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Frequency sampling digital filters for multirate applications and pipelined implementations

Sivaramakrishnan Ramaswamy
University of Nevada, Las Vegas

Frequency Sampling Digital Filters for Multirate Applications and Pipelined Implementations

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S. Ramaswamy

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UMI
300 North Zeeb Road
Ann Arbor, MI 48103

The thesis of S.Ramaswamy for the degree of Master of Science in Electrical Engineering is approved.

Chairperson, Peter Stubberud, Ph.D

Examining Committee Member, William Brogan, Ph.D

Examining Committee Member, Eugene McGaugh, Ph.D

Graduate Faculty Representative, Junichi Kanai, Ph.D

Graduate Dean, Ronald W. Smith, Ph.D

University of Nevada, Las Vegas

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ABSTRACT

Under certain circumstances, narrowband linear phase filters can be implemented more efficiently using frequency sampling filters than direct convolution filters. The desired frequency response of a frequency sampling filter is approximated by interpolating a frequency response through a set of frequency samples. The implementation of a frequency sampling filter contains unit delays in its feedback paths. Therefore, it cannot be pipelined or efficiently used in multirate applications. In this thesis, a frequency sampling filter that has only delays of length D in its feedback paths is developed. This new frequency sampling filter can be pipelined and efficiently used in multirate applications.

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Contents

ABSTRACT iii

1 **Introduction** **1**

2 **Frequency Sampling Filters** **6**

2.1 Type 1 frequency sampling filters 6

2.1.1 Type 1 Frequency Sampling Filters With Real Impulse Response . 9

2.1.2 Type 1 Frequency Sampling Filters With Linear Phase 11

2.2 Type 2 Frequency Sampling Filters 12

2.2.1 Type 2 Frequency Sampling Filters With Real Impulse Response . 14

2.2.2 Type 2 Frequency Sampling Filters With Linear Phase 16

2.3 Properties Of Frequency Sampling Filters 18

2.4 Coupled State Space Structures Used To Realize Frequency Sampling Filters 19

2.5 Sampling Rate Reduction In Frequency Sampling Filters 21

3 **Pipelined and Multirate Frequency Sampling Filters** **25**

3.1 Pipelined Type 1 Frequency Sampling Filter 26

3.1.1 Pipelined Type 1 Frequency Sampling Filter With Real Impulse Response 29

3.1.2 Pipelined Type 1 Linear Phase Frequency Sampling Filter 34

3.1.3 Properties of Multirate Type 1 Frequency Sampling Filters 36

3.2 Pipelined Type 2 Frequency Sampling Filter 39

3.2.1 Pipelined Type 2 Frequency Sampling Filter With Real Impulse Response 41

3.2.2 Pipelined Type 2 Linear Phase Frequency Sampling Filter 44

3.2.3 Properties of Multirate Type 2 Frequency Sampling Filters 49

3.3 Example: A Low-pass Frequency Sampling Filter for N Odd 51

4 **State space structures for multirate frequency sampling filters** **57**

4.1 State Space Structure for a Multirate Type 1 Frequency Sampling Filter . . 57

4.2 State Space Structure for a Multirate Type 2 Frequency Sampling Filter . . 61

4.3 Properties of Coupled State Space Structures 67

5 Conclusions **68**
BIBLIOGRAPHY 70

List of Figures

1.1a Direct form structure used to realize FIR filters. 3
1.1b Transposed Direct form structure used to realize FIR filters. 3
1.2a Structure for a Direct Form FIR filter with linear phase (N odd). 4
1.2b Structure for a Direct Form FIR filter with linear phase (N even). 4

2.1 Structure of a Type 1 frequency sampling filter. 8
2.2a Type 1 frequency sampling filter with a real impulse response of length N where N is even and $M_1 = (N/2) - 1$. 10
2.2b Structure of the kth resonator in Fig. 2.2a. 11
2.3a Type 1 frequency sampling filter with linear phase (N even) 12
2.3b Structure of the kth resonator, in Fig. 2.3a. 13
2.4 Structure of a Type 2 frequency sampling filter. 14
2.5a Type 2 frequency sampling filter with a real impulse response(N even). . . . 15
2.5b Structure of the kth resonator, where k= 0 to M_2. 16
2.6a Type 2 frequency sampling filter with linear phase(N even). 17
2.6b Structure of the kth resonator, in Fig. 2.6a. 17
2.7 Coupled state space structure for the kth resonator of a Type 1 frequency sampling filter. 22
2.8 Block diagram of a multirate filtering process. 22
2.9 Rules used to move a decimator in a flow graph. 23
2.10 Flowgraph for a direct convolution structure with the output decimated by a factor of D. 24

3.1a Pipelined Type 1 frequency sampling filter. 26
3.1b Structure of the kth resonator in Fig. 3.1a. 27
3.2a The kth resonator of a multirate Type 1 frequency sampling filter. 28
3.2b Structure,of the kth resonator, with the decimator moved towards the input side. 28
3.3a Pipelined type 1 frequency sampling filter with real impulse response. . . . 30
3.3b Structure of the resonators 1 to M_1, in Fig. 3.3a 31
3.3c Structure of resonator 0, in Fig. 3.3a 31
3.3d Structure of the resonator $N/2$, in Fig. 3.3a 32

3.4a Direct Form 1 structure of the kth resonator of a multirate Type 1 frequency sampling filter with real impulse response for k= 1 to M_1. 33
3.4b Structure of the kth resonator with the decimator moved towards the input side. 33
3.5a Pipelined Type 1 frequency sampling filter with linear phase. 35
3.5b Structure of resonators 1 to M_1, shown in Fig. 3.5a 36
3.6a Direct Form 1 structure for the kth resonator of a multirate Type 1 frequency sampling filter with linear phase. 37
3.6b Structure of the kth resonator with the decimator moved towards the input side. 37
3.7a Pipelined Type 2 frequency sampling filter. 40
3.7b Structure of the kth resonator in Fig. 3.7(a) 40
3.8a Direct Form 1 structure for the kth resonator of a multirate Type 2 frequency sampling filter. 42
3.8b Structure of the kth resonator with the decimator moved towards the input side. 42
3.9a Pipelined Type 2 frequency sampling filter with real impulse response. . . . 44
3.9b Structure of the resonators 0 to M_2 in Fig. 3.9a. 45
3.10a Direct Form 1 structure for the kth resonator of a multirate Type 2 frequency sampling filter with real impulse response. 45
3.10b Structure of the kth resonator with the decimator pushed towards the input side. 46
3.11a Pipelined Type 2 frequency sampling filter with linear phase. 47
3.11b Structure of the resonators 0 to M_2 in Fig. 3.11a. 48
3.12a Multirate Type 2 frequency sampling filter with linear phase. 49
3.12b Structure of the resonator with the decimator pushed towards the input side. 50
3.13a Magnitude spectrum of the input. 52
3.13b Magnitude spectra of the output of the multirate Type 1 frequency sampling filter, for different values of r. $D = 1$, $K = 6$, $N = 127$ 53
3.13c Phase spectra of the output of the Type 1 multirate frequency sampling filter, for different values of r. $D = 1$, $K = 6$, $N = 127$ 54
3.13d Magnitude spectra of the output of the Type 1 multirate frequency sampling filter, for different values of r. $D = 4$, $K = 6$, $N = 127$ 55
3.13e Phase spectra of the output of the Type 1 multirate frequency sampling filter, for different values of r. $D = 4$, $K = 6$, $N = 127$ 56

4.1a Structure to implement Equation (4.3), with the output, $s(n)$, decimated by a factor of D 59
4.1b Structure to implement Equation (4.4) 60
4.2 Coupled state space structure for resonators 1 to M_1. 62
4.3a Structure to implement Equation (4.3), with the output, $s(n)$, decimated by a factor of D 64
4.3b Structure to implement Equation (4.4) 65
4.4 Coupled state space structure for resonators 1 to M_2. 66

List of Tables

2.1 Multiplies required per output sample by direct convolution and frequency sampling filter structures when their outputs are decimated by a factor of D. 23

2.2 Multiplies required per output sample if the decimator in the structures in in Table 2.1 is moved toward the input. 24

3.1 Multiplies required per output sample by a direct convolution filter, a Type 1 frequency sampling filter and a pipelined Type 1 frequency sampling filter (K is the number of nonzero frequency samples, $D > 1$ and N is the length of the impulse response of the filter). 38

3.2 Multiplies required per output sample by a direct convolution filter, a Type 1 frequency sampling filter and a pipelined Type 1 frequency sampling filter with their outputs decimated (D is the decimation ratio, K is the number of nonzero frequency samples and N is the length of the impulse response of the filter). 38

3.3 Multiplies required per output sample by Type 2 frequency sampling filter and a pipelined Type 2 frequency sampling filter (K is the number of nonzero frequency samples, $D > 1$ and N is the length of the impulse response of the filter). 50

3.4 Multiplies required per output sample by Type 2 frequency sampling filter and a pipelined Type 2 frequency sampling filter with their outputs decimated (D is the decimation ratio, K is the number of nonzero frequency samples and N is the length of the impulse response of the filter). 51

4.1 Multiplies required per output sample by Direct Form structures and coupled state space structures of frequency sampling filters (K is the number of nonzero frequency samples and D is the decimation ratio of the output). . . 67

Chapter 1

Introduction

Many digital signal processing applications require linear phase frequency selective filters to bandlimit the spectrum of a signal. When designing frequency selective filters, the desired filter characteristics are specified in the frequency domain in terms of the desired magnitude and phase response of the filter. Many design processes are available for determining system functions that can closely approximate desired frequency response characteristics. After determining a system function, the filter is implemented using difference equations. Two types of difference equations exist for implementing digital filters, recursive and non-recursive. A difference equation is recursive if the current output is a function of the past outputs. For example, in a recursive filter the functional relationship between the input and the output can be expressed as $y(n) = F[y(n-1), y(n-2)..., x(n), x(n-1), x(n-2)...]$, where $x(n)$ is the system's input, $y(n)$ is the system's output and F is a function relating the output to the input. A difference equation is nonrecursive if the output is a not a function of past outputs. For example, in a nonrecursive filter the output can be expressed as $y(n) = F[x(n), x(n-1), x(n-2)...]$. Digital filters can be classified as finite impulse response (FIR) filters or infinite impulse response (IIR) filters. A recursive difference equation can implement an FIR or an IIR filter. A nonrecursive difference equation can implement only an FIR filter [1]. In this thesis, recursive implementations of FIR filters called frequency sampling filters that are well suited for pipelining and multirate applications are developed. The computational requirements of these filters are compared to direct convolution implementations.

In applications requiring a frequency selective filter, frequency dispersion due to non-linear phase can distort desired signal information. For such applications it is desirable to design linear phase filters. For a causal FIR digital filter to have linear phase, it must have an impulse response, $h(n)$, with the property $h(n) = h^*(N-1-n)$ for $0 \leq n \leq N-1$ where $h^*(n)$ is the complex conjugate of $h(n)$ [2].

Two common methods used to design linear phase FIR digital filters are the optimal filter design method and the window design method. In the optimal design method, the maximum approximation error between the desired frequency response and the actual frequency response is minimized. This design procedure generates an FIR, where $h(n) = h^*(N-1-n)$. In the window design method, the unit impulse response, $h_d(n)$, is determined from the desired frequency response specification, $H_d(e^{j\omega})$, using the inverse Fourier transform relation,

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}d\omega. \tag{1.1}$$

In general, $h_d(n)$ is infinite in duration and is truncated to length N by multiplying $h_d(n)$ by a finite length window sequence, $w(n)$, of length N. If the desired impulse response has the property, $h_d(n) = h_d^*(N-1-n)$ for $0 \leq n \leq N-1$, and if the window has the property, $w(n) = w^*(N-1-n)$ for $0 \leq n \leq N-1$, then $h_d(n)w(n) = h_d^*(N-1-n)w^*(N-1-n)$ for $0 \leq n \leq N-1$, and the filter obtained is a linear phase FIR filter.

Once an impulse response is determined from the optimal or window design methods, the filter can be implemented by one of the nonrecursive direct convolution structures shown in Figs. 1.1a and 1.1b. The structures in Fig. 1.1 are called direct convolution structures because they implement convolution directly. These filter structures require $N-1$ adds and N multiplies per output sample, where N is the length of the filter's impulse response. If the filter has linear phase and a real impulse response then, $h(n) = h(N-1-n)$ for

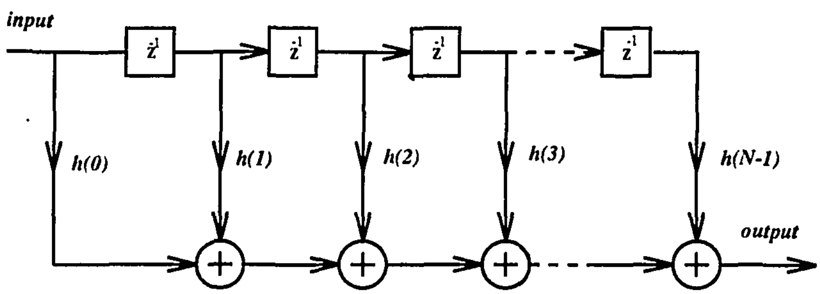


Figure 1.1a: Direct form structure used to realize FIR filters.

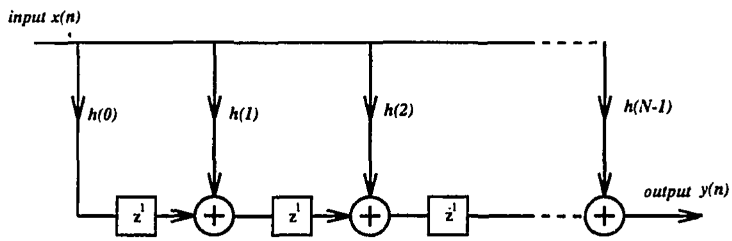


Figure 1.1b: Transposed Direct form structure used to realize FIR filters.

$0 \leq n \leq N-1$. and the structures shown in Fig. 1.2a and 1.2b can be used. These filter structures require $(N+1)/2$ multiplies and $N-1$ adds per output sample if N is odd and $N/2$ multiplies and $(N-1)$ adds per output sample if N is even. Assuming that multiplies are computationally more complex than adds, the filter structures shown in Fig. 1.2 require approximately half the number of computations required by the structures in Fig. 1.1

Unlike the previous design methods, frequency sampling filters implement FIR filters recursively. Under certain circumstances, narrowband filters can be implemented more efficiently using frequency sampling filters than using direct convolution filters [1]. Filters implemented by the frequency sampling technique are recursive. The frequency sampling filters discussed in this thesis are a Type 1 frequency sampling filter, which interpolates a frequency response through N evenly spaced frequency samples taken at frequencies $\omega = \frac{2\pi}{N}k$ for $k \in A$ where $A = \{0, 1 \ldots, N-1\}$. and a Type 2 frequency sampling filter, which interpolates a frequency response through N evenly spaced frequency samples taken at frequencies $\omega = \frac{2\pi}{N}\left(k + \frac{1}{2}\right)$ for $k \in A$ [3] [4] [5] .

Pipelining is an implementation technique where sequential instructions are over-

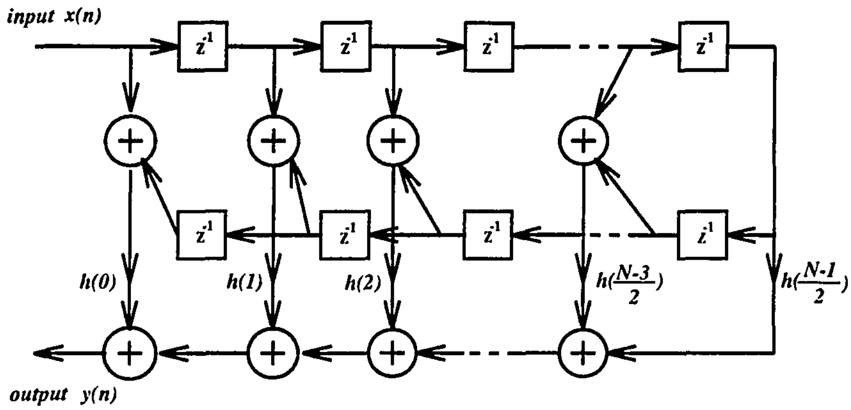


Figure 1.2a: Structure for a Direct Form FIR filter with linear phase (N odd).

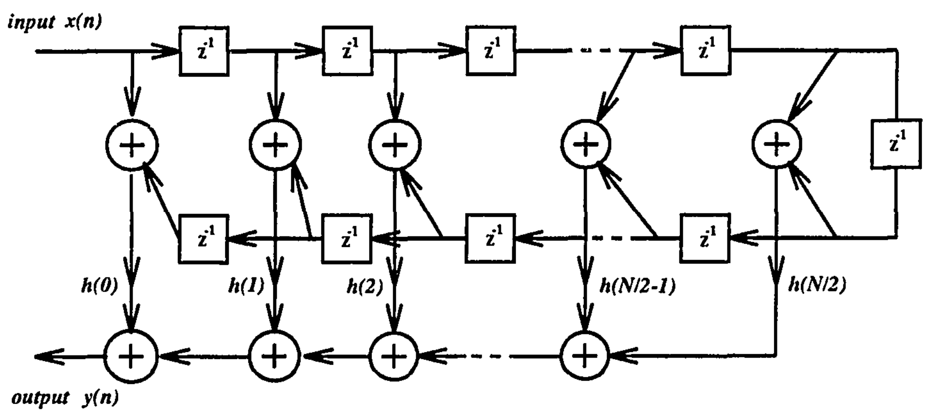


Figure 1.2b: Structure for a Direct Form FIR filter with linear phase (N even).

lapped in execution. The filter represented by Figs. 1.1a and 1.1b can be pipelined by adding registers in series with the multipliers and adders. The throughput of a digital filter can be dramatically increased by pipelining because several outputs can be processed simultaneously. Recursive structures, such as the frequency sampling structure, are not amenable to pipelining because the present output needs to be obtained before computing the next output values. A scheme was suggested by Soderstrand and Sinha in [6] and extended by Sinha and Loomis in [7] for pipelining IIR filters. This technique designs system functions which use pipelinable delays in the feedback path of the filter.

If different sampling frequencies are used in a system, then it is called a multirate system. Crooke and Craig have shown in [8] that for narrowband filtering applications, computational efficiencies of direct convolution filters can be gained by decimating the output of a nonrecursive filter. For recursive filters with unit delays in the feedback path, such as the frequency sampling filter, the process of decimation does not reduce the number of computations significantly. Martinez and Parks in [9] have discussed a class of recursive filters which have a system function with only a delay of D in the feedback path. These filters are suitable for decimation by a factor of D.

In this thesis, a frequency sampling filter is developed that has only delays of D in the feedback paths. Chapter 3 shows that these filters can be pipelined and a significant saving can be achieved when the outputs of these filters are decimated. Coupled state space implementations of these filters are discussed in Chapter 4 because they exhibit lower sensitivity to finite word length effects than Direct Form structures.

Chapter 2

Frequency Sampling Filters

A FIR filter can be realized by recursive or nonrecursive structures. Frequency sampling filters are recursive implementations of FIR filters. The recursive structures used to realize frequency sampling filters are simple to program and are very efficient under certain circumstances [10]. In this chapter, we will develop system functions and Direct Form realizations for Type 1 and Type 2 frequency sampling filters. The number of computations per output sample required by frequency sampling filters will be compared to that required by multirate frequency sampling filters. A coupled state space realization of a frequency sampling filter will be discussed.

