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A Front-fixing finite element method for the valuation of American options

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A FRONT-FIXING FINITE ELEMENT METHOD FOR
THE VALUATION OF AMERICAN OPTIONS

by

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2002

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A dissertation submitted in partial fulfillment of
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THE GRADUATE COLLEGE

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ABSTRACT

A Front-Fixing Finite Element Method for the Valuation of American Options

by

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American options are the most commonly traded options in the market. They are used to mitigate risk, speculate about the future, and are the key components of complex trading strategies. In this dissertation, we propose a new front-fixing finite element method for the valuation of American options. One of the attractive qualities of our front-fixing finite element method is that the early exercise boundaries and the option prices can be computed simultaneously with very high accuracy.

We study in detail our front-fixing finite element method for the valuation of American options on stocks, American options on zero-coupon bonds under a class of one-factor models of the short interest rate, and American options on stocks under a regime-switching model. In all three cases we establish stability, present numerical results, examine our method, and compare it with others.

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CHAPTER 1

INTRODUCTION

1.1 The American option

A financial derivative is a contract whose value is derived from the value of some underlying asset. This underlying asset can theoretically be anything, but in practice it is usually an equity, commodity, or bond. Derivatives can be used to hedge, or mitigate, the risk of economic loss arising from changes in the value of the underlying asset. Derivatives can also be used to acquire risk or to speculate about the future. Despite their abuse by some speculative investors, derivatives remain a vital and necessary tool in the financial arena.

In this dissertation we shall study a very popular type of financial derivative called an option. An option is a contract between two parties about trading the underlying asset at some time in the future. One party is the writer, who fixes the terms of the option contract and sells the option. The other party is the holder, who purchases the option at market price. There are two basic types of options referred to as the call option and the put option. The call option gives the holder the right to purchase the underlying asset for an agreed upon price by a certain date. The put option gives the holder the right to sell the underlying asset for an agreed upon price by a certain date. The date specified in the contract is known as the expiration or maturity date. The price specified in the contract is known as the exercise or strike price. It is important to note that the holder of the option is not obligated to exercise this right to buy/sell the underlying asset, which is what distinguishes options from forward and futures contracts. This degree of freedom does not come without a price. It costs nothing to

enter into a forward or futures contract, but there is a cost, or premium, to acquiring an option contract. This premium is the market price of the option.

There are two basic option exercise styles usually traded on the market called European and American options. In the European option, the holder can only exercise his right to buy or sell on the expiration date. An American option can be exercised any time before or on the expiration date. This increase in available time with which to exercise the option also comes with an increase in price; thus the price of an American option is greater than the price of a European option. The holder of an option has the choice of three possible actions:

- Sell the option at its current market price on some options exchange.
- Exercise the option.
- Retain the option and let it expire worthless.

The writer of the option is in a much different, and riskier, position. If the holder decides to exercise the option, then the writer has the obligation to sell (or purchase) the underlying asset for the agreed upon strike price. The up-front premium the holder pays for the option compensates the writer for potential liabilities in the future. For the following examples, we will assume the underlying asset is a common stock.

We say an investor is long on the option if he has purchased it and short on the option if he has sold or written it. An investor who is long on a call option is expecting the stock price to increase. The profit is the current price of the stock minus the strike price and the option premium, i.e., $S - K - c$. The potential gain in this situation is virtually unlimited and the potential loss is limited to the option

premium. In contrast, an investor who is short on a call option is expecting the stock price to decrease. The profit is the option premium. If the option buyer exercises the option, then the short position must sell the stock for the strike price. In many cases, the writer does not actually own the stock and must buy the stock at market price in order to meet the obligation of the option contract. The potential loss in this situation is virtually unlimited, and the potential gain is limited to the option premium. Figure 1.1 gives a graphical representation of the two call option positions. An investor who is long on a put option is expecting the stock price to decrease. The

Figure 1.1. Payoffs for the positions in call options.

potential profit is the strike price minus the stock price and the option premium, i.e., $K - S - p$. An investor who is short on a put option is expecting the stock price to increase. The profit is the option premium. If the option buyer exercises the option, then the short position must buy the stock from the holder for the strike price. The potential loss in this situation is limited to the strike price (if the option becomes worthless). Figure 1.2 gives a graphical representation of the two put option

positions.

Figure 1.2. Payoffs for the positions in put options.

There are many trading strategies involving options (see [30]). An investor might take a short position in a call option and simultaneously take a long position in the underlying stock (Figure 1.3). This strategy is known as writing a covered call, and locks in a fixed profit if the stock price rises above the strike price. If an investor owns a stock and wants to protect himself against a decreasing stock price, he might consider a protective put strategy. This strategy is realized when the investor simultaneously buys a put option and the underlying stock (Figure 1.3). This has the effect of limiting the potential loss to the premium of the option while allowing for unlimited gain if the price of the stock increases. It is also possible to use a protective put and covered call simultaneously in what is known as a collar strategy, which effectively puts a lower and upper bound on the value of the portfolio.

A spread trading strategy often involves taking a position in two or more options of the same type with different strike prices. A bull spread is designed to take advantage

Figure 1.3. Trading strategies involving stock and option.

Figure 1.4. Spread trading strategies.

of a rise in the stock price, and a bear spread is designed to take advantage of a decline in the stock price. A common bull spread (Figure 1.4) can be created by buying a call option on a stock with a certain strike price and selling a call option on the same stock with a higher strike price. A common bear spread (Figure 1.4) can be created by buying a put with one strike price and selling a put with another strike price, such that the strike price of the option purchased is greater than the strike price of the

option sold.

A combination trading strategy involves taking a position in both calls and puts on the same stock. One type of combination is called a straddle and involves buying a call

Figure 1.5. Combination trading strategies.

and put with the same strike price and expiration date (Figure 1.5). This portfolio profits if the stock price moves sharply in either direction. Another combination which profits when the stock price moves sharply in either direction is a strangle, which involves buying a put and a call with the same expiration dates and different strike prices (Figure 1.5).

These are just a few of the simpler strategies involving options that are implemented by investors. One can see that an investor is limited only by his imagination when it comes to creating complex trading strategies built upon seemingly simple financial instruments. It is not surprising that option trading has become a multibillion dollar market.

Some consider the trading of options in financial markets as nothing more than a

global casino. This viewpoint may be justified in some cases, but an investor who is acting with intelligence based on research is not usually considered a gambler. Legend has it that the poor Greek philosopher Thales of Miletus was the first to make use of an option contract. After predicting the weather and thus a better than average olive harvest for the year, he bought options for the use of many of the olive presses in the area. When his predictions proved to be correct, he rented out the presses for a much higher price than he obtained them via his option contracts, thus realizing significant profit on his initial investment. Options are certainly used for this type of speculative investing, but they are also used as defensive tools to protect an investor from unforeseen future events. Consider an orange juice producer who purchases his supply of oranges from the farmer via a future contract to ensure that he will have oranges at a reasonable price, even in the event of a poor harvest. This producer may also purchase puts on the oranges with a strike price equal to his future price. In the event of an exceptionally good harvest, the market price of the oranges may be much lower than the price in the future contract. The producer is obligated to buy the oranges at the future price, but can exercise his put options, sell the oranges at the strike price, and then turn around and buy them at the lower market price.

Financial markets are by definition the place where people with capital are paired with people who need capital. A company raising money for a research project might sell bonds to an investor who wants to earn a better than average rate of return on his capital. The investor might purchase puts on the bonds to protect against default, or he might purchase calls on the company's stock to take advantage of the potential growth the company may realize from the research. Both bearish and bullish investors

can agree that options are a very flexible and powerful financial tool that can be used for a number of offensive and defensive trading strategies.

Before an investor can implement a trading strategy with an option, he must purchase it. For this reason, knowledge about an option's mathematical value and price are essential to the smart and rational investor. In this dissertation, we propose a new, fast, and accurate valuation method and give case studies on the valuation of three types of American options.

1.2 The Black-Scholes model

American options are the most commonly traded options in the market. Yet their valuation, even in the standard case of a lognormal process for the underlying asset, remains a topic of active research. This situation stems from the nature of the solution, which requires the valuation of the option as well as determination of the optimal exercise strategy. In contrast, the European option has been valued by the celebrated Black-Scholes equation [13] [48]. Figure 1.6 shows how the price of a European option with a 1-year expiration and \$50 strike price on a stock with 25% volatility relates to the price of the underlying stock as calculated with the Black-Scholes equation.

