Feedback linearization of nonminimum phase systems and control of aeroelastic systems and undersea vehicles

Francis M Chockalingam

University of Nevada, Las Vegas

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FEEDBACK LINEARIZATION OF NONMINIMUM PHASE SYSTEMS AND
CONTROL OF AEROElastIC SYSTEMS
AND UNDERSEA VEHICLES

by

Francis M. Chockalingam

Bachelor of Engineering
Anna University, Madras, India
1996

A thesis submitted in partial fulfillment
of the requirements for the degree of

Master of Science

in

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Department of Electrical Engineering
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Francis M. Chockalingam

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ABSTRACT

Feedback Linearization of Nonminimum Phase Systems and Control of Aeroelastic Systems and Undersea Vehicles

by

Francis M. Chockalingam

Dr. Sahjendra N. Singh, Examination Committee Chair
Professor of Electrical Engineering
University of Nevada, Las Vegas

The thesis presents the design of feedback control systems for a class of nonminimum phase single input-single output nonlinear systems. The linearized system is assumed to have one unstable zero. Since asymptotic or exact tracking of output trajectory cannot be accomplished, an approximate output is derived by neglecting the unstable zero. Based on the inversion of the new input-output map, a feedback linearizing control is derived.

These results are applied to control an aeroelastic system and a small undersea vehicle. For pitch angle control and plunge motion regulation, an inverse control system is designed for the aeroelastic system. Simulation results are shown for the pitch controller and the design is found to be robust to variation in the parameters. Dive plane control of an undersea vehicle is accomplished using an inverse control law. To attenuate the effect of the surface waves, a servocompensator has been designed. Later, a controller is also designed using the sliding mode control technique, to make the system more robust.
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## LIST OF SYMBOLS

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<th>Definition</th>
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<tr>
<td>a</td>
<td>nondimensionalized distance from the midchord to the elastic axis</td>
</tr>
<tr>
<td>A</td>
<td>System matrix</td>
</tr>
<tr>
<td>b</td>
<td>semichord of the wing</td>
</tr>
<tr>
<td>B</td>
<td>Input matrix</td>
</tr>
<tr>
<td>c_h</td>
<td>structural damping coefficient in plunge due to viscous damping</td>
</tr>
<tr>
<td>c_l_a</td>
<td>lift coefficient per angle of attack</td>
</tr>
<tr>
<td>c_m_a</td>
<td>moment coefficient per angle of attack</td>
</tr>
<tr>
<td>c_l_b</td>
<td>lift coefficient per control surface deflection</td>
</tr>
<tr>
<td>c_m_b</td>
<td>moment coefficient per control surface deflection</td>
</tr>
<tr>
<td>c_a</td>
<td>structural damping coefficient in pitch due to viscous damping</td>
</tr>
<tr>
<td>CB</td>
<td>Center of buoyancy</td>
</tr>
<tr>
<td>CG</td>
<td>Center of gravity</td>
</tr>
<tr>
<td>D</td>
<td>Matrix of dissipative(damping) terms</td>
</tr>
<tr>
<td>h</td>
<td>plunge displacement</td>
</tr>
<tr>
<td>I_x</td>
<td>Moment of inertia about x-axis</td>
</tr>
<tr>
<td>I_y</td>
<td>Moment of inertia about y-axis</td>
</tr>
<tr>
<td>I_z</td>
<td>Moment of inertia about z-axis</td>
</tr>
<tr>
<td>I_xy</td>
<td>Product of inertia about x- and y-axes</td>
</tr>
<tr>
<td>I_xz</td>
<td>Product of inertia about x- and z-axes</td>
</tr>
<tr>
<td>I_yz</td>
<td>Product of inertia about y- and z-axes</td>
</tr>
<tr>
<td>I_a</td>
<td>mass moment of inertia of the wing about the elastic axis</td>
</tr>
<tr>
<td>J</td>
<td>Cost function</td>
</tr>
<tr>
<td>k_h</td>
<td>structural spring constant in plunge</td>
</tr>
<tr>
<td>k_a</td>
<td>structural spring constant in pitch</td>
</tr>
<tr>
<td>L</td>
<td>length of the vehicle, aerodynamic lift</td>
</tr>
<tr>
<td>m</td>
<td>mass</td>
</tr>
<tr>
<td>m_d</td>
<td>moment acting on the vehicle</td>
</tr>
<tr>
<td>M</td>
<td>Aerodynamic moment</td>
</tr>
<tr>
<td>p</td>
<td>Angular velocity component of ( \dot{q} ) about x-axis(roll)</td>
</tr>
<tr>
<td>q</td>
<td>Angular velocity component of ( \dot{q} ) about y-axis(pitch)</td>
</tr>
<tr>
<td>( \dot{q} )</td>
<td>Vehicle-fixed vector of linear and angular velocities components</td>
</tr>
<tr>
<td>r</td>
<td>Angular velocity component of ( \dot{q} ) about z-axis(yaw)</td>
</tr>
<tr>
<td>t</td>
<td>Time</td>
</tr>
<tr>
<td>u</td>
<td>Linear velocity component of ( \dot{q} ) in x-direction(surge)</td>
</tr>
<tr>
<td>u_c</td>
<td>Control input vector</td>
</tr>
<tr>
<td>U</td>
<td>freestream velocity</td>
</tr>
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</table>
\( v \)  
Linear velocity component of \( \dot{q} \) in y-direction (sway)

\( w \)  
Linear velocity component of \( \dot{q} \) in z-direction (heave)

\( x \)  
Surge position referred to the earth-fixed reference frame

\( x \)  
State vector

\( x_B \)  
The x-coordinate of CB

\( x_G \)  
The x-coordinate of CG

\( x_a \)  
nondimensionalized distance measured from the elastic axis to the center of mass

\( y \)  
Sway position referred to the earth-fixed reference frame

\( z \)  
Heave position (depth) referred to the earth-fixed reference frame

\( \dot{z}_f \)  
Fluid velocity

\( z_r \)  
Desired output trajectory

\( z_B \)  
The z-coordinate of CB

\( z_G \)  
The z-coordinate of CG

\( \alpha \)  
Pitch angle

\( \beta, \beta_1, \beta_2 \)  
Flap deflection

\( \rho \)  
Density of air or water

\( \delta \)  
Rudder (actuator) angle

\( \theta \)  
Angle of pitch

\( \phi \)  
Angle of roll

\( \psi \)  
Angle of yaw

\( \omega \)  
Disturbance due to the wave
ACKNOWLEDGMENTS

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CHAPTER 1

INTRODUCTION

Output tracking control of non-minimum phase systems has been encountered in many practical engineering applications. Some of the well-known non-minimum phase control problems include the aircraft altitude control problem, rocket trajectory tracking problem, tip trajectory tracking problem for flexible-link manipulators, aeroelastic wing-rock problem, dorsal fin control led undersea vehicles, and many more.

Exact asymptotic tracking for an open set of output trajectories for a system with internal stability requires it to have stable zero-dynamics. The control of non-minimum phase systems is a topic of active current research. One interesting approach is the output-redefinition method, whose principle is to redefine the output function so that the resulting zero-dynamics is stable. Provided that the new output function is defined in such a way that, it is essentially the same as the original output function in the frequency range of interest, exact tracking of the new output function then also implies good tracking of the original output. The key result used to prove this property is that, given a square, invertible, linear system with one or more real zeros, one can construct a new output function which eliminates any subset of the real zeros of the original system and does not add any additional zeros.

In this work, the theory that is developed for the nonlinear control of nonminimum phase systems are applied to two major problems. First, it is applied to the flutter control problem in an aeroelastic wing. Later, the same principle is also applied to the dive plane control of small undersea vehicles using dorsal fins.
1.1 Past Work

A clear and compact summary of input-output linearization has been presented by Hauser [1], which also discusses weakly non-minimum phase systems. The output redefinition method was suggested by Hedrick and Gopalswamy in [2]. Sastry [3] and, Kokotovic and Sussmann [4] further discuss the relation between stability of the zero-dynamics and overall system stability. The nonlinear output regulation theory developed by Isidori [6] ensures internal stability with asymptotic output tracking for a class of nonlinear systems with reference trajectories generated by an exosystem. The control law uses a feed-forward input plus feedback stabilization of certain state trajectory. Both the feed-forward input and the state trajectory are obtained by solving a set of nonlinear partial differential equation (PDE). The nonlinear regulator, however, encounters the difficulty of solving a set of nonlinear PDEs. Another problem with the nonlinear regulator is that transient errors cannot be controlled precisely and are usually large for non-minimum phase systems. An algorithm for approximate input-output decoupling of nonlinear MIMO systems that are either numerically ill-posed or exhibit nearly singular behavior in the application of decoupling algorithms is presented in [7]. This does not cancel the far off zeros of the open-loop system, thereby providing reasonable gain. This approach also performs an approximate asymptotic tracking.

The notion of stable inversion and use it to develop a new approach for output tracking control of nonminimum phase systems is introduced in [8]. In this method, a stable but non-causal inverse is obtained off-line that can be incorporated into a stabilizing controller for dead-beat output tracking. Though it demonstrates the value of stable inversion in achieving high-precision stable output tracking, the computation of the stable inverse is fairly complicated. Grizzle [5] has proposed a method of removing right-half plane zeros, while retaining the other zeros in their previous
location. This principle can be used to obtain an approximate minimum phase system, for which the exact asymptotic tracking theory can be applied.

Nonlinear aeroelastic systems have rich dynamic behavior and exhibit a variety of phenomena including instability, limit cycle oscillation, and even chaotic vibrations [9, 10, 11]. Flutter is an oscillatory aeroelastic instability caused by unsteady aerodynamic loads. For flutter suppression, feedback control laws have been developed by researchers [12, 13, 14]. In these studies, linear control theory has been used for the design of controllers. Based on a linear deterministic autoregressive moving average aeroservoelastic model, a digital adaptive controller for active flutter suppression has been presented in [15]. Nonlinear aeroelastic models and control systems have been developed in [15, 16, 17]. Nonlinear structural stiffness plays a dominant role in causing the onset of flutter in aeroelastic systems, though nonlinearities arising from control saturation, free play, hysteresis, structural stiffness, and stability derivatives are encountered. In a series of interesting papers, Ko et al. have considered control of this aeroelastic system based on feedback linearization theory [18] and adaptive control technique [19].

Considerable research has been done in the area of modeling of underwater vehicles [20]-[26]. Dynamic stability of submarines in the dive plane has been examined in [24]. Several studies have also been made for designing control systems for submerged vehicles [25]-[33]. Fossen [25] has considered variety of control systems including decoupling, sliding mode and adaptive control laws, and used these for the control of the NEROV underwater vehicle. A self-tuning autopilot for a remotely operated underwater vehicle has been designed in [26]. A supervisory controller for the Jason ROV has been described in [27]. Sliding mode controllers for nonlinear vehicle models have been presented in [25],[28]-[32]. An adaptive sliding mode control system and a variable structure model reference control system have been designed in [32].
quaternion feedback approach for the control of underwater vehicles has been provided in [33].

In depth study of Fish morphology and locomotion [34] is in progress at the Naval Undersea Warfare Center (NUWC), Division Newport. The hydrodynamics of cambering fins were studied by Bandyopadhyay and co-researchers. They have designed dorsal fin-like control surfaces for the low-speed maneuvering using experiments conducted on several species of fish [34]. The measurement of forces and moments produced by the control surfaces have been done and the flow pattern and vortices formed have been recorded in tests performed in tow tanks and water tunnels. But the control systems synthesis using only dorsal fins has not been yet accomplished, although some theoretical study for vehicles with dorsal and caudal fins has been done [35, 36].

Tracking a trajectory is an important concept used in many control system problem. The next section gives a brief overview of a tracking problem with respect to nonlinear systems.

### 1.2 Tracking Problem

Generally, the tasks of control systems can be divided into two categories: stabilization (or regulation) and tracking (or servo). In stabilization problems, a control system, called a stabilizer (or a regulator), is to be designed so that the state of closed-loop system will be stabilized around an equilibrium point. Examples of stabilization tasks are temperature control of refrigerators, altitude control of aircraft and position control of robot arms. In tracking control problems, the design objective is to construct a controller, called a tracker, so that the system output tracks a given time-varying trajectory. Problems such as making an aircraft fly along a specified path or making a robot hand draw straight lines or circles are typical tracking control
tasks.

Normally, tracking problems are more difficult to solve than stabilization problems, because in tracking problems the controller should not only keep the whole state bounded but also drive the system output toward the desired output. However, from a theoretical point of view, tracking design and stabilization design are often related. The stabilization problems can often be regarded as a special case of tracking problems, with the desired trajectory being a constant.

The asymptotic tracking problem can be defined as follows. Given a nonlinear dynamics system described by

\[ \dot{x} = f(x) + bu \]
\[ y = Cx \]

and a desired output trajectory \( z_r \), find a control law for the input \( u \) such that starting from any initial state in a region \( \Omega \), the tracking errors \( y(t) - z_r(t) \) go to zero, while the whole state \( x \) remains bounded. Asymptotic tracking control always requires feed-forward actions to provide the forces necessary to make the required motion. Many of the tracking controllers can be written as a sum of feed-forward and the feedback part. The feed-forward part intends to provide the necessary input for following the specified motion trajectory and canceling the effects of the known disturbances. The feedback part then stabilizes the tracking error dynamics.

Consider the polynomial \( X(s) = \alpha_n s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_0 \). \( X(s) \) is monic if \( \alpha_n = 1 \) and \( X(s) \) is Hurwitz if all the roots of \( X(s) = 0 \) are located in \( \Re[s] < 0 \). Consider a system with a transfer function,

\[ G(s) = \frac{Z(s)}{R(s)} \]

It is referred to as minimum phase if \( Z(s) \) is Hurwitz; it is referred to as stable if \( R(s) \) is Hurwitz.
For non-minimum phase systems, perfect tracking and asymptotic tracking cannot be achieved[37]. The inability of perfect tracking for a non-minimum phase linear system has its roots in its inherent tendency of “undershooting” in its step response. One approach is the output-redefinition method, which remains the focus of this thesis. An approximate tracking for a nonlinear system whose linearization possesses real right-half plane zeros is shown in [5]. The method is guaranteed to remove the right-half plane zeros while the other zeros remain in their previous location. The main idea is to modify the output of the nonlinear system based on a transformation performed on the Jacobian linearization of the system. This transformation yields a minimum phase system whose left-half plane zeros remain in their original positions. The key result used to prove this property is that, given a square, invertible, linear system with one or more real zeros, one can construct a new output function which eliminates any subset of the real zeros of the original system and does not add any additional zeros. Moreover, the transfer matrices of the original system and the constructed system are close for all frequencies less than the minimum of the magnitudes of the eliminated real zeros.

