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A classification of groups satisfying the converse of Lagrange's theorem

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A CLASSIFICATION OF GROUPS SATISFYING THE CONVERSE OF LAGRANGE'S THEOREM

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ABSTRACT

This paper is a classification of finite groups satisfying the converse of Lagrange's Theorem. We begin by showing a series of inclusions of classes of finite groups: p-groups $\subseteq$ nilpotent $\subseteq$ supersolvable $\subseteq$ polycyclic $\subseteq$ solvable. The crucial point of the paper consists of the proof that the class of supersolvable groups is contained in the class of converse Lagrange groups while the class of polycyclic groups is not. We also show that finite cyclic groups and finite abelian groups are included in the class of converse Lagrange groups. Finally, we give an example to show that the class of converse Lagrange groups is not contained in the class of supersolvable groups.

TABLE OF CONTENTS

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CHAPTER ONE

INTRODUCTION

We begin with some background information which we assume to be known to the reader. First we make some definitions:

DEFINITION 1.1: *A finite sequence of subgroups $\{H_i\}$ of a given group, G, beginning with the trivial subgroup and terminating with the group itself with the property that each H_i is normal in its successor is denoted as a* sub-normal series *while such a series with the property that each H_i is normal in G is denoted as a* normal series.

DEFINITION 1.2: *A group with order a power of p, some prime, is a* p-group. *A* Sylow p-subgroup *is the subgroup of a given finite group with order a maximal prime power.*

DEFINITION 1.3: *The* index *of a subgroup H in a group G is the number of left cosets of H in G, written* $|G:H|$.

DEFINITION 1.4: *The* center *of a group G, written Z(G), is the set of all elements in G which commute with every member of the group. The* centralizer of $x \in G$ *is the subgroup of G, denoted C(x), consisting of the elements that commute with x.*

DEFINITION 1.5: *The* normalizer of $H < G$ *is the set of elements* $g \in G$ *such that for* $H < G$, $g^{-1}Hg = H$. The normalizer is denoted N(H).

DEFINITION 1.6: *Two elements, x and y in group G, are* conjugate *if they are related by* $y = a^{-1}xa$ *for some a also in G. Two subsets, S and T in G are* conjugate *if they are related by* $S = a^{-1}Ta$. *The* conjugacy class of $a \in G$, *denoted cl(a), is the set of all elements in the group that are conjugate to element a in the group.*

We should note here that the order of the conjugacy class in a finite group is the index of the centralizer of one such element in the group. ($|cl(a)| = |G:C(a)|$)

DEFINITION 1.7: *The element* $[x,y] = x^{-1}y^{-1}xy$ *for x and y in a group G is called the* commutator *and is definied by the rule* $[x_1, \cdots, x_{n-1}, x_n] = [(x_1, \cdots, x_{n-1}), x_n]$.

DEFINITION 1.8: *Let* $\Gamma_1(G) = G$ *for some group G. Let* $\Gamma_1(G) = G$ *and* $\Gamma_k(G) = \{[x_1, ..., x_k]\}$ = *the group generated by the commutator, for arbitrary* $x_i \in G$. *The series* $G = \Gamma_1(G) > \Gamma_2(G) > \cdots$ *is called the* lower central series *of G while* $<1> < Z_1(G) < Z_2(G) < \cdots$ *is called the* upper central series *of G, where we define* $Z_{i+1}(G)$ *by the rule:* $Z_{i+1}(G)/Z_i(G)$ *is the center of* $G/Z_i(G)$.

DEFINITION 1.9: *A* characterisitic subgroup *is a subgroup of a given group which is mapped by all automorphisms of the group onto itself. The center is an example of a characterisitic subgroup. A characteristic subgroup is a normal subgroup.*

These definitions will be needed throughout the paper. Next we state some theorems:

FUNDAMENTAL THEOREM OF FINITE ABELIAN GROUPS: *Every finite abelian group is a direct product of cyclic groups of prime-power order. Moreover, the factorization is unique except for rearrangement of the factors.*

The proof of this theorem is extremely lengthy and will not be presented here. We refer the reader to Gallian, chapter 11, for a complete proof.