Consider a risky asset, e.g., a stock, whose market price we will denote by S . We assume that the asset price obeys the linear stochastic differential equation

$$dS = (\mu(t) - q(t))Sdt + \sigma(t)SdW, \quad (1.2.1)$$

where W is a Wiener process, $\mu(t)$ is the drift rate, $q(t)$ is the dividend rate, and $\sigma(t)$

Figure 1.6. European option price as it depends on stock price.

is the volatility. Now suppose that f is the price of a put option. By Itô's lemma, we have

$$df = \left(\frac{\partial f}{\partial S}(\mu(t) - q(t))S + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma(t)^2 S^2 \right) dt + \frac{\partial f}{\partial S} \sigma(t) S dW. \quad (1.2.2)$$

Letting ΔS and Δf denote discrete changes in S and f over a small time interval Δt , the discrete versions of equations (1.2.1) and (1.2.2) are

$$\Delta S = (\mu(t) - q(t))S\Delta t + \sigma(t)S\Delta W \quad (1.2.3)$$

and

$$\Delta f = \left(\frac{\partial f}{\partial S}(\mu(t) - q(t))S + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma(t)^2 S^2 \right) \Delta t + \frac{\partial f}{\partial S} \sigma(t) S \Delta W. \quad (1.2.4)$$

We make the following assumptions, as in Black-Scholes [13] and Merton [48],

- all securities are perfectly divisible (i.e., we can buy and sell any number, not necessarily an integer, of the underlying asset),
- short selling is permitted (i.e., we may sell assets that we do not own),

- there are no transaction costs or taxes,
- security trading is continuous,
- and there are no arbitrage opportunities.

Under these assumptions, we want to choose a portfolio consisting of the underlying asset and the option so that the randomness or risk can be removed from the value of the portfolio. This portfolio will consist of selling (writing) one option contract and buying $\partial f / \partial S$ units of the underlying asset. We will denote the value of the portfolio by Π so that

$$\Pi = -f + \frac{\partial f}{\partial S} S, \quad (1.2.5)$$

and the change in the value of the portfolio is given by

$$\Delta \Pi = -\Delta f + \frac{\partial f}{\partial S} \Delta S + q(t) S \Delta t. \quad (1.2.6)$$

Substituting (1.2.3) and (1.2.4) into (1.2.6) yields

$$\Delta \Pi = \left(-\frac{\partial f}{\partial t} - \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma(t)^2 S^2 + q(t) \frac{\partial f}{\partial S} S \right) \Delta t. \quad (1.2.7)$$

Note that this equation does not involve ΔW and, therefore, can be considered riskless during the time dt . Now we invoke the “no arbitrage” assumption, which implies

$$\Delta \Pi = r(t) \Pi \Delta t, \quad (1.2.8)$$

where $r(t)$ is the risk-free interest rate. In words, the portfolio has the same rate of return as any other short-term, risk-free security, the risk-free interest rate. Why does the no-arbitrage assumption imply this? If the portfolio earned more, then one

could borrow money to buy the portfolio, collect the earnings from the portfolio, and then repay the loan from these earnings keeping the difference as riskless profit. If the portfolio earned less, one could short (sell) the portfolio, use the proceeds to buy risk-free securities, collect the earnings from the securities, and pay out the earnings of the portfolio from these, keeping the difference as riskless profit. These scenarios are called arbitrage opportunities, and we assume that in a free market when one arises, the market instantaneously compensates and removes it. A common expression in the world of trading is, “In a free market there is no free lunch.”

Now substituting (1.2.5) and (1.2.7) into (1.2.8), we obtain

$$\left(-\frac{\partial f}{\partial t} - \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma(t)^2 S^2 + q(t) \frac{\partial f}{\partial S} S\right) \Delta t = r(t) \left(-f + \frac{\partial f}{\partial S} S\right) \Delta t,$$

so that

$$\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma(t)^2 S^2 + (r(t) - q(t)) S \frac{\partial f}{\partial S} = r(t) f. \quad (1.2.9)$$

Equation (1.2.9) is known as the Black-Scholes-Merton equation [57]. Any derivative whose price depends only on the current value of S and on t , and which is paid for up-front, must satisfy the Black-Scholes-Merton equation (or a variant incorporating time-dependent parameters).

It should be noted that when the price of the underlying asset $S = 0$, the asset is not tradable, and the option becomes worthless. This fact prompts us to impose the boundary condition

$$f(0, t) = 0, \quad 0 \leq t \leq T.$$

Recall that the American option gives the holder the right to exercise at any time

before or on the expiration date. The decision to exercise depends on the price paid for the option $f(S, t)$, the current price of the underlying asset S , and the contract strike price K . We define the payoff $g(S, t)$ of the option as

$$g(S, t) = \max(K - S, 0) \quad \text{for a put,} \quad (1.2.10)$$

and is the profit generated by a difference in the market price and the strike price.

The no-arbitrage assumption implies that for $0 < t \leq T$

$$f(S, t) \geq g(S, t). \quad (1.2.11)$$

To see this, suppose that in the case of the put, we have $f(S, t) < \max(K - S, 0)$.

Then the arbitrage opportunity is to buy the asset for price S and the option for price f , and then immediately exercise the option, thereby selling the asset at price K . In doing so one would make a riskless profit of $K - S - f$.

The price of the underlying asset which prompts the early exercise of the option is called the early exercise price $S^*(t)$, and is known as the free boundary. This early exercise price is not known *a priori* and, therefore, makes the valuation of the American option more complicated, since at each time t we must determine not only the option value, but also, for each value of S , whether or not it should be exercised. The free boundary is characterized by

$$f(S, t) = g(S, t) \quad \text{for } S \leq S^*(t), \quad (1.2.12)$$

$$f(S, t) > g(S, t) \quad \text{for } S > S^*(t). \quad (1.2.13)$$

We call $(0, S^*(t))$ the continuation region and $[S^*(t), \infty)$ the exercise region. Since we do not know the position of $S^*(t)$ *a priori*, we must impose two boundary conditions

at $S^*(t)$ in order to uniquely determine the option value:

$$f(S^*(t), t) = g(S^*(t), t), \quad (1.2.14)$$

$$\frac{\partial f}{\partial S}(S^*(t), t) = -1 \quad \text{for the put.} \quad (1.2.15)$$

It should be noted that the Neumann boundary condition and the Dirichlet boundary condition are independent. The Dirichlet condition is the payoff. The Neumann condition is the result of the no-arbitrage assumption. To see this, suppose in the case of the put that $\frac{\partial f}{\partial S}(S^*(t), t) < -1$. We know that at $S^*(t)$ the option value $f(S^*(t), t) = g(S^*(t), t)$ and $\frac{\partial g}{\partial S} = -1$. As S increases from $S^*(t)$, we will have $f(S, t) < g(S, t)$ in the exercise region, which contradicts (1.2.11) and (1.2.13). Now suppose that $\frac{\partial f}{\partial S}(S^*(t), t) > -1$. As S decreases from $S^*(t)$, we will have $f(S, t) > g(S, t)$, which is an arbitrage opportunity and contradicts (1.2.12).

To summarize, the Black-Scholes-Merton model for the American put option is

$$f_t + \frac{1}{2}\sigma^2 S^2 f_{SS} + (r - q)Sf_S - rf = 0, \quad S > S^*(t), \quad 0 < t \leq T, \quad (1.2.16)$$

$$\lim_{S \rightarrow \infty} f(S, t) = 0, \quad 0 \leq t \leq T, \quad (1.2.17)$$

$$f(S^*(t), t) = K - S^*(t), \quad 0 < t \leq T, \quad (1.2.18)$$

$$f_S(S^*(t), t) = -1, \quad 0 < t \leq T, \quad (1.2.19)$$

$$f(S, T) = \max(K - S, 0), \quad S \geq S^*(T). \quad (1.2.20)$$

Similarly, the Black-Scholes-Merton model for the American call option is

$$f_t + \frac{1}{2}\sigma^2 S^2 f_{SS} + (r - q)Sf_S - rf = 0, \quad 0 < S < S^*(t), \quad 0 < t \leq T, \quad (1.2.21)$$

$$f(0, t) = 0, \quad 0 \leq t \leq T, \quad (1.2.22)$$

$$f(S^*(t), t) = S^*(t) - K, \quad 0 < t \leq T, \quad (1.2.23)$$

$$f_S(S^*(t), t) = 1, \quad 0 < t \leq T, \quad (1.2.24)$$

$$f(S, T) = \max(S - K, 0), \quad 0 \leq S \leq S^*(T). \quad (1.2.25)$$

1.3 Interest rate derivatives

In many cases the underlying asset of an option is a bond, or more simply a zero-coupon bond. Since any coupon bearing bond can be replicated by a portfolio of zero-coupon bonds we only consider the case of an option on a zero-coupon bond. There are some differences between bonds and stocks which make modeling the price of a bond more difficult than a stock. Stock prices are functions of only one variable, time. Bond prices depend on time and the maturity date of the bond. The pricing model needs to match the initial data. For a stock this is just the current price. In the case of bonds the whole initial term structure is given, imposing more restrictions on the model. Also, bonds become nonrandom at maturity, which is clearly not the case for stocks. The fact that a bond gives a fixed payoff at maturity must be included in the model.

In a similar way that we modeled the stock price, let us assume that the short interest rate r is governed by a stochastic differential equation of the form

$$dr(t) = a(r(t), t)dt + b(r(t), t)dW(t), \quad (1.3.1)$$

where $W(t)$ is a Wiener process under a risk neutral measure, $a(r(t), t)$ is the risk adjusted drift, and the functional forms of $a(r(t), t)$ and $b(r(t), t)$ determine the particular short rate model and will be discussed later.