Another practical approximation may be, when performing input-out linearization using successive differentiations of the output, to simply neglect the terms containing the input and keep differentiating the selected output a number of times equal to the system order, so that there is approximately no zero-dynamics. This approach can only be meaningful if the coefficients of $u$ at the intermediate steps are small, that is if the systems are weakly non-minimum phase. The approach is conceptually similar to neglecting fast right-half plane zeros in linear systems.

Another approach to dealing with non-minimum phase systems is to modify the plant itself. In linear systems, while poles can be placed using feedback, zeros are intrinsic properties of the plant and the selected output, and can be changed only
by modifying the plant or the choice of output. Similarly, in nonlinear systems, the zero-dynamics is a property of the plant, the output, and the desired trajectory. It can be made stable by changing the desired trajectory directly, although this is rarely practical if the system is supposed to perform a variety of pre-specified tasks. Finally, it can be made stable by changing the plant design itself. This may involve relocation or addition of actuators and sensors, or modifying the physical construction of the plant.

1.3 Objective and the scope of the thesis

The objective of the thesis is to design a feedback control system for a class of nonminimum phase single input-single output nonlinear systems. For the design of control systems based on inversion theory for exact or asymptotic tracking, the zero dynamics must be stable. The zero dynamics are defined as the residual dynamics of the system when the output is identically zero [5, 6, 37]. To overcome the obstruction created by unstable zero dynamics, borrowing an idea from [5], an approximate minimum phase system is obtained by essentially eliminating the unstable zero from the transfer function. A new output variable for control is derived [38] using the linearized model of the vehicle and a new state variable representation of the vehicle is obtained for the design of control systems. For asymptotic trajectory control, an inverse control law is designed [39, 40, 41]. It is shown that in the closed loop system, state trajectory is uniformly bounded and the tracking error is ultimately confined to a small set.

These techniques are applied to two systems as an illustration. First, it is applied to an aeroelastic system, where a feedback controller for the primary pitch control is implemented. Then using the same idea the depth control of a small undersea vehicle, under the presence of surface waves are presented. In order to attenuate these
oscillations, a servo-compensator is designed based on the internal model principle [42, 43].

For the depth trajectory control of the undersea vehicle, sliding mode control law [37, 44, 45] is also designed for the continuous cambering of the dorsal fins in the presence of seawaves. The sliding mode control law is nonlinear and discontinuous in the state space and has an excellent insensitivity property with respect to disturbance and parameter variations.

1.4 Organization of the thesis

The thesis has been organized into five chapters. Chapter 1 gives the introduction. The derivation of a minimum phase approximate system and a new controlled output variable is described in Chapter 2. Chapter 3 presents the design and the simulation of the feedback controller for the aeroelastic system. The design of the dive plane control for the small undersea vehicle is given in Chapter 4. Finally the conclusions and the results are summarized in Chapter 5.
CHAPTER 2

INVERSE CONTROL OF A CLASS OF NONMINIMUM PHASE SYSTEMS

2.1 Introduction

Feedback linearization technique is important for the design of control systems for nonlinear systems. By feedback and coordinate transformation, some nonlinear systems are transformed into Brunowsky canonical form. However, for this transformation the system must satisfy some stringent conditions. Often, control systems are designed using input-output linearization which gives only a part of the dynamics in a linear form and the remaining dynamics are nonlinear. The residual dynamics is called zero dynamics.

For exact or asymptotic output trajectory tracking, the zero dynamics must be stable. For many practical systems including aeroelastic systems and underwater vehicles, the zero dynamics are unstable. For such system, it is not possible to design control systems using nonlinear inversion (input-output feedback linearization) technique. The systems with unstable zero dynamics are called nonminimum phase systems.

In this chapter control of a class of single input- single output nonlinear systems are considered for which the zero dynamics are unstable. The approach is to derive a new output variable so that the new system is minimum phase. Based on the new system a feedback linearizing control law is derived.
2.2 Nonlinear system

Consider a class of single input-single output nonlinear systems of the form

\[ \dot{x} = Ax + g(x) + bu \]
\[ y = cx \quad (2.1) \]

where the state vector \( x \in \Omega \), subset of \( \mathbb{R}^n \), the input \( u \in \mathbb{R} \), the output \( y \in \mathbb{R} \). The nonlinear vector function \( g(x) \) is of order \( O(x)^2 \). The function \( r(x) \) is said to be of order \( O(x)^2 \) if

\[ \lim_{\|x\| \to 0} \frac{r(x)}{\|x\|} \text{ exists and is } \neq 0 \]

where \( \|\cdot\| \) denotes the Euclidean norm of a vector.

The transfer function of the linear system obtained from (2.1) by setting \( g(x) = 0 \) is given by

\[ H(s) = c(sI - A)^{-1}b \quad (2.2) \]
\[ = k_p \frac{n_p(s)}{d_p(s)} \]

where \( d_p(s) = \det(sI - A) \), and \( n_p(s) = c \, \text{adj}(sI - A)b \).

Although, the approach of this work is applicable to systems with multiple unstable zeroes, for simplicity it is assumed that \( H(s) \) is of the form

\[ H(s) = k_p \frac{n_{p1}(s)(s - \mu)}{d_p(s)} \quad (2.3) \]

where \( n_p(s) = n_{p1}(s)(s - \mu) \), \( n_{p1}(s) \) is a Hurwitz polynomial and \( \mu > 0 \). \( H(s) \) has one unstable zero at \( s = \mu \), it is nonminimum phase transfer function. This implies that the nonlinear system (2.1) is also nonminimum phase.
For this system an input-output feedback linearizing control cannot be derived for the trajectory control of \( y(t) \). It is well known that in the closed-loop system including a feedback linearizing control law, for output trajectory tracking, the residual dynamics (zero-error dynamics) are unstable and when the trajectory evolves on the zero-error manifold, the states associated with the zero dynamics diverge [6].

Now in order to obtain tracking with internal stability in the closed-loop system, an approximate transfer function is considered which possesses only stable zero. The approximate transfer function \( H_a(s) \) is chosen as

\[
H_a(s) = k_p \frac{n_p(s)(s - \mu)}{(1 - \frac{s}{\mu})d_p(s)} = \frac{\mu k_p n_p(s)}{d_p(s)} \quad (2.4)
\]

In the new transfer function stable zeros and all the poles of \( H(s) \) are retained and it is a minimum phase transfer function.

### 2.3 Redefining Output Variable

Now a new state-space representation of system (2.1) is obtained. In view of the transfer functions in (2.3) and (2.4), there exists a row vector \( \check{C} \) such that

\[
H_a(s) = \check{C}(sI - A)^{-1}b = -\frac{\mu k_p n_p(s)}{d_p(s)} \quad (2.5)
\]

where \( A \) and \( b \) are given in (2.1).

Let the numerator polynomial of \( H_a(s) \) be

\[
n_a(s) = q_{n-r}s^{n-r} + q_{n-r-1}s^{n-r-1} + \cdots + q_1s + q_0 \quad (2.6)
\]

for some real numbers \( q_i \), where \( r \) is the relative degree of \( H_a(s) \).

**Lemma 2.1** Suppose that the matrix pair \( (A, b) \) is controllable and the relative degree of \( H_a(s) \) is \( r \). Then there exists a row-vector \( \check{C} \) and a matrix \( L \) such that

\[
\check{C} = [0_{1 \times (r-1)}, q_{n-r}, \ldots, q_0]L^{-1} \quad (2.7)
\]
such that $H_a(s) = \tilde{C}(sI - A)^{-1}b$.

**Proof:** Since $H_a(s)$ is of relative degree $r$, one has

$$\tilde{C}A^kb = 0, \; k = 0, 1, \ldots, r - 2$$

$$\tilde{C}A^{r-1}b \neq 0 \quad (2.8)$$

In view of (2.8), the numerator polynomial of $H_a$ can be written as [38]

$$n_a(s) = (s^n - a_{n-1}s^{n-r-1} + a_{n-2}s^{n-r-2} + \cdots + a_r)\tilde{C}A^{r-1}b + (s^{n-r-1} - a_{n-1}s^{n-r-2} + a_{n-2}s^{n-r-3} + \cdots + a_{r+1})\tilde{C}A^rb + (s^{n-r-2} - a_{n-1}s^{n-r-3} + a_{n-2}s^{n-r-4} + \cdots + a_{r+2})\tilde{C}A^{r+1}b + \cdots + (s + a_{n-1})\tilde{C}A^{n-2}b + \tilde{C}A^{n-1}b \quad (2.9)$$

if

$$d_p(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0$$

Equating the coefficients of $s^i$, $i = n - r, \ldots, 0$, in (2.9) and (2.6), one obtains

$$\tilde{C}A^{r-1}b \triangleq \tilde{C}L_{r+1}b = q_{n-r}$$

$$\tilde{C}(a_{n-1}A^{r-1} + A^r)b \triangleq \tilde{C}L_{r+2}b = q_{n-r-1}$$

$$\tilde{C}(a_{n-2}A^{r-1} + a_{n-1}A^r + A^{r+1})b \triangleq \tilde{C}L_{r+3}b = q_{n-r-2}$$

$$\vdots$$

$$\tilde{C}(a_{r+1}A^{r-1} + a_{r+2}A^r + \cdots + A^{n-2})b \triangleq \tilde{C}L_{n-1}b = q_1$$

$$\tilde{C}(a_rA^{r-1} + a_{r+1}A^r + \cdots + a_{n-1}A^{n-2} + A_{n-1})b \triangleq \tilde{C}L_nb = q_0$$

Using (2.8) and (2.11), one has

$$\tilde{C}L = Q \quad (2.11)$$
where \( Q = [0_{1 	imes (r-1)}, q_{n-r}, \ldots, q_0], L = [L_1, \ldots, L_n] \) and

\[
L_{i+1} = \tilde{C} A^i b, \ i = 0, 1, \ldots, r - 2 \tag{2.12}
\]

and the remaining elements of \( L \) are defined in (2.11). Since the matrix pair \((A, b)\) is controllable, in view of the form of \( L \), it is nonsingular and solving (2.11) gives

\[
\tilde{C} = Q L^{-1}
\]

This completes the proof of Lemma 2.1.

Now a new controlled output variable can be defined as

\[
y_a = \tilde{C} x
\]

associated with the nonlinear system (2.1). For \( g(x) = 0, \)

\[
Y_a(s) = \tilde{C}(sI - A)^{-1} bU = -\frac{\mu k_p n_{p1}(s)}{d_p(s)} U
\]

In view of (2.3), (2.5) and (2.14), it follows that

\[
Y(s) = -\frac{1}{\mu} (s - \mu) Y_a(s)
\]

which implies that

\[
y(t) = y_a(t) - \frac{1}{\mu} \dot{y}_a(t)
\]

Thus if \( y_a(t) \) and \( \dot{y}_a(t) \to 0 \), then \( y(t) \) also converges to zero as \( t \to \infty \). Now it will be useful to obtain a normal form representation of system (2.1) using the new output equation (2.13).

2.4 Normal Form Representation and Inverse Control

It is interesting to obtain a normal form representation of the nonlinear system which is useful for stability analysis and the design of an inverse control system. Lie
derivatives of a function \( \alpha(x) \) with respect to the vector field \( f(x) \) are defined as

\[
L_f(\alpha)(x) = \frac{\partial \alpha}{\partial x} f(x)
\]

\[
L^j_f(\alpha)(x) = L_f(L^{j-1}_f(\alpha))(x)
\]  \( (2.17) \)

\[
L_b^k L^j_f(\alpha)(x) = \frac{\partial L^k_f(\alpha)}{\partial x} b
\]

where

\[
f(x) \triangleq Ax + g(x)
\]  \( (2.18) \)

Consider nonlinear functions

\[
\phi_1 = \tilde{C} x = y_a
\]

\[
\phi_k = L^{k-1}_f(\tilde{C} x)(x), \, k = 2, \ldots, r
\]  \( (2.19) \)

Define function \( \alpha_i(x) \) recursively as

\[
\alpha_0(x) = 0
\]

\[
\alpha_{k+1}(x) = \tilde{C} g(x) + L_f(\alpha_k)(x), \, k = 0, 1, \ldots, r - 1
\]  \( (2.20) \)

Then in view of \( (2.18) \), one has \( (k = 0, \ldots, r - 1) \)

\[
\phi_{k+1} = \tilde{C} A^k x + \alpha_k(x)
\]  \( (2.21) \)

where \( \alpha_k(x) \) is of order \( O(x)^2 \).

Define \( \Phi = [\phi_1, \phi_2, \ldots, \phi_r]^T \). The differential \( d\Phi \) of \( \Phi \) evaluated at \( x = 0 \) gives,

\[
d\Phi(0) = \begin{bmatrix}
\frac{\partial \phi_1(0)}{\partial x} \\
\frac{\partial \phi_2(0)}{\partial x} \\
\vdots \\
\frac{\partial \phi_r(0)}{\partial x}
\end{bmatrix} = \begin{bmatrix}
\tilde{C} \\
\tilde{C} A \\
\vdots \\
\tilde{C} A^{r-1}
\end{bmatrix} \triangleq M_0
\]  \( (2.22) \)

In view of \( (2.8) \), the rows of \( M_0 \) are independent. Thus \( r \times n \) matrix \( d\Phi(0) \) has rank \( r \).

This implies that \( \phi_i \) \( (i = 1, 2, \ldots, k) \) are independent functions in the neighborhood.
of \( x = 0 \). According to (2.8), the vectors \( \tilde{C}A^i(i = 0, 1, \ldots, r - 2) \) are orthogonal to \( b \). Therefore, there exists a row-vector \((n - r) \times n\) matrix \( C_0 \) such that

\[
C_0 b = 0 \tag{2.23}
\]

and the rows of \( M_0 \) and \( C_0 \) are independent (the computation of \( C_0 \) is not necessary for controller synthesis). Let

\[
\eta(x) = C_0 x, \eta \in \mathbb{R}^{n-r} \tag{2.24}
\]

Define new coordinates \((\xi, \eta)\), by

\[
\begin{bmatrix}
\xi_1 \\
\vdots \\
\xi_r \\
\eta
\end{bmatrix} =
\begin{bmatrix}
\phi_1(x) \\
\vdots \\
\phi_r(x) \\
\eta(x)
\end{bmatrix} = \Phi(x) \tag{2.25}
\]

where \( \xi = (\xi_1, \ldots, \xi_r)^T \).

