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Anti-swing control of overhead cranes

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ANTI-SWING CONTROL OF OVERHEAD CRANES

by

Murali Srinivas Kathari

A thesis submitted in partial fulfillment
of the requirements for the degree of

Master of Science

in

Mechanical Engineering

Department of Mechanical Engineering
University of Nevada, Las Vegas
December 1995
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ABSTRACT

Overhead cranes are widely used for transporting large and heavy suspended objects such as shipping containers. Acceleration and deceleration of the crane generally induce swinging motion in the suspended payload. A method is presented to find the trajectory for an overhead crane that will ensure the transfer of the payload in the shortest time and with minimum swinging along a specified path. The overhead crane and the suspended payload are modeled as a double pendulum with motion in three dimensional space. The equations of motion of the overhead crane and the payload are transformed in terms of a single path parameter which represents single degree of motion along the path. The resulting set of equations defines the phase space of admissible motion constrained by the path geometry and the forces exerted by the crane. By applying dynamic programming principles to the transformed set of equations of motion, the trajectory with the shortest time and with minimum swinging is determined.
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CHAPTER 1

INTRODUCTION

Overhead cranes are widely used in industrial plants, warehouses, harbors and construction sites where large and heavy loads have to be transferred over long distances. Usually they are manually controlled and the skill of the operator is relied upon for safe and efficient transport of the payload. Acceleration or deceleration of the crane induce undesirable swinging of the suspended payload. This swinging decreases the safety and the operational efficiency of the system. The swinging and total reliance on the human operator skills also inhibits remote transport of the payload.

Primary approaches towards providing robust overhead crane control are feedback control and programmed open-loop control. Control through feedback involves real-time feedback of the payload swing angle, trolley position, velocity and acceleration. The following papers address the use of some type of feedback crane control. Caron et al. used speed control with desired bang-bang acceleration profile. The acceleration profile is based on the linearized period of the suspended object and is designed to result in swing-free stops of the transported payload. Ridout used variable damping linear feedback control combined with contour mapping. Damping of the feedback loop is changed as a function of the error
signal. Through this approach, faster transport times were achieved over simple feedback. Yoon et al.\textsuperscript{3}, used a hybrid control approach combining feedback control with a programmed velocity profile to provide swing-free transport and stop. The programmed velocity profile is divided into four periods and, like open loop methods, is designed to eliminate oscillation during the deceleration period. Moustafa and Ebied\textsuperscript{4} derived a nonlinear dynamical model of an overhead crane. A linearized state space model was obtained by perturbing the system about its equilibrium state. They also controlled motion in both the travel and transverse directions and considered rope twist. Aunreig and Toger\textsuperscript{5} considered hoisting of load in the time optimal control of overhead cranes, while providing swing-free stop. An optimal crane control strategy was presented by Vaha and Martinnen\textsuperscript{6} for a suspended mass modeled as a simple pendulum. The control scheme is a cascading type for minimizing swing. To avoid modeling errors and parameter sensitivities a minimum time criterion is used at the beginning of the motion and a quadratic criterion at the end. Virkkunen et al.\textsuperscript{7} presented studies of crane control using PID, minimum time, pole placement and adaptive control. They also considered the changes in the rope length. Yoshida and Kawabe\textsuperscript{8} proposed a saturating control law using a unique guaranteed cost control method. This control law is more effective in the sense of a quadratic cost than a linear cost. Yasunobu and Hasegawa\textsuperscript{9} applied predictive fuzzy control to a shipyard crane control problem using safety, stop-gap accuracy, minimum swinging, and minimum time as the performance criteria. A simple pendulum model was used and only motion in one plane was considered. Trabia and Nalley\textsuperscript{10} used a distributed fuzzy logic controller to achieve swing damped transport of suspended payloads carried by a overhead crane. A swing damped motion profile based on cubic
acceleration/deceleration profile is used and the controller is divided into displacement and swing controllers. The crane was modeled as a double pendulum to account for payloads that are considerably long and massive.

In programmed control approach, the trolley is forced to follow a desired trolley velocity trajectory that is precomputed to minimize payload swing. The following papers use some variation of programmed control. Starr\textsuperscript{11} used a path controlled manipulator to achieve swing free motion. The model is based on a simple pendulum using small angle approximations. The trajectory consists of an initial acceleration to an intermediate velocity, then a secondary acceleration to final velocity. The secondary acceleration is timed such that the swinging induced by it exactly cancels the swinging induced by the first one, thus resulting in swing free transport. This profile provides for both swing-free transport and swing-free stop. Jones et al.\textsuperscript{12} used two different acceleration profiles to move simply suspended payloads. The first method uses a constant acceleration profile, similar to that used by Starr\textsuperscript{11}. The second method uses a ramped acceleration profile. This profile offers some practical advantage over the constant acceleration profile, but provides only for swing-free stop and not swing-free transport. Werner et al.\textsuperscript{13} added to the above paper by introducing an adaptive swing-free planner. A batch nonlinear least square estimator is used to predict the parameters of the simply suspended object which are necessary for swing-free motion. Noakes and Jansen\textsuperscript{14} developed an open loop control strategy based on the natural frequency of the suspended object. They used a specific case from a general control technique involving shaping of inputs to dampen vibration.

Mason\textsuperscript{15} used an open loop optimal control algorithm with the objective of
minimizing time and providing a swing-free stop. This model is also based on a simple pendulum with a bang-bang acceleration profile. Karihaloo and Parbery\textsuperscript{16} used a simple pendulum based model to provide optimal control with the objective of minimizing energy and to provide swing-free stop, given the distance over which and the time in which the mass has to be transported. Sakawa and Shindo\textsuperscript{17} used optimal control to determine a trajectory that minimizes the swing of the suspended payload during transfer and provides swing-free stop. The total motion of the crane is divided in sections and hoisting is considered. Baharova \textit{et al.}\textsuperscript{18} used optimal control to determine energy optimal speed references for pilot crane systems. The path planning problem is divided into sections and hoisting is considered.

In most of the papers reviewed, the payload is modeled as a lumped mass single pendulum system. In many cases, such as construction sites and nuclear facilities, the payloads are of considerable length. In such cases, the payload cannot be modeled as a simple pendulum since its orientation can change with respect to the cable. Modeling the cable suspended payload as a double pendulum would be much more realistic in such cases. Therefore in this study, the overhead crane carrying the long cable suspended payload will be modeled as a double pendulum with motion in three dimensional space to account for the dynamic behavior of the payload. A overhead crane carrying a suspended payload can be considered similar to a two link mobile manipulator, with no actuators at the joints. Hence, the optimal control problem of overhead cranes can be considered similar to the optimal control problem of manipulators.

The following is a partial survey of the research in the area of optimal control path planning of manipulators. Khan and Roth\textsuperscript{27} were among the first to apply the principles of
optimal control to the problem of minimum time. Fisher and Mujtaba\textsuperscript{23} presented a graphical representation to determine the minimum path traversal time using bang-bang acceleration profile. Shiller and Dubowsky\textsuperscript{28} obtained the time-optimal trajectories for robotic manipulators considering the robot nonlinearities, its actuator saturation limits, and effect of obstacles. Tan and Potts\textsuperscript{24} proposed a minimum time trajectory planner for a discrete dynamic robot model, considering the joint torque, joint velocity and joint jerk constraints. Pfeiffer and Johanni\textsuperscript{19} presented a way to determine the optimal trajectory of an arbitrary manipulator following a prescribed path by transforming the manipulator degrees of freedom into path degrees of freedom. Shiller and Lu\textsuperscript{20, 21} presented a robust algorithm for computing time optimal trajectories of manipulators along specified paths, considering the effects of extreme velocity and acceleration. Huang and McClamroch\textsuperscript{22} the problem of time optimal control for a robotic contour following problem taking into account the inequality and the contact forces. Impact between the end effector and the work piece at entry time was also considered. Shin and McKay\textsuperscript{25} used dynamic programming principles to solve the problem of optimal trajectory planning of manipulators along a specified path. Shin and McKay\textsuperscript{26} derived a lower bound on the time to move a manipulator from one point to another, and determined the form of the path which minimizes this lower bound.