From the structure of equation (1.3.1), it is clear that $r(t)$ is a Markov process. The price at time t of a zero-coupon bond that pays 1 dollar at maturity T is given by ([11], [16], [49], [53])

$$B(r(t), t, T) = \mathbb{E} \left(\exp \left(- \int_t^T r(s) ds \right) \middle| r(t) = r \right). \quad (1.3.2)$$

It is not difficult to show that the discounted bond price process

$$P(r(t), t) := \exp \left(- \int_0^t r(s) ds \right) B(r(t), t, T)$$

is a martingale. By Itô's lemma,

$$dP(t, r(t)) = \left(\frac{\partial P}{\partial t} + a(r, t) \frac{\partial P}{\partial r} + \frac{b(r, t)^2}{2} \frac{\partial^2 P}{\partial r^2} \right) dt + b(r, t) \frac{\partial P}{\partial r} dW.$$

Since $P(r(t), t)$ is a martingale, the drift term must be identically equal to zero. Thus we have

$$\frac{\partial B}{\partial t} + a(r, t) \frac{\partial B}{\partial r} + \frac{b(r, t)^2}{2} \frac{\partial^2 B}{\partial r^2} - r(t)B = 0, \quad (1.3.3)$$

which is the fundamental PDE for pricing a zero-coupon bond ([11], [16], [49], [53]).

There are many popular models for the interest rate process that are based on the above analysis, including the Hull-White and Cox-Ingersoll-Ross models. In this dissertation, we will consider a class of models of the short interest rate r that are governed by a stochastic differential equation of the form

$$dX(t) = (\phi(t) - \psi(t)X(t))dt + \sigma(t)dW(t), \quad r(t) = \zeta(X(t)), \quad (1.3.4)$$

where $\zeta(x)$ is a twice continuously differentiable invertible function on $\mathbb{R}$; $\phi(t)$, $\psi(t)$, and $\sigma(t)$ are known functions; and $W(t)$ is a Wiener process under a risk-neutral measure.

In this way, the form of $\zeta(x)$ determines the particular short rate model. For example, with $\zeta(x) = x$ we have the Hull-White model and for $\zeta(x) = e^x$ we have the Black-Karasinski model. It should be pointed out that $\zeta(x)$ can be chosen to be any bounded invertible function from $(-\infty, +\infty)$ to $(0, 1)$, e.g.,

$$\zeta(x) = \frac{e^x}{1 + e^x}. \quad (1.3.5)$$

For such a choice of $\zeta(x)$, the interest rates will not take unrealistic values more than 1. We are referred to [32] and [41] for other possible choices of $\zeta(x)$ and the calibration of such one-factor models.

Letting $x = \eta(r)$ be the inverse function of $r = \zeta(x)$ and by using Ito's formula, we can obtain the stochastic differential equation (1.3.1) for the short rate process $r(t)$ where

$$a(r, t) = \zeta'(\eta(r))(\phi(t) - \psi(t)\eta(r)) + \frac{1}{2}\sigma(t)^2\zeta''(\eta(r)), \quad b(r, t) = \sigma(t)\zeta'(\eta(r)).$$

Furthermore, we can rewrite the PDE for B (1.3.3) into the following PDE for $V(x, t) = B(\zeta(x), t)$:

$$V_t + \frac{1}{2}\sigma(t)^2V_{xx} + (\phi(t) - \psi(t)x)V_x - \zeta(x)V = 0. \quad (1.3.6)$$

Now let us consider an American put option on a T^* -maturity zero-coupon bond. The option expiration date is T ($< T^*$) and its exercise price is K . Since the option can be exercised at any time up to its expiration date, there is a critical interest rate $r^*(t)$ which is referred to as the early exercise interest rate. Denote the option price by $p(r, t)$. Let $x^*(t) = \eta(r^*(t))$ and $\tilde{p}(x, t) = p(\zeta(x), t)$. Then $\tilde{p}(x, t)$ and $x^*(t)$ solve

the following free boundary problem:

$$\tilde{p}_t + \frac{1}{2}\sigma(t)^2\tilde{p}_{xx} + (\phi(t) - \psi(t)x)\tilde{p}_x - \zeta(x)\tilde{p} = 0, \quad -\infty < x < x^*(t), \quad 0 \leq t \leq T, \quad (1.3.7)$$

$$\tilde{p}(x^*(t), t) = g(x^*(t), t), \quad 0 \leq t \leq T, \quad (1.3.8)$$

$$\tilde{p}_x(x^*(t), t) = g_x(x^*(t), t), \quad 0 \leq t \leq T, \quad (1.3.9)$$

$$\tilde{p}(x, T) = g(x, T), \quad -\infty < x < \infty, \quad (1.3.10)$$

where $g(x, t) = \max(K - \tilde{P}(x, t; T^*), 0)$ is the payoff of the put option and $\tilde{P}(x, t; T^*)$ is the bond price when $r = \zeta(x)$ at time t .

1.4 Regime-switching model for a stock option

One of the motivating factors behind the regime-switching model is that from time to time, the market may switch between a stable low-volatility state and a more unstable high-volatility state. These periods of high volatility may arise because of short-term political or economic uncertainty.

Our original model for the price of a risky asset assumed that the price dynamics depended only on time and a random component. A simple way to incorporate stochastic volatility into the model is to assume that volatility takes one of a finite set of values, switching between these values randomly. This approach maintains the tractability of our original model while it more accurately captures the more extreme behavior observed in stock prices. We model the state of the economy with a finite state Markov chain $X(t)$.

For simplicity, we consider the case that there are only two states of the economy. We denote them by $e_1 = (1, 0)'$ and $e_2 = (0, 1)'$. Then the stock price process $S(t)$ is

assumed to follow the stochastic differential equation

$$dS(t) = S(t) (\mu_{X(t)}dt + \sigma_{X(t)}dW(t)), \quad t > 0, \quad (1.4.1)$$

where $W(t)$ is a standard Brownian motion under the risk-neutral measure, and μ_{e_i} and σ_{e_i} are constant when the economy is in the i -th state at time t , i.e., when $X(t) = e_i$.

Now we consider an American call option on a stock with strike price K and expiry date T years. As usual, there is an early exercise price $S_i^c(t)$ such that the option should be exercised when the stock price $S \geq S_i^c(t)$ at time t in the i -th state of economy. Otherwise, the option should be held. Thus $[S_i^c(t), \infty) \times [0, T]$ is known as the exercise region and $(0, S_i^c(t)) \times [0, T]$ is called the continuation region. Denote by $C_i(S, t)$ the call price in the i -th state. Then $C_i(S, t)$ and $S_i^c(t)$ solve the free boundary value problem ([62]): for $i = 1, 2$,

$$C_{i,t}(S, t) + \mathcal{A}_i C(S, t) = 0, \quad 0 \leq S < S_i^c(t), \quad 0 < t \leq T, \quad (1.4.2)$$

$$C_i(0, t) = 0, \quad 0 \leq t \leq T, \quad (1.4.3)$$

$$C_i(S, t) = S - K, \quad S \geq S_i^c(t), \quad 0 \leq t \leq T, \quad (1.4.4)$$

$$C_{i,S}(S(t), t) = 1, \quad 0 < t \leq T, \quad (1.4.5)$$

$$C_i(S, T) = (S - K)^+, \quad S \geq 0, \quad (1.4.6)$$

where $C(S, t) = (C_1(S, t), C_2(S, t))^T$,

$$\mathcal{A}_i C(S, t) = \frac{1}{2} \sigma_i^2 S^2 C_{i,SS}(S, t) + \mu_i S C_{i,S}(S, t) - r_i C_i(S, t) + \langle Q C(S, t), e_i \rangle,$$

Q is the rate matrix of $X(t)$, $\langle \cdot, \cdot \rangle$ denotes the usual inner product on R^2 , and constant r_i is the interest rate in the i -th state of economy.

Similarly, for the put option we have for $i = 1, 2$,

$$P_{i,t}(S, t) + \mathcal{A}_i P(S, t) = 0, \quad S > S_i^p(t), \quad 0 < t \leq T, \quad (1.4.7)$$

$$P_i(0, t) = 0, \quad 0 \leq t \leq T, \quad (1.4.8)$$

$$P_i(S, t) = K - S, \quad 0 < S \leq S_i^p(t), \quad 0 \leq t \leq T, \quad (1.4.9)$$

$$P_{i,S}(S(t), t) = -1, \quad 0 < t \leq T, \quad (1.4.10)$$

$$P_i(S, T) = (K - S)^+, \quad S \geq 0. \quad (1.4.11)$$

1.5 Front-fixing finite element methods

American option problems are essentially nonlinear due to the existence of the early exercise boundary. The valuation of American options has been extensively studied for more than thirty years. Various methods have been exploited for fast and accurate pricing of such important financial derivatives in practice. These methods can be classified into analytic approximations [8] [25] [36] [45] [58], Monte Carlo simulations [14] [44] [54], and numerical solutions [3] [4] [15] [17] [29] [34].

