,0,0)$ vs.* A, 2, *and* 3, *each of which would have a* 0 *in the last column.*

nos of As, 2s, 3s	up	bs	alt	corr. ρ	index	departure criterion	$10^6\times$ ave EV
$(0,0,2)$	A	Spl	S	-0.986	-0.718	≤ -1	16,214
$(0,0,2)$	2	Spl	S	-0.944	-5.44	≤ -6	250
$(0,0,2)$	3	Spl	S	-1.000	-4.36^*	≤ -5	688
$(0,1,1)$	A	H	S	0.503	25.4	–	0
$(0,1,1)$	2	H	S	0.033	91.7	–	0
$(0,1,1)$	3	H	S	-0.149	$-184.^*$	–	0
$(1,1,0)$	A	H	S	-0.472	-22.5	–	0
$(1,1,0)$	A	H	D	0.837	4.28	$\ge +5$	5,229
$(1,1,0)$	2	D	S	-0.940	-5.63	≤ -6	88
$(1,1,0)$	2	D	H	-0.661	-3.06	≤ -4	650
$(1,1,0)$	3	D	S	-1.000	-4.36^*	≤ -5	687
$(1,1,0)$	3	D	H	-0.909	-0.147^*	≤ -1	10,547

benefits of having a hand-computable toy model of blackjack outweighed the drawbacks of a significant player advantage and a largely upcard-independent basic strategy. Actually, more important than hand-computability are the explicit formulas available for basic strategy expectations with arbitrary shoe compositions. This allows exact computation of quantities that can only be estimated at blackjack.

What then have we learned about blackjack from its computable toy model, snackjack?

- The derivation of basic strategy at blackjack is conceptually very simple, despite its computational complexity. Basic strategy for blackjack is now so well known and understood that there is little insight to be gained by deriving basic strategy for snackjack or grayjack. Nevertheless, perhaps surprising to some is the conceptual simplicity of the basic strategy derivation, as illustrated by the tree diagram in Figure 1. The corresponding tree diagram for standing with a pair of tens vs. a playable ace in six-deck blackjack would have 8,496 terminal vertices (Griffin, 1999, p. 158) instead of four, but *conceptually* it is the same thing.

- It is truly remarkable, as has been noted elsewhere (Griffin, 1999, p. 17), that single-deck blackjack (under classic Las Vegas Strip rules), which was played long before it was analyzed, turned out to be an essentially fair game, with a player advantage of about four hundredths of 1%. The present study emphasizes the sensitivity of basic strategy expectations to minor rules changes. For example, the rule "A player natural pays even money, with the exception that it loses to a dealer natural," reduces the player advantage at double-deck snackjack from $+16.3\%$ to -0.0959%. The less extreme rules change in blackjack in which untied player naturals pay 6 to 5 instead of 3 to 2 has a smaller but still significant effect, as every advantage player will acknowledge.

- A formula for basic strategy expectation in six-deck blackjack as a function of shoe composition, if found, would likely be highly impractical. A polynomial in three variables of degree 8 or less has at most $\binom{8+3}{3} = 165$ terms, so the 147 terms of the polynomial in (14) is not surprising. The analogous polynomial in blackjack would have at most $\binom{m+10}{10}$ terms, where m is the maximum number of cards needed to complete a round. In six-deck blackjack that number is at least 24 (e.g., two hands of A, A, A, A, A, A, 6, A, A, A, A, 5), even before considering splits, and $\binom{24+10}{10} = 131{,}128{,}140$. The actual number of terms in the blackjack basic strategy expectation polynomial would likely be somewhat smaller but still highly impractical.

- With Z_n being the player's conditional expectation when n cards have been seen, $\mathrm{E}[(Z_n)^+]$ (or $\mathrm{E}[(Z_n - \nu)^+]$) can be computed directly for the game of red-and-black (see (12)) and for 39-deck snackjack (see (20)), assuming basic strategy. As we have explained, such computations for six-deck blackjack are likely impossible, but perhaps computer simulation would give the best results. Another potential approach would be to approximate Z_n by its linearization $\widehat{Z}_n$ based on EoRs, and then use a normal approximation involving the UNLLI function, much as we did in (43). There are some things in blackjack that simply cannot be known exactly.