In all the papers mentioned above, the manipulator links were assumed to be rigid, thus neglecting any vibrations in the manipulator links. These vibrations, especially in cases of manipulators with light weight links or manipulators handling heavy loads, will effect the performance of the manipulator. Dissanayake and Phan-Thien\textsuperscript{29} proposed a method to derive near minimum time trajectories for positioning single link flexible robot arm such that there
is no residual structural vibration at the end of the move. Hecht and Junkins\textsuperscript{30} used a Liapunov controller to make the flexible manipulator track a reference maneuver while eliminating the flexible motion. Bang-bang control of a rigid link is used to generate the reference control torques.

The objective of this study is to determine a trajectory, using dynamic programming principles, which minimizes the swinging of the payload and the time for an overhead crane to travel between two points along a specified path. In chapter 2, mathematical model for a overhead crane carrying a long cable suspended payload will be developed as a double pendulum with motion in three dimensional space. In chapter 3, a method for determining optimal trajectories for manipulators will be presented. This method will be then applied to the overhead crane problem with some modifications in chapter 4 and a numerical example is presented. And the finally in chapter 5 the significance of the results will be discussed.
CHAPTER 2

MATHEMATICAL MODELING OF OVERHEAD CRANE

In this chapter, mathematical model for the overhead crane carrying a suspended payload as shown in Figure 1 is developed. The crane is assumed to be rigid and both the trolley and the crossbeam are assumed to run on straight frictionless rails. The payload used here is long and cylindrical. It is assumed that the payload does not twist during motion and that it is suspended on permanently taut inextensible and massless rope. The presence of external forces such as the wind forces is not considered. The crane motion can be described using six degrees of freedom which are,

\[ q = (d_1, d_2, \alpha_1, \beta_1, \alpha_2, \beta_2) \]

2.1

Table 1 and Figure 2 and 3 show the notations for the dynamical model. Since it is assumed that the payload does not twist during motion and there are no external forces acting on the payload, the swinging of the payload in the \( x-z \) plane and the \( x-y \) plane are independent of each other.
Figure 1. Schematic Diagram of an Overhead Crane Carrying Suspended Payload.
Table 1. Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_1$</td>
<td>cross-beam displacement in the X-Z plane</td>
</tr>
<tr>
<td>$d_2$</td>
<td>trolley displacement in the X-Y plane</td>
</tr>
<tr>
<td>$F_1$</td>
<td>force needed to move the cross-beam</td>
</tr>
<tr>
<td>$F_2$</td>
<td>force needed to move the trolley</td>
</tr>
<tr>
<td>$g$</td>
<td>gravitational acceleration</td>
</tr>
<tr>
<td>$l_p$</td>
<td>length of payload</td>
</tr>
<tr>
<td>$l_r$</td>
<td>length of rope</td>
</tr>
<tr>
<td>$m_b$</td>
<td>mass of cross-beam</td>
</tr>
<tr>
<td>$m_t$</td>
<td>mass of trolley</td>
</tr>
<tr>
<td>$r_p$</td>
<td>location of a point along the payload axis</td>
</tr>
<tr>
<td>$T$</td>
<td>kinetic energy</td>
</tr>
<tr>
<td>$U$</td>
<td>potential energy</td>
</tr>
<tr>
<td>$\alpha_1$</td>
<td>rotation of the rope about the vertical in X-Y plane</td>
</tr>
<tr>
<td>$\alpha_2$</td>
<td>rotation of the payload w.r.t the rope in X-Y plane</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>rotation of the rope about the vertical in X-Z plane</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>rotation of the payload w.r.t the rope in X-Z plane</td>
</tr>
<tr>
<td>$\rho$</td>
<td>linear density of the payload</td>
</tr>
</tbody>
</table>
2.1 MOTION IN X-Z PLANE

The motion of the crossbeam and the payload in the x-z plane can be described using \(d_1, \beta_1,\) and \(\beta_2\) as the generalized coordinates. For a point \(P,\) on the payload axis, in the x-z plane as shown in Figure 2 displacement in \(x\) direction is,

\[x = - l_r \cos(\beta_1(t)) - r_p \cos(\beta_1(t) + \beta_2(t))\]  \(2.2\)

and the displacement in \(z\) direction is,

\[z = l_r \sin(\beta_1(t)) + r_p \sin(\beta_1(t) + \beta_2(t)) + d_1(t)\]  \(2.3\)

Differentiating equation 2.2 and 2.3 with respect to \(t,\)

\[
\dot{x} = l_r \sin(\beta_1(t)) \dot{\beta}_1(t) + r_p \sin(\beta_1(t) + \beta_2(t)) (\dot{\beta}_1(t) + \dot{\beta}_2(t)) \]
\[2.4\]

\[
\dot{z} = l_r \cos(\beta_1(t)) \dot{\beta}_1(t) + r \cos(\beta_1(t) + \beta_2(t)) (\dot{\beta}_1(t) + \dot{\beta}_2(t)) + \dot{d}_1(t) \]
\[2.5\]

Velocity in the x-z plane is,

\[\mathbf{v}_{xz}^2 = \dot{x}^2 + \dot{z}^2\]  \(2.6\)

The kinetic energy \((T)\) of the crossbeam and the payload is given by,
FIGURE 2. Motion of Payload in X-Z Plane.
\[ T = \frac{1}{2} \left( (m_r + m_b) \dot{\alpha}_1^2(t) + \int_0^{l_p} \rho \, v_{\infty}^2 \, dr_p \right) \quad 2.7 \]

substituting equation 2.4 and 2.5 into equation 2.7,

\[ T = \left[ \frac{1}{6} \rho \, I_p^3 + \frac{1}{2} \rho \, l_r \, l_p \left( l_p \cos(\beta_1(t)) \cos(\beta_1(t) + \beta_2(t)) \right) \right] \beta_1^2(t) \]
\[ + \, \rho \, I_p^2 \, \dot{\beta}_1(t) \, \dot{\beta}_2(t) \left[ \frac{1}{2} l_r \sin(\beta_1(t)) \sin(\beta_1(t) + \beta_2(t)) + \frac{2}{3} l_p \right] \]
\[ + \, \rho \, I_p \, \dot{\beta}_1(t) \, \dot{d}_1(t) \cos(\beta_1(t) + \beta_2(t)) \]
\[ + \, \rho \, I_p^2 \, \dot{\beta}_2(t) \, \dot{d}_1(t) \cos(\beta_1(t) + \beta_2(t)) \]
\[ + \, \frac{1}{2} \, d_1^2(t) [m_b + m_r + \rho l_p] \quad 2.8 \]