The front-fixing method is widely employed for solving free boundary problems. It is based on a transform under which the free boundary becomes the known line for a one-dimensional problem and is usually employed in combination with a finite difference method. The transformation was first introduced by Landau [40]. For the usual front-fixing method, the problem is restricted on a bounded domain. Then the Landau transformation is employed to realize the fixed domain. In addition to fixing the problem on a rectangular domain, another advantage of the front-fixing method is that the free boundary is computed directly, in contrast to a fixed-domain method

where the free boundary is determined by an *a posteriori* procedure. Also, when the free boundary is smooth or monotone (which is the case for American options), the front-fixing method yields more accurate results. We are referred to [22] for a detailed account in this aspect. In [59], a front-fixing finite difference method is proposed by using a three level explicit scheme, and the numerical results show that the method has accuracy compatible to that of the binomial method. We are also referred to [27], [50], [51], [52], and [55] for other recent work in this aspect for American stock options.

Figure 1.7. Domain transformations.

In this dissertation we are proposing a new front-fixing finite element method based on a front-fixing transformation which simplifies the computational solution, resulting in a faster and more accurate valuation of the American option. We use a linear transformation while the problem is truncated on a variable domain. This transformation does not affect the coefficient of the leading term in the partial differential equation as the Landau transformation does. For our method, we first transform the Black-Scholes equation into a parabolic equation over an unbounded domain, as in

Figure 1.7.

Then we introduce an artificial boundary and propose a new approximate problem over a bounded domain with two free boundaries. The front-fixing method is then realized by the linear translation $y = x + L - x^*$ into a fixed rectangular domain $[0, L] \times [0, T]$. The resulting nonlinear problem can then be solved using a finite element method.

1.6 The structure of the dissertation

This dissertation is composed of three research papers which are each presented as a chapter. In Chapter 2 we study the front-fixing finite element method for American options on stocks. The chapter begins with a statement of the problem, and using a far field estimate, we propose an approximate problem on a bounded variable domain. We then introduce the front-fixing variable transformation to reformulate the variable domain problem into a nonlinear problem on a fixed rectangular domain. From the nonlinear problem, we formulate the associated variational problem and finite element approximation.

Practitioners often use ratios known as the greeks to make decisions about particular options. We include a section detailing their computation within the framework of our solution. Details of our method's implementation are given, as well as the algorithm used to obtain the approximate solutions. Stability and nonnegativity of the approximate solutions are also established under some appropriate assumptions. We conclude the chapter by presenting numerical results, an examination of our method,

and a comparison of our method with the binomial method.

In Chapter 3 we study the front-fixing finite element method for American options on zero-coupon bonds. The chapter begins with a brief description of the interest rate process and the free boundary problem associated with a bond option. The front-fixing finite element method is then developed for the problem, and the stability is analyzed. Implementation of the method is discussed in detail. Since there are no known analytic formulas for the bond price problem for most interest rate models, we first consider how to compute zero-coupon bond prices and their derivatives within the framework of our problem. Numerical results for one-year American put options on bonds under the Hull-White and Vasicek models are presented to examine the front-fixing finite element method and compare it with other methods. We conclude the chapter by comparing our method to the usual finite element method and observe that our new front-fixing finite element method provides more accurate approximations, especially for early exercise interest rates.

In Chapter 4 we study the front-fixing finite element method for American options on stocks under the regime-switching model. Here we extend the model used in Chapter 2 so that the price of the underlying asset also depends on the state of the economy, which is modeled by a finite state Markov chain. After giving a description and statement of the problem, we formulate the variational form and finite element approximation of the problem. A theorem establishing the stability of the method is also presented. Implementation is discussed in detail, and the algorithm used to compute the approximate solutions is given. Extensive numerical results and comparisons of our method with the usual finite element method are given. Based

on our results, we are able to propose and numerically verify a conjecture about the asymptotic behavior of the early exercise price near expiration.

CHAPTER 2

AMERICAN OPTIONS ON STOCKS

2.1 Introduction

In this chapter, we propose and study a new a front fixing finite element method for the valuation of American options on stocks.[†] In Section 2.2 we state the American call option free boundary problem, which is posed on an infinite domain with one free boundary, and then use a far field estimate to formulate an approximate problem on a variable domain. In Section 2.3 we apply variable transforms to the variable domain problem and obtain a nonlinear problem on a fixed rectangular domain. We then proceed by formulating the associated variational problem and finite element approximation, and also provide the algorithm used to compute the approximate solutions. In Section 2.4 we establish the stability of the method and nonnegativity of the solution. Section 2.5 discusses how the ratios known as greeks can be easily computed within the framework of the method. Finally, we present numerical results and comparisons of our method with other methods. Numerical results show that our method is very rapid and accurate. In particular, the semi-analytic formulas can be used to compute option prices, hedge ratios, and Thetas at any given stock price and time with high accuracy. The approach of this paper can be extended to the American option problems when parameters depend on stock price and time, especially when volatility is a function of stock price. For the latter case, it is very useful to solve

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the forward problem efficiently for the inverse problem of calibrating the volatility surface [2].

2.2 A variable domain problem for American options

For simplicity, we only consider an American call option since American put options can be treated in the same fashion [4]. Let σ , r , q , K , and T be volatility, interest rate, dividend rate, exercise price, and maturity date, respectively. It is well-known that the American call option equals the value of the corresponding European call option when $q = 0$. We shall assume that $q > 0$. The price $c(S, t)$ of the American call option is the solution of the following free boundary value problem [39] [46] [48]:

$$c_t + \frac{1}{2}\sigma^2 S^2 c_{SS} + (r - q)Sc_S - rc = 0, \quad 0 < S < B^*(t), \quad 0 < t \leq T, \quad (2.2.1)$$

$$c(0, t) = 0, \quad 0 \leq t \leq T, \quad (2.2.2)$$

$$c(B^*(t), t) = B^*(t) - K, \quad 0 < t \leq T, \quad (2.2.3)$$

$$c_S(B^*(t), t) = 1, \quad 0 < t \leq T, \quad (2.2.4)$$

$$c(S, T) = (S - K)^+, \quad 0 \leq S \leq B^*(T), \quad (2.2.5)$$

where S is the stock price and $z^+ = \max(z, 0)$. The free boundary $B^*(t)$ is known as the early exercise price. The option should be exercised if the stock price S is equal to or greater than $B^*(t)$ at time t ; otherwise, the option should be held. We are referred to [6] for the solution existence and regularity of this free boundary problem.

Consider the variable transforms:

$$c(S, t) = Ke^{-\alpha y - \beta \tau} \phi(y, \tau), \quad T - t = \tau, \quad S = Ke^y, \quad (2.2.6)$$

where α and β are constants to be determined. Then (2.2.1)–(2.2.5) become

$$\phi_\tau - \gamma\phi_{yy} + \nu\phi_y + \mu\phi = 0 \quad y < b(\tau), \quad 0 < \tau \leq T, \quad (2.2.7)$$

$$\phi(b(\tau), \tau) = g(b(\tau), \tau), \quad 0 \leq \tau \leq T, \quad (2.2.8)$$

$$\phi_y(b(\tau), \tau) = g_y(b(\tau), \tau), \quad 0 < \tau \leq T, \quad (2.2.9)$$

$$\phi(y, 0) = g(y, 0), \quad -\infty < y < b(\tau), \quad (2.2.10)$$

$$\lim_{y \rightarrow -\infty} e^{-\alpha y - \beta \tau} \phi(y, \tau) = 0, \quad 0 \leq \tau \leq T, \quad (2.2.11)$$

where

$$\begin{aligned} \gamma &= \frac{\sigma^2}{2}, \quad \nu = \gamma(1 + 2\alpha) + q - r, \quad \mu = r + \alpha(r - q) - \gamma\alpha(1 + \alpha) - \beta, \\ g(y, \tau) &= e^{\alpha y + \beta \tau} (e^y - 1)^+, \quad b(\tau) = \log(B^*(T - \tau)/K). \end{aligned}$$

In order to solve (2.2.7)–(2.2.11) on a bounded domain, we need an artificial boundary condition when y is negatively large. We have the following result concerning the far field estimate of the option prices.