2.1 Type 1 frequency sampling filters

A Type 1 frequency sampling filter interpolates a frequency response through a set of frequency samples taken at frequencies $\omega = \frac{2\pi}{N}k$ for $k \in A$ where $A = \{0, 1, \ldots N-1\}$. If we let $H(k)$ $k \in A$ represent the set of N frequency samples, then

$$H(k) = H(z)\big|_{z=e^{j\frac{2\pi}{N}k}} \text{ for } k \in A.$$

This relationship shows that $H(k)$ for $k \in A$ can be interpreted as samples of the z transform, taken at N evenly spaced points on the unit circle in the z plane. The finite length impulse response, $h(n)$, of the filter whose frequency samples are given by the discrete values, $H(k)$ for $k \in A$, can be determined from the Inverse Discrete Fourier Transform (IDFT).

$$h(n) = \begin{cases} \frac{1}{N}\sum_{k=0}^{N-1} H(k)e^{j\frac{2\pi}{N}kn} & n \in A \\ 0 & \text{otherwise} \end{cases}$$

The fiequency samples, $H(k)$ $k \in A$, can be calculated from the filter's impulse response by using the Discrete Fourier Transform (DFT).

$$H(k) = \sum_{n=0}^{N-1} h(n)e^{-j\frac{2\pi}{N}kn}.$$

Since the DFT is a form of a Fourier transform, $h(n)$ is the only impulse response of length N that interpolates a frequency response through the points, $H(k)$ $k \in A$. Therefore the frequency response of a system which has an impulse response of length N can be uniquely specified by a set of N frequency samples $H(k)$ for $k \in A$. The system function of this filter is given by

$$H(z) = \sum_{n=0}^{N-1} h(n)z^{-n} \tag{2.1}$$

where $h(n)$ is given by the DFT. Substituting the DFT into Equation (2.1),

$$H(z) = \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{k=0}^{N-1} H(k)e^{j\frac{2\pi}{N}nk}\right] z^{-n}.$$

Interchanging the order of the summations and summing up over the index n,

$$H(z) = \frac{1-z^{-N}}{N}\sum_{k=0}^{N-1}\frac{H(k)}{1-e^{j\frac{2\pi}{N}k}z^{-1}} \tag{2.2}$$

Equation (2.2) has the form of Lagrange's interpolating formula [3]. The complex polynomial, $H(z)$, interpolates a frequency response through the points, $H(k)$ $k \in A$, when

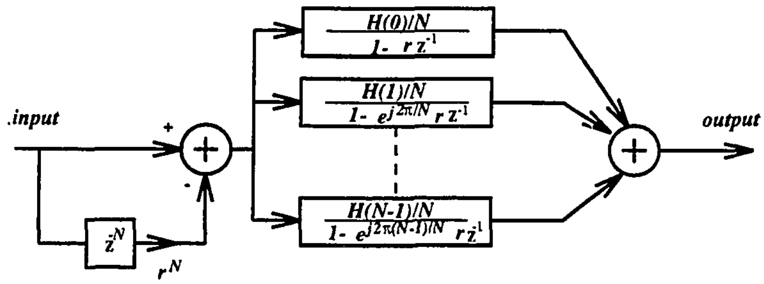


Figure 2.1: Structure of a Type 1 frequency sampling filter.

$z = e^{j\frac{2\pi}{N}k}$ for $k \in A$. Equation (2.2) shows that the filter can be realized using a cascade of a comb filter $(1 - z^{-N})$, which has zeros at $z = e^{j\frac{2\pi}{N}k}$ for $k \in A$, and a parallel connection of complex resonators whose poles occur at the zeros of the comb filter.

When a filter is implemented using special purpose hardware, the word lengths of the filter are fixed and finite. The interpolation formulas in the filter system function require the zeros of the comb filter to exactly cancel the poles of the resonator on the unit circle. The quantization of the filter's coefficients may prevent the cancelation of the poles on the unit circle making the filter unstable in the bounded input bounded output (BIBO) sense. To prevent instability in the filter, the poles of all the resonators can be moved within the unit circle, by replacing z^{-1} with rz^{-1} where $|r| < 1$ in the filter equation. For example the system function would be written as

$$H(z) = \frac{1 - r^N z^{-N}}{N} \sum_{k=0}^{N-1} \frac{H(k)}{1 - e^{j\frac{2\pi}{N}k} r z^{-1}} \tag{2.3}$$

The filter described in Equation (2.3) is called a frequency sampling filter because the filter's coefficients are a function of the frequency samples. Fig. 2.1 shows a block diagram used to implement the frequency sampling filter described in Equation (2.3). Most of the N complex resonators in the filter structure require one complex add and two complex multiplies per output sample. Each complex multiply requires two real adds and four real multiplies. Each complex add requires two real adds. Hence, the entire filter requires approximately $8N + 1$ real multiplies and $8N - 1$ real adds per output sample.

2.1.1 Type 1 Frequency Sampling Filters With Real Impulse Response

The filter can be implemented without complex arithmetic if it has a real impulse response. If the filter has a real impulse response then its frequency response, $H(e^{j\omega})$, has the property,

$$H(e^{j\omega}) = H^*(e^{-j\omega})$$

where $H^*(e^{-j\omega})$ is the complex conjugate of $H(e^{-j\omega})$.

Since $H(k) = H(e^{j\omega})|_{\omega=\frac{2\pi}{N}k}$ for $k \in A$, the frequency samples of a Type 1 frequency sampling filter will have the form

$$H(k) = H^*(N - k)$$

where $H^*(N-k)$ is the complex conjugate of $H(N-k)$. The frequency samples, $H(k)$ $k \in A$, can be written as $H(k) = |H(k)|e^{j\theta(k)}$ where $|H(k)|$ is the magnitude and $\theta(k)$ is the phase of the frequency sample. This implies that

$$|H(k)| = |H(N - k)|$$

and

$$\theta(k) = -\theta(N - k).$$

Substituting these properties into the system function in Equation (2.3),

$$\begin{aligned} H(z) &= \frac{1 - r^N z^{-N}}{N} \left\{ \frac{H(0)}{1 - rz^{-1}} + \frac{H(N/2)}{1 + rz^{-1}} + \right. \\ &\left. \sum_{k=0}^{M_1} \left[\frac{|H(k)|e^{j\theta(k)}(1 - e^{-j\frac{2\pi}{N}k} rz^{-1}) + |H(k)|e^{-j\theta(k)}(1 - e^{j\frac{2\pi}{N}k} rz^{-1})}{(1 - e^{-j\frac{2\pi}{N}k} rz^{-1})(1 - e^{j\frac{2\pi}{N}k} rz^{-1})} \right] \right\} \\ &= \frac{1 - r^N z^{-N}}{N} \left\{ \frac{H(0)}{1 - rz^{-1}} + \frac{H(N/2)}{1 + rz^{-1}} + \right. \end{aligned}$$

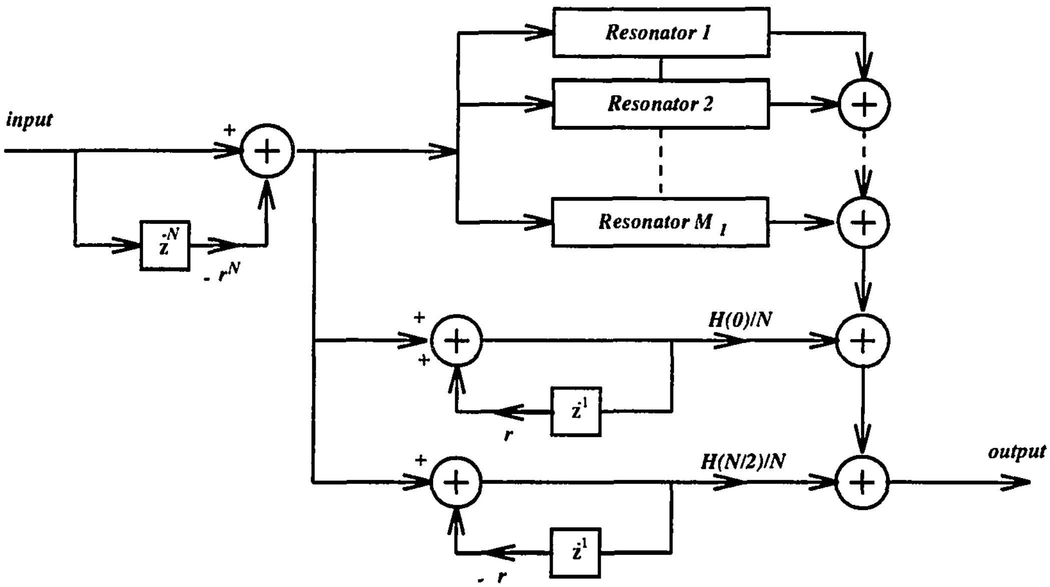


Figure 2.2a: Type 1 frequency sampling filter with a real impulse response of length N where N is even and $M_1 = (N/2) - 1$.

$$\left.\sum_{k=0}^{M_1} 2|H(k)| \left[\frac{cos(\theta(k)) - rz^{-1}cos(\theta(k) - \frac{2\pi}{N}k)}{1 - 2cos(\frac{2\pi}{N}k)rz^{-1} + r^2z^{-2}}\right]\right\} \quad (2.4)$$

where N is even and $M_1 = (N/2) - 1$. Similarly if N is odd,

$$H(z) = \frac{1 - r^N z^{-N}}{N}\left\{\frac{H(0)}{1 - rz^{-1}} + \right.$$

$$\left. + \sum_{k=0}^{M_1} 2|H(k)| \left[\frac{cos(\theta(k)) - rz^{-1}cos(\theta(k) - \frac{2\pi}{N}k)}{1 - 2cos(\frac{2\pi}{N}k)rz^{-1} + r^2z^{-2}}\right]\right\} \quad (2.5)$$

where $M_1 = (N - 1)/2$.

A block diagram for the filter described in Equation (2.4) is shown in Fig. 2.2a. Fig. 2.2b shows the structure of the kth resonator. The system function in Equation (2.4) shows that the filter consists of a comb filter in cascade with a parallel combination of $M_1 + 2$ resonators. When N is odd, the term $H(N/2)$ does not exist in the system function of the filter.

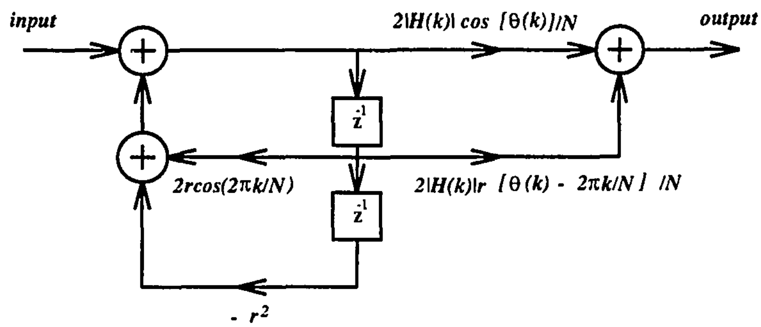


Figure 2.2b: Structure of the kth resonator in Fig. 2.2a.

The resonator in Fig. 2.2b requires 4 real multiplies and 3 real adds per output sample. Therefore, the filter requires approximately $4M_1 + 4$ real multiplies per output sample and $4M_1 + 4$ real adds per output sample.

2.1.2 Type 1 Frequency Sampling Filters With Linear Phase

The number of arithmetic operations required to implement the filters described in Equations (2.4) and (2.5) can be further reduced if the filter is also constrained to have linear phase. If the filter's frequency response has linear phase, the filter's phase, $\theta(e^{j\omega})$, has the form $\theta(e^{j\omega}) = -\left(\frac{N-1}{2}\right)\omega$ where N is the length of the filter's impulse response. Thus, the phase samples, $\theta(k)$, can be expressed as

$$\theta(k) \quad = \quad -\left[\frac{N-1}{2}\right]\frac{2\pi}{N}k.$$

It can be shown [1] that for a linear phase filter of length N, where N is even, $H(e^{j\pi}) = 0$ which implies that

$$H(N/2) = 0 \text{ when } N \text{ is even .}$$

From the expression for the phase samples,

$$cos(\theta(k)) \quad = \quad (-1)^k cos\left(\frac{\pi}{N}k\right). \tag{2.6}$$

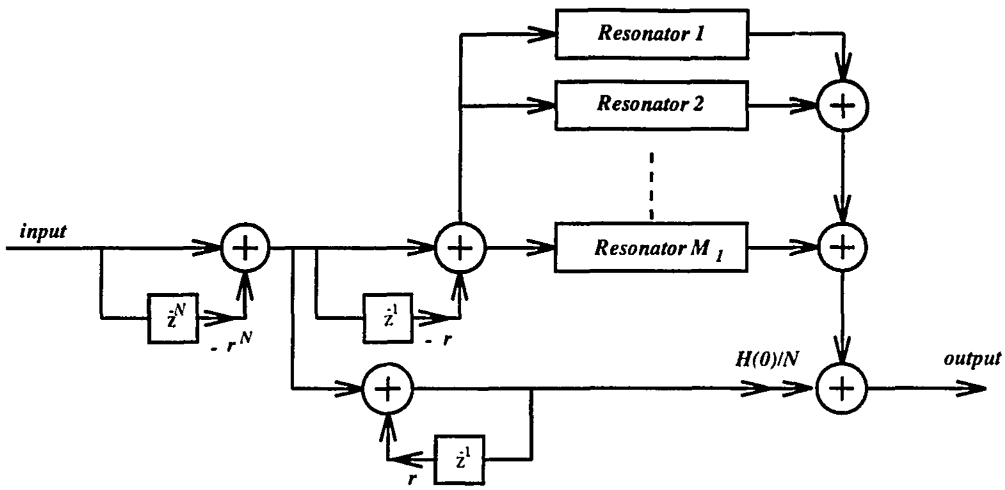


Figure 2.3a: Type 1 frequency sampling filter with linear phase (N even)

Substituting Equation (2.6) and the linear phase properties into Equations (2.4) and (2.5),

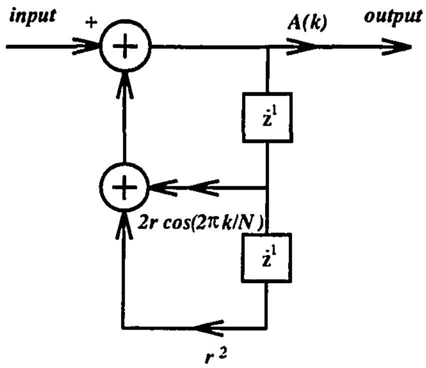
$$H(z) = \frac{1 - r^N z^{-N}}{N}\left[\frac{H(0)}{1 - rz^{-1}} + \sum_{k=1}^{M_1} \frac{(-1)^k 2|H(k)|cos(\frac{\pi}{N}k)(1 - rz^{-1})}{1 - 2cos(\frac{2\pi}{N}k)rz^{-1} + r^2 z^{-2}}\right] \qquad (2.7)$$

where $M_1 = (N/2) - 1$ when N is even, and $M_1 = (N - 1)/2$ when N is odd.

The structure of a frequency sampling filter with a real impulse response and linear phase is shown in Fig. 2.3a. The structure of the kth resonator is shown in Fig. 2.3b. Fig. 2.3b shows that the kth resonator requires 2 real adds and 3 real multiplies per output sample. Hence, the filter requires approximately $3M_1$ multiplies per output sample and $3M_1$ adds per output sample. This structure represents a frequency sampling filter requiring fewer adds and multiplies than the previously discussed structures in Figs. 2.1 and 2.2.

2.2 Type 2 Frequency Sampling Filters

A Type 1 frequency sampling filter interpolates a frequency response through a set of N equally spaced frequency samples taken from the frequency range $[0, 2\pi)$ starting at $\omega = 0$. A Type 2 frequency sampling filter interpolates a frequency response, through a set of N equally spaced frequency samples taken from the range $[0, 2\pi)$, starting at $\omega = \frac{\pi}{N}$. Thus,



where $A(k) = (-1)^k\, 2(|H(k)|/N)\cos(\pi k/N)$

Figure 2.3b: Structure of the kth resonator, in Fig. 2.3a.

the frequency samples, $H(k)$, are related to the filter's system function by

$$H(k) = H(z)\big|_{z=e^{j\frac{2\pi}{N}(k+\frac{1}{2})}} \text{ for } k \in A.$$

An impulse response of length N can be obtained from the frequency response samples by using the relationship,

$$h(n) = \frac{1}{N}\sum_{k=0}^{N-1} H(k)e^{j\frac{2\pi}{N}(k+\frac{1}{2})n}.$$

Substituting this expression for $h(n)$ into Equation 2.1,

$$H(z) = \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{k=0}^{N-1} H(k)e^{j\frac{2\pi}{N}(k+\frac{1}{2})n}\right] z^{-n}.$$

Interchanging the orders of summation and summing up over the n index,

$$H(z) = \frac{1+z^{-N}}{N}\sum_{k=0}^{N-1}\frac{H(k)}{1-e^{j\frac{2\pi}{N}(k+\frac{1}{2})}z^{-1}} \quad (2.8)$$

Equation (2.8) has the form of Lagrange's interpolation formula. The complex polynomial interpolates the frequency response through a set of points $H(z)$ where $z =$

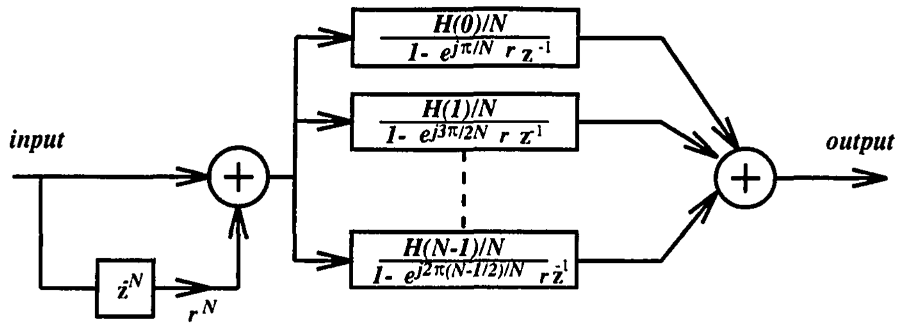


Figure 2.4: Structure of a Type 2 frequency sampling filter.

$e^{j\frac{2\pi}{N}(k+\frac{1}{2})}$ for $k \in A$. Similar to the Type 1 frequency sampling filter described in Equation (2.2), the filter described by Equation (2.8) can be realized using a cascade of a comb filter, which has N zeros and N complex resonators, which have poles that occur at the zeros of the comb filter.

To prevent instability in the filter, the poles of all the resonators can be moved within the unit circle by replacing z^{-1} with rz^{-1} where $|r| < 1$ in the filter equation. Therefore the system function of a Type 2 frequency sampling filter can be written as,

$$H(z) = \frac{1 + r^N z^{-N}}{N} \sum_{k=0}^{N-1} \frac{H(k)}{1 - e^{j\frac{2\pi}{N}(k+\frac{1}{2})} r z^{-1}} \tag{2.9}$$

Fig. 2.4 shows a block diagram which can be used to implement the frequency sampling filter described in Equation (2.9). The filter requires approximately $8N$ real multiplies and $8N - 1$ real adds per output sample.

2.2.1 Type 2 Frequency Sampling Filters With Real Impulse Response

A Type 2 frequency sampling filter can be implemented without complex arithmetic if its impulse response is real. If the impulse response of the filter is real, its frequency samples

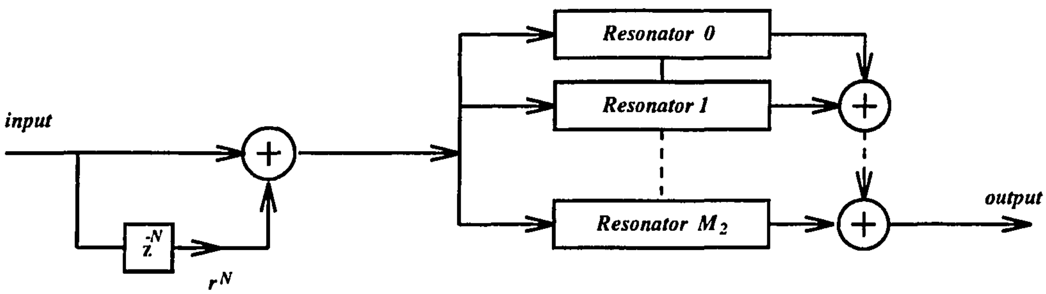


Figure 2.5a: Type 2 frequency sampling filter with a real impulse response(N even).

have the property,

$$H(k) = H^*(N-1-k)$$

which implies that

$$|H(k)| = |H(N-1-k)|$$

and

$$\theta(k) = -\theta(N-1-k).$$

Substituting this constraint into Equation (2.9), the system function of a Type 2 frequency sampling filter with a real impulse response of length N where N is even can be written as

$$H(z) = \frac{1+r^N z^{-N}}{N}\left[\sum_{k=0}^{M_2} \frac{2|H(k)|\{cos(\theta(k)) - rz^{-1}cos[\theta(k) - \frac{2\pi}{N}(k+\frac{1}{2})]\}}{1-2cos[\frac{2\pi}{N}(k+\frac{1}{2})]rz^{-1}+r^2z^{-2}}\right] \quad (2.10)$$

where $M_2 = (N/2) - 1$. When N is odd,

$$H(z) = \frac{1+r^N z^{-N}}{N}\left[\frac{H(\frac{N-1}{2})}{1+rz^{-1}}\right.$$

$$\left. + \sum_{k=0}^{M_2} \frac{2|H(k)|\{cos(\theta(k)) - rz^{-1}cos[\theta(k) - \frac{2\pi}{N}(k+\frac{1}{2})]\}}{1-2cos[\frac{2\pi}{N}(k+\frac{1}{2})]rz^{-1}+r^2z^{-2}}\right] \quad (2.11)$$

where $M_2 = (N-3)/2$.