LAGRANGE'S THEOREM (Borowski & Borwein; [1], page 328-9): *The order of every finite group G is equal to the product of the order of any subgroup H of G and the number of cosets of H in G (the index of H in G), that is,*

$$|G| = |H| \cdot |G{:}H|.$$

Proof (Kletzing; [3], page 116): Let $m = |G{:}H|$ and let $H, x_2H, \ldots, x_mH$ stand for the distinct left cosets of H in G. Let the permutation $g \to xg$ map H to the left coset xH. Thus, the cosets of H are in 1-1 correspondence with each other and hence must have the same cardinality. Since every element of G lies in some left coset and distinct left cosets are disjoint, it follows that $|G| = |H| + |x_2H| + \cdots + |x_mH|$. But all of these cosets contain the same number of elements as H. Hence, $|G| = m|H| = |H||G:H|$. ■

This paper will then be a classification of those groups which satisfy the converse of Lagrange's theorem. That is, we will discuss which groups exhibit the characteristic that given the order n of a group G, a subgroup H of G with order m can be found for all divisors m of n. This includes but is not limited to the finite classes of cyclic groups, abelian groups, p-groups, nilpotent groups, and supersolvable groups. We will denote this class of groups as "converse Lagrange."

It is known that p-groups satisfy this property, and in the course of our study we shall in fact need

SYLOW'S FIRST THEOREM (Gallian; [2], page 352-3): *Let G be a finite group and let p be a prime. If p^k divides $|G|$, then G has at least one subgroup of order p^k.*

Proof: We proceed by induction on $|G|$. If $|G| = 1$, it is trivially true. Now assume that the statement is true for all groups of order less than $|G|$. If G has a proper subgroup H such that p^k divides $|H|$, then, by our inductive assumption, H has a subgroup of order p^k and we are done. Thus, we may henceforth assume that p^k does not divide the order of any proper subgroup of G. Next, consider the class equation for G in the form

$$|G| = |Z(G)| + \Sigma|G:C(a)|,$$

where we sum over a representative of each conjugacy class cl(a), where $a \notin Z(G)$. Since p^k divides $|G| = |G:C(a)||C(a)|$ and p^k does not divide $|C(a)|$, we know that p must divide $|G:C(a)|$ for all $a \notin Z(G)$. It then follows from the class equation that p divides $|Z(G)|$. The Fundamental Theorem of Finite Abelian Groups then guarantees that Z(G) contains an element of order p, say, x. Since x is in the center of G, $\langle x \rangle$ is a normal subgroup of G, and we may form the factor group $G/\langle x \rangle$. Now observe that p^{k-1} divides $|G/\langle x \rangle|$. Thus, by the induction hypothesis, $G/\langle x \rangle$ has a subgroup of order p^{k-1} and thus this subgroup has the form $H/\langle x \rangle$, where H is a subgroup of G. Finally, note that $|H/\langle x \rangle| = p^{k-1}$ and $|\langle x \rangle| = p$ imply that $|H| = p^k$, and this completes the proof. ■

REFERENCES

[1] E.J. Borowski and J.M. Borwein, *The Harper Collins Dictionary of Mathematics,* New York: Harper Perrenial, 1991.

[2] Joseph A. Gallian, *Contemporary Abstract Algebra*, 3rd ed., Lexington, Mass.: D.C. Heath and Co., 1994.

[3] Dennis Kletzing, *Abtract Algebra,* New York: Harcourt Brace Jovanovich, 1991.

CHAPTER TWO

DEFINITIONS

DEFINITION 2.1: *A group G is* nilpotent *if it contains a chain* $<1> = H_0 < H_1 < \cdots < H_n = G$ *with* H_i/H_{i-1} *equal to the center of* G/H_{i-1}.

DEFINITION 2.2: *A group G is* supersolvable *if it contains a chain* $\langle 1\rangle = H_0 < H_1 < \cdots < H_n = G$ *of subgroups normal in G such that* H_i/H_{i-1} *is cyclic for* $i = 1, 2, \cdots, n$.