The potential energy \((U)\) is given by,

\[ U = -g \, \rho \int_0^{l_p} \left[ l_r \cos(\beta_1(t)) + r_p \cos(\beta_1(t) + \beta_2(t)) \right] \, dr_p \quad 2.9 \]

\[ U = -g \, \rho \left[ l_r \, l_p \cos(\beta_1(t)) + \frac{l_p^2}{2} \cos(\beta_1(t) + \beta_2(t)) \right] \quad 2.10 \]

and the force matrix, \(Q_i\) is given by,

\[ Q_i = \begin{pmatrix} F_1 \\ 0 \\ 0 \end{pmatrix} \quad 2.11 \]
Using Lagrangian dynamics the equations of motion can be derived as,

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = \mathcal{Q}_i
\]  \hspace{1cm} 2.12

Using small angle approximation, \textit{i.e.,} assuming,

\[
\sin(\beta_1(t)) = \beta_1(t) \quad \sin(\beta_2(t)) = \beta_2(t) \\
\cos(\beta_1(t)) = 1 \quad \cos(\beta_2(t)) = 1
\]  \hspace{1cm} 2.13

the resulting equations of motion are,

\[
(m_b + m_i + \rho \ l_p) \ddot{d}_1 + (0.5 \ \rho \ l_p^2 + \rho \ l_r \ l_p) \ \dot{\beta}_1(t) + 0.5 \ \rho \ l_p^2 \ \dot{\beta}_2(t) = F_1
\]  \hspace{1cm} 2.14

\[
\rho \ l_p(\frac{1}{2} l_p^2 l_r + l_r) \ \ddot{d}_1(t) + \rho \ l_p(\frac{1}{3} l_p^2 l_r^2 + l_r l_r) \ \dot{\beta}_1(t) + \rho \ l_p^2(\frac{1}{2} l_p^2 + \frac{1}{3} l_r) \ \dot{\beta}_2(t) \\
+ \rho \ g \ l_p(\frac{1}{2} l_p^2 l_r) \ \dot{\beta}_1(t) + \frac{1}{2} \rho \ g \ l_p^2 \ \dot{\beta}_2(t) = 0
\]  \hspace{1cm} 2.15

\[
\frac{1}{2} \rho \ l_p^2 \ \ddot{d}_1(t) + \rho \ l_p^2(\frac{1}{2} l_r^2 + \frac{1}{3} l_r) \ \dot{\beta}_1(t) + \frac{1}{3} \rho \ l_p^3 \ \dot{\beta}_2(t) \\
+ \frac{1}{2} \rho \ g \ l_p^2 \ \dot{\beta}_1(t) + \frac{1}{2} \rho \ g \ l_p^2 \ \dot{\beta}_2(t) = 0
\]  \hspace{1cm} 2.16

These equations can be written in matrix form as,
\[
[M_1] \begin{bmatrix} \ddot{d}_1(t) \\ \dot{\beta}_1(t) \\ \dot{\beta}_2(t) \end{bmatrix} + [K_1] \begin{bmatrix} \ddot{d}_1(t) \\ \dot{\beta}_1(t) \\ \dot{\beta}_2(t) \end{bmatrix} = \begin{bmatrix} F_1 \\ 0 \\ 0 \end{bmatrix}
\]

where, \( M_1 \) and \( K_1 \) are the mass and the stiffness matrices given by,

\[
M_1 = \begin{bmatrix}
m_t + p l_p & \rho l_p (l_r + \frac{1}{2} l_p) & \frac{1}{2} \rho l_p^2 \\
\rho l_p (l_r + \frac{1}{2} l_p) & \rho l_p (l_r + \frac{1}{3} l_p + l_r^2) & \rho l_p (\frac{1}{2} l_r + \frac{1}{3} l_p) \\
\frac{1}{2} \rho l_p^2 & \rho l_p (\frac{1}{2} l_r + \frac{1}{3} l_p) & \frac{1}{3} \rho l_p^3
\end{bmatrix}
\]

\[
K_1 = \begin{bmatrix}
0 & 0 & 0 \\
0 & \rho g l_p (l_r + \frac{1}{2} l_p) & \frac{1}{2} \rho g l_p^2 \\
0 & \frac{1}{2} \rho g l_p^2 & \frac{1}{2} \rho g l_p^2
\end{bmatrix}
\]

### 2.2 Motion in X-Y Plane

The motion of the crossbeam and the payload in the x-y plane can be described using \( d_2, \alpha_1 \), and \( \alpha_1 \) as the independent variables. For a point \( P \), on the payload axis, in the x-y plane...
plane as shown in Figure 3. Displacement in x direction is,

$$x = -l_r \cos(\alpha_1(t)) - r_p \cos(\alpha_1(t)+\alpha_2(t))$$  \hspace{1cm} 2.20$$

and the displacement in y direction is,

$$y = l_r \sin(\alpha_1(t)) + r_p \sin(\alpha_1(t)+\alpha_2(t)) + d_2(t)$$  \hspace{1cm} 2.21$$

Differentiating equations 2.20 and 2.21 with respect to t,

$$\dot{x} = l_r \sin(\alpha_1(t)) \dot{\alpha}_1(t) + r_p \sin(\alpha_1(t)+\alpha_2(t)) (\dot{\alpha}_1(t)+\dot{\alpha}_2(t))$$  \hspace{1cm} 2.22$$

$$\dot{y} = l_r \cos(\alpha_1(t)) \dot{\alpha}_2(t) + r_p \cos(\alpha_1(t)+\alpha_2(t)) (\dot{\alpha}_1(t)+\dot{\alpha}_2(t)) + \dot{d}_2(t)$$  \hspace{1cm} 2.23$$

Velocity in the X-Y plane is,

$$v_{xy}^2 = \dot{x}^2 + \dot{y}^2$$  \hspace{1cm} 2.24$$

The kinetic energy ($T$) is given by,

$$T = \frac{1}{2} \left[ m_t \dot{\alpha}_2^2(t) + \int_0^{l_p} \rho \ v_{xy}^2 \ dr_p \right]$$  \hspace{1cm} 2.25$$

Substituting equation 2.22 and 2.23 into equation 2.25,
FIGURE 3. Motion of Payload in X-Y Plane.
\[ T = \rho \ i_p \ \dot{\alpha}_1(t) \left[ \frac{1}{2} (i_p \sin(\alpha_1(t))\sin(\alpha_1(t) + \alpha_2(t)) \right. \]
\[ + \ i_p \cos(\alpha_1(t))\cos(\alpha_1(t) + \alpha_2(t)) \] \[ + \ \frac{1}{6} \ i_p^2 \right) + \]
\[ \rho \ i_p^2 \ \dot{\alpha}_1(t) \ \dot{\alpha}_2(t) \left[ \frac{1}{2} \ i_r \ (\sin(\alpha_1(t))\sin(\alpha_1(t) + \alpha_2(t)) \right. \]
\[ + \ \frac{1}{3} \ i_p \] \[ + \ \frac{1}{6} \ \rho \ i_p^3 \ \ddot{\alpha}_2(t) \] \[ + \ \rho \ i_p \ \dot{\alpha}_1(t) \ \dot{\alpha}_2(t) \left[ i_r \cos(\alpha_1(t)) \right. \]
\[ + \ \frac{1}{2} \ i_p \cos(\alpha_1(t) + \alpha_2(t)) \] \[ \right) + \frac{1}{2} \ i_p^2 \ \dot{\alpha}_2(t) \ (m_i + \rho \ i_p) \]