A block diagram describing Equation (2.10) is shown in Fig. 2.5. Fig. 2.5b shows

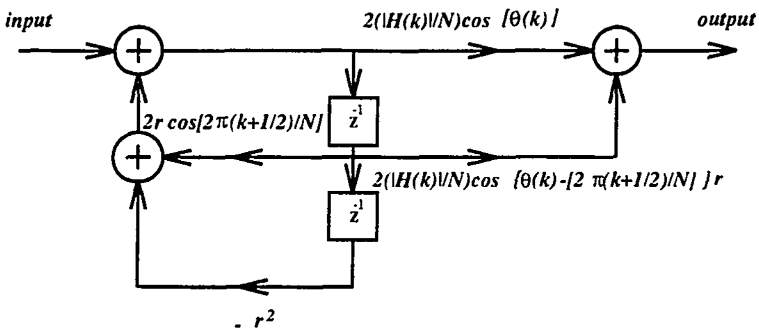


Figure 2.5b: Structure of the kth resonator, where k= 0 to M_2.

the structure of the kth resonator. The filter described by Equation (2.10) can be realized by a comb filter in cascade with $M_2 + 1$ resonators. If N is odd, the filter has a resonator due to the term $H(\frac{N-1}{2})$ in parallel with the other resonators.

Each resonator in Fig. 2.5b requires 3 real adds and 4 real multiplies per output sample. Hence, the filter requires approximately $4M_2$ real adds and $4M_2$ real multiplies per output sample.

2.2.2 Type 2 Frequency Sampling Filters With Linear Phase

The number of arithmetic operations required to realize the filters described in Equations (2.10) and (2.11) can be further reduced if the filter is also constrained to have linear phase. The phase samples, $\theta(k)$, for the linear phase filter can be expressed as

$$\theta(k) = -\left[\frac{N-1}{2}\right]\left[\frac{2\pi}{N}\left(k+\frac{1}{2}\right)\right] \tag{2.12}$$

Using equation (2.12),

$$cos[\theta(k)] = (-1)^k sin\left[\frac{\pi}{N}\left(k+\frac{1}{2}\right)\right]$$

and

$$cos\left[\theta(k) - \frac{2\pi}{N}\left(k+\frac{1}{2}\right)\right] = (-1)^k sin\left[-\frac{\pi}{N}\left(k+\frac{1}{2}\right)\right].$$

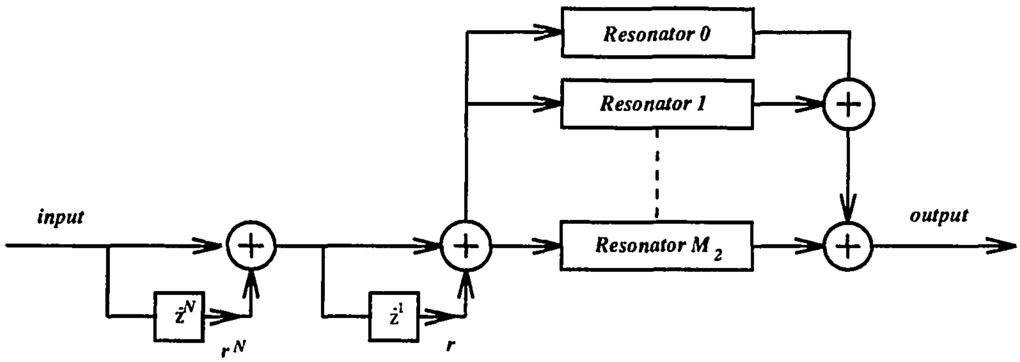


Figure 2.6a: Type 2 frequency sampling filter with linear phase(N even).

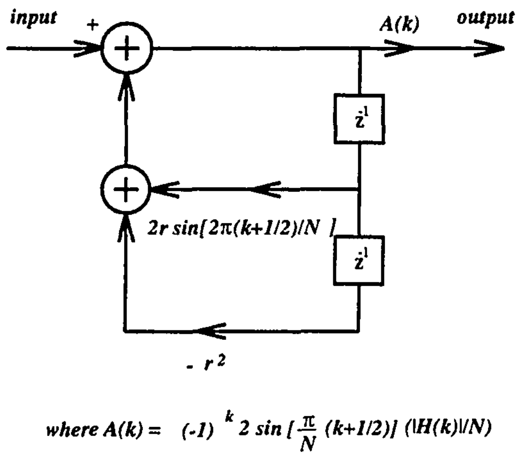


Figure 2.6b: Structure of the kth resonator. in Fig. 2.6a.

Substituting these relationships into the Equations (2.10) and (2.11) yields

$$H(z) = \frac{1+r^N z^{-N}}{N}\left\{\sum_{k=0}^{M_2}\frac{(-1)^k 2|H(k)|sin[\frac{\pi}{N}(k+\frac{1}{2})](1+rz^{-1})}{1-2cos[\frac{2\pi}{N}(k+\frac{1}{2})]rz^{-1}+r^2z^{-2}}\right\} \quad (2.13)$$

$$H(z) = \frac{1+r^N z^{-N}}{N}\left\{\frac{H\left(\frac{N-1}{2}\right)}{1+rz^{-1}}+\sum_{k=0}^{M_2}\frac{(-1)^k 2|H(k)|sin[\frac{\pi}{N}(k+\frac{1}{2})](1+rz^{-1})}{1-2cos[\frac{2\pi}{N}(k+\frac{1}{2})]rz^{-1}+r^2z^{-2}}\right\} \quad (2.14)$$

where $M_2 = (N/2) - 1$ when N is even, and $M_2 = (N-3)/2$ when N is odd.

Fig. 2.6 shows the structure of the filter described by Equation (2.13). The structure of the kth resonator is shown in Fig. 2.6b. Each resonator requires 2 real adds and 3 real multiplies per output sample. The filter in Fig. 2.6 requires approximately $3M_2$ real adds and $3M_2$ real multiplies per output sample.

2.3 Properties Of Frequency Sampling Filters

Under certain circumstances, frequency sampling structures are more efficient than direct convolution structures. For a frequency selective filter, the frequency samples in the stop band can be set to zero. If only K of the filter's frequency samples are nonzero, then only K of the filter's resonators are required since the paths along which $H(k)$ is zero need not be realized. If a filter has nonlinear phase, a direct convolution implementation of the filter requires approximately $N - 1$ adds and N multiplies per output sample to implement the filter. A corresponding frequency sampling structure requires $4K$ real adds and $4K$ real multiplies per output sample. Hence, for a nonlinear phase FIR filter, a frequency sampling structure will be more efficient than a direct convolution structure when $4K < N$ or $K < \frac{N}{4}$.

Linear phase frequency sampling filter structures which have an impulse response of length N require approximately $3K$ adds and $3K$ multiplies per output sample. A linear phase direct convolution structure requires approximately $\frac{N}{2}$ multiplies and N adds per output sample. If K of the frequency sampling filter's frequency samples are nonzero and we assume that multiplies are computationally more complex to perform than adds, then a frequency sampling structure will be more efficient than a direct convolution structure when $3K < \frac{N}{2}$ or $K < \frac{N}{6}$. Although this analysis may not be precise, it gives an indication of applications where frequency sampling filters are more efficient than direct convolution filters.

Direct convolution implementations can be easily pipelined because of their feed forward structure. Frequency sampling filters cannot be easily pipelined because of the single delay feedback path in the recursive structures in the resonators. The presence of unit delays in the feedback path of each resonator requires the previous output sample to be computed before the next output sample can be computed. This prevents the filter from being pipelined.

2.4 Coupled State Space Structures Used To Realize Frequency Sampling Filters

In this section, the state space representation of a frequency sampling filter's resonators is discussed. In addition to input and output variables, state space structures involve an additional set of variables called state variables. Mathematical equations describing the system, its input and its output are divided into two parts, a set of equations relating the state variables to the input and a set of equations relating the state variables and the current input to the output.

The system function of a Type 1 frequency sampling filter is given in Equation (2.3). The system function of the kth resonator can be written as

$$H_k(z) = \frac{S_k(z)}{X(z)}$$

where $S_k(z)$ is the z transform of the output of the kth resonator and $X(z)$ is the z transform of the input to the resonators. If N is even, the output $Y(z)$ of the filter can be expressed in terms of $S_k(z)$ as

$$Y(z) = S_0(z) + S_{N/2}(z) + \sum_{k=1, k\neq \frac{N}{2}}^{N-1} S_k(z)$$

where $S_0(z)$ and $S_{N/2}(z)$ are the outputs of the resonators corresponding to the frequency samples $H(0)$ and $H(N/2)$, respectively. If the filter has a real impulse response, then $H(k) = H^*(N-k)$ and the output $Y(z)$ of the filter can be written as

$$Y(z) = \left[\sum_{k=1}^{M_1} \frac{H(k)}{1 - e^{j\frac{2\pi}{N}k} r z^{-1}} + \sum_{k=1}^{M_1} \frac{H^*(k)}{1 - e^{-j\frac{2\pi}{N}k} r z^{-1}} + \right.$$

$$\left.\frac{H(0)}{1-rz^{-1}}+\frac{H(N/2)}{1+rz^{-1}}\right]X(z)$$

$$=\sum_{k=1}^{M_1}S_k(z)+\sum_{k=1}^{M_1}S_k^*(z)+S_0+S_{N/2}$$

where $M_1=(N/2)-1$. If N is odd, the $H(N/2)$ term does not exist, and $M_1=(N-1)/2$. $S_k(z)$ is given as

$$S_k(z) \;=\; \frac{H(k)X(z)}{1-e^{j\frac{2\pi}{N}k}rz^{-1}} \tag{2.15}$$

Taking the inverse z transform of Equation (2.15),

$$s_k(n) \;=\; e^{j\frac{2\pi}{N}k}rs_k(n-1)+H(k)x(n). \tag{2.16}$$

$s_k(n)$ can then be defined in terms of the state variables, $v_1(n)$ and $v_2(n)$, as

$$s_k(n) \;=\; v_1(n)+jv_2(n)$$

$$=\left[rcos\left(\frac{2\pi}{N}k\right)+jrsin\left(\frac{2\pi}{N}k\right)\right][v_1(n-1)+jv_2(n-1)]+$$

$$Re[H(k)]+jIm[H(k)]$$

where the real part of $H(k)$ is $Re[H(k)]$ and its imaginary part is $Im[H(k)]$. In matrix form, these state equations can be written as

$$\begin{bmatrix} v_1(n) \\ v_2(n) \end{bmatrix} = \begin{bmatrix} rcos(\frac{2\pi}{N}k) & -rsin(\frac{2\pi}{N}k) \\ rsin(\frac{2\pi}{N}k) & rcos(\frac{2\pi}{N}k) \end{bmatrix} \begin{bmatrix} v_1(n-1) \\ v_2(n-1) \end{bmatrix} + \begin{bmatrix} Re[H(k)] \\ Im[H(k)] \end{bmatrix} x(n)$$

If we let

$$Y_k(z) = S_k(z) + S_k^*(z) \text{ for } k = 1,2...M_1$$

then

$$Y(z) = \sum_{k=1}^{M_1} Y_k(z) + S_0(z) + S_{N/2}(z).$$

The output equation can be written as

$$y_k(n) = s_k(n) + s_k^*(n)$$

$$= v_1(n) + jv_2(n) + v_1(n) - jv_2(n)$$

$$= \begin{bmatrix} 2 & 0 \end{bmatrix} \mathbf{v}(n)$$

where $\mathbf{v}(n)$ is the vector of state variables.

Since coupled state space structures exhibit lower coefficient sensitivity than direct form structures [10], they can be used to implement the resonators in the frequency sampling filters to reduce the effect of finite word lengths in digital filters.

2.5 Sampling Rate Reduction In Frequency Sampling Filters

After narrowband filtering, a signal contains fewer frequencies than it did before filtering. If the original signal and the filtered signal are sampled at the Nyquist rate of the original signal, then the Nyquist rate of the filtered signal is exceeded. The system can be made more efficient if the sampling frequency of the filtered signal is reduced. The process of sampling rate reduction is called decimation. Fig. 2.8 represents a block diagram of a system where the output signal is decimated by a factor of D. This system is called a multirate system

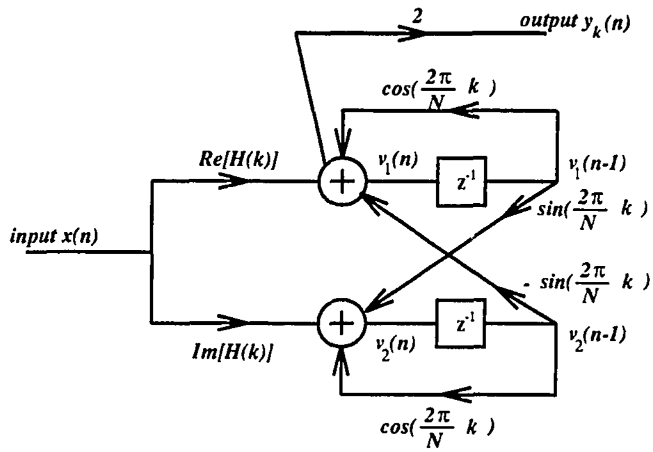


Figure 2.7: Coupled state space structure for the kth resonator of a Type 1 frequency sampling filter.

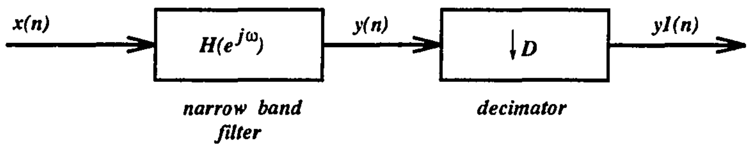


Figure 2.8: Block diagram of a multirate filtering process.

because the signals associated with the system are sampled at different frequencies.

When a signal is decimated by a factor of D, every Dth sample is preserved, and the intermediate $D-1$ samples are discarded. If the decimator can be successfully moved toward the input side of the filter, a significant computational saving can be achieved [8]. Fig. 2.9. illustrates a few rules that can be used to move the decimator in a flowgraph [11]. For example, if a direct convolution structure is decimated by a factor of D, the computational requirement of the filter can be reduced by moving the decimator toward the input side of the filter as shown in Fig. 2.10. This reduces the number of computations per output sample, and the adders and multipliers in the filter can operate D times slower while achieving the same throughput.

If the output of a frequency sampling filter is decimated by a factor of D, the presence of unit delays in the feedback path of each resonator prevent the decimator from being moved toward the input side of the filter. Hence, for frequency sampling filters little computational saving is achieved by decimating the output. Table 2.1 shows the approximate number of

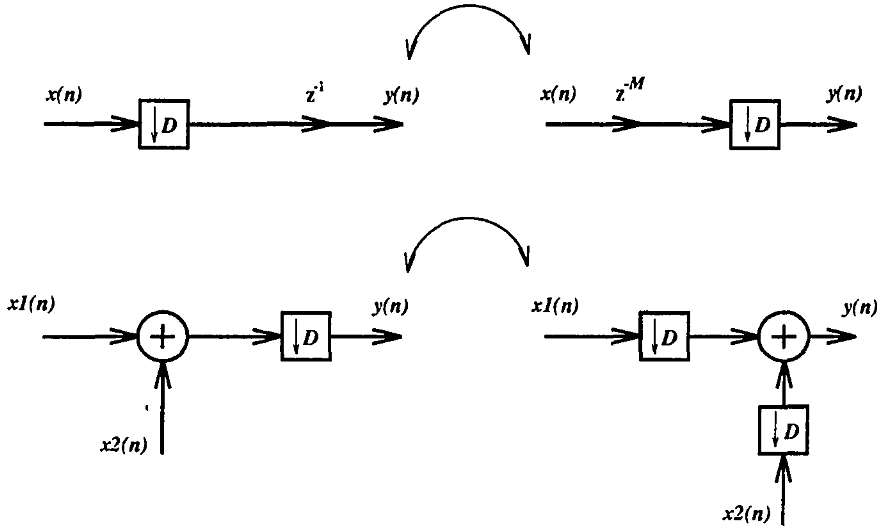


Figure 2.9: Rules used to move a decimator in a flow graph.

Filter for even N	Nonlinear phase real impulse response	Linear phase real impulse response
Direct convolution filter	ND	$N/2D$
Type 1 frequency sampling filter	$(4K+1)D$	$(3K+2)D$
Type 2 frequency sampling filter	$(4K+1)D$	$(3K+2)D$

Table 2.1: Multiplies required per output sample by direct convolution and frequency sampling filter structures when their outputs are decimated by a factor of D.

multiplies required per output sample by a direct convolution structure and by frequency sampling structures where only K of the frequency samples are nonzero, and their output is decimated by a factor of D. Table 2.2 shows the approximate number of multiplies per output sample required if the decimator is moved toward the input side of the filter.

Tables 2.1 and 2.2 illustrate that the frequency sampling filters discussed in this chapter are not well suited for decimation. In the next chapter, a class of frequency sampling filters is developed with only powers of z^{-D} in the denominators of their system functions. These filters are well suited for sampling rate reduction by a factor of D.

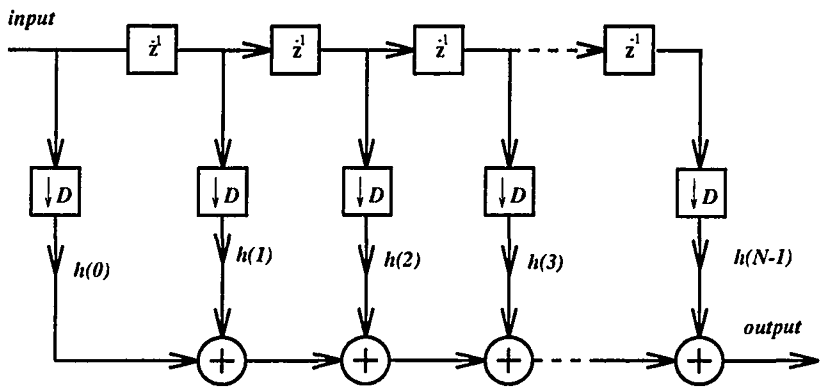


Figure 2.10: Flowgraph for a direct convolution structure with the output decimated by a factor of D.

Multirate filter for even N	Nonlinear phase real impulse response	Linear phase real impulse response
Direct convolution filter	N	$N/2$
Type 1 frequency sampling filter	$(2D+2)K+D$	$(2D+1)K+2D$
Type 2 frequency sampling filter	$(2D+2)K+D$	$(2D+1)K+2D$

Table 2.2: Multiplies required per output sample if the decimator in the structures in in Table 2.1 is moved toward the input.

Chapter 3

Pipelined and Multirate Frequency Sampling Filters

Pipelining is an implementation technique where multiple instructions are overlapped in execution like in an assembly line. A filter can be pipelined by adding registers to the multipliers and adders in the filter. The frequency sampling filters discussed in Chapter 2 have unit delays in the feedback path of each resonator. For each of these resonators, the output can be computed only if all of the previous output samples have been computed. Therefore, these structures cannot be pipelined and they are not well suited for sampling rate reduction at the output.

In this chapter, a frequency sampling filter system function is developed which has only z^{-D} terms in the denominator. This filter can be pipelined, and it is also well suited for decimation at the output. When decimating by a factor of D, the decimator can be pushed toward the input side to reduce the computational requirements of the filter.

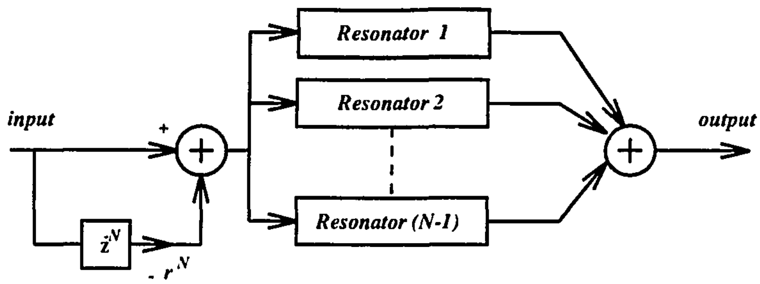


Figure 3.1a: Pipelined Type 1 frequency sampling filter.