DEFINITION 2.3: *A group G is* polycyclic *if it contains a chain* $<1> = H_0 < H_1 < \cdots < H_n = G$ *of subgroups with each* H_{i-1} *normal in* H_i *and* H_i/H_{i-1} *cyclic.*

Thus, both supersolvable and polycyclic groups contain a chain with cyclic factors with the difference being that in a supersolvable group the chain is normal while in a polycyclic group the chain need only be sub-normal.

DEFINITION 2.4: *A group G is* solvable *if it contains a sub-normal series in which every normal factor, i.e. every H_i/H_{i-1}, is abelian.*

P-GROUPS ⊂ NILPOTENT ⊂ SUPERSOLVABLE ⊂ POLYCYCLIC ⊂ SOLVABLE

In order to prove our first theorem, we will need a couple of lemmas.

LEMMA 2.1 (Hall; [1], page 14): *Let H be a subgroup in G. The number of conjugates of H under G is the index in G of the normalizer of H in G, $[G:N_G(H)]$.*

Proof: Write $N_G(H) = D$ for brevity and let

$$G = D + Dx_2 + \cdots + Dx_r. \quad r = [G:D].$$

Then $x^{-1}Hx = y^{-1}Hy$, $x, y \in G$ if, and only if,

$$H = (yx^{-1})^{-1}H(yx^{-1});$$

that is, $yx^{-1} \in D$ or $y \in Dx$. Hence two conjugates of H under G are the same if, and only if, the transforming elements belong to the same left coset of D. Hence the number of distinct conjugates is the index of D in G. ■

LEMMA 2.2 (Hall; [1], page 47-48): *The center of a finite p-group is greater than the identity alone.*

Proof: If P is a finite p-group, let us write P as a sum of conjugacy classes:

$$P = C_1 + C_2 + \cdots + C_r.$$

Here C_1 consists of the identity alone. Let h_i be the number of elements in C_i, which by Lemma 2.1 is the index of a subgroup of P and so is either 1 for an element of the center or is otherwise a power of p. But if P is of order p^m we must have

$$p^m = h_i + h_2 + \cdots + h_r. \qquad (*)$$

Here $h_1 = 1$, and consequently in (*) the remaining h's cannot all be proper powers of p and so there must be further h's equal to 1, so that the center of P is greater than the identity alone. ■

THEOREM 2.1 (Hall; [1], page 155): *The class of p-groups is included in the class of nilpotent groups.*

Proof: By Lemma 2.2, every finite p-group P has center different from the identity. Hence the upper central series for P terminates with the entire group, which gives the characteristic series. Thus, P is nilpotent. ■

THEOREM 2.2 (Hall; [1], page 152-3): *The class of nilpotent groups is included in the class of supersolvable groups.*

Proof: Let G be finitely generated and nilpotent. Let its lower central series be

$$G = \Gamma_1(G) > \Gamma_2(G) > \cdots > \Gamma_c(G) > \Gamma_{c+1}(G) = <1>$$

Since $\Gamma_c(G)$ is Abelian and finitely generated, it is the direct product of, say, m cyclic groups. Also since $\Gamma_c(G)$ is in the center of G, any subgroup of it is normal in G. Thus there is a chain

$$\Gamma_{c+1} = 1 < \{a_1\} < \{a_1, a_2\} < \cdots < \{a_1, a_2, \cdots, a_m\} = \Gamma_c(G),$$

all being normal subgroups of G and having the property that the factor group of consecutive groups is cyclic. Similarly, we may insert normal sub-groups between $\Gamma_{i+1}(G)$ and $\Gamma_i(G)$, with the property that the factor group of consecutive groups is cyclic. In this way we find a series for G which is the defining property for G to be supersolvable. ■

THEOREM 2.3: *The class of supersolvable groups is included in the class of polycyclic groups.*