The potential energy \((U)\) is given by,
\[ U = -g \ \rho \ \int_0^{i_p} \left[ i_r \cos(\alpha_3(t)) \right. \]
\[ + \ \rho \ i_r \cos(\alpha_3(t) + \alpha_4(t)) \] \[ \left. \right] \ dr_p \] \[ 2.27 \]

\[ U = -g \ \rho \left[ i_r \ i_p \cos(\alpha_1(t)) \right. \]
\[ + \ \frac{i_p^2}{2} \left. \cos(\alpha_1(t) + \alpha_2(t)) \right] \] \[ 2.28 \]

and the force matrix, \(Q_i\), is given by,
\[ Q_i = \begin{pmatrix} F_2 \\ 0 \\ 0 \end{pmatrix} \] \[ 2.29 \]

Using Lagrangian dynamics the equations of motion are derived as,
\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = Q_i \] \[ 2.30 \]
Using small angle approximations, i.e., assuming,

\[
\begin{align*}
\sin(\alpha_1(t)) &= \alpha_1(t) \\
\sin(\alpha_2(t)) &= \alpha_2(t) \\
\sin(\alpha_1(t)) &= 1 \\
\sin(\alpha_2(t)) &= 1
\end{align*}
\]

2.31

the resulting equations of motion are

\[
(m_i + p l_p) \ddot{\alpha}_2(t) + \left(\frac{1}{2} \rho l_p^2 + \rho l_r l_p\right) \ddot{\alpha}_1(t) + \frac{1}{2} \rho l_p^2 \ddot{\alpha}_2(t) = F_2
\]

2.32

\[
\frac{1}{2} \rho l_p^2 \ddot{\alpha}_2(t) + \rho l_p^2 \left(\frac{1}{2} l_r + \frac{1}{3} l_p\right) \ddot{\alpha}_1(t) + \frac{2}{3} \rho l_p^3 \ddot{\alpha}_2(t) +
\]

\[
\frac{1}{2} \rho g l_p^2 \alpha_1(t) + \frac{1}{2} \rho g l_p^2 \alpha_2(t) = 0
\]

2.33

\[
\rho l_r \left(\frac{1}{2} l_p + l_r\right) \ddot{\alpha}_2(t) + \rho l_r \left(\frac{1}{2} l_p^2 + l_r l_p + l_r l_p^2\right) \ddot{\alpha}_1(t) + \rho l_p^2 \left(\frac{1}{2} l_r + \frac{1}{3} l_p\right) \ddot{\alpha}_2(t) +
\]

\[
\rho g l_r \left(\frac{1}{2} l_p + l_r\right) \alpha_1(t) + \frac{1}{2} \rho g l_p^2 \alpha_2(t) = 0
\]

2.34

These equations can be written in the matrix form as,

\[
[M_2] \begin{pmatrix} \ddot{\alpha}_2(t) \\ \ddot{\alpha}_1(t) \\ \ddot{\alpha}_2(t) \end{pmatrix} + [K_2] \begin{pmatrix} \ddot{\alpha}_2(t) \\ \ddot{\alpha}_1(t) \\ \ddot{\alpha}_2(t) \end{pmatrix} = \begin{pmatrix} F_2 \\ 0 \\ 0 \end{pmatrix}
\]

2.35

where \( M_2 \) and \( K_2 \) are the mass and the stiffness matrices given by,
\[ M_2 = \begin{bmatrix}
    m_b m_c \rho l_p & \rho l_p (l_r - \frac{1}{2} l_p) & \frac{1}{2} \rho l_p^2 \\
    \rho l_p (l_r \frac{1}{2} l_p) & \rho l_p (l_r^2 l_p + \frac{1}{3} l_p^2) & \rho l_p^2 (\frac{1}{2} l_r + \frac{1}{3} l_p) \\
    \frac{1}{2} \rho l_p^2 & \rho l_p^2 (\frac{1}{2} l_r + \frac{1}{3} l_p) & \frac{1}{3} \rho l_p^3
\end{bmatrix} \]

\[ K_2 = \begin{bmatrix}
    0 & 0 & 0 \\
    0 & \rho g l_p (l_r + \frac{1}{2} l_p) & \frac{1}{2} \rho g l_p^2 \\
    0 & \frac{1}{2} \rho g l_p^2 & \frac{1}{2} \rho g l_p^2
\end{bmatrix} \]
CHAPTER 3

OPTIMAL PATH PLANNING FOR MANIPULATORS

In this chapter, a computational scheme for obtaining the optimal trajectories of an arbitrary manipulator following a prescribed path is described. This scheme is based on the developments by Pfeiffer and Johanni\textsuperscript{19}, Shiller and Lu\textsuperscript{20, 21}. This procedure with some modifications is later used to determine the optimal trajectory of an overhead crane carrying suspended payloads.

3.1 Minimum Time Trajectory Planning for Manipulators

The dynamics of an \textit{n-joint} manipulator can be described by,

\[
M \ddot{q} + \dot{q}^T H \dot{q} + G = T
\]

3.1

where, \( q \) is \((n \times 1)\) vector of the joint coordinates, \( M \) the \((n \times n)\) mass matrix, \( G \) the \((n \times 1)\) vector of gravitation potential, \( H \) the is the \((n \times n)\) vector of coriolis and the centrifugal forces, and \( T \) the \((n \times 1)\) vector of joint torques/forces. Equation 3.1 can be rewritten as,
\[
M_i \ddot{q}_i + \dot{q}_i^T \left[ \frac{\partial M_i^T}{\partial q} - \frac{1}{2} \frac{\partial M_i}{\partial q_i} \right] \dot{q}_i + G_i = T_i \quad i = 1..n
\]

The desired path of a manipulator end effector can be parameterized in terms of \( s \) such that \( s_0 \leq s \leq s_f \) where \( s_0 \) and \( s_f \) correspond to the starting and the final points of the path. Therefore, the motion of an \( n \)-degree of freedom manipulator can be described in terms of single path parameter \( s \), i.e.,

\[
q_i = f(s) \quad \dot{q}_i = q_i' \quad \ddot{q}_i = q_i'' + q_i' \dot{s} \quad \dddot{q}_i = q_i''\dot{s} + q_i' \ddot{s} + q_i'' \dddot{s}
\]

where \( q' = dq/ds \quad q'' = d^2q/ds^2 \quad \dot{s} = ds/dt \quad \ddot{s} = d^2s/dt^2 \).

Using equation 3.3, equation 3.2 may be transformed to,

\[
A_i(s) \dddot{s} + B_i(s) \dddot{s}^2 + C_i(s) = T_i(s) \quad i = 1..n
\]

where,

\[
A_i(s) = M_i \quad q'
\]

\[
B_i(s) = M_i \quad q'' + q''^T \left[ \frac{\partial M_i^T}{\partial q} - \frac{1}{2} \frac{\partial M_i}{\partial q_i} \right] q'
\]

\[
C_i(s) = G_i
\]

Equation 3.4 is a set of second-order differential equation with time as independent and the path coordinate \( s \) as dependent variable. It may be transformed to a first order differential equation in \(( \ddot{s}^2, \dot{s} )\) using the relation,
Substituting equation 3.6 into equation 3.4,

\[ A_j(s) \left(\dot{s}^2\right)' + 2 B_j(s) \dot{s}^2 + 2 C_j(s) = T_j(s) \]  

Equation 3.7 can be rewritten as,

\[ h' + \frac{2 B_j(s)}{A_j(s)} h = \frac{2 \left(T_j(s) - C_j(s)\right)}{A_j(s)} \]  

where, \( h = \dot{s}^2 \).