3.1 Pipelined Type 1 Frequency Sampling Filter

The system function of a Type 1 frequency sampling filter is given in Equation (2.3) as

$$H(z) = \frac{1 - r^N z^{-N}}{N} \sum_{k=0}^{N-1} \frac{H(k)}{1 - e^{j\frac{2\pi}{N}k} r z^{-1}}$$

From the rules of geometric progression,

$$\sum_{l=0}^{D-1} (e^{j\frac{2\pi}{N}k} r z^{-1})^l = \frac{1 - e^{j\frac{2\pi}{N}kD} r^D z^{-D}}{1 - e^{j\frac{2\pi}{N}k} r z^{-1}}$$

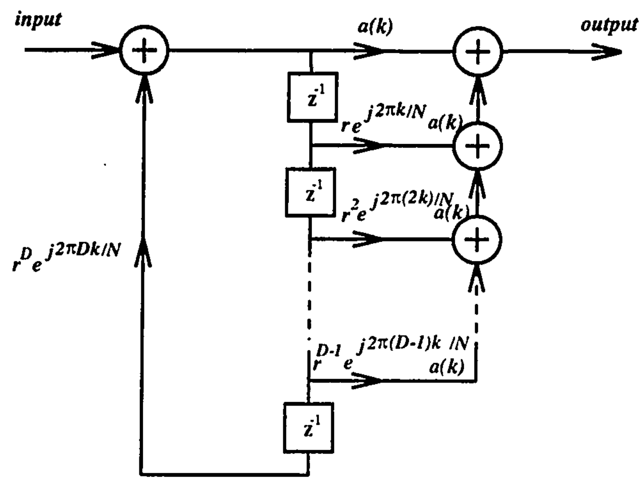
which implies that

$$1 - e^{j\frac{2\pi}{N}k} r z^{-1} = \frac{1 - e^{j\frac{2\pi}{N}kD} r^D z^{-D}}{\sum_{l=0}^{D-1} (e^{j\frac{2\pi}{N}k} r z^{-1})^l} \quad (3.1)$$

Substituting Equation (3.1) into Equation (2.3),

$$H(z) = \frac{1 - r^N z^{-N}}{N} \sum_{k=0}^{N-1} \frac{H(k) \sum_{l=0}^{D-1} (e^{j\frac{2\pi}{N}k} r z^{-1})^l}{1 - e^{j\frac{2\pi}{N}kD} r^D z^{-D}} \quad (3.2)$$

The block diagram for the filter described in Equation (3.2) is shown in Fig. 3.1a where the structure of the kth resonator is shown in Fig. 3.1b. The system function of the filter given in Equation (3.2) shows that the filter can be realized by a comb filter with N



where $a(k) = H(k)/N$

Figure 3.1b: Structure of the kth resonator in Fig. 3.1a.

zeros in cascade with N parallel resonators. Each resonator has D poles and $D-1$ zeros. The resonator shown in Fig. 3.1b requires $D+1$ complex multiplies and D complex adds per output sample. Hence, the filter requires approximately $(4D+4)N+1$ real multiplies and $(4D+4)N-1$ real adds per output sample.

Fig. 3.1a represents a filter with a significant increase in the hardware compared to the filter shown in Fig. 2.1. However, if we assume that multiplies are the only source of bottleneck involved in the implementation of the filter and require less than D time periods to execute, then the presence of only delays of D in the feedback paths makes it possible to pipeline this filter resulting in an improved throughput.

Multirate Type 1 Frequency Sampling Filter

The system function in Equation (3.2) is also well suited for multirate applications. Consider the filter described by Equation (3.2) where the sampling rate of the output signal is reduced by a factor of D. Instead of computing all the output samples, a computational reduction can be achieved by moving the decimator toward the input side of the filter.

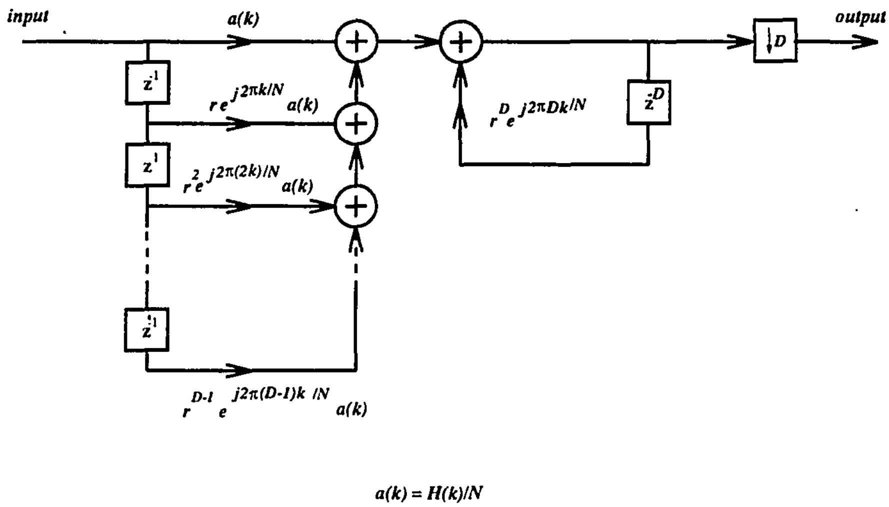


Figure 3.2a: The kth resonator of a multirate Type 1 frequency sampling filter.

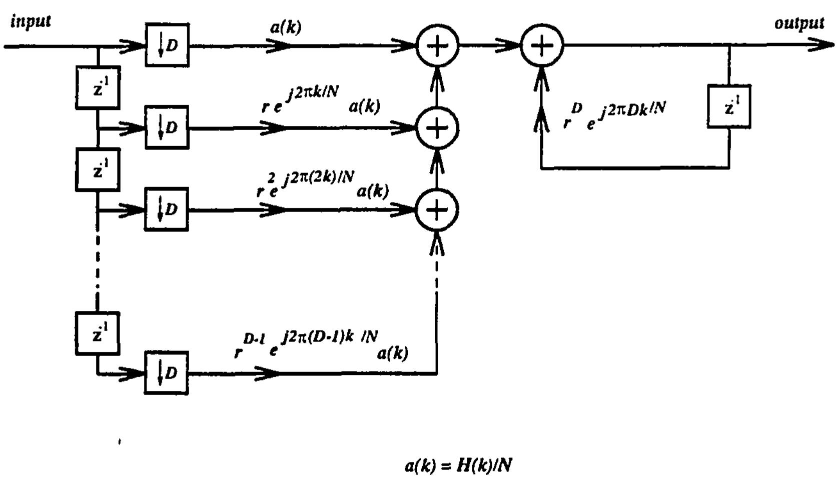


Figure 3.2b: Structure of the kth resonator, with the decimator moved towards the input side.

Fig. 3.2a represents a Direct Form 1 structure of the resonator in Fig. 3.1b, where the output is decimated by a factor of D. Fig. 3.2b represents an equivalent structure with the decimator moved towards the input side of the filter. Fig. 3.2b requires D times fewer adds and multiplies per output sample than the structure in Fig. 3.2a. Thus, Fig. 3.2b represents a frequency sampling filter structure with a reduced computational requirement.

3.1.1 Pipelined Type 1 Frequency Sampling Filter With Real Impulse Response

The frequency sampling filter described by Equation (3.2) can be implemented without complex arithmetic if it has a real impulse response. The system function in Equation (3.2) can be expressed as

$$H(z) = \frac{1-r^N z^{-N}}{N}\left[\frac{H(0)\sum_{l=0}^{D-1}(rz^{-1})^l}{1-r^D z^{-D}} + \frac{H(N/2)\sum_{l=0}^{D-1}(e^{j\pi}rz^{-1})^l}{1-e^{j\pi D}r^D z^{-D}} + \right.$$

$$\left. \sum_{k=1}^{\frac{N}{2}-1}\frac{|H(k)|e^{j\theta(k)}\sum_{l=0}^{D-1}(e^{j\frac{2\pi}{N}k}rz^{-1})^l}{1-e^{j\frac{2\pi}{N}kD}r^D z^{-D}} + \sum_{k=\frac{N}{2}+1}^{N-1}\frac{|H(k)|e^{j\theta(k)}\sum_{l=0}^{D-1}(e^{j\frac{2\pi}{N}k}rz^{-1})^l}{1-e^{j\frac{2\pi}{N}kD}r^D z^{-D}}\right]$$

when N is even. If the filter has a real impulse response, the frequency samples have the property, $H(k) = H^*(N-k)$ which implies

$$|H(k)| = |H(N-k)|$$

and

$$\theta(k) = -\theta(N-k).$$

Substituting these relationships into the system function of the filter,

$$H(z) = \frac{1-r^N z^{-N}}{N}\left\{\frac{H(0)\sum_{l=0}^{D-1}(rz^{-1})^l}{1-r^D z^{-D}} + \frac{H(N/2)\sum_{l=0}^{D-1}(e^{j\pi}rz^{-1})^l}{1-e^{j\pi D}r^D z^{-D}} + \right.$$

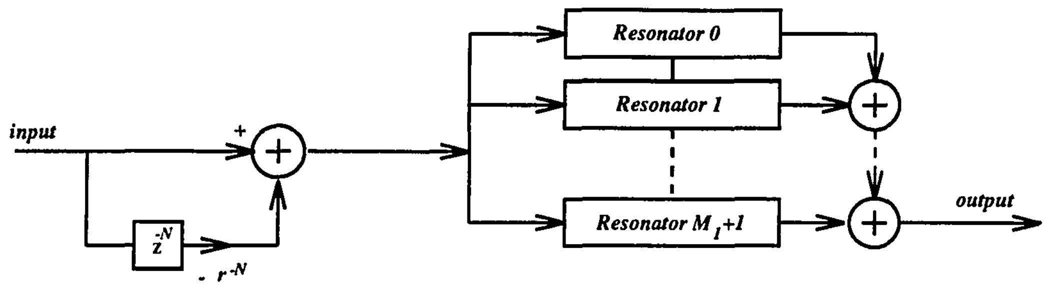


Figure 3.3a: Pipelined type 1 frequency sampling filter with real impulse response.

$$\sum_{k=1}^{M_1}\left[\frac{|H(k)|e^{j\theta(k)}\sum_{l=0}^{D-1}(e^{j\frac{2\pi}{N}k}rz^{-1})^l}{1-e^{j\frac{2\pi}{N}kD}r^Dz^{-D}}+\frac{|H(k)|e^{-j\theta(k)}\sum_{l=0}^{D-1}(e^{-j\frac{2\pi}{N}k}rz^{-1})^l}{1-e^{-j\frac{2\pi}{N}kD}r^Dz^{-D}}\right]\Bigg\}$$

$$= \frac{1-r^Nz^{-N}}{N}\left\{\frac{H(0)\sum_{l=0}^{D-1}(rz^{-1})^l}{1-r^Dz^{-D}}+\frac{H(N/2)\sum_{l=0}^{D-1}(e^{j\pi}rz^{-1})^l}{1-e^{j\pi D}r^Dz^{-D}}+\right.$$

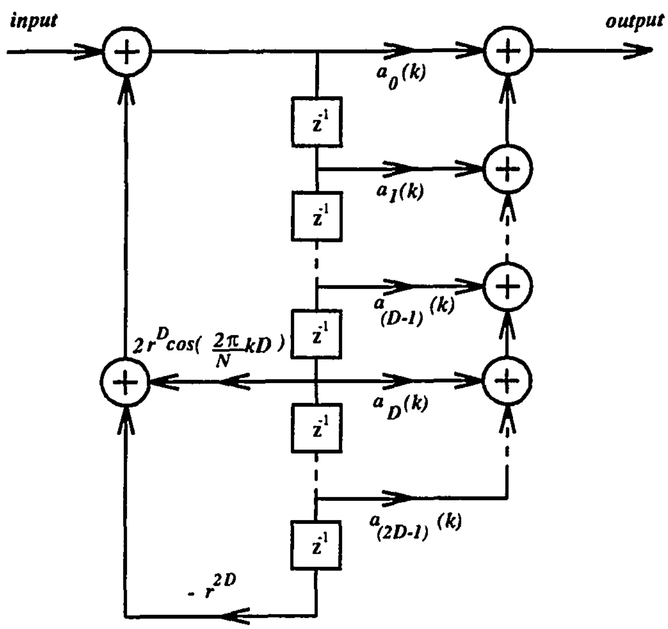
$$\sum_{k=1}^{M_1}2|H(k)|\left[\frac{\sum_{l=0}^{D-1}cos(\theta(k)+\frac{2\pi}{N}lk)(rz^{-1})^l}{1-2cos(\frac{2\pi}{N}kD)r^Dz^{-D}+r^{2D}z^{-2D}}+\right.$$

$$\left.\left.\frac{\sum_{l=0}^{D-1}cos(\theta(k)+\frac{2\pi}{N}(D-l)k)(rz^{-1})^{-D-l}}{1-2cos(\frac{2\pi}{N}kD)r^Dz^{-D}+r^{2D}z^{-2D}}\right]\right\} \tag{3.3}$$

where $M_1 = (N/2) - 1$ if N is even. If N is odd, then the filter's system function does not have a $H(N/2)$ term and $M_1 = (N-1)/2$.

The system function of the filter described in Equation (3.3) shows that it consists of a comb filter in cascade with $M_1 + 2$ parallel resonators. Fig. 3.3a shows a block diagram of the filter described by Equation (3.3) where the structure of resonators 1 to M_1 is shown in Fig. 3.3b. Each of the resonators, 1 to M_1, requires $2D+2$ real multiplies and $2D+1$ real adds per output sample. Figs. 3.3c and 3.3d show the block diagrams for resonator 0, and resonator $N/2$ respectively. Each of these resonators require D real adds and $D+1$ real multiplies per output sample. Hence, the filter requires approximately $(2D+2)(M_1+1)+1$ real multiplies and $(2D+1)(M_1+1)+M_1$ real adds per output sample.

Fig 3.3 represents a frequency sampling filter with a significant increase in the hard-



$$a_i(k) = r^i \cos(\theta(k)+2\pi ik/N)\, 2|H(k)|/N$$

where $i=0,1...2D-1$

Figure 3.3b: Structure of the resonators 1 to M_1, in Fig. 3.3a

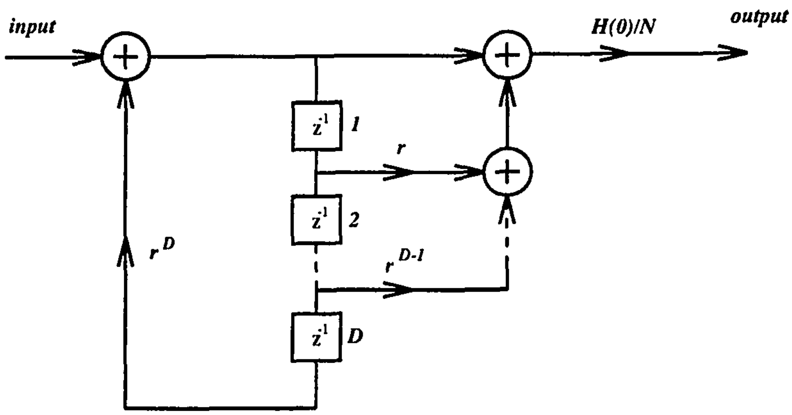


Figure 3.3c: Structure of resonator 0, in Fig. 3.3a

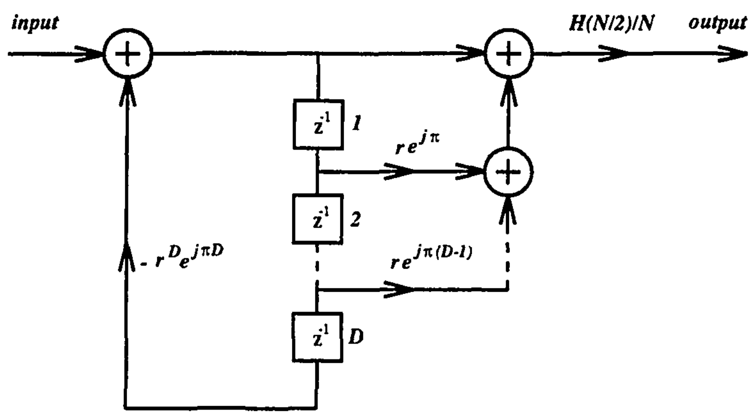


Figure 3.3d: Structure of the resonator $N/2$, in Fig. 3.3a

ware required to implement the filter, compared to the frequency sampling filter in Fig. 2.2. However, the filter structure can be pipelined because only delays of D are in the feedback paths of the filter. If we assume that the multipliers are the only source of bottleneck in pipelining the filter and if the multiplier takes less than D time periods to execute, then the filter structure in Fig. 3.3 can be pipelined.

Multirate Type 1 Frequency Sampling Filter With a Real Impulse Response

If the output of the filter described by Equation (3.3) is decimated by a factor of D, a significant computational saving can be achieved by moving the decimator toward the input side of the filter. Fig. 3.4a represents a Direct Form 1 structure for resonators 1 to M_1 where the output of each resonator is decimated by a factor of D. Fig. 3.4b represents an equivalent structure with the decimator moved toward the input side of the filter. The structure in Fig. 3.4a requires D times as many adds and multiplies per output sample as the structure in Fig. 3.4b. For an equivalent throughput, the structure shown in Fig. 3.3b requires adders and multipliers that are D times faster than the corresponding units in Fig. 3.4b. Thus, the filter structure shown in Fig. 3.4b represents a filter with a reduced computational requirement.

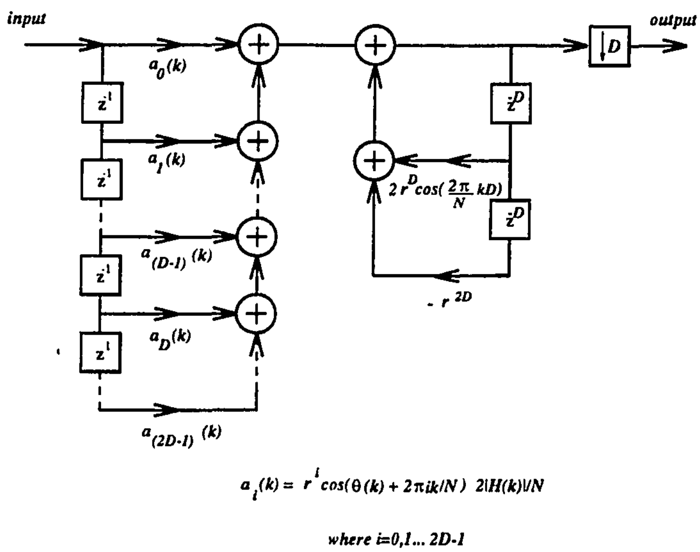


Figure 3.4a: Direct Form 1 structure of the kth resonator of a multirate Type 1 frequency sampling filter with real impulse response for k= 1 to M_1.

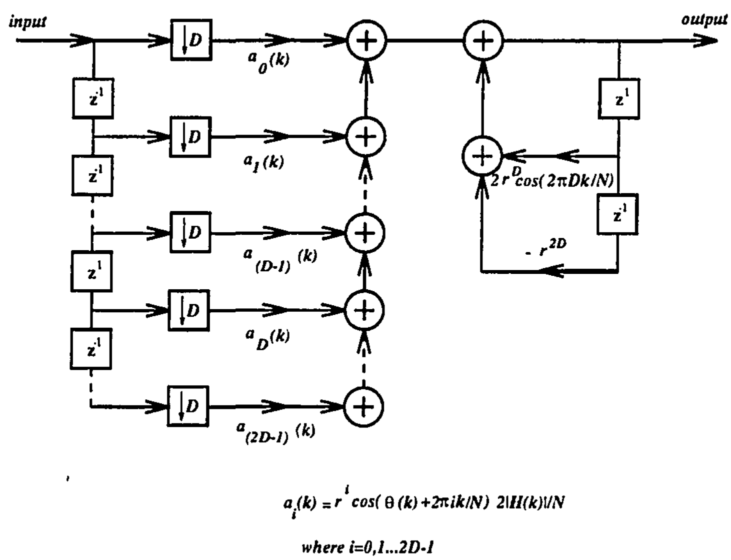


Figure 3.4b: Structure of the kth resonator with the decimator moved towards the input side.