Proof: Let $<1> = H_o < H_1 < \cdots < H_n = G$ be a chain in group G. Let each $H_i \lhd G$. Then for $g \in G$, $h_i \in H_i$, we have $g^{-1}h_ig$ is in H_i, and $g^{-1}h_{i-1}g$ is in H_{i-1}. Also,

$$(g^{-1}h_ig)^{-1}g^{-1}h_{i-1}g(g^{-1}h_ig) =$$

$$g^{-1}h_i^{-1}gg^{-1}h_{i-1}gg^{-1}h_ig =$$

$$(h_ig)^{-1}h_{i-1}(h_ig) \in H_{i-1}$$

since $h_i g \in G$ and $(h_i g)^{-1} \in G$ and $H_{i-1} \lhd G$. Thus we see that a normal series implies a subnormal series and hence that supersolvable implies polycyclic. ■

THEOREM 2.4: *The class of polycyclic groups is included in the class of solvable groups.*

Proof: Let G be a polycyclic group. Let $xH, yH \in H_i/H_{i-1}$ and aH a generator for H_i/H_{i-1}. Then $xH = (aH)^n$ and $yH = (aH)^m$ for some integers n,m, and therefore $xHyH = xyH = a^n a^m H = a^{n+m}H = a^m a^n H = yxH$ $=yHxH$. Hence, H_i/H_{i-1} is abelian and G is solvable. ■

Thus we obtain the desired series of inclusions: p-groups $\subset$ nilpotent $\subset$ supersolvable $\subset$ polycyclic $\subset$ solvable, where all groups considered are finite. Our next step will be to establish a cut-off point for those groups which are converse Lagrange. This point occurs between supersolvable and polycyclic groups. Thus, supersolvable groups (and the groups preceding them in the series of inclusions) will be included in the class of converse lagrange groups while polycyclic groups (and the groups following them) will not.

REFERENCE

[1] Marshall Hall, *The Theory of Groups*, New York: Macmillan, 1959.

CHAPTER THREE

SUPERSOLVABLE $\subset$ CONVERSE LAGRANGE

In order to prove that the class of supersolvable groups is included in the class of converse Lagrange groups, we will need several lemmas which outline the basic properties of supersolvable groups.

LEMMA 3.1: (Hall; [4], page 30-31): *Given a group G, suppose S is a subgroup of G and T a normal subgroup. Then $S \cap T$ is a normal subgroup of S and the factor groups ST/T (where ST denotes all elements of the form st for $s \in S$ and $t \in T$) and $S/S \cap T$ are isomorphic.*

Proof: Let $u \in S \cap T$, $s \in S$. Then $s^{-1}us \in S$. Also, since T is normal in G and $u \in T$, $s^{-1}us \in S \cap T$, and so $S \cap T$ is normal in S.

Let us write $D = S \cap T$ for brevity.

$$S = D + Ds_2 + \cdots + Ds_r \qquad (*)$$

Then we assert

$$ST = T + Ts_2 + \cdots + Ts_r \qquad (**)$$

using the same coset representatives in (**) as in (*). Here, if $Ts_i = Ts_j$, then $s_is_j^{-1} \in T$. But $s_is_j^{-1} \in S$, whence

$s_i s_j^{-1} \in S \cap T = D$, contrary to (*). Hence the cosets Ts_i in (**) are all distinct. Moreover, since T is a normal subgroup, ST is of the form $Ts = Tds_i$, with $s = ds_i$ from (*). But as $d \in T$, $Tds_i = Ts_i$, and so the cosets in (**) will exhaust ST. The correspondence

$$Ds_i \rightleftarrows Ts_i \qquad (***)$$

is a 1-1 correspondence between the cosets in (*) and (**), and thus a 1-1 correspondence between the elements of S/D and those of ST/T. Also, if $s_i s_j = ds_k$ with $d \in D$, since $D \leq T$, we shall have both $Ds_iDs_j = Ds_k$ and $Ts_iTs_j = Ts_k$. Thus the rule (***) is an isomorphism between the factor groups S/D and ST/T. ■