Equation 3.8 is a first order nonhomogeneous linear differential equation which may be solved as (shown in Appendix I),

\[ \dot{s}^2(s) = \left[ \dot{s}^2(s_0) + \int_{s_0}^{s} \frac{2B_j du}{A_j} - \frac{\int_{s_0}^{s} 2B_j du}{A_j} \right] e^{-\int_{s_0}^{s} \frac{2B_j du}{A_j}} \]  

Therefore, manipulator motion along a prescribed trajectory is constrained by the geometry of the path and by the limits on the joint torques. The geometrical constraints have already been considered in the above equations. The torque constraints are given by,

\[ T_{i, \text{min}} \leq T_i \leq T_{i, \text{max}} \]
Combining equations 3.4 and 3.10,

\[ A_i(s) \ddot{s} + B_i(s) \dot{s}^2 \leq T_{i,\text{max}} - C_i(s) \]

\[ A_i(s) \ddot{s} + B_i(s) \dot{s}^2 \geq T_{i,\text{min}} - C_i(s) \quad i = 1..n \]

The equality form of equations 3.11 may be regarded as the equations of two parallel lines in the \( \dot{s}^2-\dot{s} \) plane for each path point \( s \). For each joint \( i \) such a pair of straight lines is obtained. These lines form a polygon in that plane as shown in Figure 4. To ensure non-trivial solution, the line \( \dot{s} = 0 \) is added to the polygon. Motion can take place only in the interior of this polygon. For each path point \( s \), a different polygon is obtained. The maximum possible path velocity \( \dot{s}_{\text{max}} \) at this point is given by the right most vertex of the polygon, as shown in Figure 4. Plotting \( \dot{s}_{\text{max}} \) for every point along the path forms the velocity limit curve in the \( s-\dot{s} \) plane, as shown in Figure 6. For each \( \dot{s} \) less than \( \dot{s}_{\text{max}} \), there are two extreme values for \( \dot{s} \) (upper bound \( \dot{s}_a \) and lower bound \( \dot{s}_d \) as shown in Figure 4), which are the maximum and the minimum possible accelerations/decelerations within the limits of geometrical and force constraints for the path point considered. The velocity at any path point is given by equation 3.9. Usually, only one joint torque is saturated at the maximum acceleration or deceleration, except at points where the two torque limits intersect, as shown in Figure 4 at point \( a \) and at the velocity limit. The corresponding joint for which the torque is saturated, called the control joint, will have the maximum joint velocity.

The slope of a trajectory at a given point in the \( s-\dot{s} \) plane can be associated with the acceleration at that point as,
Figure 4. Admissible Range in the $\ddot{S}-\dot{S}$ Plane at a Regular Point.
Figure 5. Admissible Range in the $\dot{S}$-$\ddot{S}$ Plane at a Critical Point.
\[
\frac{d\dot{s}}{ds} = \frac{d\dot{s}}{dt} \frac{dt}{ds} = \frac{\dot{s}}{\dot{\dot{s}}} \tag{3.12}
\]

and the time at any path point is given by,

\[
t(s) = \int_{s_0}^{s} \frac{1}{\dot{s}(s)} \, ds \tag{3.13}
\]

The time-minimum problem, therefore, can be reduced to the problem of finding a curve in the \(s-\dot{s}\) plane which is as high as possible to gain time optimality, the derivative of which nowhere exceeds the limits given by the extreme values of \(\dot{s}\), and to find the switching points between maximum acceleration and maximum deceleration.

Transferring the extreme values of \(\dot{s}\) to the \(s-\dot{s}\) plane for all possible combinations of \((s, \dot{s})\) results in a gradient field of extremals, which represent the curves with maximum possible acceleration/deceleration. This field of extremals is confined by the \(\dot{s}_{\text{max}}\) curve. Generally, the acceleration at \(\dot{s}_{\text{max}}\) reduces to a single value, and the acceleration at the \(\dot{s}_{\text{max}}\) curve is generally unique, as shown in Figure 4, except at critical points at which the acceleration is not uniquely determined. This case occurs when one of the \(A_i(s) = 0\), as shown in Figure 5. The limit curve acts as a trajectory source or trajectory sink, except at the critical points. At a trajectory sink, point \(a\) on the limit curve in Figure 6, a unique admissible acceleration will force the trajectory into the forbidden region above the limit curve, while at trajectory source, point \(h\), the trajectory is directed away from the forbidden region. The trajectory can touch the limit curve only at points where a trajectory sink
Figure 6. A Typical Velocity Limit Curve and a Time Optimal Trajectory in the Phase Plane.
Figure 7. Typical Acceleration along the Velocity Limit Curve at Critical, Tangency, and Singular Points.
switches to a trajectory source. This occurs at tangency points, point $b$, and critical points, point $e$ in Figure 6.

At tangency points the acceleration at the limit curve is unique, i.e., $\dot{s} = \ddot{s}_a$. If $A_i(s) = 0$ for some joint $i=j$, then equation 3.11 reduces to

$$T_{j, \min} \leq B_j(s) \dot{s}^2 + C_j(s) \leq T_{j, \max}$$

Equation 3.14 represents a vertical line in the $\dot{s}^2$-$\ddot{s}$ plane. If this line intersects the feasible region determined by all the other constraints, then the acceleration at the velocity limit spans a finite range, as shown in Figure 5. The critical points represent a discontinuity in the slope of the $\dot{s}_{\text{max}}$ curve and a discontinuity in the acceleration at the velocity limit. The acceleration range and the typical sharp corner at the critical points allow trajectories with various slopes to touch the limit curve at these points without crossing into the forbidden region. Figure 7 shows two types of critical points $a$ & $c$ and a tangency point $b$. At $a$ and $b$, any feasible trajectory will not cross the limit curve using either the maximum acceleration or deceleration. At point $c$, the maximum acceleration drives the trajectory across the limit curve. This point is called as singular critical point.

The following algorithm obtains the time optimal trajectory along specified paths, considering singularity points and arcs.

I. From the initial point construct the forward extremal. This is done by integrating forward the maximum acceleration. If it reaches the final point, go to step V. If the trajectory hits the limit curve at some point $s_m$, go to step II.
II. Search forward for the nearest critical or tangency point, $s_b > s_a$.

III. From $s_b$, integrate backward the maximum feasible deceleration until the trajectory crosses the previous trajectory at some point $s_c < s_a$. At this point, the trajectory switches from acceleration to deceleration.

IV. From $s_b$ integrate forward the maximum feasible acceleration until it hits the limit curve again. If it reaches the final point goto step V, otherwise, goto step II.

V. From the final point integrate backward the maximum deceleration, to construct the reverse extremal, until crossing the previous trajectory.

Having determined the time optimal trajectory in the phase plane, i.e., in the form $\dot{s}_{opt} = \dot{s}(s)$, time in dependence of $s$ is computed using equation 3.13.

3.2 General Optimization of Manipulator Trajectory

In some cases, it might be desirable to optimize the motion along a given trajectory according to additional criteria. For example, consider minimization of square of velocity and joint torques in addition to time. Square of velocity is proportional to kinetic energy and minimization of joint torques produces a smoothing effect favorable for the joint motors and helps avoid exciting elastic vibrations in the system. These three criteria may be combined by weighting coefficients,

$$cost = w_1 \frac{1}{\dot{s}} + w_2 \dot{s}^2 + w_3 \sum \left( \frac{T_i}{T_{i,\text{max}}} \right)^2$$

3.15
The algorithm described in section 3.1 is used to determine the optimal trajectory. In this case, the forward and the reverse extremals would be the trajectories with minimum value of the cost function given by equation 3.15. Using equations 3.9 the velocity distribution along a given path can be calculated, which depends on the joint torques. One of these torques is optimized according to equation 3.15. The following procedure is used to determine control joint.