- The section on bet variation contains several computations for 39-deck snackjack that cannot be replicated for six-deck blackjack. If they could be, we would likely reach the same conclusions as we do at snackjack. Specifically, we computed the L^1 distances between Z_n and its linearization $\widehat{Z}_n$ based on EoRs and its linearization Z_n^* based on the chosen card-counting system. We find from Table 8 that, for the first 2/3 of the shoe, the bulk of the error in approximating Z_n by Z_n^* is explained by the use of the level-one deuces-minus-aces point count in place of the EoRs; the nonlinearity effect is relatively inconsequential. Another finding, based on limited evidence, was that the betting efficiency of a card-counting system is well approximated by the betting correlation, that is, the correlation between the EoRs and the numbers of the point count. The latter is computable for blackjack, whereas the former is not (except by simulation). There is of course a theoretical reason for this (Griffin, 1999, p. 51).

- A surprise to us was the extent to which the distribution of the true count at snackjack departs from normality. This is likely true at blackjack as well, but less easy to verify. For snackjack, it is a consequence of Figure 3, which shows that the rounded true count fails to be discrete normal for some choices of n. The $n = 260$ case is what we expected, whereas the $n = 234$ case illustrates what can happen. A more complete analysis than that done for the figure shows that the distribution of the rounded true count is bimodal if and only if $105 \leq n \leq 138$ or $209 \leq n \leq 255$. Theory tells us that the true count is asymptotically normal but of course this lacks rigor because we never let N (the number of cards in the shoe) tend to infinity; instead, N is fixed at 312. As Griffin (1999, p. 38) put it, "the proof of the pudding is in the eating."

- As we mentioned in the section on strategy variation, it is conventional in blackjack to compute the effects of removal based on a 52-card deck, then multiply them by a conversion factor for multiple-deck applications. The justification for this is based on the following observation. Let $\mathrm{EoR}_N(i)$ denote the effect of removal of card value i from a deck of N cards on basic strategy expectation. Then it can be shown that $\lim_{N\to\infty}(N-1)\mathrm{EoR}_N(i)$ exists, and thus $\mathrm{EoR}_{312}(i)$ is approximately equal to $(51/311)\mathrm{EoR}_{52}(i)$, for example. We can use snackjack to investigate how accurate we can expect this approximation to be for six-deck blackjack. For snackjack, Table 12 displays the relevant data. The correlation between the $N = 52$ EoRs and the $N = 312$ EoRs is 0.999135. The result is that the approximate effects of removal based on a 52-card ($6\frac{1}{2}$-deck) pack, instead of a 312-card (39-deck) shoe, are considerably less accurate than our level-six point count but substantially more accurate than our level-one point count. This can also confirmed in terms of L^1 distances, as in Table 8.

Table 12
Effects of removal on snackjack's basic strategy expectation. For simplicity we used (14) *to evaluate these numbers, even though basic strategy in the case* $N = 52$ *differs slightly from the strategy implicit in* (14). *The entries for* $N = 312$ *coincide with those of Table 7.*

N	$(N-1)\mathrm{EoR}_N(1)$	$(N-1)\mathrm{EoR}_N(2)$	$(N-1)\mathrm{EoR}_N(3)$
52	−0.516148	0.711619	−0.0977352
104	−0.490108	0.702236	−0.106064
312	−0.473605	0.696413	−0.111404
∞	−0.465576	0.693604	−0.114014

- In snackjack we have seen that some strategy variation decisions (such as standing instead of hitting a hard 5) are not well suited to the deuces-minus-aces count. Similarly, and it is well known to experts, some strategy variation decisions in blackjack are not well suited to the Hi-Lo count (14 vs. 10 and 16 vs. 7 are two examples mentioned by Schlesinger, 2018, p. 57).

In summary, the basis for card counting is linearization of a nonlinear function. And this paper provides theoretical support for a conclusion for which abundant anecdotal evidence exists, namely that card counting works.

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