The Jacobian matrix \( d\Phi(x) \) evaluated at \( x = 0 \) is given by

\[
d\Phi(0) = \begin{bmatrix}
\tilde{C} \\
\vdots \\
\tilde{C}A^{r-1} \\
C_0
\end{bmatrix} \tag{2.26}
\]

Since rank of \( d\Phi(x) \) at \( x = 0 \) is \( n \), the transformation \( \Phi : x \rightarrow [\phi_1, \ldots, \phi_r, \eta^T]^T \) is a local diffeomorphism in the neighborhood \( x = 0 \).

A representation of the nonlinear system in the normal form in \((\xi, \eta)\) coordinates can be obtained \([6]\). Differentiating \( \xi_i \) gives

\[
\begin{align*}
\dot{\xi}_1 &= L_f(\tilde{C}x)(x) \\
\dot{\xi}_k &= L_f^2(\tilde{C}x)(x) + L_b(\alpha_1)(x)u \\
&= \xi_{k+1} + L_b(\alpha_{k-1})(x)u, \quad k = 1, \ldots, r - 1 \\
\dot{\xi}_r &= L_f^r(\tilde{C}x)(x) + [L_b(\alpha_{r-1})(x)]u + \tilde{C}A^{r-1}bu \\
\dot{\eta} &= L_f(C_0x)(x)
\end{align*} \tag{2.27}
\]
Using \( x = \Phi^{-1}(\xi, \eta) \), one writes (2.28) in the matrix notation as
\[
\begin{bmatrix}
\dot{\xi}_1 \\
\dot{\xi}_2 \\
\vdots \\
\dot{\xi}_r \\
\dot{\eta}
\end{bmatrix} =
\begin{bmatrix}
\xi_2 \\
\xi_3 \\
\vdots \\
a^*(\xi, \eta) + b^*(\xi, \eta)u \\
q_0(\xi, \eta)
\end{bmatrix} +
\begin{bmatrix}
\psi_u(\xi, \eta) \\
0
\end{bmatrix}u
\tag{2.28}
\]

where \( \Phi^{-1}(\xi, \eta) \) is substituted for \( x \).

\[
a^* = L_f(\tilde{C}x)(x)
\]
\[
b^* = L_h(\alpha_{r-1})(x) + \tilde{C}A^{-1}b
\tag{2.29}
\]
\[
q_0 = C_0Ax + C_0g(x)
\]

The \( r \) vector function \( \psi_u \) is given by
\[
\psi_u =
\begin{bmatrix}
0 \\
L_h(\alpha_1)(x) \\
\vdots \\
L_h(\alpha_{r-1})(x)
\end{bmatrix}
\tag{2.30}
\]

It can be noted that \( \psi_uu \) is of order \( O(\xi, \eta, u)^2 \).

For small values of \( \xi, \eta, \) and \( u \), function \( \psi_uu \) can be neglected to obtain an approximate representation of the nonlinear system as
\[
\begin{bmatrix}
\dot{\xi}_1 \\
\dot{\xi}_2 \\
\vdots \\
\dot{\xi}_r \\
\dot{\eta}
\end{bmatrix} =
\begin{bmatrix}
\xi_2 \\
\xi_3 \\
\vdots \\
a^*(\xi, \eta) + b^*(\xi, \eta)u \\
q_0(\xi, \eta)
\end{bmatrix}
\tag{2.31}
\]

For a nonlinear system relative degree is defined to be the smallest order \( k \) of the derivative of the output such that the input appears explicitly for the first time. The approximate system (2.31) with output \( \xi_1 \) has relative degree \( r \), but the relative degree of the true system (2.28) is not well defined at \( x = 0 \), since \( \psi_u(0) = 0 \) but \( \psi_u(x) \) is not zero in the neighborhood of \( x = 0 \).

Suppose that \( \psi_{ui} = 0, i = 2, \ldots, l-1, \) and \( \psi_{ul} \neq 0 \). It is of interest to derive an inverse control law for the system (2.28) which requires the inverse of the function
ψ_{ui}. Such an inverse control law for the control of $z_a = \xi_1$ will require infinitely large control magnitude as $x$ tends to zero. To overcome this difficulty, we shall derive an inverse control law based on the approximate nonlinear system (2.31).

In view of (2.30), at $x = 0$, $b^* = \tilde{C}A^{-1}b \neq 0$, therefore, in the neighborhood of the origin $b^*(\xi, \eta)$ is nonzero and its inverse exists.

In view of (2.31), an inverse control law $u = u_{inv}$ is obtained which is of the form

$$u_{inv} = (b^*(\xi, \eta))^{-1}[-a^*(\xi, \eta) + z_r^{(r)} - p_r \tilde{\xi}_r - \cdots - p_1 \tilde{\xi}_1]$$  \hspace{1cm} (2.32)

where $p_i > 0$, $z_r$ is the reference trajectory to be tracked and

$$\tilde{\xi}_i = \xi_i - \frac{d^{i-1}z_r}{dt^{i-1}}, \hspace{0.2cm} i = 1, \ldots, r$$  \hspace{1cm} (2.33)

Substituting the control law (2.32) in (2.31), the $r^{th}$ row of (2.31) gives

$$\dot{\xi}_r + p_r \dot{\xi}_r + \cdots + p_1 \dot{\xi}_1 = 0$$  \hspace{1cm} (2.34)

Noting from (2.33) that $\dot{\xi}_{k+1} = \dot{\xi}_k$, (2.34) gives

$$\ddot{\xi}_1^{(r)} + p_r \ddot{\xi}_1^{(r-1)} + \cdots + p_1 \ddot{\xi}_1 = 0$$  \hspace{1cm} (2.35)

where $\ddot{\xi}_i^{(j)} = \frac{\partial^{(j)} \dot{\xi}_i}{\partial \xi^j}$. The characteristic polynomial $\Pi(s)$ associated with (2.35) is given by

$$\Pi(s) = (s^r + p_r s^{r-1} + \cdots + p_1)$$  \hspace{1cm} (2.36)

The coefficients $p_i$ of $\Pi$ are chosen such that it is a Hurwitz polynomial, that is, the roots of $\Pi(s) = 0$ have negative real parts.

For a choice of Hurwitz polynomial $\Pi(s)$, it follows that the system (2.36) is exponentially stable and $\tilde{\xi}_1 = (y_a - z_r)$ tends towards zero exponentially and the output asymptotically tracks the reference trajectory $z_r$. Furthermore, for $z_r$ chosen for a set point control such that $z_r \rightarrow z_r^*$, a constant and the derivatives of $z_r(t)$ tend to zero, $(y_a(t), \dot{y}_a(t)) \rightarrow (z_r^*, 0)$ as $t \rightarrow \infty$ and from (2.16) it follows that
Thus for the approximate system (2.31), the inverse control law accomplishes output trajectory control to a desired value and follows the reference path \( z_r(t) \) accurately.

Since the transfer function \( H_a(s) \) is minimum phase, it follows that the equilibrium point \( \eta = 0 \) of the system

\[
\dot{\eta} = q_0(0, \eta)
\]  

is exponentially stable. This implies that the trajectory of

\[
\dot{\eta} = q_0(\xi, \eta)
\]  

remains bounded as \( \xi \) evolves in a small neighborhood of the origin. Thus the inverse control accomplishes tracking with local internal stability in the closed-loop system.

For the actual system (2.28) with the control law (2.35), gives

\[
\begin{bmatrix}
\dot{\xi} \\
\dot{\eta}
\end{bmatrix} = A_c \begin{bmatrix}
\dot{\xi} \\
\dot{\eta}
\end{bmatrix} + \begin{bmatrix}
0 \\
q_0(\bar{\xi} + Z_r, \eta)
\end{bmatrix} + \left( \begin{array}{c}
\psi_c(\bar{\xi} + Z_r, \eta, u)
\end{array} \right)
\]  

(2.39)

where \( Z_r = [z_r, z_r, \ldots, z_r^{(r)}]^T \), \( \psi_c = \psi_u u \).

\[
\psi_c(\bar{\xi} + Z_r, \eta, u) = \psi_u(\bar{\xi} + Z_r, \eta) u
\]

\[
A_c = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-p_1 & -p_2 & -p_3 & \cdots & -p_r
\end{bmatrix}
\]  

(2.40)

It is interesting to examine the tracking ability of the controller (2.32) when it is synthesized for the original system (2.1). It is reasonable to expect that the inverse control system will have accurate trajectory following capability as long as the state trajectory remains in a small neighborhood of the origin. Indeed, the following result can be stated relating the system (2.1).
Theorem 2.1: Consider the closed-loop system (2.39) which represents the system (2.1) including the inverse control law (2.32). Suppose that the reference trajectory $z_r$ and its derivatives are sufficiently small, then the closed-loop system has the following properties,

P1: There exists a neighborhood $U_e$ of $x = 0$ such that the trajectory beginning in $U_e$ are bounded.

P2: For $z_r(t) \equiv 0$, the equilibrium point $(\xi, \eta) = 0$ is exponentially stable.

Proof:

The proof is based on the Lyapunov theory and follows the steps outlined in a related problem of approximate linearization treated in [1]. First we note that since $A_c$ is a stable matrix, there exists a positive definite symmetric matrix $P$ which satisfies the Lyapunov equation

$$A_c^T P + PA_c = -I$$

(2.41)

where $I$ is the identity matrix.

Furthermore, since the zero dynamics is exponentially stable, by the converse theorem of Lyapunov, there exists a Lyapunov function for the system [46]

$$\dot{\eta} = q_0(0, \eta)$$

(2.42)

satisfying

$$k_1|\eta|^2 \leq V_1(\eta) \leq k_2|\eta|^2$$

$$\frac{\partial V}{\partial \eta} q_0(0, \eta) \leq -k_3|\eta|^2$$

(2.43)

$$\left|\frac{\partial V}{\partial \eta}\right| \leq k_4|\eta|$$
for some positive constants $k_i (i = 1, \ldots, 4)$.

To establish stability result, consider a positive definite composite Lyapunov function

$$V(\xi, \eta) = \xi^T P \dot{\xi} + \lambda V_1(\eta)$$

(2.44)

where $\lambda > 0$ is an appropriate constant to be determined later.

Then the derivative of $V$ along the solution of (2.44) is given by

$$\dot{V} = \xi^T (P A_{c} + A_{c} P) \dot{\xi} + 2 \xi^T P \psi_c + \frac{\partial V_1}{\partial \eta} q_0(\xi, \eta)$$

(2.45)

$$= -\|\xi\|^2 - k_3 \|\eta\|^2 + 2 \xi^T P \psi_c + \frac{\partial V_1}{\partial \eta} [q_0(\xi, \eta) - q_0(0, \eta)]$$

Since $\psi_c(x, u)$ and $u_c(x, Z_r)$ are locally Lipschitz in $x$ and $\psi_c$ depends linearly on $u$ and it follows that for some $k_u$ and $k_{\psi_1}$,

$$|u_c(x, Z_r)| \leq k_u (\|x\| + b_r)$$

(2.46)

$$\|P \psi_c(x, u)\| \leq k_{\psi_1} \|x\| \|u\|$$

(2.47)

where it is assumed that $\|Z_r\| \leq b_r$. Since $(\xi, \eta)$ is a local diffeomorphism of $x$, for some $k_x$,

$$\|x\| \leq \|\xi\| \leq k_x (\|\xi\|^T, \eta)^T$$

(2.48)

$$\leq k_x (\|\xi\| + b_r + |\eta|)$$

Since $q_0(\xi, \eta)$ is Lipschitz in $\xi$, for some $k_q$, one has

$$|q_0(\xi, \eta) - q_0(0, \eta)| \leq k_q \|\xi\| \leq k_q (\|\xi\| + b_r)$$

(2.49)

Using (2.46) to (2.49) in (2.46), and using the steps outlined in [1], it can be similarly shown that for sufficiently small $z^{(j)} (j = 0, 1, \ldots, 3)$, there exists a neighborhood
U_\varepsilon of origin x = 0 (i.e., \|x\| < \varepsilon) in which the derivative of V along the trajectory of the closed-loop system (2.39) satisfies
\[ \dot{V} \leq -\zeta_1 \|\dot{\xi}\|^2 - \zeta_2 |\eta|^2 + k_0 \zeta_3 (Z_r) \] (2.50)
for some \zeta_i > 0 and where \zeta_3 \to 0, as \|\tilde{Z}_r\| tends to zero. This shows that for \xi and \eta in a small neighborhood of (\xi, \eta) = 0, \dot{V} < 0 and therefore the trajectory beginning in \(U_\varepsilon (\xi, \eta)\) is uniformly bounded and it is ultimately confined in a small ball around the origin. Note that size of this ball can be made sufficiently small if \(Z_r, \tilde{z}_r^{(r)}\) is chosen small enough. Since the proof can be completed following [1], the details are not provided here.

For proving property P2, it can be noted that for \(z_r \equiv 0\), the function \(\psi_c\) in (2.39) simplifies to \(\psi_c = [0, \psi_u(\xi, \eta)u, 0, 0]^T\), and one has \(u = u(\xi, \eta)\). The function \(\psi(\xi, \eta)u(\xi, \eta)\) is of order \(O(\xi, \eta)^2\). One notes that the linearized system about the origin obtained from (2.39) is exponentially stable since \(A_c\) is a stable matrix and that the zero dynamics is exponentially stable since the transfer function \(H_A\) is minimum phase. Then exponential stability of (\(\xi, \eta) = 0\) follows easily since the perturbation functions \(\psi_c\) and \(c_0 g\) are of order \(O(\xi, \eta)^2\) [46].

\textbf{Remark 2.1} A special class of systems of interest is that for which \(\psi_u = 0\). In this case, the nonlinear system and the linearized system have the same relative degree. For such systems in the closed-loop system, one has
\[ \ddot{\xi} = A_c \ddot{\xi} \] (2.51)

Thus as long as the trajectory of the zero-error dynamics are bounded, \(\ddot{\xi}(t) \to 0\) for any \(z_r\) and initial condition \(\ddot{\xi}(0)\) as \(t \to \infty\).
2.5 Summary

A method by which the output can be redefined for constructing an approximate minimum phase system for general nonlinear nonminimum phase system was explained in this chapter. Later using the modified output, the inverse controller design for the general nonlinear system is also presented. Further, the asymptotic stability and the boundedness of the control input was proved using the Lyapunov approach. In the next chapter, the same approach is applied for the control of flutter in an aeroelastic system.
CHAPTER 3

CONTROL OF AN AEROELASTIC SYSTEM

3.1 Introduction

Aeroelasticity is concerned with the interaction among inertial, elastic and aerodynamic forces. Aeroelastic phenomena significantly affect the stability and control performance of aerospace vehicle. Flutter is an oscillatory aeroelastic instability caused by unsteady aerodynamic loads.