LEMMA 3.2 (Hall; [4], page 158): *Subgroups and factor groups of supersolvable groups are supersolvable.*

Proof: Let G be supersolvable and

$$\langle 1 \rangle = H_o < H_1 < \cdots < H_n = G$$

be a normal series with every H_i/H_{i-1} a cyclic group. Let us define a homomorphism $\eta : G \to G/K$ by $\eta(H_i) = B_i$. Then, for a factor group $G/K = T$, the homomorphic images B_i of the H_i will form a normal series $\langle 1 \rangle = B_0 < B_1 < \cdots < B_n = T$ where, if we delete repetitions of the same group, consecutive terms B_i, B_{i-1} will have a cyclic factor group B_i/B_{i-1}, since every homomorphic image of a cyclic group is cyclic or the identity. For a subgroup A take

$$\langle 1 \rangle = C_0 < C_1 < \cdots < C_n = A,$$

where $C_i = A \cap H_i$. For every i, $A \cap H_i$ is normal in A, and by Lemma 3.1, we have

$$C_{i+1}/C_i = A \cap H_{i+1}/A \cap H_i \cong H_i(A \cap H_{i+1})/H_i.$$

But the right-hand side of this is a subgroup of H_{i+1}/H_i, and hence cyclic or the identity. Thus C_{i+1}/C_i is cyclic or the identity, and so A is supersolvable. ■

LEMMA 3.3 (Hall; [4], page 159): *A supersolvable group G has a normal series*

$$<1> = H_0 < H_1 < \cdots < H_n = G,$$

in which every H_i/H_{i-1} is either infinite cyclic or cyclic of prime order, and if H_i/H_{i-1} and H_{i+1}/H_i are of prime orders p_i and p_{i+1}, we have $p_i \leq p_{i+1}$.

Proof: Let $<1> = A_0 < A_1 < \cdots < A_n = G$ be a normal series, with each A_i/A_{i-1} cyclic. If A_j/A_{j-1} is of finite order $p_1p_2\cdots p_s$, where $p_1, p_2, \cdots, p_s$ are primes (not necessarily distinct), then A_j/A_{j-1} has unique cyclic subgroup of each of the orders $p_1, p_1p_2, \cdots, p_1\cdots p_{s-1}$, and these are characteristic subgroups. Hence the s-1 corresponding subgroups between A_{j-1} and A_j are normal in G, and the factor groups of consecutive groups are cyclic of prime order. Refining in this way every factor group A_j/A_{j-1} of finite order, we obtain the normal series of the theorem in which every factor group is either infinite cyclic or cyclic of prime order.

Now to prove the second assertion, if A_i/A_{i-1} and A_{i+1}/A_i are of prime orders q and p, respectively, with $q > p$, then A_{i+1}/A_{i-1} is of order pq, with $p < q$, and this has a characteristic subgroup of order q whose inverse image A_i^{-1} will be normal in G. If we replace A_i by A_i^{-1}, then A_{i+1}/A_i^{-1} will be of order p and A_i^{-1}/A_{i-1} will be of order q. Continuing this process, which does not alter the length of the normal series, we shall ultimately get a series in which the orders of consecutive factor groups of prime order do not increase in magnitude, as stated in the theorem. ■

COROLLARY: *If G is a finite supersolvable group of order*

$p_1 p_2 \cdots p_r$, where $p_1 \leq p_2 \leq \cdots \leq p_r$ are primes, then G has a normal series $\langle 1 \rangle = H_1 < H_2 < \cdots < H_r = G$ in which every H_i is a maximal normal subgroup contained in H_{i+1}, where H_{i+1}/H_i is of order p_i.

LEMMA 3.4 (Hall; [4], page 141): *If S is a solvable group of order mn, (m, n) = 1, then S contains a sub-group of order m.*

Proof: Note that if $m = p^a$, a prime power, then the property is given by the first Sylow theorem. We shall proceed by induction on the order of G, being trivially true if the order of G is a power of a prime.