3.1.2 Pipelined Type 1 Linear Phase Frequency Sampling Filter

The phase, $\theta(e^{j\omega})$, of a FIR linear phase filter that has an impulse response of length N can be expressed as

$$\theta(e^{j\omega}) = -\omega\left(\frac{N-1}{2}\right).$$

Thus, the phase samples for a linear phase Type 1 frequency sampling filter are

$$\theta(k) = -\left[\frac{N-1}{2}\right]\frac{2\pi}{N}k$$

It can be shown [1] that for a Type 1 linear phase filter of length N, $H(N/2) = 0$. From the expression for the phase samples,

$$cos\left(\theta(k) + \frac{2\pi}{N}lk\right) = (-1)^k cos\left(\frac{(2l+1)\pi k}{N}\right) \tag{3.4}$$

and

$$cos\left(\theta(k) - \frac{2\pi}{N}lk\right) = (-1)^k cos\left(\frac{(2l-1)\pi k}{N}\right) \tag{3.5}$$

Substituting Equations (3.4) and (3.5) into the Equation (3.3), an expression for the system function of a pipelined linear phase frequency sampling filter can be obtained.

$$H(z) = \frac{1 - r^N z^{-N}}{N}\left\{\frac{H(0)\sum_{l=0}^{D-1}(rz^{-1})^l}{1 - r^D z^{-D}}\right.$$

$$\left. + \sum_{k=1}^{M_1} 2|H(k)|\left[\frac{\sum_{l=0}^{D-1}(-1)^k cos\left(\frac{(2l+1)\pi k}{N}\right)(rz^{-1})^l + \sum_{l=1}^{D-1}(-1)^k cos\left(\frac{(2l-1)\pi k}{N}\right)(rz^{-1})^{2D-l}}{1 - 2cos(\frac{2\pi}{N}kD)r^D z^{-D} + r^{2D}z^{-2D}}\right]\right\}$$

$$= \frac{1 - r^N z^{-N}}{N}\left\{\sum_{k=1}^{M_1} 2|H(k)|\left[\frac{\sum_{l=0}^{D-1}(-1)^k cos\left(\frac{(2l+1)\pi k}{N}\right)\left((rz^{-1})^l - (rz^{-1})^{2D-1-l}\right)}{1 - 2cos(\frac{2\pi}{N}kD)r^D z^{-D} + r^{2D}z^{-2D}}\right]\right.$$

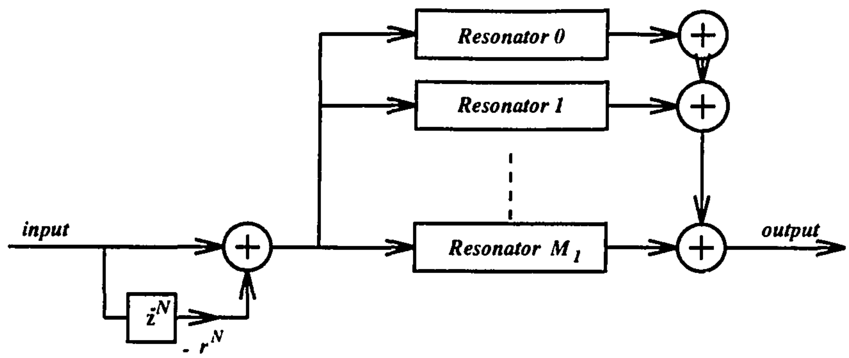


Figure 3.5a: Pipelined Type 1 frequency sampling filter with linear phase.

$$+\frac{H(0)\sum_{l=0}^{D-1}(rz^{-1})^l}{1-r^Dz^{-D}}\Bigg\} \quad (3.6)$$

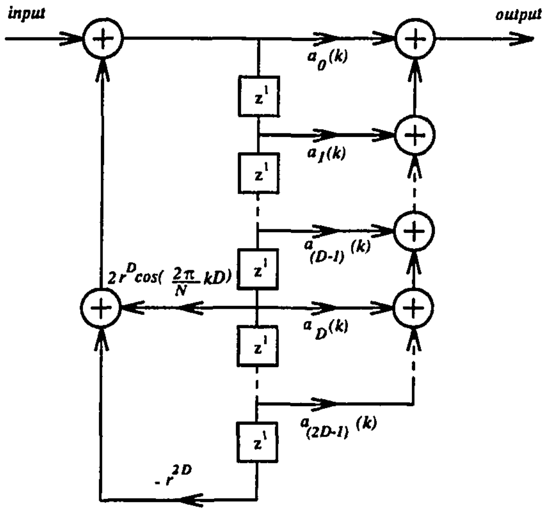
where $M_1 = (N/2) - 1$ if N is even and $M_1 = (N - 1)/2$ if N is odd.

The structure of the frequency sampling filter described by the system function in Equation (3.6) is shown in Fig. 3.5a where Fig. 3.5b shows the structure of resonators 1 through M_1. The structure of resonator 0 is shown in Fig. 3.3c. Each of these resonators requires $2D + 2$ real multiplies and $2D + 1$ real adds per output sample. Hence, the filter requires $(2D + 2)M_1 + D + 2$ real multiplies and $(2D + 2)M_1 + D$ real adds per output sample.

Fig 3.5a represents a frequency sampling structure with a significant increase in required hardware compared to the structure in Fig. 2.3a. However, this filter can be pipelined because only delays of D are in the feedback paths of the resonators.

Multirate Type 1 Frequency Sampling Filter With Linear Phase

After frequency selective filtering, the output of the frequency sampling filter can be decimated. Fig. 3.6a represents a Direct Form 1 structure used to implement the resonators 1 through M_1 of the filter, described by Equation (3.6), with the output decimated by a factor of D. To achieve computational reduction, the decimator is moved toward the input side of the filter. Fig. 3.6b represents an equivalent structure with the decimator moved



$$a_l(k) = r^l(-1)^k \cos((2l+1)k\pi/N)\,2|H(k)|/N \quad \text{for } l=0 \text{ to } D-1.$$

$$a_l(k) = -r^l(-1)^k \cos([2(2D-1-l)+1]k\pi/N)\,2|H(k)|/N \quad \text{for } l=D \text{ to } 2D-1$$

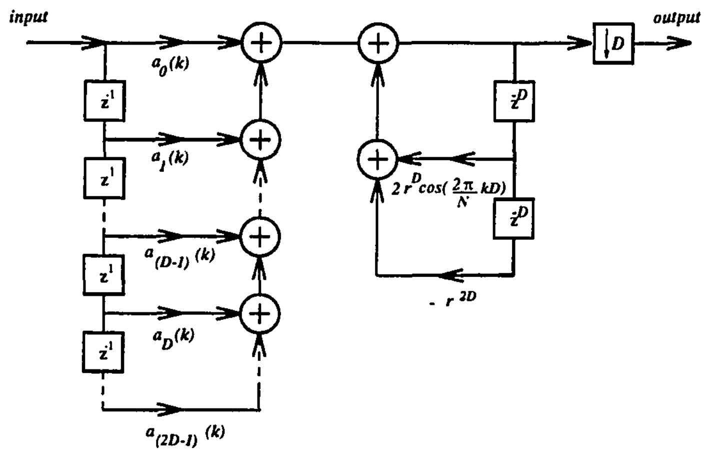
Figure 3.5b: Structure of resonators 1 to M_1, shown in Fig. 3.5a

toward the input side. Fig. 3.6b represents a structure with a significant computational saving over the structure in Fig. 3.6a

3.1.3 Properties of Multirate Type 1 Frequency Sampling Filters

Assuming that only K of the frequency samples of the filter are nonzero. a comparison of the multiplies required per output sample by a direct convolution filter. a Type 1 frequency sampling filter and a pipelined Type 1 frequency sampling filter is shown in Table 3.1. Table 3.2 shows the computational requirements of these filters if their output is decimated by a factor of D.

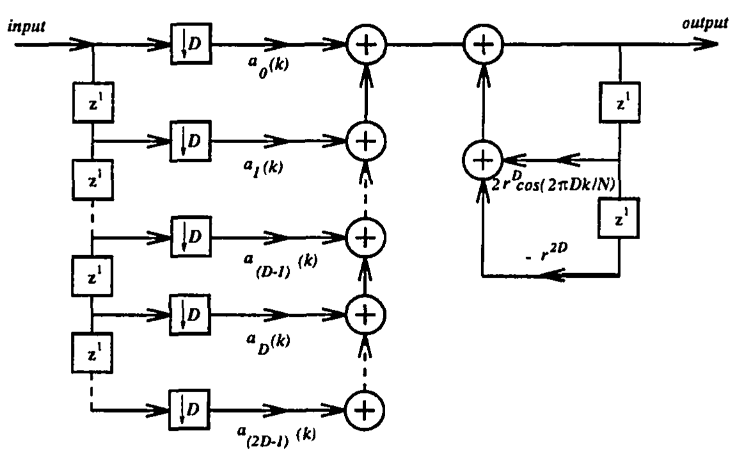
Table 3.2 shows that the system function of the Type 1 frequency sampling filter. developed in this section is more suited for multirate systems when compared to the Type 1 frequency sampling filter discussed in Chapter 2. The linear phase Type 1 frequency sampling filter developed in this section with the output decimated by a factor of D is more



$$a_l(k) = r^l (-1)^k \cos((2l+1)\pi k/N) 2|H(k)|/N \quad \text{for } l = 0 \text{ to } D-1$$

$$a_l(k) = -\, r^l (-1)^k \cos\left(\frac{[2(2D-1-l)+1]k\pi}{N}\right) 2|H(k)|/N \quad \text{for } l = D \text{ to } 2D-1$$

Figure 3.6a: Direct Form 1 structure for the kth resonator of a multirate Type 1 frequency sampling filter with linear phase.



$$a_l(k) = r^l (-1)^k \cos((2l+1)\pi k/N) 2|H(k)|/N \quad \text{for } l = 0 \text{ to } D-1.$$

$$a_l(k) = -\, r^l (-1)^k \cos([2(2D-1-l)+1] k \pi/N) 2|H(k)|/N \quad \text{for } l = D \text{ to } 2D-1$$

Figure 3.6b: Structure of the kth resonator with the decimator moved towards the input side.

Filter for even N	Nonlinear phase real impulse response	Linear phase real impulse response
Direct convolution filter	N	$N/2$
Type 1 frequency sampling filter	$4K$ $+1$	$3K$ $+2$
Pipelined Type 1 frequency sampling filter	$(2D+2)K$ $+1$	$(2D+2)K$ $+1$

Table 3.1: Multiplies required per output sample by a direct convolution filter, a Type 1 frequency sampling filter and a pipelined Type 1 frequency sampling filter (K is the number of nonzero frequency samples, $D > 1$ and N is the length of the impulse response of the filter).

Filter for even N	Nonlinear phase real impulse response	Linear phase real impulse response
Direct convolution filter with the output decimated	N	$N/2$
Type 1 frequency sampling filter with the output decimated	$(2D+2)K$ $+D$	$(2D+1)K$ $+D$
Pipelined Type 1 frequency sampling filter with the output decimated	$(2D+2)K$ $+D$	$(2D+2)K$ $+D$

Table 3.2: Multiplies required per output sample by a direct convolution filter, a Type 1 frequency sampling filter and a pipelined Type 1 frequency sampling filter with their outputs decimated (D is the decimation ratio, K is the number of nonzero frequency samples and N is the length of the impulse response of the filter).

efficient than a linear phase direct convolution filter. when $(2D+2)K+1 < N/2$ or

$$K < \frac{\frac{N}{2}-1}{2D+2} \simeq \frac{N}{4D+4}.$$

A nonlinear phase Type 1 frequency sampling filter developed in this section with the output decimated by a factor of D is more efficient than a nonlinear phase direct convolution filter. when $K < \frac{N}{2D+2}$.

3.2 Pipelined Type 2 Frequency Sampling Filter

A Type 2 frequency sampling filter can be developed similar to the Type 1 frequency sampling filter in Section 3.1. The system function of a Type 2 frequency sampling filter was given in Equation (2.9) as

$$H(z) = \frac{1+r^N z^{-N}}{N} \sum_{k=0}^{N-1} \frac{H(k)}{1-e^{j\frac{2\pi}{N}(k+\frac{1}{2})} r z^{-1}}$$

From the rules of geometric progression,

$$\sum_{l=0}^{D-1} (e^{j\frac{2\pi}{N}(k+\frac{1}{2})} r z^{-1})^l = \frac{1-e^{j\frac{2\pi}{N}(k+\frac{1}{2})D} r^D z^{-D}}{1-e^{j\frac{2\pi}{N}(k+\frac{1}{2})} r z^{-1}}$$

which implies that

$$1-e^{j\frac{2\pi}{N}(k+\frac{1}{2})} r z^{-1} = \frac{1-e^{j\frac{2\pi}{N}(k+\frac{1}{2})D} r^D z^{-D}}{\sum_{l=0}^{D-1} (e^{j\frac{2\pi}{N}(k+\frac{1}{2})} r z^{-1})^l} \quad (3.7)$$

Substituting Equation (3.7) into the Equation (2.9),

$$H(z) = \frac{1+r^N z^{-N}}{N} \sum_{k=0}^{N-1} \frac{H(k) \sum_{l=0}^{D-1} (e^{j\frac{2\pi}{N}(k+\frac{1}{2})} r z^{-1})^l}{1-e^{j\frac{2\pi}{N}(k+\frac{1}{2})D} r^D z^{-D}} \quad (3.8)$$

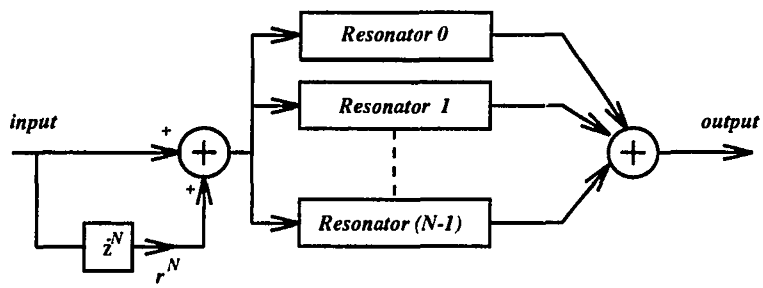
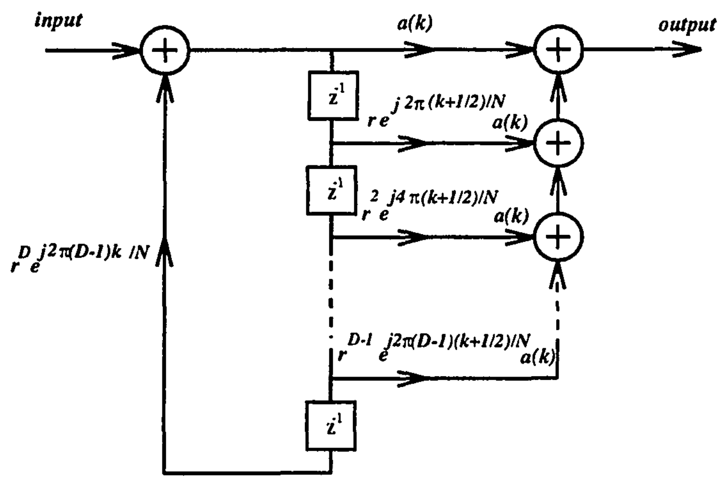


Figure 3.7a: Pipelined Type 2 frequency sampling filter.



where a(k) = H(k)/N.

Figure 3.7b: Structure of the kth resonator in Fig. 3.7(a)

The block diagram of the filter described in Equation (3.8) is shown in Fig. 3.7a where the structure of the kth resonator is shown in Fig. 3.7b. The system function of the filter given in Equation (3.8) consists of a comb filter with N zeros in cascade with a parallel combination of $N-1$ resonators. Each of the resonators has D poles and $(D-1)$ zeros. The resonator shown in Fig. 3.7b requires $D+1$ complex multiplies and D complex adds per output sample . Hence, the filter requires approximately $(4D+4)N-1$ real adds and $(4D+4)N+1$ multiplies per output sample. If we assume that multiplies are at least D times slower than adds, the filter described by Equation (3.8) can be pipelined because of the presence of only z^{-D} in the denominator of the system function of each resonator.

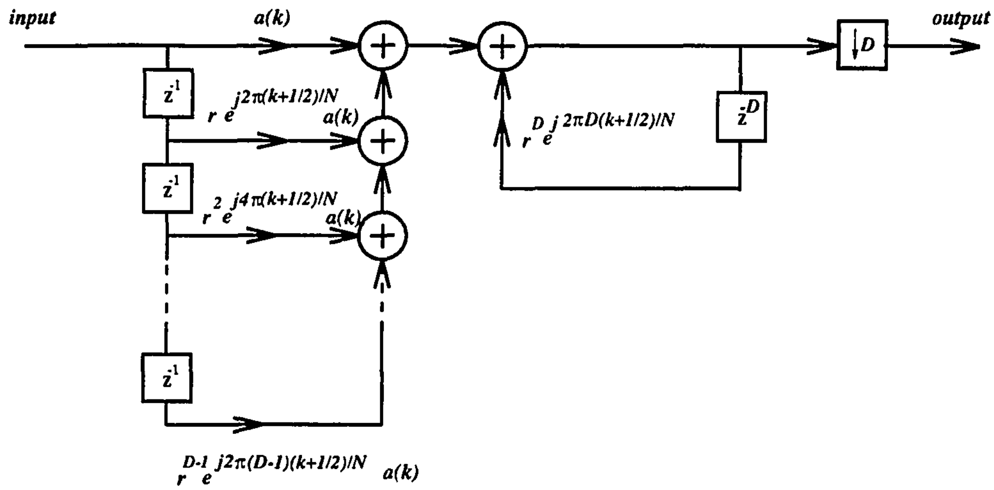
Multirate Type 2 Frequency Sampling Filter

After narrowband filtering, the output of the frequency sampling filter can be decimated. The system function in Equation (3.8) is well suited for multirate applications where the output is decimated by a factor of D. Instead of computing all the output samples, a computational reduction can be achieved by moving the decimator toward the input side of the filter.

Fig. 3.8a represents a Direct Form 1 structure for the resonator in Fig. 3.7b. Fig. 3.8b illustrates an equivalent structure for the filter with the decimator moved towards the input side of the filter. To achieve the same throughput, the structure in Fig. 3.7b needs to have adders and multipliers that are D times faster than the corresponding units in the structure in Fig. 3.8b. Thus, Fig. 3.8b represents a filter structure with a reduced computational requirement compared to the structure in Fig. 3.7b.

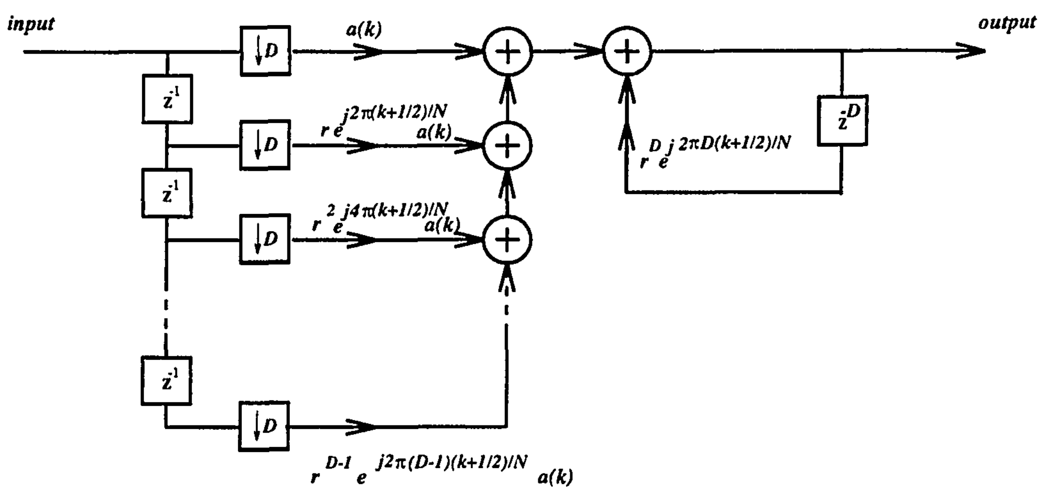
3.2.1 Pipelined Type 2 Frequency Sampling Filter With Real Impulse Response

The filter described by Equation (3.8) can be implemented without complex arithmetic , if the filter's impulse response is real. The system function in Equation (3.8) can be expressed



where $a(k) = H(k)/N$

Figure 3.8a: Direct Form 1 structure for the kth resonator of a multirate Type 2 frequency sampling filter.



where $a(k) = H(k)/N$

Figure 3.8b: Structure of the kth resonator with the decimator moved towards the input side.

as

$$H(z) = \frac{1+r^N z^{-N}}{N}\sum_{k=0}^{\frac{N}{2}-1}\left[\frac{|H(k)|e^{j\theta(k)}\sum_{l=0}^{D-1}(e^{j\frac{2\pi}{N}(k+\frac{1}{2})}rz^{-1})^l}{1-e^{j\frac{2\pi}{N}(k+\frac{1}{2})D}r^D z^{-D}} + \sum_{k=\frac{N}{2}}^{N-1}\frac{|H(k)|e^{j\theta(k)}\sum_{l=0}^{D-1}(e^{j\frac{2\pi}{N}(k+\frac{1}{2})}rz^{-1})^l}{1-e^{j\frac{2\pi}{N}(k+\frac{1}{2})D}r^D z^{-D}}\right] \quad (3.9)$$

where N is even. If a Type 2 filter is constrained to have a real impulse response, then its frequency response samples have the property,

$$|H(k)| = |H(N-1-k)|$$

and

$$\theta(k) = -\theta(N-1-k)$$

Substituting these relationships into Equation (3.9),

$$\begin{aligned} H(z) &= \frac{1+r^N z^{-N}}{N}\sum_{k=0}^{M_2}\left[\frac{|H(k)|e^{j\theta(k)}\sum_{l=0}^{D-1}(e^{j\frac{2\pi}{N}(k+\frac{1}{2})}rz^{-1})^l}{1-e^{j\frac{2\pi}{N}(k+\frac{1}{2})D}r^D z^{-D}} + \frac{|H(k)|e^{-j\theta(k)}\sum_{l=0}^{D-1}(e^{-j\frac{2\pi}{N}(k+\frac{1}{2})}rz^{-1})^l}{1-e^{-j\frac{2\pi}{N}(k+\frac{1}{2})D}r^D z^{-D}}\right] \\ &= \frac{1+r^N z^{-N}}{N}\sum_{k=0}^{M_2} 2|H(k)|\left[\frac{\sum_{l=0}^{D-1}\cos\left(\theta(k)+\frac{2\pi}{N}l(k+\frac{1}{2})\right)rz^{-1}}{1-2\cos(\frac{2\pi}{N}(k+\frac{1}{2})D)r^D z^{-D}+r^{2D}z^{-2D}} - \frac{\sum_{l=0}^{D-1}\cos\left(\theta(k)+\frac{2\pi}{N}(D-l)(k+\frac{1}{2})\right)(rz^{-1})^{D+l}}{1-2\cos(\frac{2\pi}{N}(k+\frac{1}{2})D)r^D z^{-D}+r^{2D}z^{-2D}}\right] \quad (3.10) \end{aligned}$$

where $M_2 = (N/2) - 1$.