In this chapter a control design is presented for the control of a nonlinear aeroelastic system. Interestingly, the linear transfer function relating the pitch angle and the control input is nonminimum phase. The theory developed in the previous chapter is applied and a control law is designed for the control of the pitch angle and the regulation of the plunge displacement.

3.2 Aeroelastic Model

This section gives the equations of motion for the aeroelastic model. Consider the prototypical aeroelastic wing section as shown in Fig. 3.1. The governing equations of motion are given by [18]

\[
\begin{bmatrix}
  m & mx_\alpha b \\
  mx_\alpha b & I_\alpha
\end{bmatrix}
\begin{bmatrix}
  \ddot{h} \\
  \ddot{\alpha}
\end{bmatrix} + 
\begin{bmatrix}
  c_h & 0 \\
  0 & c_\alpha
\end{bmatrix}
\begin{bmatrix}
  \dot{h} \\
  \dot{\alpha}
\end{bmatrix} + 
\begin{bmatrix}
  k_h & 0 \\
  0 & k_\alpha(\alpha)
\end{bmatrix}
\begin{bmatrix}
  h \\
  \alpha
\end{bmatrix} = 
\begin{bmatrix}
  -L \\
  M
\end{bmatrix}
\]

(3.1)

where \( h \) is the plunge displacement and \( \alpha \) is the pitch angle. In equation (3.1), \( m \) is the mass of the wing; \( c_\alpha \) and \( c_h \) are the pitch and plunge damping coefficients,
Figure 3.1: Aeroelastic model
respectively; and $M$ and $L$ are the aerodynamic lift and moment. The input $\beta$ is the control surface deflection. It is assumed that the quasi-steady aerodynamic force and moment are of the form

$$ L = \rho U^2 b c_{l \alpha} [\alpha + (\dot{h} / U) + \left(\frac{1}{2} - a\right)b(\dot{\alpha} / U)] + \rho U^2 b c_{l \beta} \beta \quad (3.2) $$

$$ M = \rho U^2 b^2 c_{m \alpha} [\alpha + (\dot{h} / U) + \left(\frac{1}{2} - a\right)b(\dot{\alpha} / U)] + \rho U^2 b^2 c_{m \beta} \beta $$

where $c_{l \alpha}$ and $c_{m \alpha}$ are the lift and moment coefficient; and $c_{l \beta}$, $c_{m \beta}$ are control coefficients. The nonlinear stiffness $k_{\alpha}(\alpha)$ is such that

$$ a k_{\alpha}(\alpha) = a k_{\alpha 0} - k_{\alpha}(\alpha) \quad (3.3) $$

where $k_{\alpha 0}$ is a constant and $k_{\alpha}(\alpha)$ is the nonlinear part of $a k_{\alpha}$. The trajectories of system (3.1) in a bounded region $\Omega \subset \mathbb{R}^4$ surrounding the origin will be of considerable interest. The nonlinear stiffness term $k_{\alpha}(\alpha)$ is obtained by curve fitting the measured displacement-moment data for a nonlinear spring as [17]

$$ k_{\alpha}(\alpha) = 2.82(1 - 22.1\alpha + 1315.5\alpha^2 - 8580\alpha^3 + 17289.7\alpha^4) $$

Defining the state variable $\mathbf{x} = (h, \alpha, \dot{h}, \dot{\alpha})^T$, one obtains a representation of (3.1) in the form

$$ \dot{x} = Ax + g(\alpha) + bu \quad (3.4) $$

where

$$ A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k_1 & -k_2 U^2 + 2.82(-m x_\alpha b / d) & -c_1 & -c_2 \\ -k_3 & -k_4 U^2 + 2.82(m / d) & -c_3 & -c_4 \end{bmatrix} $$

$$ g(\alpha) = \begin{bmatrix} 0 \\ 0 \\ -m x_\alpha b / d \\ m / d \end{bmatrix} (2.82)(-22.1\alpha^2 + 1315.5\alpha^3 - 8580\alpha^4 + 17289.7\alpha^5) $$

$$ b = \begin{bmatrix} 0 \\ 0 \\ g_3 \\ g_4 \end{bmatrix} U^2 $$

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The system variables are given by

\[ d = m(I_a - mx^2_a b^2) \]

\[ k_1 = I_a k_h / d \]

\[ k_2 = (I_a \rho bc_{1a} + mx_a b^3 \rho c_{ma}) / d \]

\[ k_3 = -mx_a b k_h / d \]

\[ k_4 = (-mx_a b^2 \rho c_{1a} - m \rho b^2 c_{ma}) / d \]

\[ c_1 = \left[ I_a (c_h + \rho Ub_{1a}) + mx_a \rho Ub^3 c_{ma} \right] / d \]

\[ c_2 = \left[ I_a \rho Ub^2 c_{1a} \left( \frac{1}{2} - a \right) - mx_a bc_{a} + mx_a \rho Ub^4 c_{ma} \left( \frac{1}{2} - a \right) \right] / d \]

\[ c_3 = (-mx_a b c_h - mx_a \rho Ub^2 c_{1a} - m \rho Ub^2 c_{ma}) / d \]

\[ c_4 = [mc_a - mx_a \rho Ub^3 c_{1a} \left( \frac{1}{2} - a \right) - m \rho Ub^3 c_{ma} \left( \frac{1}{2} - a \right)] / d \]

\[ g_3 = (-I_a \rho bc_{1s} - mx_a b^3 \rho c_{ms}) / d \]

\[ g_4 = (mx_a b^2 \rho c_{1s} + m \rho b^2 c_{ms}) / d \]

Consider a reference trajectory \( z_r \) which is a prescribed pitch angle trajectory \( \alpha_m \) converging to zero. It is of interest to derive a controller so that \( \alpha \) tracks \( \alpha_m \) asymptotically, and in the closed-loop system the state vector \( (h, \alpha, \dot{h}, \dot{\alpha})^T \) converges to zero as \( t \to \infty \). Here \( T \) denotes transposition. The above nonlinear model will be very useful in getting the transfer function as well as for the simulation of the controller.

### 3.3 Input-Output Representations

The transfer function relating the flap deflection \( \beta \), which is the input and the pitch angle \( \alpha \) which is the output, is derived in this section. The linearized system (3.1) can be written using (3.3)

\[ D(s) \begin{pmatrix} h \\ \alpha \end{pmatrix} = \begin{pmatrix} -c_{1s} \\ b_{c_{ms}} \end{pmatrix} \rho b U^2 \beta \]
where \( s \) denotes the Laplace variable or the differential operator, the elements of the 2 \(
\times 2\) matrix \( D = (d_{ij}) \) are
\[
\begin{align*}
\quad d_{11} &= ms^2 + (c_h + \rho Ub cl_a)s + k_h \\
\quad d_{12} &= mx_cbs^2 + \rho Ub^2 cl_a(0.5 - a)s + \rho Ub cl_a \\
\quad d_{21} &= mx_cbs^2 - \rho Ub^2 cl_{ma}s \\
\quad d_{22} &= I_a s^2 + [c_a - \rho Ub^3 cl_{ma}(0.5 - a)]s + k_{ao} - \rho Ub^2 b^2 cl_{ma}
\end{align*}
\]

Solving (3.6) for \( \alpha \), gives
\[
\alpha = (d_{21}(s)ci_\beta + d_{11}(s)bc_{m\beta})\Delta^{-1}(s)\rho bU^2 \beta
\]
where \( \Delta(s) = d_{11}d_{22} - d_{12}d_{21} \). This input-output representation for \( \alpha \) is useful for the design of controller.

The input (\( \beta \))-output (\( \alpha \)) representation obtained from (3.8) can be written as
\[
\alpha = G_p(s)\beta
\]
where the transfer functions \( G_p \) is given by
\[
\begin{align*}
G_p &= [m(x_a ci_\beta + c_{m\beta})s^2 + (c_{m\beta}(c_h + \rho Ub cl_a) - \rho Ub cl_{ma} c_{i_\beta})s \\
&+ k_h c_{m\beta}]\rho b^2 U^2[m(I_a - m b^2 x_a^2)R_p]^{-1} \\
&\triangleq k_{p1} \frac{Z_p(s)}{R_p(s)}
\end{align*}
\]
where
\[
k_{p1} = \frac{(x_a ci_\beta + c_{m\beta})\rho b^2 U^2}{(I_a - m b^2 x_a^2)}
\]
Here \( Z_p(s) \) and \( R_p(s) \) are monic polynomials given by
\[
Z_p(s) = s^2 + k_1s + k_0s
\]
\[
R_p(s) = (d_{11}(s)d_{22}(s) - d_{12}(s)d_{21}(s))[m(I_a - m b^2 x_a^2)]^{-1}
\]
where

\[ k_1 = \frac{[c_m(c_h + \rho Ubc_l) - \rho Ubc_m c_l]}{k_d} \]
\[ k_0 = k_h c_m / k_d \]
\[ k_d = m(x_c c_l + c_m) \]

For the chosen parameters,

\[ Z_p(s) = (s - \mu_1)(s + \mu_2) \quad (3.11) \]

where \( \mu_1, \mu_2 > 0 \). Thus the transfer function \( G_p \) is nonminimum phase.

### 3.4 Redefining Output

For the design of control system, an output variable \( y_a = \alpha_a \) is chosen and the corresponding new transfer function is given by

\[ G_a(s) = \frac{k_p(s - \mu_1)(s + \mu_2)}{(1 - \frac{s}{\mu_1})R_p(s)} = \frac{-\mu_1 k_p(s + \mu_2)}{R_p(s)} \quad (3.12) \]

Let

\[ y_a = \tilde{C}x \]

The computation of the row vector is easily done using Lemma 2.1 of the previous chapter. It can be noted that for the linear system, the new transfer function has relative degree 3.

Now a state space representation of system (3.4) is obtained. In view of the transfer functions (3.11) and (3.12), it follows that there exists a row vector \( \tilde{C} \in \mathbb{R}^4 \) such that [47],

\[ G_a(s) = \tilde{C}(sI - A)^{-1}b = \frac{-\mu_1 k_p(s + \mu_2)}{d_p(s)} \quad (3.13) \]

where \( A \) and \( b \) are as defined in (3.4).

The row vector \( \tilde{C} \) can be obtained using the following result.
Lemma 3.1: For the controllable system (3.4), there exists a unique row-vector \( \tilde{C} \)

\[
\tilde{C} = -\mu_1 k_p [0, 0, 1, \mu_2] L_c^{-1}
\] (3.14)

such that \( G_a(s) = \tilde{C}(sI - A)^{-1}b \), where

\[
L_c = [b, Ab, A^2b, m_3 A^2b + A^3b]
\] (3.15)

where

\[
det(sI - A) = s^4 + m_3 s^3 + m_2 s^2 + m_1 s + m_0.
\]

Proof: Using the result of [38], it can be shown that the transfer function given in (3.13) can be expanded as

\[
G_a(s) = [(s^3 + m_3 s^2 + m_2 s + m_1)\tilde{C}b + (s^2 + m_3 s + m_2)\tilde{C}Ab \\
+ (s + m_3)\tilde{C}A^2b + \tilde{C}A^3b]/d_p(s)
\] (3.16)

where \( m_i \) are the coefficients of \( d_p(s) \) given in (3.11).

Since \( G_a(s) \) has relative degree (the difference in degrees of the denominator and numerator polynomials of \( G_a \)) 3 according to (3.12), in view of (3.16), one must have

\[
\tilde{C}b = 0 \\
\tilde{C}Ab = 0 \\
\tilde{C}A^2b \neq 0
\] (3.17)

Now comparing the remaining terms of (3.13) and (3.16), gives

\[
\tilde{C}A^2b = -\mu_1 k_p \\
m_3 \tilde{C}A^2b + \tilde{C}A^3b = -k_p \mu_1 \mu_2
\] (3.18)

Using (3.18), (3.18) and the definition of \( L_c \) in (3.15), one has

\[
\tilde{C}L_c = [0, 0, -\mu_1 k_p, -\mu_1 \mu_2 k_p]
\] (3.19)
The system (3.4) is controllable, therefore $L_c$ is nonsingular and there exists a unique solution of (3.4) given by (3.14).

Now a new controlled output variable can be defined as

$$z_a = \alpha_a = \tilde{C}x$$

(3.20)

associated with the nonlinear system (3.4). One has

$$Z_a(s) = \tilde{C}(sI - A)^{-1}bU = \frac{-\mu_1 k_p(s + \mu_2)}{d_p(s)}U$$

(3.21)

In view of (3.11), (3.12), and (3.21), it follows that

$$Z(s) = -\frac{1}{\mu_1}(s - \mu_1)Z_a(s)$$

(3.22)

which implies that

$$\alpha(t) = \alpha_a(t) - \frac{1}{\mu_1}\dot{\alpha}_a(t)$$

(3.23)

Now it will be useful to obtain a new representation of system (3.4).

### 3.5 Inverse Control Law

A normal form representation of the nonlinear system which is useful for stability analysis and the design of an inverse control system can be obtained as discussed in Section 2.4. The same approach has been used for this system also, which has a relative degree 3. Lie derivatives of a function $\alpha(x)$ with respect to the vector field $f(x)$ are defined as in (2.18).

Consider nonlinear functions

$$\phi_1 = \tilde{C}x = \alpha_a$$

$$\phi_2 = L_f(\tilde{C}x)(x)$$

$$\phi_3 = L_f^2(\tilde{C}x)(x)$$

(3.24)

The nonlinear function $f(x)$ is defined as
\[ f(x) = Ax + g(x) \]  

(3.25)

where \( g(x) \) are of order \( O(x)^2 \).

In view of (3.25), one has

\[
\begin{align*}
\phi_2 &= \tilde{C}Ax + L_g(\tilde{Cx}) \\
\phi_3 &= \tilde{C}A^2x + \tilde{C}Ag(x) + L_fL_g(\tilde{Cx})
\end{align*}
\]

(3.26)

where \( L_g(\tilde{Cx}) \) and \( L_fL_g(\tilde{Cx}) \) are of order \( O(x)^2 \).