Case 1: Let G have a proper normal subgroup H of order $m_1 n_1$ and index $m_2 n_2$, where $m = m_1 m_2$, $n = n_1 n_2$, and $n_1 < n$. Then G/H by induction contains a subgroup of order m_2 which corresponds to a subgroup D of G or order mn_1. D by induction contains a subgroup of order m.

Case 2: Let G contain a unique minimal normal subgroup K (where "minimal" denotes the least in the chain) of order $n = p^a$. Then let L be a minimal normal subgroup properly containing K. Then L/K is of order q^b with $q \neq p$. Let Q be a Sylow subgroup of L of order q^b, and let M be the normalizer of Q in G. Consider $M \cap K = T$. T is a normal subgroup of M and, as a subgroup of K, is elementary Abelian (in other words, K is a direct product of cyclic groups each of order some p, a prime). Every element of T permutes with every element of Q, since a commutator of an element in Q and an element in T lies in $T \cap Q = 1$. Hence T belongs to the center C of L, which, as a characteristic subgroup of L, is a normal subgroup of G. Since K is minimal and unique, $C = K$ or $C = 1$. If $C = K$, then $L = K \times Q$, and Q is a normal subgroup of G contrary to the uniqueness of K. Hence $T = C = 1$. Thus Q is its own normalizer in L and has as many conjugates in L as its index in L; that is, Q has $n = p^a$ conjugates in L. Any conjugate of Q in G lies in L, since L is normal. Hence Q has $n = p^a$ conjugates in G, whence M is of index $n = p^a$ in G and hence of order m. ■

THEOREM 3.1 (Deskins [2]): *A group G is supersolvable if and only if each subgroup $H \leq G$ contains a subgroup of order d for each divisor d of $|H|$.*

The following alternate formulation, clearly equivalent, is more easily treated.

THEOREM 3.1′ (Deskins [2]): *A group G is supersolvable if and only if each subgroup $H \leq G$ contains a subgroup of index p for each prime divisor p of $|H|$.*

Proof. First suppose that G is supersolvable. Then the following facts are known about G:

(i) Its subgroups and factor groups are supersolvable (Lemma 3.2).

(ii) If p is the maximal prime factor of $|G|$ and p^r is the order of the p-Sylow subgroups of G, then G contains a normal subgroup P of order p^{r-1}. Moreover the p-Sylow subgroup G_p of G is normal (corollary to Lemma 3.3).

We prove the necessity now by induction on $|G|$. If H is a proper subgroup of G then it is supersolvable by (i) and so by the induction hypothesis contains a subgroup of prime index for each prime factor of $|H|$. Thus we need only show that G contains a subgroup of index q whenever q is a prime factor of $|G|$. If $q < p$, consider the group $G' = G/G_p$, where G_p is the subgroup described in (ii). By (i) G′ is supersolvable and since $|G'| < |G|$, G′ contains a subgroup K′ of index q. Clearly K, the pre-image of K′ in G, has index q. If on the other hand $q = p$, then we use Lemma 3.4.

Since G is certainly solvable it contains a subgroup T of order $|G|/p^r$. Then the subgroup PT has index p in G.

The sufficiency is proved by induction on the order of G also. Let q be the least prime factor of $|G|$. By hypothesis, G contains a subgroup K of index q in G, and by a standard exercise (Burnside; [1], page 45) K is normal in G. Since a subgroup of K is also a subgroup of G we conclude form the induction hypothesisthat K is supersolvable. Then from (ii) above we know that the p-Sylow subgroup K_p of K is normal in

K and hence also in G, where p is the maximal prime factor of $|G|$. If $p = q$ then G is a p-group and hence supersolvable, so we need only consider the case $q \neq p$. Then K_p is the p-Sylow subgroup of G.