Theorem 2.1. *For a given positive number $\epsilon \in (0, 1)$, we have*

$$c(S, t) \leq \epsilon, \quad \forall 0 \leq S \leq Ke^{-Y}, \quad 0 \leq t \leq T,$$

where

$$Y = 2.5\gamma T\alpha_0 + 0.5\sqrt{25\gamma^2 T^2 \alpha_0^2 - 20\gamma T \log(\epsilon/(\sqrt{5}K))}, \quad \alpha_0 = \frac{r - q}{\sigma^2} - \frac{1}{2}.$$

Proof. Let $\alpha = \alpha_0$, $\beta = \gamma\alpha_0^2 + r$ in (2.2.6). Then we have $\nu = \mu = 0$, i.e., equation (2.2.7) becomes the heat equation. Recall that ([39])

$$B^*(0) \leq B^*(t) \leq KX,$$

where

$$B^*(0) = K \max(r/q, 1), \quad X = \frac{1}{2} + \frac{r + \gamma + \sqrt{(r - q - \gamma)^2 + 4\gamma r}}{2q}.$$

We have

$$0 \leq b(\tau) \leq \log(X), \quad \forall 0 \leq \tau \leq T,$$

which means that equation (2.2.7) holds when $y < 0$. Hence, by Theorem 19.3.2 of [19], we have the following integration representation of ϕ :

$$\phi(y, \tau) = -\frac{y}{\sqrt{4\gamma\pi}} \int_0^\tau (\tau - s)^{-\frac{3}{2}} e^{-\frac{y^2}{4\gamma(\tau-s)}} \phi(0, s) ds, \quad y < 0.$$

Since the American call price is always less than or equal to the stock price, we have

$\phi(0, s) \leq e^{\beta s}$. Thus, for $y < 0$, we get

$$\begin{aligned} \phi(y, \tau) &\leq \frac{-ye^{\beta\tau}}{\sqrt{4\gamma\pi}} \int_0^\tau (\tau - s)^{-\frac{3}{2}} e^{-\frac{y^2}{4\gamma(\tau-s)}} ds \\ &= \frac{2e^{\beta\tau}}{\sqrt{\pi}} \int_{\frac{-y}{\sqrt{4\gamma\tau}}}^\infty e^{-\xi^2} d\xi \quad \left(\xi = \frac{-y}{\sqrt{4\gamma(\tau-s)}} \right) \\ &\leq \frac{e^{\beta\tau - \frac{y^2}{5\gamma\tau}}}{\sqrt{\pi}} \int_0^\infty e^{-\frac{\xi^2}{5}} d\xi \\ &= \sqrt{5} e^{\beta\tau - \frac{y^2}{5\gamma\tau}}. \end{aligned}$$

Then we have for $y < 0$

$$c(Ke^y, t) = Ke^{-\alpha y - \beta\tau} \phi(y, \tau) \leq \sqrt{5} K e^{-\alpha y - \frac{y^2}{5\gamma\tau}} \leq \sqrt{5} K e^{-\alpha y - \frac{y^2}{5\gamma T}}.$$

It is easy to verify that

$$\sqrt{5} K e^{-\alpha y - \frac{y^2}{5\gamma T}} \leq \epsilon$$

if and only if $y \leq -Y$. The proof is completed. $\square$

Remark 2.1. We are referred to [37] for the far field boundary conditions for general European options.

Let

$$L = Y + \log(X), \quad a(\tau) = b(\tau) - L. \quad (2.2.12)$$

It follows from Theorem 1 that

$$c(S, t) \leq \epsilon, \quad \forall 0 \leq S \leq Ke^{a(T-t)}, \quad 0 \leq t \leq T.$$

Therefore, we can propose the following approximate problem to (2.2.7)–(2.2.11):

$$\psi_\tau - \gamma\psi_{yy} + \nu\psi_y + \mu\psi = 0, \quad a(\tau) \leq y < b(\tau), \quad 0 < \tau \leq T, \quad (2.2.13)$$

$$\psi(a(\tau), \tau) = 0, \quad 0 \leq \tau \leq T, \quad (2.2.14)$$

$$\psi(b(\tau), \tau) = g(b(\tau), \tau), \quad 0 \leq \tau \leq T, \quad (2.2.15)$$

$$\psi_y(b(\tau), \tau) = g_y(b(\tau), \tau), \quad 0 < \tau \leq T, \quad (2.2.16)$$

$$\psi(y, 0) = g(y, 0), \quad a(\tau) < y < b(\tau). \quad (2.2.17)$$

Here (2.2.13), (2.2.14), (2.2.16), and (2.2.17) form a variable domain problem, which defines a mapping from $b(\tau)$ to $\psi(b(\tau), \tau)$. Then by (2.2.15), we have a nonlinear equation for determining the free boundary $b(\tau)$. In the next section, we shall consider a front-fixing finite element method for the variable domain problem and an efficient Newton's method for the resulting nonlinear systems.

Remark 2.2. We should point out that $b(\tau)$ in the above problem is essentially different from the one in problem (2.2.7)–(2.2.11). It is still an open problem to estimate the errors between the solutions of the two problems.

Remark 2.3. It is well-known that the call option price $c(S, t)$ is an increasing function of stock price S and a decreasing function of time t . Then

$$\phi(y, \tau) = \frac{1}{K} e^{\alpha y + \beta \tau} c(K e^y, T - \tau)$$

is an increasing function of y and τ when $\alpha \geq 0$ and $\beta \geq 0$, and thus $\psi(x, \tau)$ should also be an increasing function of x and τ .

2.3 A front-fixing finite element method

In this section, we shall apply our front-fixing finite element method to the variable domain problem (2.2.13)–(2.2.17) and propose an algorithm for solving the resulting nonlinear systems.

Using the variable transforms:

$$x = y - a(\tau), \quad u(x, \tau) = \psi(a(\tau) + x, \tau), \quad (2.3.1)$$

we can reformulate problem (2.2.13)–(2.2.17) into the following nonlinear problem:

$$u_\tau - \gamma u_{xx} + (\nu - b'(\tau))u_x + \mu u = 0, \quad 0 < x < L, \quad 0 < \tau \leq T, \quad (2.3.2)$$

$$u(0, \tau) = 0, \quad 0 \leq \tau \leq T, \quad (2.3.3)$$

$$u(L, \tau) = g(b(\tau), \tau), \quad 0 \leq \tau \leq T, \quad (2.3.4)$$

$$u_x(L, \tau) = g_y(b(\tau), \tau), \quad 0 < \tau \leq T, \quad (2.3.5)$$

$$u(x, 0) = u_0(x), \quad 0 \leq x \leq L, \quad (2.3.6)$$

where $u_0(x) = g(a(0) + x, 0)$. As mentioned in Section 2.2 we want to integrate the Neumann boundary condition (2.3.5) into the variational problem and use the

Dirichlet boundary condition (2.3.4) to determine the free boundary $b(\tau)$. Define the bilinear form

$$\mathcal{A}(b'(\tau), u, v) = \gamma(u_x, v_x) + (\nu - b'(\tau))(u_x, v) + \mu(u, v),$$

where $(\cdot, \cdot)$ denotes the inner product of $L^2(\Omega)$, the space of square integrable functions on $\Omega = (0, L)$. Let $H_E^1(\Omega)$ be the closure of $\{v \in C^\infty(\overline{\Omega}) : v(0) = 0\}$ in the usual Sobolev space $H^1(\Omega)$ ([1], [43]), and let $H^{-1}(\Omega)$ be the dual space of $H_E^1(\Omega)$. Then the variational form for problem (2.3.2)–(2.3.6) is: Find $u \in L^2(0, T; H_E^1(\Omega))$ and $b \in C([0, T]) \cap C^1((0, T])$ such that $u_\tau \in L^2(0, T; H^{-1}(\Omega))$, $u(0) = u_0$, and

$$(u_\tau, v) + \mathcal{A}(b'(\tau), u, v) = f(b(\tau), \tau)v(L), \quad \forall v \in H_E^1(\Omega), \quad \text{a.e. } 0 \leq \tau \leq T, \quad (2.3.1)$$

$$u(L, \tau) = g(b(\tau), \tau), \quad 0 \leq \tau \leq T, \quad (2.3.2)$$

where $f(y, \tau) = \gamma g_y(y, \tau)$.

Let $\Delta_\tau : 0 = \tau_0 < \tau_1 < \dots < \tau_M = T$ and $\Delta_x : 0 = x_0 < x_1 < \dots < x_N = L$ be partitions of $[0, T]$ and $[0, L]$, respectively, where M, N are positive integers. Let V_h be the piecewise linear element subspace of $H_E^1(\Omega)$ with respect to the partition Δ_x , where $h = \max_{1 \leq j \leq N} (x_j - x_{j-1})$. Denote the basis functions of V_h by $\varphi_1, \varphi_2, \dots, \varphi_N$, i.e., $\varphi_j \in V_h$ and $\varphi_j(x_i) = \delta_{ij}$ for $j = 1, 2, \dots, N$ and $i = 0, 1, \dots, N$, where δ_{ij} is the Kronecker delta. Let

$$u_h^0 = \sum_{j=1}^N u_0(x_j) \varphi_j(x), \quad b_0 = b(0) = \log(\max(r/q, 1)).$$

The finite element approximation to the variational problem (2.3.1)–(2.3.2) is: For