Define \( \Phi = [\phi_1, \phi_2, \phi_3]^T \). The differential \( d\Phi \) of \( \Phi \) evaluated at \( x = 0 \) gives,

\[
d\Phi(0) = \begin{bmatrix}
\frac{\partial \phi_1(0)}{\partial x} \\
\frac{\partial \phi_2(0)}{\partial x} \\
\frac{\partial \phi_3(0)}{\partial x}
\end{bmatrix} = \begin{bmatrix}
\tilde{C} \\
\tilde{CA} \\
\tilde{CA}^2
\end{bmatrix}
\]

(3.27)

The \( 3 \times 4 \) matrix \( d\Phi(0) \) has rank 3 in view of (3.18). This implies that \( \phi_i \) (\( i=1,2,3 \)) are independent functions in the neighborhood of \( x = 0 \). According to (3.18), the vectors \( \tilde{C} \) and \( \tilde{CA} \) are orthogonal to \( b \). Therefore, there exists a row-vector \( C_0 \) such that \( C_0b = 0 \) and \( \tilde{C}, \tilde{CA}, \tilde{CA}^2 \) and \( C_0 \) are independent (the computation of \( C_0 \) is not necessary for controller synthesis). Let \( \eta(x) = C_0x \). Define new coordinates, \( (\xi, \eta) \), by

\[
\begin{bmatrix}
\xi_1 \\
\xi_2 \\
\xi_3 \\
\eta
\end{bmatrix} = \begin{bmatrix}
\phi_1(x) \\
\phi_2(x) \\
\phi_3(x) \\
\eta(x)
\end{bmatrix} = \Phi(x)
\]

(3.28)

The Jacobian matrix \( d\Phi(x) \) evaluated at \( x = 0 \) is given by

\[
d\Phi(0) = \begin{bmatrix}
\tilde{C} \\
\tilde{CA} \\
\tilde{CA}^2 \\
C_0
\end{bmatrix}
\]

(3.29)

Since rank of \( d\Phi(x) \) at \( x = 0 \) is 4, the transformation \( \Phi : x \to [\phi_1, \phi_2, \phi_3, \eta]^T \) is a local diffeomorphism in the neighborhood \( x = 0 \).
A representation of the nonlinear system in the normal form in \((\xi, \eta)\) coordinates can be obtained [6]. Differentiating \(\xi_i\) gives

\[
\begin{align*}
\dot{\xi}_1 &= L_f(\tilde{C}x)(x) = \xi_2 \\
\dot{\xi}_2 &= L_f^2(\tilde{C}x)(x) = \xi_3 \\
\dot{\xi}_3 &= L_f^3(\tilde{C}x)(x) + [L_\beta L_f^2(\tilde{C}x)(x)]u \\
\dot{\eta} &= L_f(C_0x)(x)
\end{align*}
\tag{3.30}
\]

Using \(x = \Phi^{-1}(\xi, \eta), \xi = [\xi_1, \xi_2, \xi_3]^T\), one writes (3.31) in the matrix notation as

\[
\begin{pmatrix}
\dot{\xi}_1 \\
\dot{\xi}_2 \\
\dot{\xi}_3 \\
\dot{\eta}
\end{pmatrix} =
\begin{bmatrix}
\xi_2 \\
\xi_3 \\
a^*(\xi, \eta) + b^*(\xi, \eta)u \\
q_0(\xi, \eta)
\end{bmatrix}
\tag{3.32}
\]

where \(\Phi^{-1}(\xi, \eta)\) is substituted for \(x\),

\[
\begin{align*}
a^* &= L_f^3(\tilde{C}x)(x) \\
b^* &= L_\beta L_f^2(\tilde{C}x)(x) \\
q_0 &= C_0Ax + C_0g(x)
\end{align*}
\tag{3.33}
\]

The system (3.32) with output \(\xi_1\) has relative degree 3.

It can be verified that

\[
b^* = \tilde{C}A^2b + \left[\frac{\partial}{\partial x}(\tilde{C}Ag + L_f(\tilde{C}g(x)))\right]b
\]

At \(x = 0\), \(b^* = \tilde{C}A^2b \neq 0\), therefore, in the neighborhood of the origin \(b^*(\xi, \eta)\) is nonzero and its inverse exists.

In view of (3.32), an inverse control law \(u = u_{inv}\) is obtained which is of the form

\[
u_{inv} = (b^*(\xi, \eta))^{-1}\left[-a^*(\xi, \eta) + z_r^{(3)} - p_3(\xi_3 - \dot{z}_r) - p_2(\xi_2 - \dot{z}_r)
\right.
\]

\[
\left.-p_1(\xi_1 - z_r) - p_0x_x\right]
\tag{3.35}
\]
where $z_r$ is a reference pitch angle trajectory, $p_i > 0$ and

$$\dot{\xi}_s = \xi_1 - z_r \quad (3.36)$$

The integral feedback is introduced here in order to obtain robustness in the control system. Define for $i = 1, 2, 3$,

$$\dot{\xi}_i = \xi_i - \frac{d^{i-1}z_r}{dt^{i-1}} \quad (3.37)$$

Substituting the control law (3.35) in (3.32), the third row of (3.32) gives

$$\dot{\xi}_3 + p_3 \dot{\xi}_3 + p_2 \dot{\xi}_2 + p_1 \dot{\xi} + p_0 x_s = 0 \quad (3.38)$$

Noting from (3.37) that $\dot{\xi}_2 = \dot{\xi}_1$, $\dot{\xi}_3 = \ddot{\xi}_1$, differentiating (3.38) once, gives

$$\dddot{\xi}_1 + p_3 \dddot{\xi}_1 + p_2 \ddot{\xi}_1 + p_1 \dot{\xi}_1 + p_0 \xi_1 = 0 \quad (3.39)$$

where $\dddot{\xi}_1 = \frac{d^3 \xi}{dt^3}$. The characteristic polynomial $\Pi(s)$ associated with (3.39) is given by

$$\Pi(s) = (s^4 + p_3 s^3 + p_2 s^2 + p_1 s + p_0) \quad (3.40)$$

The coefficients $p_i$ of $\Pi$ are chosen such that it is a Hurwitz polynomial, that is, the roots of $\Pi(s) = 0$ have negative real parts.

For a choice of Hurwitz polynomial $\Pi(s)$, it follows that the system (3.39) is exponentially stable and $\dot{\xi}_1 = (\alpha_a - z_r)$ tends towards zero exponentially and the output asymptotically tracks the reference trajectory $z_r$. Furthermore, for $z_r$ chosen for a set point control such that $z_r \to 0$, a constant and the derivatives of $z_r(t)$ tend to zero, $(\alpha_a(t), \dot{\alpha}_a(t)) \to (0, 0)$ as $t \to \infty$ and from (3.23) it follows that

$$\alpha(t) = (\alpha_a(t) - \frac{1}{\mu_1} \dot{\alpha}_a(t)) \to \alpha_a(t) \to 0$$

Thus for the approximate system (3.32), the inverse control law accomplishes control of pitch angle to a desired value and follows the reference path $z_r(t)$ accurately.
Since the transfer function (3.12) is minimum phase, it follows that the equilibrium point $\eta = 0$ of the system is exponentially stable. This implies that the trajectory of

$$\dot{\eta} = q_0(\xi, \eta) \tag{3.41}$$

remains bounded as $\xi$ evolves in a small neighborhood of the origin. Thus the inverse control accomplishes tracking with local internal stability in the closed-loop system.

For the system (3.32) with the control law (3.39), gives

$$\begin{pmatrix} \dot{\xi}_s \\ \dot{\eta} \end{pmatrix} = \begin{bmatrix} A_c \xi_s \\ 0 \end{bmatrix} + \begin{pmatrix} 0 \\ q_0(\bar{\xi} + Z_r, \eta) \end{pmatrix} \tag{3.42}$$

where $\xi_s = [\xi^T, x_s]^T$, $Z_r = [z_r, \dot{z}_r, \ddot{z}_r]^T$, $\bar{Z}_r = [Z_r^T, z_r^{(3)}]^T$,

$$A_c = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -p_1 & -p_2 & -p_3 & -p_0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{3.43}$$

According to Remark 2.1, it follows that the closed-loop system (3.42) is asymptotically stable if $z_r = 0$, since the zero dynamics is exponentially stable.

Theorem 2.1 provides a sufficient condition for the boundedness of trajectory. In fact, simulation results which will be presented later shows that even in the presence of parameter uncertainty, the tracking error $(\alpha_a - z_r)$ tends to zero and bounded tracking error is obtained. The property $(P2)$ implies that the inverse controller accomplishes regulation of the trajectory to the origin exponentially when $z_r \equiv 0$.

### 3.6 Simulation Results

This section presents the results of the digital simulation of the controller designed in this chapter. The system parameters to be used are as follows [18].

$$b = 0.135m$$
\begin{align*}
\text{span} & = 0.6m \\
k_h & = 2844.4N/m \\
c_h & = 27.43Ns/m \\
c_a & = 0.036Ns \\
\rho & = 1.224kg/m^3 \\
c_{\alpha} & = 6.28 \\
c_{m\alpha} & = 3.358 \\
c_{\alpha\beta} & = (0.5 + a)c_{\alpha} \\
c_{m\beta} & = -0.635 \\
m & = 12.387Kg \\
I_a & = 0.065 \\
r_{cg} & = 0.0873 - (b + ab) \\
x_{\alpha} & = r_{cg}/b
\end{align*}

The equations of motion (3.1) and in turn the state model (3.4) are dependent on the freestream velocity \( U \) and also on the elastic axis location \( a \). The controller was designed \( U = 15m/s \) and \( a = -0.6 \). The simulations were also done considering some variations in the velocity \( U \). The initial conditions for the simulations were chosen as \( \alpha = 0.1\text{rad} \) and \( y = 0.01m \). For the chosen system parameters, the poles and zeros of the transfer function relating the flap deflection \( \beta \) (input) and the pitch \( \alpha \) (output) are as follows. Poles are at \( 3.2142 \pm 12.2890i, -5.2514 \pm 11.7595i \). The zeros are given by \( \mu_1 = 32.4594 \) and \( \mu_2 = 23.2386 \), where \( \mu_1 \) is an unstable zero. The gain of the transfer function \( k_p = 0.0842 \).

The open loop response of the system is as shown in Fig. 3.2. As discussed in [18], pp p, which is the case with the open loop response shown below.

The design of the controller was done using MATLAB and the simulation was done
Figure 3.2: Open loop response: (a) pitch, (b) plunge

using SIMULINK. The block diagram of the closed-loop controller for the pitch angle is as shown in Fig. 3.3.

A fourth order smooth reference trajectory generator was used to produce the command trajectory $z_r$ as the third derivative of $z_r$ is required for the controller. The command generator is of the form

$$(s + \lambda_c)^4 z_r = 0$$

where $\lambda_c > 0$ is appropriately chosen to obtain desirable reference trajectories. The inverse controller gain are chosen as $p_3 = 4.2$, $p_2 = -19.2$, $p_1 = 16$. The integrator was not used in the feedback to simplify the simulation.

Define the tracking error for the pitch variable as $z_e = z - z_r = \alpha - \alpha_r$. Let $z_{em}$, and $\beta_m$ be the maximum magnitudes of $z_e$, and the flap deflection $\beta$, respectively.

3.6.1 Pitch control: nominal system

The complete nominal closed-loop system (3.4) was simulated. The $\lambda_c$ of the command generator was chosen to be 5. The responses are as shown in Fig. 3.4. A smooth control of the pitch can be observed in about 2 seconds.
Figure 3.3: Simulation of the Pitch controller

The maximum tracking error is $z_{em} = 0.122 rad = 7^\circ$. The maximum flap deflection $\beta_m = 0.5 rad = 28.6^\circ$. It can be noted that both pitch as well as the plunge settles to zero in about 3 seconds.

Simulation was also done for the model as in Fig. 3.4, but higher flow velocity $U = 20 m/s$ was assumed. The poles and zeros of the transfer function relating the flap deflection $\beta$ (input) and the pitch $\alpha$ (output) are as follows. Poles are at $6.0565 \pm 12.5209i, -8.1878 \pm 11.3185i$. The zeros are given by $\mu_1 = 32.8394$ and $\mu_2 = 22.9698$, where $\mu_1$ is an unstable zero. The gain of the transfer function $k_p = 0.0842$. Selected responses are shown in Fig. 3.5. A larger control input is required in this case. The maximum required flap deflection is $\beta_m = 0.8 rad = 45^\circ$. The maximum tracking error $z_{em} = 0.06 rad = 3.4^\circ$.

The variation of the location of the zeros with the variation in the midchord distance from the elastic axis is shown in [18]. It is seen that for a value of $a > -0.55$, there are stable zeros. Thus for these cases, the modification of the output
Figure 3.4: Simulation results of the Pitch controller with nominal parameters and with $U = 15\text{m/s}$
Figure 3.5: Simulation results of the Pitch controller with nominal parameters and with $U = 20m/s$
variable is unnecessary as the system is minimum phase. The poles and zeros of the transfer function relating the flap deflection $\beta$(input) and the pitch $\alpha$(output) are as follows. Poles are at $-3.9502 \pm 11.1459i, 2.0383 \pm 11.1434i$. The zeros are given by $\mu_1 = 124.5477$ and $\mu_2 = 46.0802$, where $\mu_1$ is an unstable zero. The gain of the transfer function $k_p = 0.0101$. The responses for the system with a slight variation in the midchord distance $a$ is shown in Fig. 3.6. $U = 15m/s$ and $a = -0.55$ was chosen. The maximum required flap deflection is $\beta_m = 0.5rad = 28^\circ$. The maximum tracking error $z_{em} = 0.04rad = 2.3^\circ$.

Figure 3.6: Simulation results of the Pitch controller with nominal parameters and with $U = 15m/s$ and $a = -0.55$
In order to examine the robustness of control system, the closed-loop system was simulated with variation of ±25% in the aeroelastic system parameters. The initial conditions were chosen the same as in the previous cases. The flow velocity $U = 15\text{m/s}$ and the midchord axis distance $a = -0.6$. Selected responses for +25% parameter perturbations are shown in Fig. 3.7. Though the plunge response converged in about 2 seconds, the pitch response took a longer time (7 seconds). The maximum required flap deflection is $\beta_m = 0.9\text{rad} = 51.5^\circ$. The maximum tracking error $z_{em} = 0.03\text{rad} = 1.7^\circ$.