1. At any point, the joint for which the relative velocity gains the maximum value becomes the control joint, \( i.e. \)

\[
q_j' = \max_{i=1}^{n} q_i' 
\]  

2. During the motion another joint torque may exceed the maximum possible value \( |T_k| > |T_{k, \text{max}}| \). Then the control is transferred to joint \( k \).

3. Given the control joint \( j \), its torque is optimized by varying the torque to result in the minimum value of the cost function. From equations 3.4 and 3.9 for \( i=j \), the path velocity \( \dot{s} \) and path acceleration \( \ddot{s} \) is calculated. From equation 3.4 for \( i \neq j \), the joint torques are calculated.

The optimization problem is solved using dynamic programming approach. The path between \( s = s_p \) and \( s_f \) is divided into small segments each of length \( \Delta s \). At any point \( s \), a linear search is done for the the control joint torque between \( T_{j, \text{max}} \) and \( T_{j, \text{min}} \) regarding equations 3.4 and 3.9 to determine the torque which results in the minimum value of the cost
function given by equation 3.15. By combining the algorithm in section 3.1 and the procedure described above for deciding the control joint, the trajectory of the manipulator, which optimizes the chosen performance criteria, may be determined.
CHAPTER 4

OPTIMAL PATH PLANNING FOR OVERHEAD CRANE

In this chapter the scheme developed in the previous chapter for optimal path planning of manipulators is applied to an overhead crane carrying a suspended payload.

The overhead crane with the suspended payload which is modeled as a double pendulum can be considered similar to a two link manipulator with six degree of freedom, i.e. the base of the manipulator can move in the y-z plane and each of the links can rotate about the y and the z axis, which has no actuators at the joints. Thus, the two links represent the rope and the payload and the base representing the motion of the trolley and the cross beam. Since there is no control on the swing angles, $\alpha$'s and $\beta$'s, of the rope and the payload the swing angles are not function of $s$, but rather are function of time $t$ and the motion of the trolley and the crossbeam, in contrast to the case of manipulators described in section 3.1, where the joint angles are known as a function of $s$ for a given path.

4.1 Optimal Trajectory Planning

The path of the crane can be parameterized in terms of $s$ such that $s_0 \leq s \leq s_f$, where
s₀ and sₜ correspond to the starting and the final points of the path. Therefore, the coordinates of a point on the path can be described as,

\[ d_i = f_i(s) \quad i = 1 \ldots 2 \quad 4.1 \]

Differentiating equation 4.1 with respect to \( t \),

\[ \dot{d}_i = d_i' \dot{s} \quad \ddot{d}_i = d_i' \ddot{s} + d_i'' s^2 \quad i = 1 \ldots 2 \quad 4.2 \]

where \( d_i' = \frac{d}{ds}d_i \) and \( d_i'' = \frac{d^2}{ds^2}d_i \).

Rearranging the equations of motions (equations 2.17 and 2.35) of the crane,

\[ \ddot{d}_1 = \frac{\rho \ g \ l_p \ \beta_1(t) + F_1}{m_b + m_t} \quad 4.3 \]

\[ \ddot{d}_2 = \frac{\rho \ g \ l_p \ \alpha_1(t) + F_2}{m_t} \quad 4.4 \]

Substituting equation 4.2 into equations 4.3 and 4.4,

\[ (m_b + m_t)d_1' \dot{s} + (m_b + m_t)d_1'' s^2 - \rho g l_p \beta_1(t) = F_1 \quad 4.5 \]

\[ m_t \ d_2' \dot{s} + m_t \ d_2'' \dot{s}^2 - \rho g l_p \alpha_1(t) = F_2 \quad 4.6 \]
These equations can be written as,

\[ A_i(s) \dot{s} + B_i(s) s^2 + C_i = F_i \quad i = 1..2 \quad 4.7 \]

where,

\[ A_1(s) = (m_b+m_t) d_1' \quad B_1(s) = (m_b+m_t) d_1'' \quad C_1 = -\rho g l_p \beta_1(t) \quad 4.8 \]

\[ A_2(s) = (m_t) d_2' \quad B_2(s) = (m_t) d_2'' \quad C_2 = -\rho g l_p \alpha_1(t) \]

The above equations are similar to those developed earlier for manipulators.

To find a trajectory with minimum swinging and optimum traversal time for the overhead crane, the performance criteria is chosen as time, square of the swing angle, and square of the angular velocity of the payload. These three criteria may be combined using weighting coefficients,

\[ \text{cost} = w_1 \frac{1}{\dot{s}} + w_2 \left( \alpha_1^2(t) + \alpha_2^2(t) + \beta_1^2(t) + \beta_2^2(t) \right) + w_3 \left( \dot{\alpha}_1^2(t) + \dot{\alpha}_2^2(t) + \dot{\beta}_1^2(t) + \dot{\beta}_2^2(t) \right) \quad 4.9 \]

The forward and the reverse extremals would be the acceleration and deceleration curves, respectively, with minimum value of the cost function given by equation 4.9, and not with maximum acceleration/deceleration as in case of time optimal trajectories as described in the previous chapter. The forward and the reverse extremals, constructed using the
algorithm described in the previous chapter, result in different values of the independent variable \( t \) at the switching points. Since the swing angles, \( \alpha 's \) and \( \beta 's \) are functions of the independent variable \( t \), and not that of the dependent variable \( s \), switching from the forward to the reverse extremal will result in discontinuities in the swing angles at these points. Hence, the algorithm will be used with some modifications to avoid these discontinuities at the switching points.

4.1.1 Algorithm for Construction of Optimal Trajectories

I. From the initial point construct the forward extremal. The procedure for construction of the extremals is described below. If it reaches the final point, goto step V. If the trajectory hits the limit curve at some point \( s_a \), go to step II.

II. Search forward for the nearest critical or tangency point, \( s_b > s_a \).

III. From \( s_b \), construct the reverse extremal until the trajectory crosses the previous trajectory at some point \( s_c < s_a \). At this point, the trajectory switches from acceleration to deceleration curve.

IV. From \( s_e \), construct the forward extremal until it hits the limit curve again. If it reaches the final point goto step V, otherwise, go to step II.

V. From the final point construct the reverse extremal until crossing the previous trajectory.

4.1.2 Procedure for construction of the extremals
1. At any point, the force corresponding to the direction in which the relative velocity gains the maximum value, *i.e.*, 

\[ d'_j = \max d'_i \quad i = 1\ldots2 \tag{4.10} \]

becomes the controlling force.

2. During the motion the force in the other direction may exceed the maximum possible value \( |F_k| > |F_{k_{\text{max}}}| \). Then the control is transferred to force in direction \( k \).

3. The coefficients \( A_i(s), B_i(s) \) and \( C_i \) in equation 4.7 are calculated using the relations given by equation 4.8. Assuming the swing angles to be constant over \( \Delta s \), their values at the previous path point are used to calculate \( C_i \) at the current path point.

4. Given the control direction \( j \) the force in that direction is optimized by varying the control force to result in the minimum value of equation 4.9. From equations 3.9 and 4.7 for \( i=j \), the path velocity \( \dot{s} \) and path acceleration \( \ddot{s} \) are calculated. From equation 4.7 for \( i \neq j \), the force in the other direction is calculated.

5. At any path point, the time as a function of \( s \) is given by equation 3.13 and the differential equations of motion (equations 2.17 and 2.35) are solved to find the swing angles \( \alpha \)'s and \( \beta \)'s.