$m = 1, 2, \dots, M$, find $u_h^m \in V_h$ and $b_m > 0$ such that

$$(\delta_\tau u_h^m, v) + \mathcal{A}(\delta_\tau b_m, u_h^{m-\theta}, v) = f^{m-\theta}v(L), \quad \forall v \in V_h, \quad (2.3.3)$$

$$u_h^m(L) = g(b_m, \tau_m), \quad (2.3.4)$$

where $\theta \in [0, 1]$, and

$$\begin{aligned} \delta_\tau u_h^m &= \frac{u_h^m - u_h^{m-1}}{k_m}, \quad \delta_\tau b_m = \frac{b_m - b_{m-1}}{k_m}, \quad k_m = \tau_m - \tau_{m-1}, \\ u_h^{m-\theta} &= (1 - \theta)u_h^m + \theta u_h^{m-1}, \quad f^{m-\theta} = (1 - \theta)f(b_m, \tau_m) + \theta f(b_{m-1}, \tau_{m-1}). \end{aligned}$$

Write

$$u_h^m(x) = \sum_{j=1}^N u_j^m \varphi_j(x).$$

The matrix form of (2.3.3)–(2.3.4) is:

$$A\delta_\tau U^m + (\mu A + \gamma B + (\nu - \delta_\tau b_m)C) U^{m-\theta} = F^{m-\theta}, \quad (2.3.5)$$

$$u_N^m = g(b_m, \tau_m), \quad (2.3.6)$$

for $m = 1, \dots, M$, where

$$A = ((\varphi_j', \varphi_i'))_{N \times N}, \quad B = ((\varphi_j', \varphi_i))_{N \times N}, \quad C = ((\varphi_j, \varphi_i))_{N \times N},$$

$$F^{m-\theta} = (0, \dots, 0, f^{m-\theta})^\top, \quad U^m = (u_1^m, \dots, u_N^m)^\top.$$

Now let us consider how to solve the nonlinear system (2.3.5)–(2.3.6) numerically.

Notice that (2.3.5) defines an implicit vector function U^m of b_m . We can consider u_m^N as an implicit function of b_m . Since b_m should be positive and $g(y, \tau) = e^{\alpha y + \beta \tau}(e^y - 1)$ for $y > 0$, we can rewrite (2.3.6) into

$$Q(b_m) \equiv b_m - \log(1 + e^{-\alpha b_m - \beta \tau_m} u_N^m) = 0. \quad (2.3.7)$$

In order to solve this nonlinear equation for b_m by a Newton's method, we need to compute $Q'(b_m)$. Since

$$Q'(b_m) = 1 - \frac{e^{-\alpha b_m - \beta \tau_m} \left(\frac{du_N^m}{db_m} - \alpha u_N^m \right)}{1 + e^{-\alpha b_m - \beta \tau_m} u_N^m}$$

we must compute $\frac{du_N^m}{db_m}$. Differentiating (2.3.5) with respect to b_m leads to

$$\left(\frac{1}{k_m} A + (1 - \theta)(\mu A + \gamma B + (\nu - \delta_\tau b_m) C) \right) V^m = G^{m-\theta}, \quad (2.3.8)$$

where

$$V^m = \frac{\partial U^m}{\partial b_m}, \quad G^{m-\theta} = (1 - \theta) \frac{\partial F^m}{\partial b_m} + \frac{1}{k_m} C U^{m-\theta}.$$

Systems (2.3.5) and (2.3.8) have the same tridiagonal coefficient matrix and thus can be solved simultaneously by using the Thomas algorithm. For the sake of simplicity, we may omit the sub/super-script m and shall not distinguish U^m and b_m from their approximations. For a given error tolerance $\varepsilon > 0$, we have the following algorithm for solving (2.3.5)–(2.3.6):

1. Let $b^{(0)} = b_{m-1} + k_m$.
2. For $j = 1, 2, \dots$,
 - Solve systems (2.3.5) and (2.3.8) for $U(b^{(j-1)})$ and $u'_N(b^{(j-1)})$.
 - Compute
$$b^{(j)} = b^{(j-1)} - \frac{Q(b^{(j-1)})}{Q'(b^{(j-1)})}.$$
 - If $|b^{(j)} - b^{(j-1)}| \leq \varepsilon$, then let $b_m = b^{(j)}$ and terminate the loop.
3. Solve system (2.3.5) for a better approximation to U^m .

Remark 2.4. Our numerical tests show that the maximum number of iterations in the above loop is not greater than 7 when $\varepsilon = 10^{-6}$, which is expected since $b^{(0)} = b_{m-1} + k_m$ is very closed to b_m .

To conclude this section, let us explain how to compute the approximations of option prices at any given point (S, t) . For $m = 0, 1, \dots, M$, $j = 0, 1, \dots, N$, by the inverse transforms of (2.2.6) and (2.3.1), the approximation of $c(S_j^m, \tau_m)$ is given by $c_j^m = Ke^{-\alpha(b_m - L + x_j) - \beta\tau_m} u_j^m$, where $S_j^m = Ke^{x_j - L + b_m}$. Let $\text{SPLINE}(X, Y, x)$ denote the cubic spline interpolation function which gives the value of the interpolant at the point x , where vector X specifies the nodes and vector Y specifies the corresponding function values. Then for $S \geq 0$, the approximation of $c(S, T - \tau_m)$ can be obtained by

$$c_h^m(S) = \begin{cases} 0, & S < A_m^*, \\ \text{SPLINE}(S^m, c^m, S), & A_m^* \leq S \leq B_m^*, \\ S - K, & S > B_m^*, \end{cases} \quad (2.3.9)$$

where $A_m^* = Ke^{b_m - L}$, $B_m^* = Ke^{b_m}$, $S^m = (S_0^m, \dots, S_N^m)^T$, and $c^m = (c_0^m, \dots, c_N^m)^T$.

Let $\mathcal{T} = (\tau_0, \tau_1, \dots, \tau_M)^T$ and $c_h(S) = (c_h^0(S), c_h^1(S), \dots, c_h^M(S))^T$. Finally, the approximation of $c(S, T - \tau)$ is given by

$$c_{kh}(S, T - \tau) = \text{SPLINE}(c_h(S), \mathcal{T}, \tau). \quad (2.3.10)$$

2.4 Stability and nonnegativity of approximate solutions

In this section, we consider the stability of the finite element approximations and the nonnegativity of solutions to nonlinear system (2.3.5)–(2.3.6). We need to assume that system (2.3.5)–(2.3.6) has a unique solution and that

$$0 < \delta_\tau b_m \leq C \frac{1 + \kappa \sqrt{|\log(\tau_m)|}}{\sqrt{\tau_m}}, \quad m = 1, 2, \dots, M, \quad (2.4.1)$$

where $\kappa = 0$ if $r > d$ and $\kappa = 1$ if $r \leq d$, and C is a positive constant independent of h, k and u_h^m (we shall use C to denote such a generic positive constant). The first inequality implies that $\{b_m\}_0^M$ is an increasing sequence. As a matter of fact, it is true

that $b(\tau)$ is an increasing function of t (see [33]). The second inequality in (2.4.1) is motivated by the properties of the early exercise boundary $B^*(t)$ near the expiration date (see [23]):

$$B^*(T - \tau) \sim \begin{cases} K + K\sigma\sqrt{\tau \log(\sigma^2/(8\pi(r-d)^2\tau))}, & \text{if } r < d, \\ K + K\sigma\sqrt{2\tau \log(1/(4\sqrt{\pi}r\tau))}, & \text{if } r = d, \\ \frac{r}{d}K \left(1 + \sigma c_0\sqrt{2\tau}\right), & \text{if } r > d, \end{cases} \quad (2.4.2)$$

for $0 < \tau \ll 1$, where c_0 is some constant. Since $b(\tau) = \log(B^*(T - \tau)/K)$, we have

$$0 < b'(\tau) \leq C \frac{1 + \kappa\sqrt{|\log(\tau)|}}{\sqrt{\tau}}, \quad 0 < \tau \leq T. \quad (2.4.3)$$

Therefore, assumption (2.4.1) is the discrete version of the above inequalities.

In the following, we only consider $\theta = 0$ and $\theta = 1/2$, i.e., the Backward Euler scheme and the Crank-Nicolson scheme.

Theorem 2.2. *Assume that β is determined such that $\mu \geq 0$. Under assumption (2.4.1), the system (2.3.3)–(2.3.4) is stable when $\theta = 0$ or $\frac{1}{2}$, i.e., we have*

$$\max_{m=1}^M \|u_h^m\| \leq \|u_h^0\| + C, \quad (2.4.4)$$

where $\|\cdot\|$ denotes the norm of $L^2(\Omega)$.