Let M be a minimal normal subgroup of G which lies in the center of K_p. We shall show that $M = p$. By hypothesis G contains a subgroup N of index p. Then either $N \cap M = \langle 1 \rangle$ or $N \cap M = M$ since M is abelian and minimal normal in G. If $N \cap M = \langle 1 \rangle$ then $|M| = |G|/|N| = p$, so suppose $N \cap M = M$. Now by the induction hypothesis N is supersolvable so that M contains a subgroup M_1 which is normal in N and of order p. Furthermore, since N has index p in G there exists and element x of K_p which lies outside of N. But $M_1 \leq M \leq$ center of K_p, so that x normalizes M_1. Thus M_1 is normal in G and $M_1 = M$ because of the minimality of M. So G contains a normal subgroup M of order p in all cases.

Now consider $G^* = G/M$. If G^* is supersolvable then it is clear from the definition above that G is supersolvable. To prove G^* supersolvable we shall show that each subgroup H^* of G^* contains a subgroup of prime index for every prime divisor of $|H^*|$. If H is a proper subgroup of G then its pre-image H in G is also proper and, as noted above, supersolvable. Then H^* is supersolvable by (i) and so contains the required subgroups. Consequently we need only show that G^* contains the required maximal subgroups.

If r is a prime factor of $|G^*|$ then r divides $|G|$. Therefore G contains a subgroup R of index r. If $R \geq M$ then $R^* = R/M$ has the same index in G^* as R does in G. If $R \ngeq M$ then $R \cap M = \langle 1 \rangle$ as above, so that $G = RM$ and $G^* = RM/M \cong$. Since R is a proper subgroup of G it is supersolvable by induction, and G^* isomorphic with R clearly implies that G^* has all the required subgroups. Thus in all cases G^* is supersolvable, proving that G is supersolvable.

It should be noted that a group G is not necessarily supersolvable if the condition, "each subgroup $H \leq G$ contains a subgroup of every possible order," is replaced by "G contains a subgroup of every possible order." ∎

POLYCYCLIC $\not\subset$ CONVERSE LAGRANGE

Our next important fact is that the class of polycyclic groups is NOT included in the class of converse Lagrange groups. A counter example is given by the group A_4, the alternating group of even permutations of $\{1, 2, 3, 4\}$. Note that "the group A_4 possesses a polycyclic series

$1 < \{1, (12)(34)\} < \{1, (12)(34), (13)(24), (14)(23)\} < A_4$

with factors of orders 2, 2, 3. This series is not normal (since the subgroup $<(12)(34)>$ is not normal in A_4) although it is subnormal" (Kargapolov and Merzljakov; [5], page 34).

Since A_4 has order 12, it is sufficient to prove that it has no subgroups of order 6 and, thus, is not converse Lagrange.

THEOREM 3.2 (Gallian; [3], page 139): *The group A_4 has no subgroups of order 6.*

Proof: By the following table A_4 has eight elements of order 3. Suppose that H is a subgroup of order 6. Let a be any element of order 3 in A_4. Since H has index 2 in A_4, at most two of the cosets H, aH, and a^2H are distinct. But equality of any pair of these three implies that $aH = H$, so that $a \in H$. (For example, if $H = a^2H$, multiply on the left by a.) Thus, a subgroup of A_4 of order 6 would have to contain eight elements of order 3, which is absurd. ■

TABLE (Gallian; [3], page 94): The Alternating Group of Even Permutations of {1, 2, 3, 4}

(In this table, the permutations of A_4 are designated as $\alpha_1, \alpha_2, \ldots, \alpha_{12}$ and an entry K inside the table represents α_K. For example, $\alpha_3 \alpha_8 = \alpha_6$.)

	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}
$(1) = \alpha_1$	1	2	3	4	5	6	7	8	9	10	11	12
$(12)(34) = \alpha_2$	2	1	4	3	6	5	8	7	10	9	12	11
$(13)(24) = \alpha_3$	3	4	1	2	7	8	5	6	11	12	9	10
$(14)(23) = \alpha_4$	4	3	2	1	8	7	6	5	12	11	10	9
$(123) = \alpha_5$	5	8	6	7	9	12	10	11	1	4	2	3
$(243) = \alpha_6$	6	7	5	8	10	11	9	12	2	3	1	4
$(142) = \alpha_7$	7	6	8	5	11	10	12	9	3	2	4	1
$(134) = \alpha_8$	8	5	7	6	12	9	11	10	4	1	3	2
$(132) = \alpha_9$	9	11	12	10	1	3	4	2	5	7	8	6
$(143) = \alpha_{10}$	10	12	11	9	2	4	3	1	6	8	7	5
$(234) = \alpha_{11}$	11	9	10	12	3	1	2	4	7	5	6	8
$(124) = \alpha_{12}$	12	10	9	11	4	2	1	3	8	6	5	7