Now, the trajectory constructed using the above described procedure optimizes the chosen performance criteria. Using the control forces obtained construct the trajectory again. The coefficients \( A_i(s), B_i(s) \) and \( C_i \) are calculated as described previously. Since the control
forces are known the path velocity $\dot{s}$, path acceleration $\ddot{s}$ and the force in the other direction are calculated from equations 3.9 and 4.7.

4.2 Example

The theory presented in the previous section is applied to an overhead crane carrying suspended a payload with the specifications given in Table 2.

A straight line path is chosen such that the overhead crane moves 60 meters in the direction of the crossbeam ($d_1$) and 20 meters in the direction of the trolley ($d_2$). The motion of overhead crane is assumed to start from zero initial conditions. The path of the overhead crane in terms of the path parameter can be written as,

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_b$</td>
<td>mass of cross-beam</td>
<td>1,150 kg</td>
</tr>
<tr>
<td>$m_t$</td>
<td>mass of trolley</td>
<td>300 kg</td>
</tr>
<tr>
<td>$\rho$</td>
<td>linear density of the payload</td>
<td>1,153 kg/m</td>
</tr>
<tr>
<td>$l_r$</td>
<td>length of rope</td>
<td>3.0 m</td>
</tr>
<tr>
<td>$l_p$</td>
<td>length of payload</td>
<td>5.7 m</td>
</tr>
<tr>
<td>$F_{1\text{ max}}$</td>
<td>max. force in the direction of cross-beam travel</td>
<td>20,000 N</td>
</tr>
<tr>
<td>$F_{1\text{ min}}$</td>
<td>min. force in the direction of cross-beam travel</td>
<td>-16,000 N</td>
</tr>
<tr>
<td>$F_{2\text{ max}}$</td>
<td>max. force in the direction of trolley travel</td>
<td>13,000 N</td>
</tr>
<tr>
<td>$F_{2\text{ min}}$</td>
<td>min. force in the direction of trolley travel</td>
<td>-13,000 N</td>
</tr>
</tbody>
</table>
\[ d_1(s) = 60.0 \, \text{s} \]
\[ d_2(s) = 20.0 \, \text{s} \]

where \( s \) is any point on the path such that \( 0 \leq s \leq 1 \). From equation 4.11,

\[ d_1' = 60 \quad d_1'' = 0 \quad d_2' = 20 \quad d_2'' = 0 \]

Equation 4.7 can be written as,

\[ A_i(s) \bar{s} + C_i = F_i \quad i = 1, 2 \]

since \( B_i(s) = 0 \) from equations 4.8 and 4.12. The force constraints are given by,

\[ F_i, \text{min} \leq F_i \leq F_i, \text{max} \]

Combining equations 4.13 and 4.14,

\[ A_i(s) \bar{s} \leq F_i, \text{max} - C_i \]
\[ A_i(s) \bar{s} \geq F_i, \text{min} - C_i \quad i = 1, 2 \]

At any path point \( s \), the equality form of equation 4.15 represents two lines, for each \( i \), in the \( \bar{s}^2 - \bar{s} \) plane parallel to the \( \bar{s}^2 \) axis. Hence in this case there will be no velocity limit curve bounding the extremal curves.

At any path point \( s \), for a given control joint, the force is varied between zero and \( F_{i, \text{max}} \) and \( F_{i, \text{min}} \), in steps of \( \delta F \), to find the minimum value of the cost function given by equation 4.9, for constructing the forward and the reverse extremals respectively. The
velocity \( (\delta) \) is calculated by solving equation 3.9 numerically. Also the time is calculated by solving equations 3.13 numerically (as shown in Appendix II). To decrease the numerical error in computation, which is mainly due to the numerical integration of the equations 3.9 and 3.13, small \( \Delta s \) is chosen \((\Delta s = 0.0001)\) and the control force is divided such that,

\[
\delta F = \frac{F_{i, \text{max/min}}}{10000}
\]

4.2.1 **Optimal Path Planning with time criteria**

Figure 8 shows the optimal velocity curve with time as the only optimizing criteria, \( i.e., w_1 = 1, w_2 = 0, \) and \( w_3 = 0 \) in equation 4.9. The forces necessary to produce this trajectory are shown in Figure 9. The swing angles of the payload are shown is Figures 10-13. Due to the bang-bang nature of the forces large amplitudes of the swing angles are produced after the trajectory switches to the deceleration curve from the acceleration curve.

4.2.2 **Optimal Path Planning with time-angle criteria**

To find a trajectory with minimum swinging and optimum time for the overhead crane to travel along the chosen path, the weighting coefficients in equation 4.9 are chosen as \( w_1 = 10^{-4}, w_2 = 10^4, \) and \( w_3 = 0 \). Figure 14 and 15 show the velocity and the force curves for this case. It can be seen that this results in fluctuation of the forces. The
Figure 8. $s$ - $s$ Curve for Time Optimal Solution.

Figure 9. Force Curves for Time Optimal Solution.
Figure 10. $\alpha_1 - s$ Curve for Time Optimal Solution.

Figure 11. $\alpha_2 - s$ Curve for Time Optimal Solution.
Figure 12. $\beta_1$ vs $s$ Curve for Time Optimal Solution.

Figure 13. $\beta_2$ vs $s$ Curve for Time Optimal Solution.
corresponding swing angles are shown in Figures 16-17. To reduce these fluctuations an additional constraint is applied on the force such that the change in the control force between any two consecutive path points is not greater than \(3\delta F\), i.e.,

\[
|F(s_i) - F(s_{i-1})| \leq 3 \delta F
\]

Figure 20 and 21 show the optimal velocity curve with the force constraint and the forces needed to produce this trajectory respectively. The swing angles of the payload are shown in Figures 22-25.

4.2.2 Optimal Path Planning with time-angle-angular velocity criteria

Figure 26 shows the optimal velocity curve for this case with \(w_1 = 10^{-4}\), \(w_2 = 10^{4}\), and \(w_3 = 10^{4}\) in equation 4.9. Figures 27-31 show the corresponding force curves and the swing angles.
Figure 14. $s-s$ Curve for Time-Angle Solution without the Additional Force Constraint.

Figure 15. Force Curves for Time-Angle Solution without the Additional Force Constraint.
Figure 16. $\alpha_1 - s$ Curve for Time-Angle Solution. without the Additional Force Constraint.

Figure 17. $\alpha_2 - s$ Curve for Time-Angle Solution. without the Additional Force Constraint.
Figure 18. $\beta_1$ Curve for Time-Angle Solution, without the Additional Force Constraint.

Figure 19. $\beta_2$ Curve for Time-Angle Solution, without the Additional Force Constraint.
Figure 20. $s - s$ Curve for Time-Angle Solution.

Figure 21. Force Curves for Time-Angle Solution.
Figure 22. $\alpha_1 - s$ Curve for Time-Angle Solution.

Figure 23. $\alpha_2 - s$ Curve for Time-Angle Solution.
Figure 24. $\beta_1$ vs. $s$ Curve for Time-Angle Solution.

Figure 25. $\beta_2$ vs. $s$ Curve for Time-Angle Solution.
Figure 26. $s - s$ Curve for Time-Angle-Angular Velocity Solution.

Figure 27. Force Curves for Time-Angle-Angular Velocity Solution.
Figure 28. $\alpha_1$ - $s$ Curve for Time-Angle-Angular Velocity Solution.