A block diagram for the filter described in Equation (3.10) is shown in Fig. 3.9 where a structure of resonators 0 to M_2 is shown in Fig. 3.9b. If N is odd, $M_2 = (N-3)/2$, and the

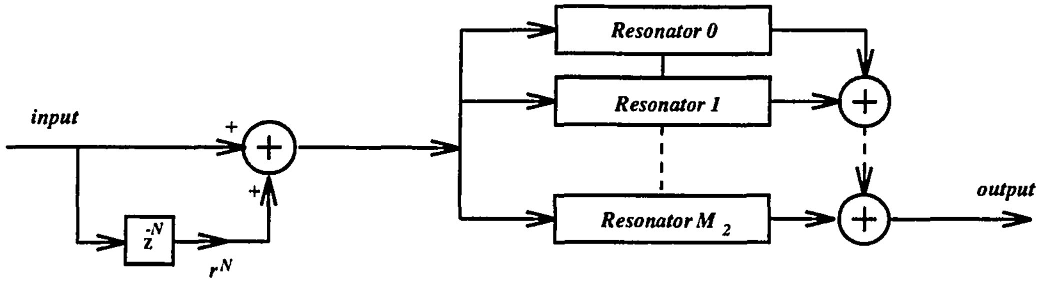


Figure 3.9a: Pipelined Type 2 frequency sampling filter with real impulse response.

filter system function has a term due to the frequency sample, $H(\frac{N-1}{2})$. Each of these resonators requires $2D+2$ real multiplies and $2D+1$ real adds per output sample. Hence, the filter requires approximately $(2D+2)(M_2+1)+1$ real multiplies and $(2D+1)(M_2+1)+M_2$ real adds per output sample. The hardware requirement of the filter in Fig. 3.9a is more than that in Fig. 2.4a. However, the structure in Fig. 3.9a can be pipelined.

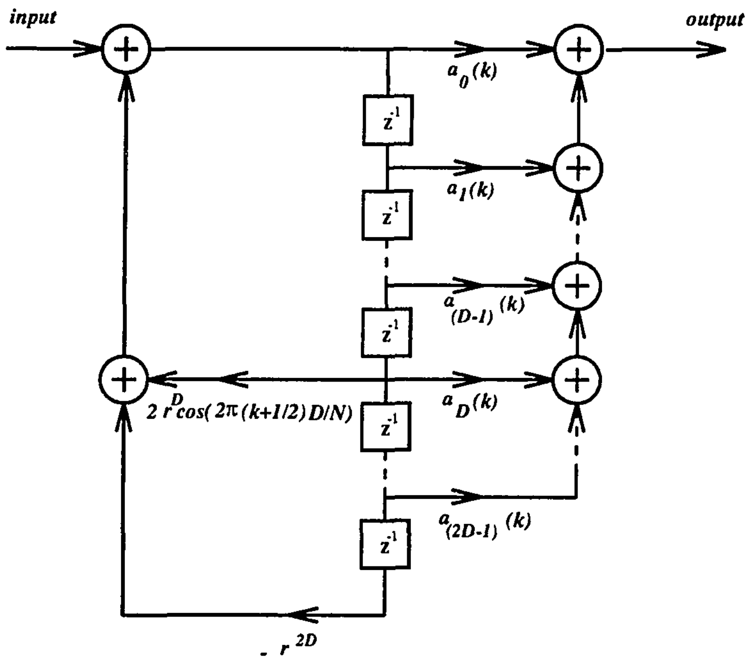
Multirate Type 2 Frequency Sampling Filter With Real Impulse Response

The filter described by Equation (3.10) is well suited for sampling rate reduction by a factor of D at the output. Fig. 3.10a represents Direct Form 1 structure for resonators 1 to M_2 where the output of each resonator is decimated by a factor of D. To achieve computational saving, the decimator should be moved toward the input side of the filter. Fig. 3.10b represents an equivalent structure with the decimator moved toward the input side of the filter. This structure requires fewer adds and multiplies per output sample than the structure in Fig. 3.10a.

3.2.2 Pipelined Type 2 Linear Phase Frequency Sampling Filter

If a Type 2 frequency sampling filter is constrained to have linear phase, then the phase samples can be written as

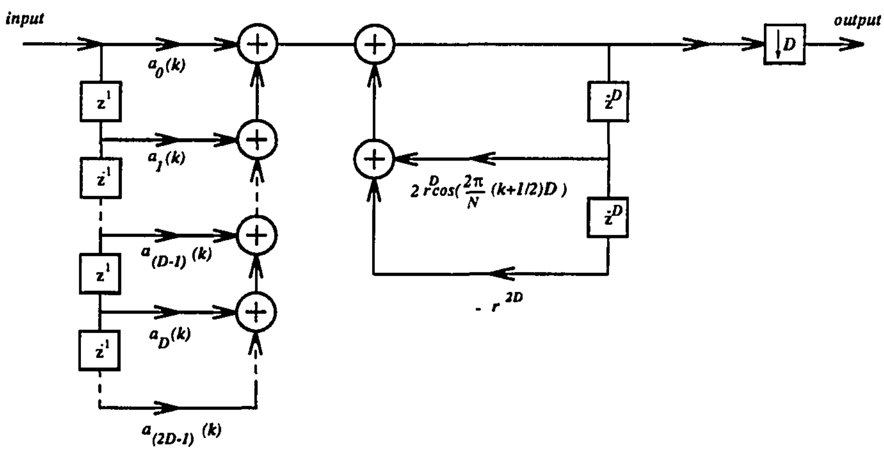
$$\theta(k) = -\frac{2\pi}{N}\left(k+\frac{1}{2}\right)\left[\frac{N-1}{2}\right]$$



$a_i(k) = r^i \cos(\theta(k)+2\pi i(k+1/2)/N)\ 2|H(k)|/N$

where $i=0,1...2D-1$

Figure 3.9b: Structure of the resonators 0 to M_2 in Fig. 3.9a.



$a_i(k) = r^i \cos(\theta(k) + 2\pi i(k+1/2)/N)\ 2|H(k)|/N$

where $i=0,1...\ 2D-1$

Figure 3.10a: Direct Form 1 structure for the kth resonator of a multirate Type 2 frequency sampling filter with real impulse response.

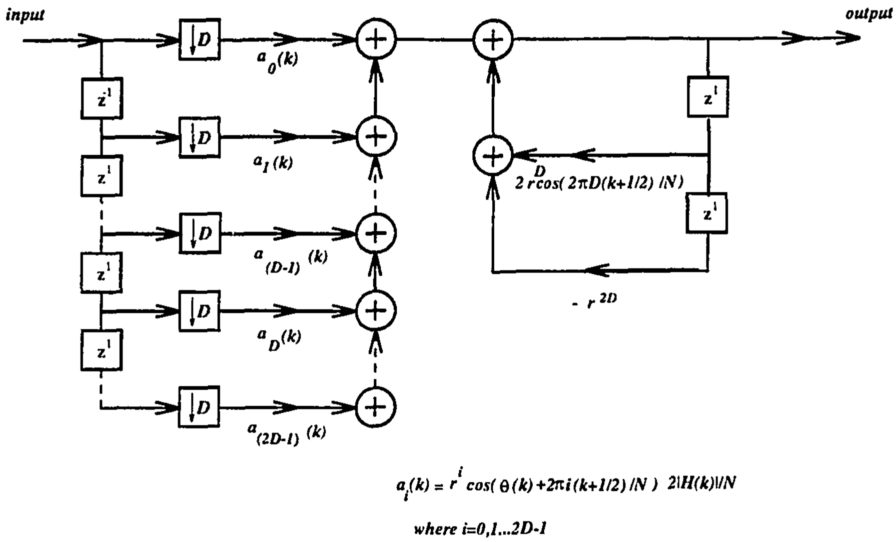


Figure 3.10b: Structure of the kth resonator with the decimator pushed towards the input side.

Therefore

$$cos\left(\theta(k) + \frac{2\pi}{N}l\left(k+\frac{1}{2}\right)\right) \quad = \quad (-1)^k sin\left(\frac{(2l+1)\pi}{N}\left(k+\frac{1}{2}\right)\right) \qquad (3.11)$$

and

$$cos\left(\theta(k) - \frac{2\pi}{N}l\left(k+\frac{1}{2}\right)\right) \quad = \quad (-1)^k sin\left(\frac{(2l-1)\pi}{N}\left(k+\frac{1}{2}\right)\right) \qquad (3.12)$$

Substituting Equations (3.11) and (3.12) in Equation (3.10),

$$H(z) = \frac{1+r^N z^{-N}}{N}\sum_{k=1}^{M_2} 2|H(k)|\left[\frac{\sum_{l=0}^{D-1}(-1)^k sin\left(\frac{(2l+1)\pi(k+\frac{1}{2})}{N}\right) rz^{-l}}{1-2cos(\frac{2\pi}{N}kD)r^D z^{-D} + r^{2D}z^{-2D}} - \right.$$

$$\left.\frac{\sum_{l=1}^{D-1}(-1)^k sin\left(\frac{(2l-1)\pi(k+\frac{1}{2})}{N}\right)(rz^{-1})^{2D-l}}{1-2cos(\frac{2\pi}{N}kD)r^D z^{-D} + r^{2D}z^{-2D}}\right]$$

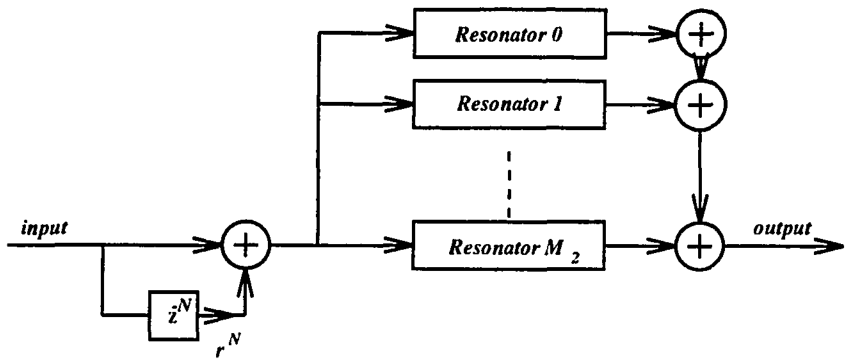


Figure 3.11a: Pipelined Type 2 frequency sampling filter with linear phase.

$$= \frac{1 + r^N z^{-N}}{N} \sum_{k=1}^{M_2} 2|H(k)| \left[\frac{\sum_{l=0}^{D-1} (-1)^k sin\left(\frac{(2l+1)\pi(k+\frac{1}{2})}{N}\right) r z^{-l}}{1 - 2cos(\frac{2\pi}{N}kD) r^D z^{-D} + r^{2D} z^{-2D}} \right.$$

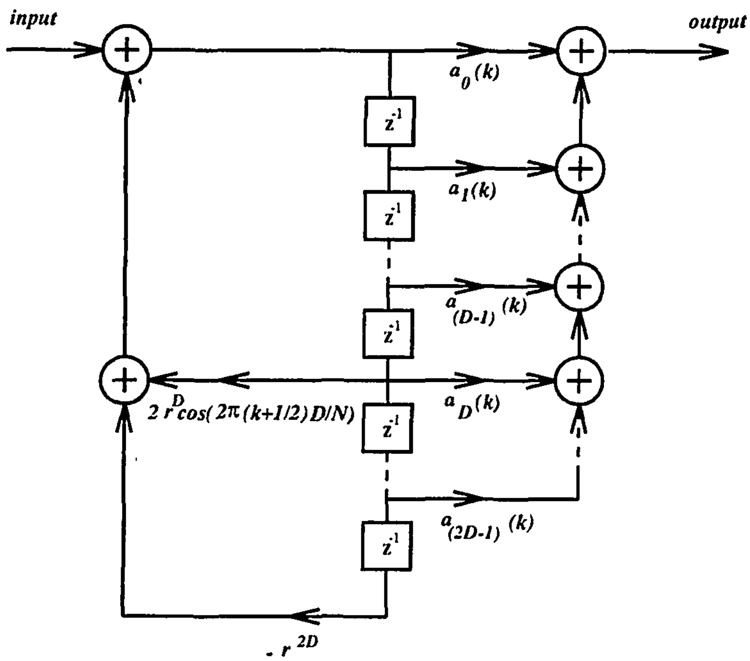
$$\left. - \frac{\sum_{l=0}^{D-1} (-1)^k sin\left(\frac{(2l+1)\pi(k+\frac{1}{2})}{N}\right) (r z^{-l})^{2D-1-l}}{1 - 2cos(\frac{2\pi}{N}kD) r^D z^{-D} + r^{2D} z^{-2D}} \right] \qquad (3.13)$$

where N is even and $M_2 = (N/2) - 1$.

The structure of the pipelined frequency sampling filter described by Equation (3.12) is shown in Fig. 3.11a where Fig. 3.11b shows the structure of resonators 1 through M_1. Each of these resonators requires $2D + 2$ real multiplies and $2D + 1$ real adds per output sample. The structure of resonator 0 is shown in Fig. 3.3c. Hence, the filter requires $(2D + 2)M_2 + D + 2$ real multiplies and $(2D + 2)M_2 + D$ real adds per output sample. The presence of only delays of D in the feedback paths of each resonator makes it possible to pipeline the filter.

Multirate Type 2 Frequency Sampling Filter With Linear Phase

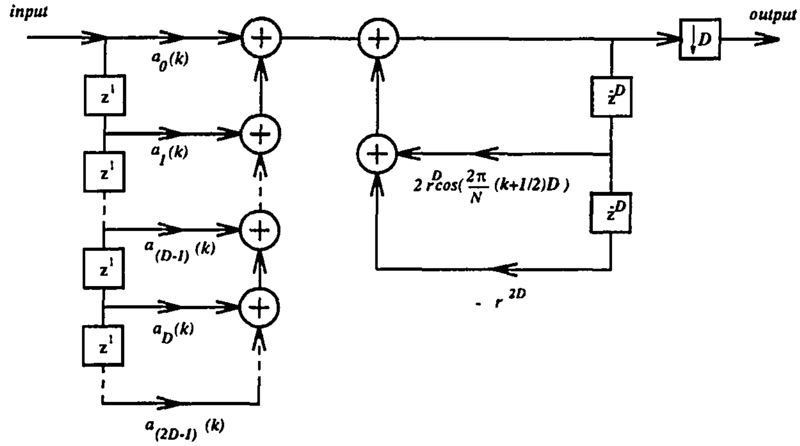
The system function of the filter given in Equation (3.13) contains only powers of z^{-D} in the denominator. This makes it well suited for sampling rate reduction at the output. by a factor of D. Fig. 3.12a shows a Direct Form 1 structure of resonators 0 to M_2 of the filter



$$a_l(k) = r^l (-1)^k \sin((2l+1)\pi(k+1/2)/N)\, 2|H(k)|/N \quad \text{for } l = 0 \text{ to } D-1$$

$$a_l(k) = - r^l (-1)^k \sin([2(2D-1-l)+1]\pi(k+1/2)/N]\, 2|H(k)|/N \quad \text{for } l = D \text{ to } 2D-1$$

Figure 3.11b: Structure of the resonators 0 to M_2 in Fig. 3.11a.



$$a_l(k) = r^l(-1)^k \sin((2l+1)\pi(k+1/2)/N)\, 2|H(k)|/N \quad \text{for } l = 0 \text{ to } D-1$$

$$a_l(k) = -r^l(-1)^k \sin([2(2D-1-l)+1]\,\pi(k+1/2)/N]\, 2|H(k)|/N \quad \text{for } l = D \text{ to } 2D-1$$

Figure 3.12a: Multirate Type 2 frequency sampling filter with linear phase.

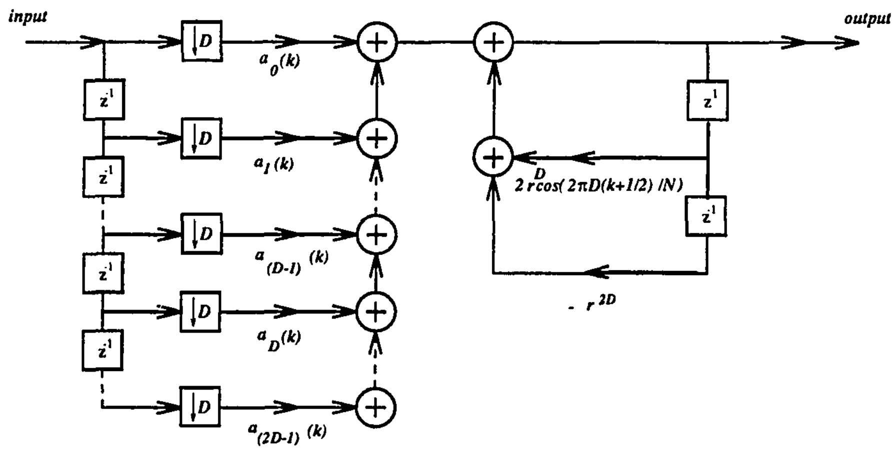
described by Equation (3.13) where the output of each resonator is decimated by a factor of D. To achieve a computational reduction, the decimator should be moved toward the input side of the filter. Fig. 3.12b represents an equivalent structure where the decimator has been moved toward the input side. Fig. 3.12b represents a resonator structure with a smaller computational requirement compared to the resonator in Fig. 3.12a.

3.2.3 Properties of Multirate Type 2 Frequency Sampling Filters

Table 3.3 shows a comparison of the multiplies required by a direct convolution filter, a Type 2 frequency sampling filter and a pipelined Type 2 frequency sampling filter. Table 3.4 shows the computational requirements of these filters if their output is decimated by a factor of D.

Similar to the Type 1 frequency sampling filter developed in Section 3.1, the linear phase Type 2 frequency sampling filter developed in this section is more efficient than a linear direct convolution structure when $(2D+2)K + 1 < N/2$ or

$$K < \frac{\frac{N}{2} - 1}{2D+2} \simeq \frac{N}{4D+4}.$$



$$a_l(k) = r^l(-1)^k \sin((2l+1)\pi(k+1/2)/N)\ 2|H(k)|/N \quad \text{for } l = 0 \text{ to } D-1$$

$$a_l(k) = -r^l(-1)^k \sin([2(2D-1-l)+1]\ \pi(k+1/2)/N)\ 2|H(k)|/N \quad \text{for } l=D \text{ to } 2D-1$$

Figure 3.12b: Structure of the resonator with the decimator pushed towards the input side.