Proof. Letting $v = u_h^{m-\theta}$ in (2.3.3), we get

$$(\delta_\tau u_h^m, u_h^{m-\theta}) + \gamma \|u_{hx}^{m-\theta}\|^2 + (\nu - \delta_\tau b_m)(u_{hx}^{m-\theta}, u_h^{m-\theta}) + \mu \|u_h^{m-\theta}\|^2 = f^{m-\theta} u_h^{m-\theta}(L).$$

By using the well-known equality

$$(\delta_\tau u_h^m, u_h^{m-\theta}) = \frac{1}{2k_m} (\|u_h^m\|^2 - \|u_h^{m-1}\|^2 + (1 - 2\theta)\|u_h^m - u_h^{m-1}\|^2),$$

and

$$(u_{hx}^{m-\theta}, u_h^{m-\theta}) = \frac{1}{2} |u_h^{m-\theta}(L)|^2,$$

we have

$$\begin{aligned} & \|u_h^m\|^2 - \|u_h^{m-1}\|^2 + 2k_m(1-2\theta)\|u_h^m - u_h^{m-1}\|^2 + \gamma\|u_h^{m-\theta}\|^2 \\ & + k_m(\nu - \delta_\tau b_m)|u_h^{m-\theta}(L)|^2 + 2k_m\mu\|u_h^{m-\theta}\|^2 = 2k_m f^{m-\theta} u_h^{m-\theta}(L). \end{aligned}$$

Hence,

$$\|u_h^m\|^2 - \|u_h^{m-1}\|^2 \leq 2k_m f^{m-\theta} u_h^{m-\theta}(L) - k_m(\nu - \delta_\tau b_m)|u_h^{m-\theta}(L)|^2.$$

Notice that assumption (2.4.1) implies that sequence $\{b_m\}_0^M$ is bounded. Thus, sequences $\{f^{m-\theta}\}$ and $\{u_h^{m-\theta}(L)\}$ are also bounded. Using (2.4.1) again, we get

$$\|u_h^m\|^2 - \|u_h^{m-1}\|^2 \leq Ck_m \frac{1 + \kappa\sqrt{|\log(\tau_m)|}}{\sqrt{\tau_m}}.$$

By summation, we obtain

$$\max_{m=1}^M \|u_h^m\|^2 \leq \|u_h^0\|^2 + C \sum_{m=1}^M k_m \frac{1 + \kappa\sqrt{|\log(\tau_m)|}}{\sqrt{\tau_m}} \leq \|u_h^0\|^2 + C \int_0^T \frac{1 + \kappa\sqrt{|\log(\tau)|}}{\sqrt{t}} d\tau.$$

The theorem is proved. $\square$

Remark 2.5. For $\theta = 1$, using $v = u_h^m$ instead of $v = u_h^{m-\theta}$ in the above proof, we can show that the system is stable when the mesh ratio k/h^2 is less than some positive constant.

Although it is difficult to show the solution existence of system (2.3.5) and (2.3.6), we have the following results concerning the nonnegativity of U^m .

Theorem 2.3. *Under the assumption (2.4.1), if $\alpha \geq -1$, $\mu \geq 0$, and*

$$\rho_1 = h \max_{m=1}^M \frac{1 + \kappa\sqrt{|\log(\tau_m)|}}{\sqrt{\tau_m}}, \quad \rho_2 = \max_{m=1}^M \frac{h^2}{k_m} \quad (2.4.5)$$

are small enough, then the solution of system (2.3.5) is nonnegative, i.e., $u_j^m \geq 0$ for $i = 1, 2, \dots, N$, $m = 1, 2, \dots, M$.

Proof. For simplicity, we only consider the Backward Euler scheme ($\theta = 0$). Write system (2.3.5) into equations:

$$\begin{cases} b_1^m u_i^m + c_1^m u_2^m = d_1, \\ a_i^m u_{i-1}^m + b_i^m u_i^m + c_i^m u_{i+1}^m = d_i^m, \quad i = 2, \dots, N-1, \\ a_{N-1}^m u_{N-1}^m + b_N^m u_N^m = d_N^m, \end{cases} \quad (2.4.6)$$

where

$$\begin{aligned} a_i^m &= \frac{1}{6}(1 + \mu k_m)h_i - \frac{\gamma k_m}{h_i} - \frac{1}{2}(\nu k_m + b_{m-1} - b_m), \quad i = 2, \dots, N, \\ b_i^m &= \frac{1}{3}(1 + \mu k_m)(h_i + h_{i+1}) + \gamma k_m \left(\frac{1}{h_i} + \frac{1}{h_{i+1}} \right), \quad i = 1, \dots, N-1, \\ b_N^m &= \frac{1}{3}(1 + \mu k_m)h_N + \frac{\gamma k_m}{h_N} + \frac{1}{2}(\nu k_m + b_{m-1} - b_m), \\ c_i^m &= \frac{1}{6}(1 + \mu k_m)h_{i+1} - \frac{\gamma k_m}{h_{i+1}} + \frac{1}{2}(\nu k_m + b_{m-1} - b_m), \quad i = 1, \dots, N-1, \\ d_i^m &= \frac{h_i}{6}(u_{i-1}^{m-1} + 2u_i^{m-1}) + \frac{h_{i+1}}{6}(2u_i^{m-1} + u_{i+1}^{m-1}), \quad i = 1, \dots, N-1, \\ d_N^m &= \frac{h_N}{6}(u_{N-1}^{m-1} + 2u_N^{m-1}) + \gamma k_m e^{\beta \tau_m + \alpha b_m} ((\alpha + 1)e^{b_m} - \alpha), \\ h_i &= x_i - x_{i-1}, \quad i = 1, \dots, N. \end{aligned}$$

When $\alpha \geq -1$, $\mu \geq 0$, and ρ_1 and ρ_2 are small enough, we can verify that

$$\begin{aligned} a_i^m &< 0, \quad i = 2, \dots, N, \\ b_i^m &> 0, \quad i = 1, 2, \dots, N, \\ c_i^m &< 0, \quad i = 1, \dots, N-1, \\ b_1^m + c_1^m &> 0, \\ a_i^m + b_i^m + c_i^m &> 0, \quad i = 2, \dots, N-1, \\ a_N^m + b_N^m &> 0. \end{aligned}$$

Hence, the coefficient matrix of system (2.4.6) is an M -matrix and thus its inverse is nonnegative ([9]). Since U^0 is nonnegative, vector $(d_1^0, \dots, d_N^0)^T$ is also nonnegative when $\alpha \geq -1$. Therefore, by induction, $(d_1^m, \dots, d_N^m)'$ and U^m are nonnegative for $m = 1, \dots, M$ when $\alpha \geq -1$ and $\mu \geq 0$. $\square$

Remark 2.6. Since $b(\tau)$ is not differentiable at $\tau = 0$ (see (2.4.3)) and u may not have the desired regularity on $[0, L] \times [0, T]$, it is hard to establish any error estimates for our front-fixing finite element method. However, numerical results in §2.6 show that the Crank-Nicolson scheme converges quadratically.

2.5 Evaluation of Greeks

In this section, we consider how to compute important Greeks such as Delta (Δ), Theta (Θ), Rho (R), and Vega (V), which are partial derivatives of the option price with respect to stock price S , time to the expiration date $\tau = T - t$, interest rate r , and volatility σ . We first consider the semi-analytic approximation of the option prices by taking advantage that our method is able to compute early exercise prices very rapidly and accurately. The semi-analytic formulas for Delta and Theta are thus derived by differentiation. Then the moving boundary problems will be formulated for computing Rho and Vega.

We start with the analytic representation of the American option price ([33], [38] and [39]):

$$c(S, T - \tau) = \begin{cases} C(S, T - \tau) + p(S, \tau), & 0 \leq S \leq B^*(T - \tau), \quad 0 \leq \tau \leq T, \\ S - K, & S \geq B^*(T - \tau), \quad 0 \leq \tau \leq T, \end{cases} \quad (2.5.1)$$

where $C(S, \tau)$ is the European call option price given by the Black-Scholes formula

and $p(S, \tau)$ is the early exercise premium given by

$$p(S, \tau) = \int_0^\tau (qSe^{-q(\tau-\xi)}N(\gamma_1) - rKe^{-r(\tau-\xi)}N(\gamma_2)) d\xi, \quad (2.5.2)$$

where N is the CDF of the standard normal distribution, and

$$\gamma_1 = \frac{\log(S/B^*(T-\xi)) + (r-q+\sigma^2/2)(\tau-\xi)}{\sigma\sqrt{\tau-\xi}}, \quad \gamma_2 = \gamma_1 - \sigma\sqrt{\tau-\xi}. \quad (2.5.3)$$