Simulation was also done with −25% parameter uncertainty. In this case also smooth depth control was accomplished. The maximum flap deflection required was a bit higher in this case ($\beta_m = 1\text{rad} = 57.5^\circ$). The maximum tracking error $z_{em} = 0.04\text{rad} = 2.3^\circ$. The pitch response took about 10 seconds to settle.

### 3.7 Summary

A mathematical model for the Aeroelastic wing rock problem is presented. For certain range of the midchord axis length, the system was found to have unstable zeros, which poses a major problem in exact asymptotic tracking. Thus a new normal form representation for the system and a modified output were derived in this chapter. Based on this new representation, a nonlinear inverse control law was designed for pitch angle regulation.

Extensive simulation of the closed-loop system was performed. The results shows that both pitch as well as the plunge were regulated. The simulation was also done for different values of flow velocity and the midchord axis distance. The system with some variation in the parameters was also simulated and the controller was found to be considerably robust.
Figure 3.7: Simulation results of the Pitch controller with +25\% off-nominal parameters and with $U = 15\text{m/s}$ and $a = -0.6$
The same principles were applied for the depth control of a small undersea vehicle in the next chapter.
CHAPTER 4

CONTROL OF AN UNDERSEA VEHICLE

4.1 Introduction

The development of a new generation of unmanned underwater vehicles as well as underwater vehicle-manipulator systems will be crucial in many applications including defense, hydrographic survey, deep sea-bed mining, study of aquaculture, inspection, maintenance, and repair of sub-sea production facilities, etc. In this chapter, the controllers using various control system tools are designed. The model that is considered is only for the dive plane motion, although this approach is applicable to yaw plane control. The vehicle is assumed to have a uniform forward velocity. The control of the depth of the vehicle under the influence of the surface waves is the main focus of this chapter.

It is of interest to design dorsal fin control systems for the precise control of the depth and the regulation of the pitch angle in the dive plane. The control systems are designed based on (1) nonlinear inversion, and (2) the sliding mode control technique. Interestingly, the transfer function which relates the input variable (force produced by dorsal fins) and the output variable (depth of the vehicle) is non-minimum phase, since it has an unstable zero. For the design of control systems based on inversion theory for exact asymptotic tracking, the zero dynamics must be stable.

To overcome the obstruction created by unstable zero dynamics, a new output variable for control is derived using the linearized model of the vehicle and a new state variable representation of the vehicle is obtained for the design of control systems as
discussed in Chapter 2. For asymptotic trajectory control, an inverse control law is obtained. It is shown that for the closed loop system, state trajectory is uniformly bounded and the tracking error is ultimately confined to a small set, inspite of the presence of waves. Larger feedback gains may be chosen to reduce the steady state tracking error. Although, the inverse controller accomplishes precise control when the surface waves are absent, small oscillations are observed when periodic waves are acting on the vehicle. In order to attenuate these oscillations, a servo-compensator is designed based on the internal model principle.

For the depth trajectory control, sliding mode control law is also designed for the continuous cambering of the dorsal fins in the presence of seawaves. The sliding mode control law is nonlinear and discontinuous in the state space and has an excellent insensitivity property with respect to disturbance and parameter variations.

4.2 Mathematical Model

The Mathematical model of undersea vehicles under the influence of the waves is given in the literature. This section gives the nonlinear model and the state variable representation of the longitudinal equations of motions describing the dive plane motion of the undersea vehicle under consideration.

The coupled nonlinear equations of the vehicle dynamics [28] are given as follows. Since only longitudinal motion is considered with a constant surge velocity, the heave motion and the pitch motion are of more importance.

**Heave and Pitch motion equation**

\[
m[w - uq - x_Gq - z_Gq^2] = \frac{\rho}{2} L^4 Z q \dot{q} + \frac{\rho}{2} L^3 [Z \dot{w} + Z u q] + \frac{\rho}{2} L^2 Z w u w + Z \delta
\]

\[-\frac{\rho}{2} C_D \int_{x_{tail}}^{x_{nose}} b(x)(w - x q)|w - x q|dx + f_d
\]

\[I_y \ddot{q} - m[x_G(\dot{w} - uq) - z_Gw q] = \frac{\rho}{2} L^5 M \dot{q} + \frac{\rho}{2} L^4 [M \ddot{w} + M u q] + \frac{\rho}{2} L^3 [M \dot{w} u w + u^2 M \delta]
\]

(4.1)
where $\dot{\theta} = q$. Here $\theta$ is the pitch angle, $w$ is the heave velocity (along body axis $z_b$), $x_{GB} = x_G - x_B$, $z_{GB} = z_G - z_B$, $\delta$ is the camber of the dorsal fins, $m_d$ and $f_d$ are the force and moment acting on the vehicle caused by the surface wave and $\dot{z}_f$ is the fluid velocity. The forward speed is assumed to be held steady ($u = U$) by a control mechanism. The dorsal fins produce only a normal force $(z_{\delta})$ proportional to the camber $\delta$ of the fins and can be continuously varied for the purpose of control.

The state variable representation is useful in the design of the controller which is presented in this section. The input for the system is the actuation command $\delta$. The depth of the vehicle is given by

$$\dot{z} = \dot{z}_f - u \sin \theta + w \cos \theta$$

The system equations (4.1) can be written in a state variable form as [36]

$$\begin{pmatrix}
\dot{z} \\
\dot{w} \\
\dot{q} \\
\dot{\theta}
\end{pmatrix} =
\begin{bmatrix}
-U \sin \theta + w \cos \theta + \dot{z}_f \\
a_{22}w + a_{23}q + a_{24}(x_{GB} \cos \theta + z_{GB} \sin \theta) + a_5(w, q) + d_1 \\
a_{32}w + a_{33}q + a_{34}(x_{GB} \cos \theta + z_{GB} \sin \theta) + a_6(w, q) + d_2 \\
q
\end{bmatrix}
+ \begin{bmatrix}
0 \\
b_2 \\
b_3 \\
0
\end{bmatrix} \delta$$

or

$$\dot{x} = f(x) + bu_c + Dd$$

(4.2)

where $x = (z, w, q, \theta)^T \in \mathbb{R}^4$ is the state vector (T denotes transposition), $u_c = \delta$ is the control input, $\dot{z}_f$ is the velocity of the fluid, disturbance $d = [\dot{z}_f, d_1, d_2]^T$, and input vector $b = [0, b_2, b_3, 0]^T$. Various functions $a_{ij}$, $a_k$, $b_l$ and $d_l$ are easily obtained by comparing (4.1) and (4.2).

The matrices $f(x)$, $D$ and $b$ are obtained by comparing equations (4.2) and (4.3). For the system (4.3), we are interested in designing a dorsal fin control system for the depth control and the pitch angle regulation.
4.3 Redefining Output

In the previous section, the mathematical model of the dive plane motion of the undersea vehicle under consideration was given. A nonlinear control technique, which takes into account the nonlinearities during the design stage itself is of more interest. But the system under consideration is non-minimum phase with one of the zeros in the right half plane. Thus a technique as described in Section 2.3, wherein the output variable is modified so as to make the modified transfer function, minimum phase.

The Jacobian linearization of the system given in (4.1) (with \(d \neq 0\)) about the equilibrium state \((z^*, w^*, q^*, \theta^*) = 0\), gives

\[
\dot{x} = Ax + bu_c + Dd
\]  

(4.4)

where \(A \in \mathbb{R}^{4 \times 4}\) and is given by

\[
A = \left[ \frac{\partial f(0)}{\partial x} \right]
\]

\[
= \begin{bmatrix}
0 & 1 & 0 & -U \\
0 & a_{22} & a_{23} & a_{24}z_G \\
0 & a_{32} & a_{33} & a_{34}z_G \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

Since depth variable \(z\) is of interest, consider an output associated with the system (4.4),

\[
z = Cx
\]  

(4.5)

where \(C = [1, 0, 0, 0]\).

Taking the laplace transform of (4.4), one obtains the transfer function relating \(z\) and \(u_c\) given by

\[
\frac{Z(s)}{U_c(s)} = H(s) = \frac{k_p(s - \mu_1)(s + \mu_2)}{d_p(s)}
\]  

(4.6)

\[
d_p(s) = s^4 + m_3s^3 + m_2s^2 + m_1s
\]

where \(s\) denotes the laplace variable or a differential operator, \(Z\) and \(U_c\) are the laplace transforms of \(z\) and \(u_c\). For the vehicle under study, \(\mu_1, \mu_2 > 0\) are the two real zeros.
of the transfer function $H(s)$, and $d_p(s)$ is a fourth order polynomial of variable $s$ having coefficients $m_i$.

Now in order to obtain tracking with internal stability in the closed-loop system, an approximate transfer function is considered as shown in Chapter 2 which possesses only stable zeros. The approximate transfer function is chosen as

$$H_a(s) = \frac{k_p(s - \mu_1)(s + \mu_2)}{(1 - \frac{s}{\mu_1})d_p(s)} = \frac{-\mu_1 k_p(s + \mu_2)}{d_p(s)}$$

In the new transfer function stable zero and all the poles have been retained and it is a minimum phase transfer function.

Now a state space representation of system (4.3) is obtained. In view of the transfer functions (4.6) and (4.7), it follows that there exists a row vector $\tilde{C} \in \mathbb{R}^4$ such that

$$H_a(s) = \tilde{C}(sI - A)^{-1}b = \frac{-\mu_1 k_p(s + \mu_2)}{d_p(s)}$$

where $A$ and $b$ are as defined in (4.4).

The row vector $\tilde{C}$ can be obtained using the result given by Lemma 3.1 which is given by

$$\tilde{C} = [0, 0, -\mu_1 k_p, -\mu_1 \mu_2 k_p] L_c^{-1}$$

The system (4.4) is controllable, therefore $L_c$ is nonsingular and there exists a unique solution of (4.4) given by (4.9).

In the next section, normal form representation of the system is obtained. This is similar to that given in Section 2.4. But here, the disturbance is also taken into consideration for the representation.

### 4.4 Normal Form Representation

It is interesting to obtain a normal form representation of the nonlinear system which is useful for stability analysis and the design of an inverse control system. Lie
derivatives of a function \( \alpha(x) \) with respect to the vector field \( f(x) \) are defined as in (2.18).

Define

\[
L_D L_f^k(\alpha)(x) = \frac{\partial L_f^k(\alpha)}{\partial x} D
\]

(4.10)

Consider nonlinear functions

\[
\begin{align*}
\phi_1 &= \tilde{C} x = z_a \\
\phi_2 &= L_f(\tilde{C} x)(x) \\
\phi_3 &= L_f^2(\tilde{C} x)(x)
\end{align*}
\]

The nonlinear function \( f(x) \) is defined as

\[
f(x) = Ax + g(x)
\]

(4.12)

where \( g(x) \) are of order \( O(x)^2 \). In view of (4.12), one has

\[
\begin{align*}
\phi_2 &= \tilde{C} A x + L_g(\tilde{C} x) \\
\phi_3 &= \tilde{C} A^2 x + \tilde{C} A g(x) + L_f L_g(\tilde{C} x)
\end{align*}
\]

(4.13)

where \( L_g(\tilde{C} x) \) and \( L_f L_g(\tilde{C} x) \) are of order \( O(x)^2 \).

The differential \( d\Phi \) of \( \Phi \) evaluated at \( x = 0 \) gives,

\[
d\Phi(0) = \begin{bmatrix}
\frac{\partial \phi_1}{\partial x}(0) \\
\frac{\partial \phi_2}{\partial x}(0) \\
\frac{\partial \phi_3}{\partial x}(0)
\end{bmatrix} = \begin{bmatrix}
\tilde{C} \\
\tilde{C} A \\
\tilde{C} A^2
\end{bmatrix}
\]

(4.14)

The \( 3 \times 4 \) matrix \( d\Phi(0) \) has rank 3 in view of (3.18). This implies that \( \phi_i \) (i=1,2,3) are independent functions in the neighborhood of \( x = 0 \). According to (3.18), the vectors \( \tilde{C} \) and \( \tilde{C} A \) are orthogonal to \( b \). Therefore, there exists a row-vector \( C_0 \) such that \( C_0 b = 0 \) and \( \tilde{C}, \tilde{C} A, \tilde{C} A^2 \) and \( C_0 \) are independent (the computation of \( C_0 \) is not
necessary for controller synthesis). Let \( \eta(x) = C_0 x \). Define new coordinates, \((x, \eta)\), by
\[
\begin{bmatrix}
\xi_1 \\
\xi_2 \\
\xi_3 \\
\eta
\end{bmatrix} = \begin{bmatrix}
\phi_1(x) \\
\phi_2(x) \\
\phi_3(x) \\
\eta(x)
\end{bmatrix} = \Phi(x)
\tag{4.15}
\]
The Jacobian matrix \(d\Phi(x)\) evaluated at \(x = 0\) is given by
\[
d\Phi(0) = \begin{bmatrix}
\tilde{C} \\
\tilde{C}A \\
\tilde{C}A^2 \\
C_0
\end{bmatrix}
\tag{4.16}
\]
Since rank of \(d\Phi(x)\) at \(x = 0\) is 4, the transformation \(\Phi : x \rightarrow [\phi_1, \phi_2, \phi_3, \eta]^T\) is a local diffeomorphism in the neighborhood \(x = 0\).