Filter for even N	Nonlinear phase real impulse response	Linear phase real impulse response
Direct convolution filter	N	$N/2$
Type 2 frequency sampling filter	$4K$ $+1$	$3K$ $+2$
Pipelined Type 2 frequency sampling filter	$(2D+2)K$ $+1$	$(2D+2)K$ $+1$

Table 3.3: Multiplies required per output sample by Type 2 frequency sampling filter and a pipelined Type 2 frequency sampling filter (K is the number of nonzero frequency samples, $D > 1$ and N is the length of the impulse response of the filter).

Filter for even N	Nonlinear phase real impulse response	Linear phase real impulse response
Direct convolution filter with the output decimated	N	$N/2$
Type 2 frequency sampling filter with the output decimated	$(2D+2)K$ $+D$	$(2D+1)K$ $+D$
Pipelined Type 2 frequency sampling filter with the output decimated	$(2D+2)K$ $+D$	$(2D+2)K$ $+D$

Table 3.4: Multiplies required per output sample by Type 2 frequency sampling filter and a pipelined Type 2 frequency sampling filter with their outputs decimated (D is the decimation ratio. K is the number of nonzero frequency samples and N is the length of the impulse response of the filter).

A nonlinear phase Type 2 frequency sampling filter developed in this section is more efficient than a nonlinear direct convolution structure when $K < \frac{N}{2D+2}$.

3.3 Example: A Low-pass Frequency Sampling Filter for N Odd

A frequency sampling filter designed in [3] is used to compare the frequency spectrum of the output of a linear phase pipelined Type 1 frequency sampling filter without decimation of the output and with decimation by a factor of D. The input. $x(n)$ to the filter is

$$x(n) = \begin{cases} cos(0.03\pi n) + cos(0.5\pi n) & \text{for } 0 \leq n \leq 200 \\ 0 & \text{elsewhere} \end{cases}$$

Fig. 3.13a shows the magnitude spectrum of the input signal. The magnitude and phase spectra of the output of the pipelined frequency sampling filter are shown by Figs. 3.13b and 3.13c respectively. The magnitude and phase spectra for the multirate frequency sampling filter are represented by Figs. 3.13d and 3.13e respectively. These plots are generated for different values of r. In order to reduce the nonlinearity in the phase of the filter, r is kept close to unity.

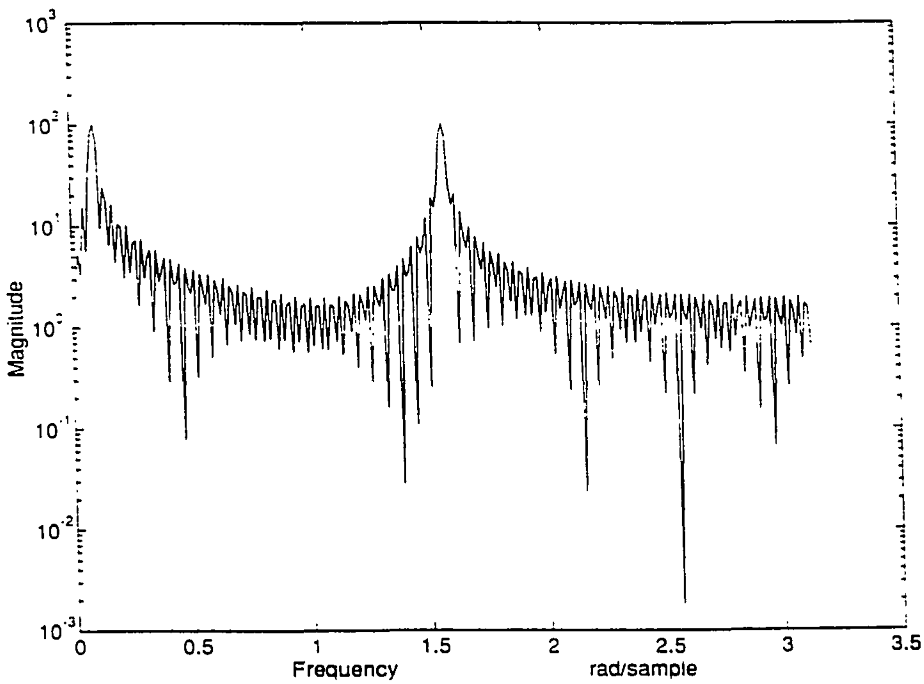


Figure 3.13a: Magnitude spectrum of the input.

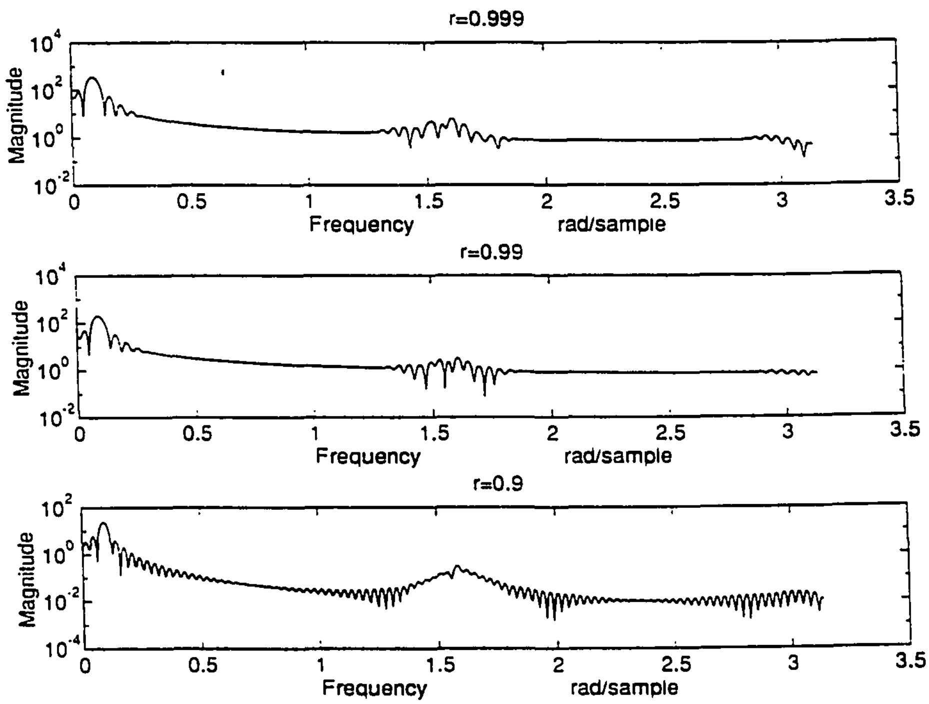


Figure 3.13b: Magnitude spectra of the output of the multirate Type 1 frequency sampling filter, for different values of r. $D = 1$, $K = 6$, $N = 127$

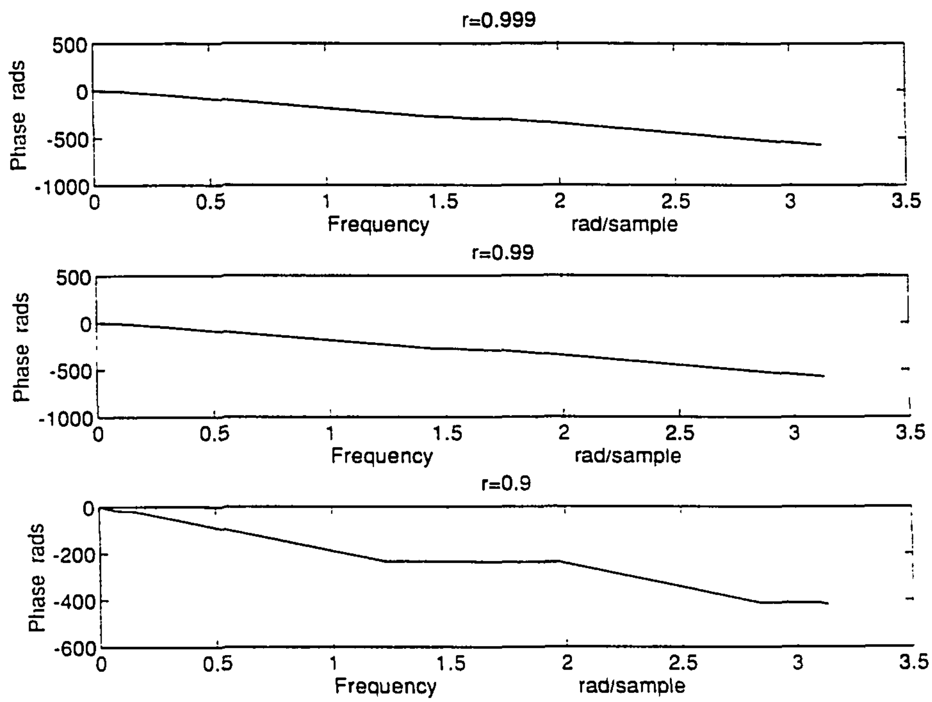


Figure 3.13c: Phase spectra of the output of the Type 1 multirate frequency sampling filter, for different values of r. $D = 1$, $K = 6$, $N = 127$

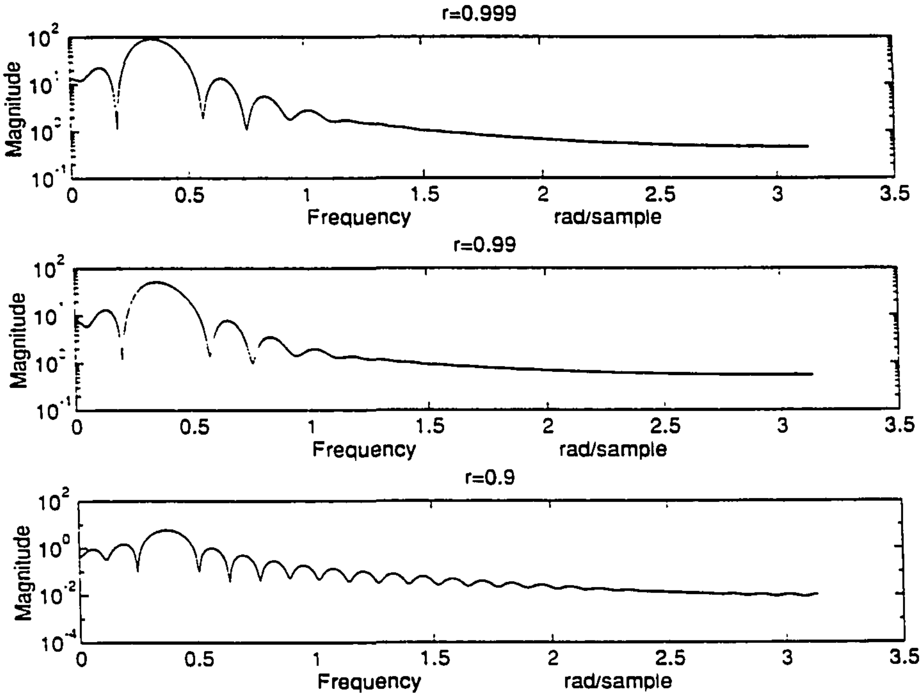


Figure 3.13d: Magnitude spectra of the output of the Type 1 multirate frequency sampling filter, for different values of r. $D = 4$, $K = 6$, $N = 127$

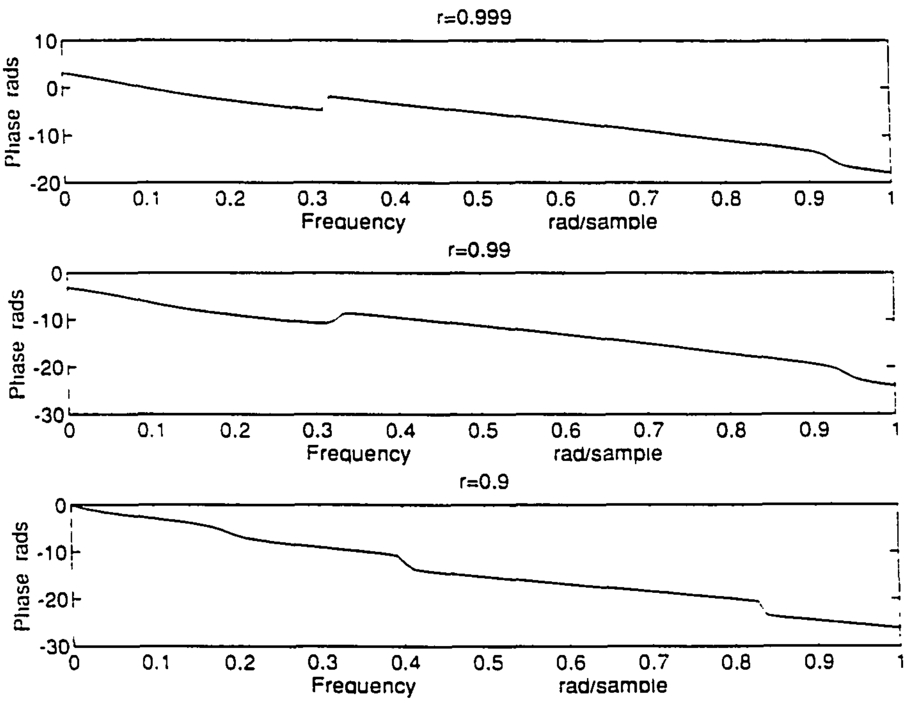


Figure 3.13e: Phase spectra of the output of the Type 1 multirate frequency sampling filter, for different values of r. $D = 4$, $K = 6$, $N = 127$

Chapter 4

State space structures for multirate frequency sampling filters

Coupled state space structures have a lower coefficient sensitivity and are less sensitive to quantization errors than Direct Form structures [6]. In this chapter, we will use coupled state space structures to implement the frequency sampling filters developed in Chapter 3.

4.1 State Space Structure for a Multirate Type 1 Frequency Sampling Filter

In this section, we will develop a coupled state space structure of a Type 1 frequency sampling filter which has its output decimated by a factor of D. The system function for a multirate Type 1 frequency sampling filter was given in the Equation (3.2) as

$$H(z) = \frac{1-r^N z^{-N}}{N} \sum_{k=0}^{N-1} \frac{H(k)\sum_{l=0}^{D-1}(e^{j\frac{2\pi}{N}k} r z^{-1})^l}{1-e^{j\frac{2\pi}{N}kD} r^D z^{-D}}.$$

If the impulse response of the filter is real, the frequency response samples have the property $H(k) = H^*(N-k)$. Using this property, we can express the filter's system function as

$$H(z) = \frac{1-r^N z^{-N}}{N} \left\{ \sum_{k=1}^{M_1} \left[\frac{H(k)\sum_{l=0}^{D-1}(e^{j\frac{2\pi}{N}k} r z^{-1})^l}{1-e^{j\frac{2\pi}{N}kD} r^D z^{-D}} + \frac{H^*(k)\sum_{l=0}^{D-1}(e^{-j\frac{2\pi}{N}k} r z^{-1})^l}{1-e^{-j\frac{2\pi}{N}kD} r^D z^{-D}} \right] + \right.$$

$$\left. \frac{H(0)\sum_{l=0}^{D-1}(r z^{-1})^l}{1-r^D z^{-D}} + \frac{H(N/2)\sum_{l=0}^{D-1}(e^{j\pi} r z^{-1})^l}{1-e^{j\pi D} r^D z^{-D}} \right\}$$

where N is even and $M_1 = (N/2) - 1$. If N is odd, the $H(N/2)$ term does not exist in the system function of the filter and $M_1 = (N-1)/2$. If we define $H_k(z)$ as

$$H_k(z) = \frac{H(k)\sum_{l=0}^{D-1}(e^{j\frac{2\pi}{N}k} r z^{-1})^l}{1-e^{j\frac{2\pi}{N}kD} r^D z^{-D}} + \frac{H^*(k)\sum_{l=0}^{D-1}(e^{-j\frac{2\pi}{N}k} r z^{-1})^l}{1-e^{-j\frac{2\pi}{N}kD} r^D z^{-D}}. \tag{4.1}$$

for $k = 1, 2, \ldots, M_1$ then

$$H(z) = \frac{1-r^N z^{-N}}{N} \left[\sum_{k=1}^{M_1} H_k(z) + \frac{H(0)\sum_{l=0}^{D-1}(r z^{-1})^l}{1-r^D z^{-D}} + \frac{H(N/2)\sum_{l=0}^{D-1}(e^{j\pi} r z^{-1})^l}{1-e^{j\pi D} r^D z^{-D}} \right].$$

If we also define $H_{kn}(z)$ as

$$H_{kn}(z) = \left[\frac{H(k)(e^{j\frac{2\pi}{N}k} r z^{-1})^n}{1-e^{j\frac{2\pi}{N}kD} r^D z^{-D}} + \frac{H^*(k)(e^{-j\frac{2\pi}{N}k} r z^{-1})^n}{1-e^{-j\frac{2\pi}{N}kD} r^D z^{-D}} \right]$$

then $H_{k0}(z)$ can be written as

$$H_{k0}(z) = \left[\frac{H(k)}{1-e^{j\frac{2\pi}{N}kD} r^D z^{-D}} + \frac{H^*(k)}{1-e^{-j\frac{2\pi}{N}kD} r^D z^{-D}} \right]$$

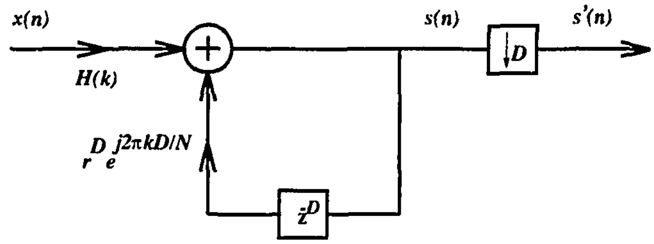


Figure 4.1a: Structure to implement Equation (4.3), with the output, $s(n)$, decimated by a factor of D .

If we let $Y_{k0}(z)$ be the output of the section of the filter given by the system function, $H_{k0}(z)$, and let $X(z)$ be the input to the resonators, then

$$Y_{k0}(z) = \left[\frac{H(k)X(z)}{1 - e^{j\frac{2\pi}{N}kD}r^D z^{-D}} + \frac{H^*(k)X(z)}{1 - e^{-j\frac{2\pi}{N}kD}r^D z^{-D}}\right].$$

If we let

$$S(z) = \frac{H(k)X(z)}{1 - e^{j\frac{2\pi}{N}kD}r^D z^{-D}} \tag{4.2}$$

then

$$Y_{k0}(z) = S(z) + S^*(z).$$

This implies that

$$y_{k0}(n) = s(n) + s^*(n).$$

The decimation of the output, $y_{k0}(n)$, is equivalent to the decimation of $s(n)$ and $s^*(n)$. Taking the inverse z transform of Equation (4.2),

$$s(n) - e^{j\frac{2\pi}{N}kD}r^D s(n - D) = H(k)x(n) \tag{4.3}$$

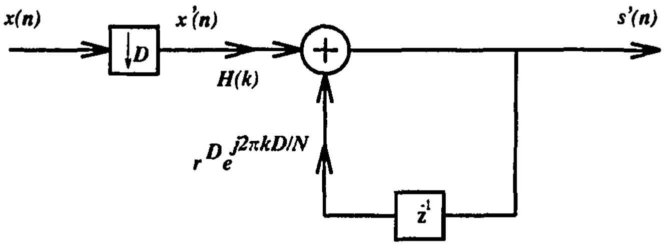


Figure 4.1b: Structure to implement Equation (4.4) .