Figure 29. $\alpha_2$ - $s$ Curve for Time-Angle-Angular Velocity Solution.
Figure 30. $\beta_1$ vs $s$ Curve for Time-Angle-Angular Velocity Solution.

Figure 31. $\beta_2$ vs $s$ Curve for Time-Angle-Angular Velocity Solution.
4.3 Comparison of Results

Table 3. Comparison of Results

<table>
<thead>
<tr>
<th>Weighting coefficients</th>
<th>Maximum Magnitudes of the swing angles (in radians)</th>
<th>Time (in sec.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_1$ $w_2$ $w_3$</td>
<td>$\alpha_1$ $\alpha_2$ $\beta_1$ $\beta_2$ $t$</td>
<td></td>
</tr>
<tr>
<td>1 0 0</td>
<td>0.39376 0.43434 1.19846 1.31469 8.75395</td>
<td></td>
</tr>
<tr>
<td>$10^{-4}$ $10^4$ 0</td>
<td>0.26796 0.25974 0.79987 0.75013 9.23659</td>
<td></td>
</tr>
<tr>
<td>$10^{-4}$ $10^4$ $10^4$</td>
<td>0.26747 0.20893 0.77963 0.63595 8.45222</td>
<td></td>
</tr>
</tbody>
</table>

Table 3 shows a comparison between the three cases presented. In the case with only time as the performance criteria, it is seen that the controlling force ($F_t$) is always maximized (Figure 9). This is because, the overhead crane will accelerate and decelerate along the externals with the maximum values of acceleration/deceleration. Due to the bang-bang nature of the forces, larger amplitudes of the swing angles are excited after the switching point as shown in Figures 10-13. In the case with time and angles as the performance criteria, the amplitudes of the swing angles are considerably reduced, as shown in Figure 22-25, compared to the time only case discussed perviously. However, the time taken for the overhead crane to traverse the path is higher due to the trade-off between the components of the cost function. In the third case, with time, angle, and the angular velocity of the payload as the performance criteria, the swing angles are damped further, as shown in Figures 28-31. The time taken for the overhead crane to traverse the path is lowest, in this
case, compared to the other two cases. Minimization of the angular velocities of the payload results in lower kinetic energy of the payload. Hence, the payload offers less resistance to the motion of the crossbeam and the trolley.
A method for evaluation of optimal trajectories of an overhead crane system is presented. This method provides swing damped transport of the payload. This method is different from the existing methods for determining the optimal trajectories of overhead crane systems, for two main reasons. First, the payload is modeled as a double pendulum instead of a simple pendulum to account for the dynamic behavior of long and massive payloads. Second, the equations of motions are transformed in terms of the path coordinate \( s \) representing the one degree of motion along the prescribed path. This transformation allows the use of the geometric properties of the transformed set of equations to determine the optimal trajectories.

In the example presented, it is seen that the solution with time as the minimization criteria results in near minimum time optimal trajectory and not the true minimum time trajectory. The time optimal solution results in exciting higher amplitudes of the swinging in the payload. Best results are obtained when all the three performance criteria, \( i.e., \) time, the angle and the angular velocity of the payload are considered. Also, the time-angle-angular velocity solution results in the lowest time and minimum values of the swing angles.
Hence, a true time minimum solution can be reached only if the swing angles remain zero throughout the motion of the overhead crane, i.e. no swinging is excited in the payload.

Further studies should aim to provide swing free stop for the payload and a smoother transfer from the acceleration to the deceleration curve which may result in lower amplitudes of swinging in the payload after the switching point.
APPENDIX I

SOLUTION OF NONHOMOGENEOUS LINEAR FIRST ORDER DIFFERENTIAL EQUATIONS

A first order differential equation is said to be linear if it can be written in the form,

\[ y' + f(x) y = r(x) \quad 1 \]

The characteristic feature of this equation is that it is linear in \( y \) and \( y' \), whereas \( f \) and \( r \) may be any functions of \( x \). If \( r(x) = 0 \), for all \( x \) in the domain of \( r \), the is said to be homogeneous, otherwise, it is said to be nonhomogeneous.

Equation 1 can be written in the form,

\[ (fy-r) \ dx + dy = 0 \quad 2 \]

and an integrating factor \( F(x) \) which depends only on \( x \) can be found. If such a factor exists, then

\[ F(x) (fy-r) \ dx + F(x) \ dy = 0 \quad 3 \]

must be exact. Hence,
Separating the variables in equation 4 and then integrating it,

\[ \ln |F| = \int f(x) \, dx \]

Hence,

\[ F(x) = e^{z(x)} \quad \text{where} \quad z(x) = \int f(x) \, dx \]

Now multiplying equation 1 by this integrating factor,

\[ e^z \left( y' + fy \right) = e^z r \]

Since \( z' = f \), equation 7 may be written as,

\[ \frac{d}{dx} (ye^z) = e^z r \]

Integrating both sides of equation 8 and rearranging,

\[ y(x) = e^{-z} \left[ \int e^z r \, dx + c \right] \quad \text{where} \quad z = \int f(x) \, dx \]

which is the general solution of equation 1.
Equation 3.9 can be written as,

\[ h' + \frac{2 B_i(s)}{A_i(s)} h = \frac{2 (T_i(s) - C_i(s))}{A_i(s)} \]

where \( h' = s^2 \). From equations 1 and 10,

\[ f = \frac{2 B_i(s)}{A_i(s)} \]

\[ r = \frac{2 (T_i(s) - C_i(s))}{A_i(s)} \]

Substituting equation 11 into equation 9, the solution of equation 3.7 is obtained as,

\[
\delta^2(s) = \left[ \delta^2(s_0) + \int_{s_0}^{s} e^{\int_{s_0}^{t} \frac{2B_i(\nu)}{A_i(\nu)} d\nu} \frac{2(T_i(t) - C_i(t))}{A_i(t)} \right] e^{\int_{s_0}^{t} \frac{2B_i(\nu)}{A_i(\nu)} d\nu}
\]
APPENDIX II

NUMERICAL INTEGRATION OF $1/\dot{s}$ CURVE

At any path point the $s$, the time $t$ is calculated by numerically integrating equation 3.13.

$$t(s) = \int_{s_0}^{s_1} \frac{1}{\dot{s}(s)} ds$$  \hspace{1cm} 3.13

Using trapezoidal method equation 3.13 can be integrated as,

$$t(s) = \frac{h}{2} \left( \frac{1}{\dot{s}(s)} + \frac{1}{\dot{s}(s_{i-1})} \right) + t(s_{i-1})$$  \hspace{1cm} 1

where $h = \Delta s$. Since $\dot{s}$ is to be zero at the first, $i.e.$ at $s=0$ equation 1 cannot be used at this point. To overcome this problem and to increase the accuracy of integration, $s$ between zero and $\Delta s$ is further divided into ten points and Simpson rule is applied at these points to integrate the $1/\dot{s}$ curve, assuming $\frac{1}{\dot{s}} = 0$ and the time $t = 0$ at $s=0$, $i.e.$,

$$t(s) = \frac{h_1}{3} \left( \frac{1}{\dot{s}(s)} + 4 \frac{1}{\dot{s}(s_{i-1})} + \frac{1}{\dot{s}(s_{i-2})} \right) + t(s_{i-2})$$  \hspace{1cm} 2
where \( h_1 = \frac{\delta s}{10} \). The \( 1/\delta \) curve is integrated similarly at \( s = s_f \).
BIBLIOGRAPHY


11. G. P. Starr, "Swing-Free Transport of Suspended Objects with a Path Controlled Robot"