A representation of the nonlinear system is now obtained in the normal form in \((\xi, \eta)\) coordinates [6]. Differentiating \(\xi_i\) gives
\[
\dot{\xi}_1 = L_f(\tilde{C}x)(x) + \tilde{C}Dd
\]
\[
\dot{\xi}_2 = L_f^2(\tilde{C}x)(x) + [L_b L_f(\tilde{C}x)(x)]u_c + [L_D L_f(\tilde{C}x)(x)]d
\tag{4.17}
\]
\[
\dot{\xi}_3 = L_f^2(\tilde{C}x)(x) + [L_b L_f^2(\tilde{C}x)(x)]u_c + [L_D L_f^2(\tilde{C}x)(x)]d
\]
\[
\dot{\eta} = L_f(C_0 x)(x) + C_0 Dd
\]
Using \(x = \Phi^{-1}(\xi, \eta)\), \(\xi = [\xi_1, \xi_2, \xi_3]^T\), one writes (4.17) in the matrix notation as
\[
\begin{bmatrix}
\dot{\xi}_1 \\
\dot{\xi}_2 \\
\dot{\xi}_3 \\
\dot{\eta}
\end{bmatrix} = \begin{bmatrix}
\xi_2 \\
\xi_3 \\
a^*(\xi, \eta) + b^*(\xi, \eta)u_c \\
q_0(\xi, \eta)
\end{bmatrix} + \begin{bmatrix}
\psi_1 \\
\psi_2(\xi, \eta) \\
\psi_3(\xi, \eta) \\
\psi_4
\end{bmatrix}d + \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix} u_c
\tag{4.18}
\]
where \(\Phi^{-1}(\xi, \eta)\) is substituted for \(x\), \(\psi_1 = \tilde{C}D\), \(\psi_4 = C_0 D\), \(\psi = [\psi_1, \psi_2, \psi_3, \psi_4]^T\).

\[
a^* = L_f^2(\tilde{C}x)(x)
\]
\[
b^* = L_b L_f^2(\tilde{C}x)(x)
\tag{4.19}
\[ q_0 = C_0 A x + C_0 g(x) \]
\[ \psi = [\psi_1, L_D L_f(\phi_1)(x), L_D L_f^2(\phi_1)(x), \psi_4]^T \]

Since \( \tilde{C} A b = 0 \), the function \( \psi_u \) is given by
\[
\psi_u = L_b L_f(\tilde{C} x)(x)
\]
\[
= \tilde{C} A b + \frac{\partial (\tilde{C} g)}{\partial x} b
\]
\[
= \frac{\partial (\tilde{C} g)}{\partial x} b
\]

It can be noted that \( \psi_u u_c \) is of order \( O(\xi, \eta, u_c)^2 \).

For small values of \( \xi, \eta, u_c \) and \( d \), functions \( \psi d \) and \( \psi u u_c \) can be neglected to obtain an approximate representation of the nonlinear system as
\[
\begin{pmatrix}
\dot{\xi}_1 \\
\dot{\xi}_2 \\
\dot{\xi}_3 \\
\dot{\eta}
\end{pmatrix} =
\begin{bmatrix}
\xi_2 \\
\xi_3 \\
a^*(\xi, \eta) + b^*(\xi, \eta) u_c \\
qu(\xi, \eta)
\end{bmatrix}
\]

The approximate system (4.21) with output \( \xi_1 \) has relative degree 3, but the relative degree of the true system (4.18) is not well defined at \( x = 0 \), since \( \psi_u(0) = 0 \) but \( \psi_u(x) \) is not zero in the neighborhood of \( x = 0 \). Using this normal form representation, the inverse control law for the undersea vehicle under consideration is presented.

4.5 Inverse Control Law

It is of interest to derive an inverse control law for the system (4.18) which requires the inverse of the function \( \psi_u \). Such an inverse control law for the control of \( z_a = \xi_1 \) will require infinitely large control magnitude as \( x \) tends to zero. To overcome this difficulty, we shall derive an inverse control law based on the approximate nonlinear system (4.21).

It can be verified that
\[
b^* = \tilde{C} A^2 b + \left[ \frac{\partial}{\partial x} (\tilde{C} A g + L_f(\tilde{C} g(x))) \right] b
\]
At \( x = 0, b^* = CA^2b \neq 0 \), therefore, in the neighborhood of the origin \( b^*(\xi, \eta) \) is nonzero and its inverse exists.

In view of (4.21), an inverse control law \( u_c = u_{cinv} \) is obtained which is of the form [6, 39]

\[
\begin{align*}
\dot{x}_s &= \xi_1 - z_r \quad (4.23)
\end{align*}
\]

The integral feedback is introduced here in order to obtain robustness in the control system. Define for \( i = 1, 2, 3 \),

\[
\ddot{\xi}_i = \xi_i - \frac{d^{i-1}z_r}{dt^{i-1}} \quad (4.24)
\]

Substituting the control law (4.22) in (4.21), the third row of (4.21) gives

\[
\begin{align*}
\dot{\xi}_3 + p_3\dot{\xi}_3 + p_2\dot{\xi}_2 + p_1\dot{\xi}_1 + p_0x_s &= 0 \\
(4.25)
\end{align*}
\]

Noting from (4.24) that \( \ddot{\xi}_2 = \ddot{\xi}_1, \ddot{\xi}_3 = \ddot{\xi}_1 \), differentiating (4.25) once, gives

\[
\begin{align*}
\dddot{\xi}_1^{(4)} + p_3\dddot{\xi}_1^{(3)} + p_2\dddot{\xi}_1^{(2)} + p_1\dddot{\xi}_1^{(1)} + p_0\dddot{\xi}_1 &= 0 \\
(4.26)
\end{align*}
\]

where \( \dddot{\xi}_i^{(j)} = \frac{\partial^{(j)}\dot{\xi}_i}{\partial t^{(j)}} \). The characteristic polynomial \( \Pi(s) \) associated with (4.26) is given by

\[
\Pi(s) = (s^4 + p_3s^3 + p_2s^2 + p_1s + p_0) \\
(4.27)
\]

The coefficients \( p_i \) of \( \Pi \) are chosen such that it is a Hurwitz polynomial, that is, the roots of \( \Pi(s) = 0 \) have negative real parts.

For a choice of Hurwitz polynomial \( \Pi(s) \), it follows that the system (4.26) is exponentially stable and \( \dddot{\xi}_1 = (z_a - z_r) \) tends towards zero exponentially and the output asymptotically tracks the reference trajectory \( z_r \). Furthermore, for \( z_r \) chosen for a set point control such that \( z_r \to z_r^* \), a constant and the derivatives of \( z_r(t) \) tend
to zero, \((z_a(t), \dot{z}_a(t)) \rightarrow (z_r^*, 0)\) as \(t \rightarrow \infty\) and from steady state condition it follows that

\[
z(t) = (z_a(t) - \frac{1}{\mu_1} \dot{z}_a(t)) \rightarrow z_a(t) \rightarrow z_r^*
\]

Thus when \(d=0\), for the approximate system (4.21), the inverse control law accomplishes control of depth to a desired value and follows the reference path \(z_r(t)\) accurately.

Since the transfer function (4.7) is minimum phase, it follows that the equilibrium point \(\eta = 0\) of the system

\[
\dot{\eta} = q_0(0, \eta)
\]

is exponentially stable. This implies that the trajectory of

\[
\dot{\eta} = q_0(\xi, \eta)
\]

remains bounded as \(\xi\) evolves in a small neighborhood of the origin. Thus the inverse control accomplishes tracking with local internal stability in the closed-loop system.

For the actual system (4.18) with the control law (4.26), gives

\[
\begin{pmatrix}
\dot{\xi}_s \\
\dot{\eta}
\end{pmatrix} = 
\begin{bmatrix}
A_c & 0 \\
0 & 0
\end{bmatrix} + 
\begin{pmatrix}
0 \\
q_0(\xi + Z_r, \eta)
\end{pmatrix} + 
\begin{pmatrix}
\psi_c(\xi + Z_r, \eta, u_c, d) \\
\psi_d
\end{pmatrix}
\]

(4.30)

where \(\dot{\xi}_s = [\xi^T, x_s]^T, Z_r = [z_r, \dot{z}_r, \ddot{z}_r]^T, Z_r = [Z_r^T, z_r^{(3)}]^T,\)

\[
\psi_{2a}(\xi + Z_r, \eta, u_c, d) = \psi_{2}(\xi + Z_r, \eta)d + \psi_u(\xi + Z_r, \eta)u_c
\]

(4.31)

\[
A_c = 
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-p_1 & -p_2 & -p_3 & -p_0 \\
1 & 0 & 0 & 0
\end{bmatrix}
\]

(4.32)

\[
\psi_c = 
\begin{bmatrix}
\psi_1d \\
\psi_2a \\
\psi_3d \\
0
\end{bmatrix}
\]

(4.33)
It is interesting to examine the tracking ability of the controller (4.22) when it is synthesized for the original system (4.3). It is reasonable to expect that in the presence of small disturbance, the inverse control system will have accurate trajectory following capability as long as the state trajectory remains in a small neighborhood of the origin. Indeed, the following result can be stated relating the system (4.3).

**Theorem 4.1**: Consider the closed-loop system (4.30) which represents the system (4.3) including the inverse control law (4.22). Suppose that the reference trajectory $z_r$ and its derivatives are sufficiently small, then the closed-loop system has the following properties,

**P1**: There exists a neighborhood $U_e$ of $x = 0$ such that the trajectory beginning in $U_e$ are bounded.

**P2**: For $z_r(t) \equiv 0$, and $d(t) \equiv 0$, the equilibrium point $(\xi, x_s, \eta) = 0$ is exponentially stable.

Proof of P1 is not given here, since it can be proved following the steps in Theorem 2.1. For proving property P2, it can be noted that for $d(t) \equiv 0$ and $z_r \equiv 0$, the function $\psi_c$ in (4.30) simplifies to $\psi_c = [0, \psi_u(\xi, \eta)u_c, 0, 0]^T$, and one has $u_c = u_c(\xi, \eta)$. The function $\psi_c(\xi, \eta)u_c(\xi, \eta)$ is of order $O(\xi, \eta)^2$. One notes that the linearized system about the origin obtained from (4.30) is exponentially stable since $A_c$ is a stable matrix and that the zero dynamics is exponentially stable since the transfer function $H_a$ is minimum phase. Then exponential stability of $(\xi, x_s, \eta) = 0$ follows easily since the perturbation functions $\psi_c$ and $c_0g$ are of order $O(\xi, \eta)^2$ [46].

Theorem 4.1 only provides a sufficient condition for the boundedness of trajectory. In fact, simulation results which will be presented later, show that for large maneuvers in the presence of parameter uncertainty, the tracking error $(z_a - z_r)$ tends to zero.
due to the integral error feedback in the inverse control law and bounded tracking error is obtained when the disturbance inputs are time-varying. The property (P2) implies that the inverse controller accomplishes regulation of the trajectory to the origin exponentially when $d \equiv 0$ and $z_r \equiv 0$.

4.6 Servo compensation for attenuation of wave effect

As shown in the previous subsection, bounded tracking error is obtained when $d$ is not zero. Assuming that wave induced forces are periodic, it is possible to minimize the effect of waves. For this purpose, the internal model principle is useful. The output considered here is the modified output given in Section 4.3. For simplicity in presentation, compensation only for sinusoidal disturbance of single frequency shall be considered. This approach can be extended by designing servo-compensators of higher order to include the modes of each disturbance component, when the disturbance input consists of sinusoidal functions of different frequencies.

Suppose the wave frequency is $\omega_0$ and the elements of $d(t)$ are a sinusoidal functions of frequency $\omega_0$. The amplitudes and phase angles of $d_i(t)$ are assumed to be unknown. Then a servo compensator of the form

$$\begin{align*}
\dot{x}_c1 &= x_c2 \\
\dot{x}_c2 &= -\omega_0^2 x_c1 + (\xi_1 - z_r)
\end{align*}
$$

(4.34)

for minimizing the effect of disturbance caused by the waves is introduced. Note that the input to the servocompensator is the tracking error $(\xi_1 - z_r)$. The system (4.34) can be written in a matrix form as

$$\dot{x}_c = A_1 \ddot{\xi}_s + A_2 x_c
$$

(4.35)

where

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix}. $$
The control input is now selected as

\[ u_c = u_{c_{inv}} + b^{-1}(\xi, \eta)v \]  

(4.36)

where \( v \) is an auxiliary control signal and \( u_{c_{inv}} \) is the inverse control law derived in (4.22). For \( d = 0 \), the linearized system obtained from (4.30) with control law (4.36) including the servocompensator (4.35), can be expressed in a compact form as

\[
\begin{pmatrix}
\dot{\xi}_s \\
\dot{x}_c
\end{pmatrix} =
\begin{bmatrix}
A_c & 0 \\
A_1 & A_2
\end{bmatrix}
\begin{pmatrix}
\xi_s \\
x_c
\end{pmatrix} +
\begin{pmatrix}
b_c \\
0
\end{pmatrix} v
\]

(4.37)

\[ \triangleq A_a(\xi_s^T, x_c^T)^T + b_a v \]

where \( b_c = [1, 0, 0, 0]^T \). The auxiliary signal \( v \) is chosen of the form

\[ v = F_1 \xi_a + F_2(x_{c1}, x_{c2})^T \triangleq F_a \xi_a \]  

(4.38)

where \( \xi_a = [\xi_s^T, x_c^T]^T \) and \( F_a = (F_1, F_2) \). The feedback gain matrices \( F_i \) are obtained using linear optimal control theory or pole placement techniques. Here optimal control technique has been used. For obtaining the linear feedback law (4.38), a performance index of the form

\[ J = \int_0^\infty (\xi_a^T Q_a \xi_a + r_a v^2) dt \]

is chosen for minimization where \( Q_a \) is a positive definite symmetric matrix and \( r_a > 0 \) are weighting parameters. The optimal control is obtained by solving the Riccati equation

\[ A_a^T S + K_a A_a - S b_a r_a^{-1} b_a^T S + Q_a = 0 \]  

(4.39)

for \( S \) and the feedback matrix \( F_a \) is given by [47],

\[ F_a = -r_a^{-1} b_a^T S \]  

(4.40)

The optimal control law (4.38) is such that the matrix \( (A_a + b_a F_a) \) is stable. Based on results of [43], it follows that in the linearized system, for any sinusoidal disturbance of frequency \( \omega_0 \), \( \tilde{\xi}_1 = z_a - z_r \to 0 \), as \( t \to \infty \).
Although, for nonlinear system, the convergence of tracking error is difficult to prove, simulation results show that good asymptotic tracking is accomplished using the control law (4.36). Note that $v$ should be small so that tracking performance of the inverse controller is not affected considerably by the auxiliary control signal $u$. The control signal $v$ is only for attenuating the effect of waves.

Since exact asymptotic tracking control of depth of a non-minimum phase system cannot be accomplished, a new normal form representation of the system and a modified output were derived. Based on this new representation, a nonlinear inverse control law was designed for depth control and pitch angle regulation. In the closed-loop system, the trajectories were shown to be bounded and the origin of the disturbance free system with zero command input was proved to be exponentially stable. A servocompensator was designed for the rejection of sinusoidal disturbance inputs caused by waves.

A Lyapunov-based feedback design technique is presented in the next section which is more effective in the presence of model uncertainties and the disturbance.

### 4.7 Sliding Mode Control

In the previous section, linear control technique was introduced for the tracking problem. It was found that the inverse controller along with the servocompensator was effective in attenuating the sinusoidal wave-effect as well as, less susceptible to parameter variation. However, it is essential to know the frequency of the wave for design. It is interesting to design control system which is insensitive to large parameter uncertainty and time-varying disturbance input and not necessarily only for sinusoidal disturbance inputs.