Fig. 4.1a shows a structure for implementing Equation (4.3), with the output, $s(n)$, decimated by a factor of D. Fig 4.1b shows an equivalent structure with the decimator moved toward the input side of the filter. The difference equation for this equivalent structure is given as

$$s'(n) = e^{j\frac{2\pi}{N}kD} r^D s'(n-1) + H(k)x'(n) \qquad (4.4)$$

where $s'(n)$ and $x'(n)$ are obtained by decimating $s(n)$ and $x(n)$, respectively, by a factor of D. $s'(n)$ can be expressed in terms of state variables, $v_1(n)$ and $v_2(n)$ as

$$s'(n) = v_1(n) + jv_2(n).$$

The frequency samples, $H(k)$ can be written as

$$H(k) = Re[H(k)] + jIm[H(k)]$$

where $Re[H(k)]$ is the real part and $Im[H(k)]$ is the imaginary part of $H(k)$. Therefore, $v_1(n)$ and $v_2(n)$ can be written in matrix form as

$$\mathbf{v}(\mathrm{n}) = \begin{bmatrix} v_1(n) \\ v_2(n) \end{bmatrix} = \begin{bmatrix} cos(\frac{2\pi}{N}kD)r^D & -sin(\frac{2\pi}{N}kD)r^D \\ sin(\frac{2\pi}{N}kD)r^D & cos(\frac{2\pi}{N}kD)r^D \end{bmatrix} \begin{bmatrix} v_1(n-1) \\ v_2(n-1) \end{bmatrix} + \begin{bmatrix} Re[H(k)] \\ Im[H(k)] \end{bmatrix} x'(n).$$

The output equation can be written as

$$y_{k0}(n) = s(n) + s^*(n)$$

$$= [v_1(n) + jv_2(n)] + [v_1(n) - jv_2(n)]$$

$$= \begin{bmatrix} 2 & 0 \end{bmatrix} \mathbf{v}(n)$$

The state equations and output equations for the other terms in $H_k(z)$ can be derived similarly. The output, $y_k(n)$, is calculated by adding up the outputs due to all the terms in $H_k(z)$.

The structure of a multirate Type 1 frequency sampling filter with a real impulse response is shown in Fig. 3.3a. Fig. 4.2 shows the structure of the resonators 1 to M_1 if the output of the frequency sampling filter is decimated by a factor of D. Each resonator requires $2D+5$ real multiplies per output sample and $2D-2$ real adds per output sample. This is approximately as many computations as that required for a Direct Form structure shown in Fig. 3.3b. The filter requires $(2D+5)M_1+2D$ real multiplies and $(2D-2)M_1+2D$ real adds per output sample.

4.2 State Space Structure for a Multirate Type 2 Frequency Sampling Filter

In this section, we will develop a coupled state space structure for a Type 2 frequency sampling filter which has its output is decimated by a factor of D. The system function for a pipelined Type 2 frequency sampling filter is given by the Equation (3.8) as

$$H(z) = \frac{1 + r^N z^{-N}}{N} \sum_{k=0}^{N-1} \frac{H(k) \sum_{l=0}^{D-1} (e^{j\frac{2\pi}{N}(k+\frac{1}{2})} r z^{-1})^l}{1 - e^{j\frac{2\pi}{N}(k+\frac{1}{2})D} r^D z^{-D}}$$

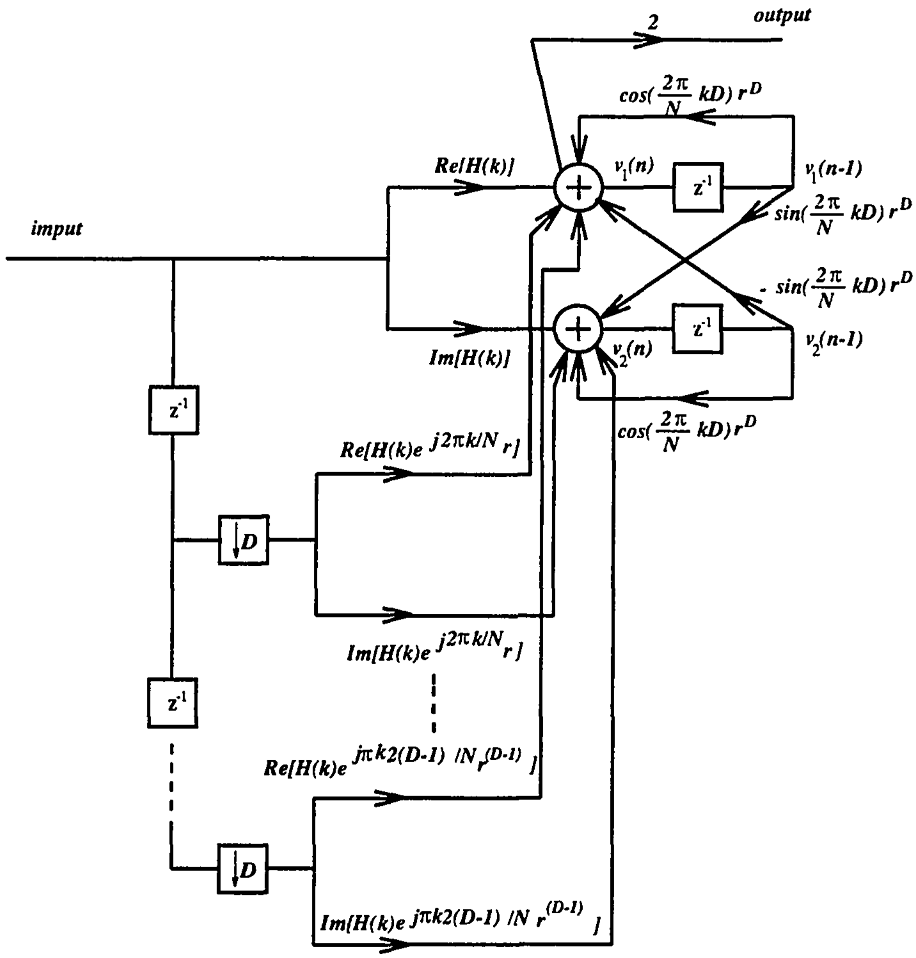


Figure 4.2: Coupled state space structure for resonators 1 to M_1.

If the impulse response of the filter is real, the frequency response samples, $H(k)$ have the property $H(k) = H^*(N-1-k)$. Using this property the system function of the filter can be written as

$$H(z) = \frac{1+r^N z^{-N}}{N}\left[\sum_{k=0}^{M_2} \frac{H(k)\sum_{l=0}^{D-1}(e^{j\frac{2\pi}{N}(k+\frac{1}{2})}rz^{-1})^l}{1-e^{j\frac{2\pi}{N}(k+\frac{1}{2})D}r^D z^{-D}} + \frac{H^*(k)\sum_{l=0}^{D-1}(e^{-j\frac{2\pi}{N}(k+\frac{1}{2})}rz^{-1})^l}{1-e^{-j\frac{2\pi}{N}(k+\frac{1}{2})D}r^D z^{-D}}\right]$$

where N is even, and $M_2 = (N/2)-1$. If N is odd, the system function has a term due to the filter coefficient $H(\frac{N-1}{2})$, and $M_1 = (N-3)/2$. If we define $H_k(z)$ as

$$H_k(z) = \frac{H(k)\sum_{l=0}^{D-1}(e^{j\frac{2\pi}{N}(k+\frac{1}{2})}rz^{-1})^l}{1-e^{j\frac{2\pi}{N}(k+\frac{1}{2})D}r^D z^{-D}} + \frac{H^*(k)\sum_{l=0}^{D-1}(e^{-j\frac{2\pi}{N}(k+\frac{1}{2})}rz^{-1})^l}{1-e^{-j\frac{2\pi}{N}(k+\frac{1}{2})D}r^D z^{-D}}. \quad (4.5)$$

for $k = 1, 2, \ldots M_2$ then

$$H(z) = \frac{1+r^N z^{-N}}{N}\left[\sum_{k=1}^{M_2} H_k(z)\right].$$

If we also define $H_{kn}(z)$ as

$$H_{kn}(z) = \left[\frac{H(k)(e^{j\frac{2\pi}{N}(k+\frac{1}{2})}rz^{-1})^n}{1-e^{j\frac{2\pi}{N}(k+\frac{1}{2})D}r^D z^{-D}} + \frac{H^*(k)(e^{-j\frac{2\pi}{N}(k+\frac{1}{2})}rz^{-1})^n}{1-e^{-j\frac{2\pi}{N}(k+\frac{1}{2})D}r^D z^{-D}}\right]$$

then $H_{k0}(z)$ can be written as

$$H_{k0}(z) = \left[\frac{H(k)}{1-e^{j\frac{2\pi}{N}(k+\frac{1}{2})D}r^D z^{-D}} + \frac{H^*(k)}{1-e^{-j\frac{2\pi}{N}(k+\frac{1}{2})D}r^D z^{-D}}\right]$$

If we let $Y_{k0}(z)$ be the output of the section of the filter given by the system function, $H_{k0}(z)$, and let $X(z)$ be the input to the resonators, then

$$Y_{k0}(z) = \left[\frac{H(k)X(z)}{1-e^{j\frac{2\pi}{N}(k+\frac{1}{2})D}r^D z^{-D}} + \frac{H^*(k)X(z)}{1-e^{-j\frac{2\pi}{N}(k+\frac{1}{2})D}r^D z^{-D}}\right].$$

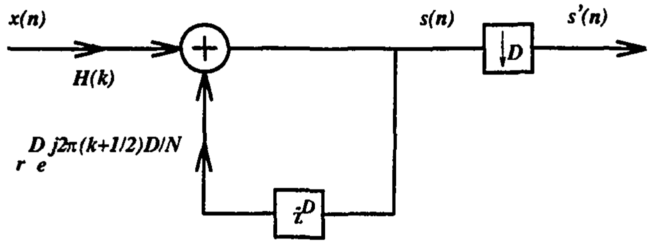


Figure 4.3a: Structure to implement Equation (4.3), with the output, $s(n)$, decimated by a factor of D .

If we let

$$S(z) \;=\; \frac{H(k)X(z)}{1 - e^{j\frac{2\pi}{N}(k+\frac{1}{2})D}r^{D}z^{-D}} \tag{4.6}$$

then

$$Y_{k0}(z) \;=\; S(z) + S^{*}(z).$$

This implies that

$$y_{k0}(n) = s(n) + s^{*}(n).$$

Taking the inverse z transform of Equation (4.6),

$$s(n) - e^{j\frac{2\pi}{N}(k+\frac{1}{2})D}r^{D}s(n-D) \;=\; H(k)x(n) \tag{4.7}$$

Fig. 4.3a shows the structure for implementing the Equation (4.7) with the output decimated by a factor of D. Fig. 4.3b shows an equivalent structure with the decimator moved toward the input side of the filter. The difference equation for this equivalent structure is given as

$$s'(n) \;=\; e^{j\frac{2\pi}{N}(k+\frac{1}{2})D}r^{D}s'(n-1) + H(k)x'(n) \tag{4.8}$$

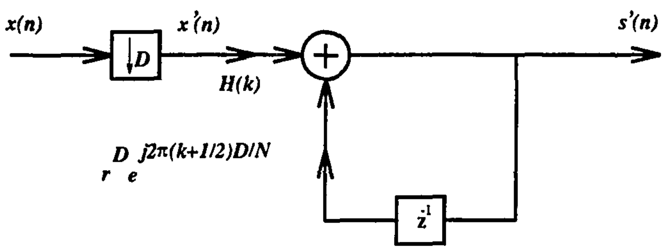


Figure 4.3b: Structure to implement Equation (4.4) .

where $s'(n)$ and $x'(n)$ are obtained by decimating $s(n)$ and $x(n)$ respectively, by a factor of D. $s'(n)$ can be expressed in terms of the state variables $v_1(n)$ and $v_2(n)$ as

$$s'(n) = v_1(n) + jv_2(n).$$

In matrix form, $v_1(n)$ and $v_2(n)$ can be written as

$$\mathbf{v}(n) = \begin{bmatrix} v_1(n) \\ v_2(n) \end{bmatrix} = \begin{bmatrix} cos\left[\frac{2\pi}{N}(k+\frac{1}{2})D\right] r^D & -sin\left[\frac{2\pi}{N}(k+\frac{1}{2})D\right] r^D \\ sin\left[\frac{2\pi}{N}(k+\frac{1}{2})D\right] r^D & cos\left[\frac{2\pi}{N}(k+\frac{1}{2})D\right] r^D \end{bmatrix} \begin{bmatrix} v_1(n-1) \\ v_2(n-1) \end{bmatrix}$$

$$+ \begin{bmatrix} Re[H(k)] \\ Im[H(k)] \end{bmatrix} x'(n)$$

The output equation can be written as follows

$$y_{k0}(n) = s(n) + s^*(n)$$

$$y_{k0}(n) = \begin{bmatrix} 2 & 0 \end{bmatrix} \mathbf{v}(n)$$

The state equations for the other terms in $H_k(z)$ can be derived similarly. The output, $y_k(n)$, is calculated by adding up the outputs due to all the terms in $H_k(z)$.

The structure of a multirate Type 2 frequency sampling filter with a real impulse response is shown in Fig. 3.9a. Fig. 4.4 shows the structure of resonators 1 to M_2 if the

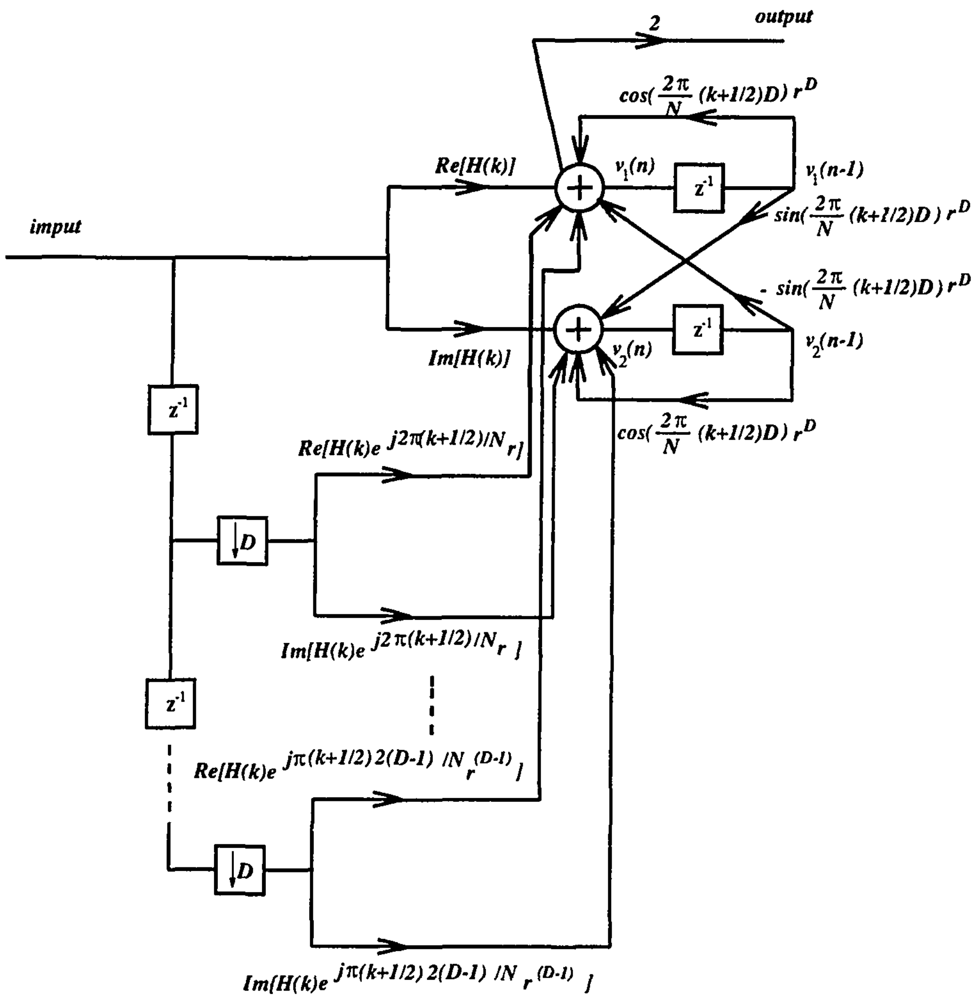


Figure 4.4: Coupled state space structure for resonators 1 to M_2.

Filter for even N	Nonlinear phase real impulse response
Direct Form 1 structure for a multirate Type 1 frequency sampling filter	$(2D+2)K$ +1
Coupled state space structure for a multirate Type 1 frequency sampling filter	$(2D+5)K$
Direct Form 1 structure for a multirate Type 2 frequency sampling filter	$(2D+2)K$ +1
Coupled state space structure for a multirate Type 2 frequency sampling filter	$(2D+5)K$

Table 4.1: Multiplies required per output sample by Direct Form structures and coupled state space structures of frequency sampling filters (K is the number of nonzero frequency samples and D is the decimation ratio of the output).

output of the filter is decimated by a factor of D. Each resonator performs $2D+5$ real multiplies per output sample, and $2D-2$ real adds per output sample. This is approximately same as that for a Direct Form structure, shown in Fig. 3.9b. The filter requires $(2D+5)M_2+D$ real multiplies and $(2D-2)M_2+D$ real adds per output sample.

4.3 Properties of Coupled State Space Structures

Table 4.1 shows that a state space structure for a multirate frequency sampling filter requires the same number of computations per output sample as a Direct Form structure. State space structures can be used to implement multirate frequency sampling filters, with a reduced computational requirement and low coefficient sensitivity.

Chapter 5

Conclusions

Under certain circumstances, narrowband filters can be implemented more efficiently using frequency sampling filters than direct convolution filters [1]. Frequency sampling filters are recursive. The system functions of a Type 1 and Type 2 frequency sampling filter are given by Equation (2.3) and Equation (2.9) respectively. These system functions have z^{-1} in the denominator, which implies that the filter structure has unit delays in the feedback paths. The presence of these unit delays in the feedback paths prevents pipelining and for multirate applications where the output is decimated, no computational saving is achieved.

The problem in pipelining a frequency sampling filter can be addressed by changing the system function of the filter so that it contains only powers of z^{-D} in its denominator. If we assume that multiplies are computationally more complex than adds and if the multipliers take less than D time periods to perform each multiplication then the frequency sampling filter can be pipelined.

The new frequency sampling filter system function that has only powers of z^{-D} in the denominator, makes the filter amenable for sampling rate reduction at the output of the filter. If the output of this filter is decimated by a factor of D, the computational requirements of the adders and multipliers in the filter can be reduced.

Table 3.2 and Table 3.4 compare the number of multiplies required by direct convolution filters, frequency sampling filters and pipelined frequency sampling filters when their outputs are decimated by a factor of D. It is shown in these tables that the frequency sampling filters developed in Chapter 3 require the same number of multiplies per output sample as those mentioned in Chapter 2, when used in multirate systems. Pipelined Type 1 and Type 2 frequency sampling filters are more efficient than a direct convolution filter if for a linear phase filter $K < \frac{N}{4D+4}$ and for a nonlinear phase filter if $K < \frac{N}{2D}$

Coupled state space structures can also be used to implement the resonator in frequency sampling filters. They have a lower coefficient sensitivity than Direct Form structures [10]. These structures require approximately the same number of computations as Direct Form structures, hence they can be used to implement frequency sampling filters economically.

Bibliography

[1] PETER A. STUBBERUD, "Frequency sampling filters, " *Ph. D.dissertation, Junc 1990, UCLA.*

[2] L. R. RABINER AND B. GOLD, "Theory and application of digital signal processing," *Englewood Cliffs, NJ: Prentice Hall, 1975.*

[3] PETER A. STUBBERUD AND CORNELIUS T. LEONDES, "The design of frequency sampling filters by the method of Lagrange multipliers, " *IEEE Trans. Circuits and Systems, vol. 40, no. 1 , pp.51-54, Jan. 93.*

[4] L. R. RABINER AND R. W. SCHAFER, "Recursive and nonrecursive realizations of digital filters designed by frequency sampling techniques, " *IEEE Trans. Audio Eloctroacoust., vol. AU-19, pp 200-207, Sept. 1971.*

[5] T. J. MCCREARY, "On frequency sampling digital filters," *IEEE Trans. Audio Electroacoust., vol. 20, pp. 222-223, Aug. 1972.*

[6] MICHAEL A. SODERSTRAND AND BHASKAR SINHA, "A pipelined recursive residue number system digital filter," *IEEE Trans. Circuits and Systems, vol. CAS-31, no. 4, Apr. 1984.*

[7] H. H. LOOMIS AND B. SINHA, "High speed recursive filter realizations," *Asilomar Confcrencc in Signal Processing.*

[8] A. W. CROOKE AND JOHN. W. CRAIG, "Digital filters for sample rate reduction," *IEEE Trans. on Audio Electroacoust., vol. AU-20, No. 4, Oct. 1972.*

[9] H. G. MARTINEZ AND T. W. PARKS, "A class of infinite duration impulse response digital filters for sampling rate reduction," *IEEE Trans. ASSP., vol. ASSP-24, No. 2, pp. 154-162,.Apr. 1979.*

[10] PROAKIS AND MANOLAKIS "DSP. theory, algorithms and applications," *Macmillan, second edition, 1990*

[11] R. E. CROCHIERE AND L. R. RABINER "Multirate DSP," *Englewood Cliffs, NJ: Prentice Hall, 1988*