FINITE CYCLIC GROUPS AND ABELIAN GROUPS ARE CONVERSE LAGRANGE

THEOREM 3.3 (Kletzing; [6], page 72): *Let G be a finite cyclic group of order n. Then G has a unique subgroup of order d for every divisor d of n.*

Proof: Let d be any positive divisor of n. To construct a subgroup of order d, we first set n = qd for some integer q and observe that $|x^q| = n/\gcd(q,n) = qd/\gcd(q,qd) = d$. Therefore, $\langle x^q \rangle$ is a subgroup of order d. To show that $\langle x^q \rangle$ is the only subgroup of order d, let H be a subgroup of G such that |H| =

d. Since H is cyclic, $H = \langle x^s \rangle$ for some positive integer s, and hence $d = |H| = |x^s| = n/\gcd(s,n) = qd/\gcd(s,n)$, from which it follows that $\gcd(s,n) = q$. Hence, q divides s, say $s = kq$. Then $x^s = \langle x^q \rangle^k \in \langle x^q \rangle$, and it follows that $H \leq \langle x^q \rangle$. Since $|H| = d = |x^q|$, we conclude that $H = \langle x^q \rangle$. Thus, G has a unique subgroup of order d, and the proof is complete. ■

THEOREM 3.4 (Gallian; [3], PAGE 188): *If m divides the order of a finite Abelian group G, then G has a subgroup of order m.*

Proof: Corollary to the Fundamental Theorem of Finite Abelian Groups.

REFERENCES

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CHAPTER FOUR

Finally, we present an example to show that the class of converse Lagrange groups is not included in the class of supersolvable groups.

EXAMPLE (Weinstein, [1], Page 128-9)): The Octahedral Group

Let $G = G(X,Y \mid X^3 = 1, Y^2 = 1, (XY)^4 = 1)$

The proof of the fact that the Octahedral group satisfies the desired property is extremely lengthy and so we refer the reader to Weinstein's *Examples of Groups* [1] for a complete proof; however, we shall outline some of the involved facts here.

The Octahedral Group, G, is isomorphic to the symmetric group on four letters, S_4. Thus, its cardinality is equal to 24. It clearly has subgroups of cardinalities 1 and 24. Let SL(2,F) signify the group of 2 by 2 matrices of determinant 1 with entries in F. Let PSL(2,F) = SL(2,F)/Z(SL(2,F)). Then PSL(2,Z_3) can be embedded in G. Since PSL(2,Z_3) has subgroups of cardinalities 2,3,4, and 12, so too does G. Finally, S_3 and D_4 can be embedded in G, and thus G has subgroups of cardinalities 6 and 8 respectively.

For the proof that G is not supersolvable we refer you again to Weinstein, examples 4.6 and 4.9, which show that PSL(2,Z_3) can be embedded into G and that PSL(2,Z_3) is not supersolvable. Thus, G is not supersolvable.

G is an important counterexample since it separates the classes of supersolvable and converse lagrange groups. It further shows that the class of converse lagrange groups is not subgroup closed since its derived group is not converse Lagrange (Weinstein; [1], page 131).

CONCLUSION

In sum, we see that finite cyclic groups, abelian groups, p-groups, nilpotent group, and supersolvable groups all satisfy the converse of Lagrange's theorem. However, we also see that these do not comprise the whole of this class. We also see that polycyclic groups, solvable groups, and arbitrary groups are not necessarily contained in this class and that the class of converse Lagrange groups is not subgroup closed.

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