Many feedback control techniques are based on the idea of designing the feedback control in such a way that a Lyapunov function, or more specifically the derivative of a
Lyapunov function, has certain properties that guarantee boundedness of trajectories and convergence to an equilibrium point or an equilibrium set. In this section, a Lyapunov-based design approach, namely Sliding mode control is introduced. In sliding mode control, trajectories are forced to reach a sliding manifold in finite time, and stay on the manifold for all future time. This method allows the model to be more imprecise and permits attenuation of disturbance caused by arbitrary waves.

A control law using the sliding mode control concept is derived for the depth control of the undersea vehicles, which is later simulated.

4.7.1 Derivation of control law

In this subsection, a dorsal fin control system is designed for depth control assuming that there is no disturbance. As is the case with any of the Lyapunov-based design, sliding mode controller requires that the system be non-minimum phase for achieving perfect tracking [37]. Thus as described in Sec. 4.3, a new output variable is constructed and the controller is designed for that output variable. Later it is shown that there is a bounded error in the presence of the waves. The reference trajectory that is considered is the same as that used for the previous two techniques which is given by a fourth order command generator of the form

\[(s + \lambda_c)^4 z_r - \lambda_c^4 z_{r*} = 0 \tag{4.41}\]

where \(\lambda_c > 0\).

The normal form representation for the system under consideration with \(d = 0\), which is derived in Section 4.4, is given by

\[
\begin{pmatrix}
\dot{\xi}_1 \\
\dot{\xi}_2 \\
\dot{\xi}_3 \\
\dot{\eta}
\end{pmatrix} = 
\begin{bmatrix}
\xi_2 \\
\xi_3 \\
a^*(\xi, \eta) + b^*(\xi, \eta)u_c \\
q_0(\xi, \eta)
\end{bmatrix}
\tag{4.42}
\]
where the new coordinates \((\xi, \eta)\), the functions \(q_0, a^*\) and \(b^*\) are as defined in Sec. 4.4.

In view of (4.42), a sliding mode control law \(u_c = u_{c \text{slide}}\) is obtained which is of the form

\[
u_{c \text{slide}} = (b^*(\xi, \eta))^{-1}[-a^*(\xi, \eta) + z^{(3)} + v] \tag{4.43}
\]

where \(p_i > 0\) and \(v\) is given by the sliding surface. Define for \(i = 1, 2, 3\),

\[
\dot{\xi}_i = \xi_i - \frac{d^{i-1}z_r}{dt^{i-1}} \tag{4.44}
\]

Substituting the control law (4.43) in (4.42), the third row of (4.42) gives

\[
\ddot{\xi}_3 = v \tag{4.45}
\]

The sliding surface \(S = 0\) is defined as

\[
S = \dot{\xi}_3 + \lambda_2 \dot{\xi}_2 + \lambda_1 \dot{\xi}_1 + \lambda_0 \int_0^t \dot{\xi}_1 d\tau \tag{4.46}
\]

which can be written as

\[
S = \ddot{\xi}_1 + \lambda_2 \dot{\xi}_2 + \lambda_1 \dot{\xi}_1 + \lambda_0 \int_0^t \dot{\xi}_1 d\tau \tag{4.47}
\]

The parameters \(\lambda_i\) are chosen such that the polynomial \(\Pi(s) = s^3 + \lambda_2 s^2 + \lambda_1 s + \lambda_0\) is hurwitz.

Using (4.47) and (4.45),

\[
\dot{S} = \lambda_2 \ddot{\xi}_1 + \lambda_1 \dot{\xi}_1 + \lambda_0 \dot{\xi} + v \tag{4.48}
\]

\(v\) is given by [37, 45]

\[
v = -\lambda_2 \ddot{\xi}_1 - \lambda_1 \dot{\xi}_1 - \lambda_0 \dot{\xi} - k_1 S - k_2 \text{sgn}(S) \tag{4.49}
\]

where \(k_1\) and \(k_2\) are chosen appropriately to get a lesser bounded error.
One approach to eliminate chattering is to use a continuous approximation of the discontinuous sliding mode controller. By using continuous approximation, the theoretical difficulties associated with discontinuous controllers are avoided. Instead of using the signum nonlinearity, its saturation function approximation is used.

Thus after smoothening, $v$ is given by

$$ v = -\lambda_2 \ddot{x}_1 - \lambda_1 \dot{x}_1 - \lambda_0 \dot{x}_1 - k_1 S - k_2 \text{sat}(S/\epsilon) \tag{4.50} $$

where the saturation function is defined by

$$ \text{sat}(y) = \begin{cases} y, & \text{if } |y| \leq 1 \\ \text{sgn}(y), & \text{if } |y| > 1 \end{cases} \tag{4.51} $$

and $\epsilon$ is a positive constant. The slope of the linear portion of $\text{sat}(S/\epsilon)$ is $1/\epsilon$. A good approximation requires the use of small $\epsilon$. In the limit, as $\epsilon \to 0$, the saturation nonlinearity $\text{sat}(S/\epsilon)$ approaches the signum nonlinearity $\text{sgn}(S)$. When $\epsilon$ is too small, the high-gain feedback in the linear portion of the saturation function may excite unmodeled high-frequency dynamics. Therefore, the choice of $\epsilon$ is a tradeoff between accuracy on one hand and robustness to unmodeled high-frequency dynamics on the other hand.

Thus the sliding mode control law is given by

$$ u_{\text{slide}} = (b^*(\xi, \eta))^{-1}[-a^*(\xi, \eta) + z_r^{(3)} - \lambda_2 (\xi_3 - \dot{z}_r) - \lambda_1 (\xi_2 - \dot{z}_r) - \lambda_0 (\xi_1 - z_r) - k_1 S - k_2 \text{sat}(S/\epsilon)] \tag{4.52} $$

For a choice of the polynomial $\Pi(s)$ be Hurwitz, $\dot{\xi}_1 = (z_a - z_r)$ tends towards zero exponentially and the output asymptotically tracks the reference trajectory $z_r$. Furthermore, for $z_r$ chosen for a set point control such that $z_r \to z_*^r$, a constant and the derivatives of $z_r(t)$ tend to zero, $(z_a(t), \dot{z}_a(t)) \to (z_*^r, 0)$ as $t \to \infty$ and from (3.23) it follows that

$$ z(t) = (z_a(t) - \frac{1}{\mu_1} \dot{z}_a(t)) \to z_a(t) \to z_*^r $$

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Thus when $d=0$, for the approximate system (4.42), the sliding control law accomplishes control of depth to a desired value and follows the reference path $z_r(t)$ accurately.

In the next subsection, the effect of the disturbance in this control system is explored.

### 4.7.2 Effect of Disturbance

In this subsection, it is shown that the error is bounded in the presence of sinusoidal disturbance for the linearized model. Consider the model,

\[
\dot{x} = Ax + bu + D\omega
\]

\[
y = Cx
\]

(4.53)

where $\omega$ is the disturbance. For the linear model,

\[
\begin{bmatrix}
\dot{\xi}_1 \\
\dot{\xi}_2 \\
\dot{\xi}_3
\end{bmatrix} =
\begin{bmatrix}
Cx - y_r \\
CAx - y_r \\
CA^2x - y_r
\end{bmatrix}
\]

(4.54)

Differentiating (4.54) and using (4.53), it can be shown that

\[
\begin{align*}
\dot{\xi}_1 &= \dot{\xi}_2 + CD\omega \\
\dot{\xi}_2 &= \dot{\xi}_3 + CAD\omega \\
\dot{\xi}_3 &= CA^3x - y_r^{(3)} + CA^2bu + CA^2D\omega
\end{align*}
\]

(4.55)

Thus,

\[
u = (CA^2b)^{-1}[-CA^3x + y_r^{(3)} + v]
\]

(4.56)

The sliding surface that has been considered for the control law is of the form

\[
S = \dot{\xi}_3 + \lambda_2\dot{\xi}_2 + \lambda_1\dot{\xi}_1
\]
\[
\begin{align*}
\dot{\xi}_2 &= -CAD\omega + \lambda_2(\dot{\xi}_1 - CD\omega) + \lambda_1\dot{\xi}_1 \\
\dot{\xi}_1 &= -CD\omega - CAD\omega + \lambda_2(\dot{\xi}_1 - CD\omega) + \lambda_1\dot{\xi}_1 \\
&= (\dot{\xi}_1 + \lambda_2\dot{\xi}_1 + \lambda_1\dot{\xi}_1) - CAD\omega - CD\omega - \lambda_2 CD\omega
\end{align*}
\]

It can be noted that for simplicity, the integral term in \( S \) has been ignored here.

Define \( g = (CAD\omega + CD\omega + \lambda_2 CD\omega) \). Consider now

\[
\frac{1}{2} \frac{d}{dt} |S|^2 = S\dot{S}
\]

\[
= S[v + CA^2D\omega + \lambda_2(\dot{\xi}_3 + CAD\omega) + \lambda_1(\dot{\xi}_2 + CAD\omega)]
\]

Choose,

\[
v = -\lambda_2\dot{\xi}_3 - \lambda_1\dot{\xi}_2 - k \text{sgn}(S)
\]

\( k \) is given by

\[
k = k_{\text{max}} + \mu
\]

and

\[
k_{\text{max}} \geq |(CA^2D + \lambda_2 CAD + \lambda_1 CD)||\omega|
\]

Furthermore,

\[
|S| \frac{d}{dt} |S| \leq -k \text{sgn}(S) S + |S|k_{\text{max}}
\]

Therefore,

\[
\frac{d}{dt} |S| \leq -\mu
\]
This implies that $|S| \to 0$ in finite time. During the sliding phase when $S = 0$, for a sinusoidal disturbance of frequency $\omega_0$, one has

$$\ddot{\xi}_1 + \lambda_2 \dot{\xi}_1 + \lambda_1 \xi_1 = S + g(t)$$

$$= A\sin(\omega_0 t + \phi)$$

After a finite time $t_*$, where $g(t) - A\sin(\omega_0 t + \phi)$ and $A$ and $\phi$ are arbitrary constants. The steady state value is given by,

$$\ddot{\xi}_{1*} = |H(j\omega_0)|A\sin(\omega_0 t + \phi + \angle H(j\omega_0))$$

where $H(s) = 1/[s^2 + \lambda_2 s + \lambda_1]$ 

Amplitude of $\ddot{\xi}_1 = A|H(j\omega_0)|$.

$$|H(j\omega_0)| = \frac{1}{\sqrt{(\lambda_1 - \omega_0^2)^2 + (\lambda_2 \omega_0)^2}}$$  \hspace{1cm} (4.62)

Thus the amplitude of the tracking error which is proportional to $|H| \to 0$ as $\lambda_1, \lambda_2 \to \infty$. Though there is a finite tracking error, it is bounded and can be decreased by increasing $\lambda_1$ and $\lambda_2$.

Using the modified output derived, a sliding mode controller was developed for the depth control of the undersea vehicle under consideration.

4.8 Summary

In this chapter, the theory formulated in the Chapter 2 were applied to a small undersea vehicle, whose dive plane motion is non-minimum phase. A mathematical model for the longitudinal equations of motion is presented. For an effective trajectory tracking, the controller design was done using the inversion principle and sliding mode controller principle. Both required the system to be minimum phase. Thus a modified output was constructed, for which the controller were designed.
CHAPTER 5

CONCLUSION AND RECOMMENDATIONS

5.1 Conclusion

Non-minimum phase systems pose a major bottleneck for the trajectory tracking problems. The focus of this thesis is to obtain an approximate minimum phase system corresponding to a class of non-minimum phase systems. This was done using the output-redefinition method. Further using the modified output, the feedback linearization principle for the exact trajectory tracking of a nonlinear system is presented. Later these theory were applied for two practical systems, which were non-minimum phase having unstable zeros.

Flutter, which is an oscillatory aeroelastic instability caused by unsteady aerodynamic loads, is a common occurrence for the aircraft wings. The first controller was designed for the control of the pitch angle and the regulation of the plunge displacement for an Aeroelastic system. For certain range of the midchord distance from the elastic axis, the model was found to exhibit unstable zero dynamics. Thus a modified output was constructed so as to obtain an approximate minimum phase system. For this system, the feedback linearization technique was applied to design a control law. Extensive simulation of the closed-loop system showed that the inverse controller accomplished accurate depth trajectory tracking even in the presence of parameter uncertainty. Simulation results were also shown for different values of the freestream velocity and the midchord distances. The transient tracking error can be attributed to the unmatched initial conditions of the reference trajectory generator.
The matched initial conditions required a large flap deflection. For this work, the reference trajectory was fixed based on the modified output, which is not the same as the original output in the transient part. Generation of a reference trajectory for the modified output based on a specified trajectory for the original output will possibly improve the transient tracking error.

Maneuvering of a small undersea vehicle in the presence of free surface wave was considered in the later part of the thesis. To achieve a nearly perfect tracking in the dive plane, the inverse control law was derived for the nonlinear model. Since exact asymptotic tracking control of depth of a non-minimum phase system cannot be accomplished, a new normal form representation of the system and a modified output were derived. Based on this new representation, a nonlinear inverse control law was designed for depth control and pitch angle regulation. In the closed-loop system, the trajectories were shown to be bounded and the origin of the disturbance free system with zero command input was proved to be exponentially stable. A servocompensator was designed for the rejection of sinusoidal disturbance inputs caused by waves. Since the servocompensator is not essential for control, it can be turned off in the disturbance free environment. Finally, sliding mode controller was also designed for the nonlinear model. The new representation with the modified output giving a minimum phase system was used for the derivation of the control law. This technique is more effective in the presence of the parameter uncertainty and the disturbance input compared to the controllers designed using the servomechanism theory and nonlinear inversion.

5.2 Future Recommendations

Although, the controller design accomplishes trajectory control, the transient performance is very much dependent on the location of unstable zeros and the nonlinear-
ity in the system. Research is needed to obtain a qualitative assessment of tracking performance of the designed controller based on the modified output. It seems that as the nonlinearity gets stronger, tracking error increases. Furthermore, even for the linearized system, the effect of location of unstable zeros on transient performance needs to be examined. Trajectory planning is also essential to decrease the transient tracking error.

The plunge displacement control for the aeroelastic system with unstable zero dynamics needs to be done. Control of the undersea vehicle with both caudal and dorsal fin remains to be done. Also yaw plane control is yet to be explored. Design of complex maneuvering ability for this vehicles will also be of great importance. Control using only measured variables is of considerable interest.
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