Let $B_{kh}^*(t)$ be the cubic spline interpolation of the approximations of $B^*(t)$ at $T - \tau_0, \dots, T - \tau_M$. Then the semi-analytic approximation of $c(S, T - \tau)$ is given by

$$c_{kh}(S, T - \tau) = \begin{cases} C(S, T - \tau) + p_{kh}(S, \tau), & 0 \leq S \leq B_{kh}^*(T - \tau), \quad 0 \leq \tau \leq T, \\ S - K, & S \geq B_{kh}^*(T - \tau), \quad 0 \leq \tau \leq T, \end{cases} \quad (2.5.4)$$

where

$$p_{kh}(S, \tau) = \int_0^\tau (qSe^{-q(\tau-\xi)}N(\gamma_{1kh}) - rKe^{-r(\tau-\xi)}N(\gamma_{2kh})) d\xi, \quad (2.5.5)$$

and γ_{1kh} and γ_{2kh} are defined by (2.5.3) with replacing B^* by B_{kh}^* . Differentiating (2.5.4) gives the approximations of Delta and Theta as follows:

$$\Delta_{kh} = \Delta_C + \frac{\partial p_{kh}(S, \tau)}{\partial S}, \quad \Theta_{kh} = \Theta_C + \frac{\partial p_{kh}(S, \tau)}{\partial \tau}, \quad 0 \leq S \leq B_{kh}^*(\tau), \quad 0 \leq \tau \leq T, \quad (2.5.6)$$

where

$$\Delta_C = \frac{\partial C(S, T - \tau)}{\partial S} = e^{-q\tau}N(\gamma_3), \quad (2.5.7)$$

$$\Theta_C = \frac{\partial C(S, T - \tau)}{\partial \tau} = \frac{\sigma Se^{-q\tau - \frac{\gamma_3^2}{2}}}{2\sqrt{2\pi\tau}} + rKe^{-r\tau}N(\gamma_4) - qSe^{-q\tau}N(\gamma_3), \quad (2.5.8)$$

are the Delta and Theta of the European option and

$$\gamma_3 = \frac{\log(S/K) + (r-q+\sigma^2/2)\tau}{\sigma\sqrt{\tau}}, \quad \gamma_4 = \gamma_3 - \sigma\sqrt{\tau}.$$

It follows from (2.5.5) that

$$\frac{\partial p_{kh}(S, \tau)}{\partial S} = \int_0^\tau q e^{-q(\tau-\xi)} N(\gamma_{1kh}) d\xi + \int_0^\tau \frac{e^{-q(\tau-\xi) - \frac{\gamma_{1kh}^2}{2}}}{\sigma \sqrt{2\pi(\tau-\xi)}} \left(q - \frac{rK}{B_{kh}^*(T-\xi)} \right) d\xi, \quad (2.5.9)$$

$$\begin{aligned} \frac{\partial p_{kh}(S, \tau)}{\partial \tau} &= \int_0^\tau (r^2 K e^{-r(\tau-\xi)} N(\gamma_{2kh}) - q^2 S e^{-q(\tau-\xi)} N(\gamma_{1kh})) d\xi \\ &\quad + \frac{S}{2\sigma\sqrt{2\pi}} \int_0^\tau \frac{e^{-q(\tau-\xi) - \frac{\gamma_{1kh}^2}{2}}}{\sqrt{\tau-\xi}} \left(\nu_1 - \frac{\nu_2 K}{B_{kh}^*(T-\xi)} \right) d\xi \\ &\quad - \frac{S}{2\sigma\sqrt{2\pi}} \int_0^\tau \frac{e^{-q(\tau-\xi) - \frac{\gamma_{1kh}^2}{2}}}{(\tau-\xi)^{\frac{3}{2}}} \left(q - \frac{rK}{B_{kh}^*(T-\xi)} \right) \log \left(\frac{S}{B_{kh}^*(T-\xi)} \right) d\xi \end{aligned} \quad (2.5.10)$$

where $\nu_1 = q(r - q + \sigma^2/2)$ and $\nu_2 = r(r - q - \sigma^2/2)$.

To sum up, we can compute the approximations of the call price, Delta, and Theta at any given point (S, τ) by (2.5.4) and (2.5.6). The integrals in (2.5.5), (2.5.9) and (2.5.10) will be computed by numerical integrations.

Remark 2.7. It follows from (2.4.2), (2.5.1) and (2.5.8) that $c_t(S, T - \tau)$ is unbounded near the point $(S, \tau) = (K, 0)$. Hence, one can not establish any error estimates over the domain containing this point for a finite difference scheme since c_{tt} is at least needed to be bounded. However, c_t is in L^2 over such a domain, which is good enough for finite element methods (see [4]).

Next, let us consider Rho and Vega. If we differentiate (2.5.4) with respect to r or σ , we must have $\frac{\partial S_{kh}^*}{\partial r}$ or $\frac{\partial S_{kh}^*}{\partial \sigma}$ in the resulting formula for Rho or Vega, but they are not available from the front-fixing finite element solver for c and S^* . However, we can formulate two variable domain problems for computing these Greeks.

Let $H = \frac{\partial c}{\partial r}$ or $H = \frac{\partial c}{\partial \sigma}$. Differentiating (2.2.1), (2.2.2), (2.2.3), and (2.2.5) with

respect to r or σ , we obtain the following variable domain problem:

$$H_t + \frac{1}{2}\sigma^2 S^2 H_{SS} + (r - q)SH_S - rH = G(S, t), \quad 0 < S < S^*(t), \quad 0 < t \leq T, \quad (2.5.11)$$

$$H(0, t) = 0, \quad 0 \leq t \leq T, \quad (2.5.12)$$

$$H(S^*(t), t) = 0, \quad 0 < t \leq T, \quad (2.5.13)$$

$$H(S, T) = 0, \quad 0 \leq S \leq S^*(T), \quad (2.5.14)$$

where $G(S, t) = c - Sc_s$ or $G(S, t) = -\sigma S^2 c_{SS}$, and (2.2.4) was used. As in Section 2.2 by using the variable transforms (2.2.6) and (2.3.1), we have the approximation to the above problem:

$$w_\tau - \gamma w_{xx} + (\nu - b'(\tau))w_x + \mu w = g, \quad 0 \leq x < L, \quad 0 < \tau \leq T, \quad (2.5.1)$$

$$w(0, t) = 0, \quad 0 \leq t \leq T, \quad (2.5.2)$$

$$w(L, t) = 0, \quad 0 \leq t \leq T, \quad (2.5.3)$$

$$w(y, 0) = 0, \quad 0 \leq x \leq L, \quad (2.5.4)$$

where

$$g = \begin{cases} u_x - (1 + \alpha)u, & \text{for Rho;} \\ \sigma(\alpha(1 + \alpha)u - (1 + 2\alpha)u_x + u_{xx}), & \text{for Vega.} \end{cases}$$

The above variable domain problem can be solved by the front-fixing finite element method proposed in Section 2.3. Let w_{kh} be the finite element approximation of w .

Then the approximation of Rho or Vega is thus given by

$$H_{kh}(S, T - \tau) = \begin{cases} Ke^{-\alpha y - \beta \tau} w_{kh}(y + L - b(\tau), \tau), & Ke^{a(\tau)} \leq S \leq Ke^{b(\tau)}, \quad 0 \leq \tau \leq T, \\ 0, & \text{otherwise,} \end{cases}$$

where $y = \log(S/K)$.

2.6 Numerical results

In this section, we shall present numerical results to examine our method and to compare it with other methods. Our programs were written in C++ and run on a computer with a Intel Core 2 CPU of 2 GHz. In the following, we only consider the Crank-Nicolson method which has been shown to be the most accurate by numerical tests.

Let us consider one-year American call options. The exercise price is \$100. The stock volatility is 0.1, 0.2, or 0.3. Interest rates and dividend rates are 0.05 or 0.1. Therefore, we shall have 12 combinations of parameters. We take $\alpha = 0$ and $\beta = r$ and thus have $\nu = \gamma + d - r$ and $\mu = 0$. The error tolerance ϵ is set to be the same both for estimating Y in Theorem 2.1 and for the Newton's method.

Since $b(\tau)$ behaves as $\sqrt{\tau}$ or $\sqrt{\tau \log(1/\tau)}$ when τ goes to zero, we shall use variable step sizes in time. For a given positive integer M , let

$$\tau_m = \frac{Tm^2}{M^2}, \quad m = 0, 1, \dots, M.$$

Then the step size at the m -th step is $k_m = \tau_m - \tau_{m-1} = T(2m - 1)/M^2$, which increases from T/M^2 to about $2T/M$. In this way, more steps will concentrate near $\tau = 0$, for example, 10% and 50% of steps in $[0, 0.01]$ and $[0, 0.25]$, respectively. We shall use uniform partitions in the spatial variable x such that the mesh size h is almost the same as $\min_{1 \leq m \leq M} \sqrt{k_m} = \sqrt{T}/M$.

First we examine how the error tolerance ϵ affects the truncation errors. In Table 2.1, we display the average maximum truncation errors (AMTE) for 12 combinations of parameters. The maximum truncation errors are computed between the approxi-

mations of option prices for $\epsilon = 10^{-10}$ and the ϵ in the table. The average maximum numbers of iterations for the Newton's method (I) are also presented in this table. We observe that AMTEs have almost the same order for a given number of steps in time, which means that L defined by (2.2.12) for the computational domain is very accurate. We have the same numerical results for Rho and Vega. It is also shown that Newton's method attains the desired accuracy within most 8 iterations.

Table 2.1. Average maximum truncation errors and maximum numbers of iterations

M	256		512		1024		2048		4096	
ϵ	AMTE	I	AMTE	I	AMTE	I	AMTE	I	AMTE	I
10^{-5}	$1.22e-4$	6	$2.69e-5$	6	$8.16e-6$	6	$2.82e-6$	6	$5.89e-7$	6
10^{-6}	$1.07e-4$	6	$3.34e-5$	6	$6.42e-6$	7	$3.53e-6$	7	$4.31e-7$	7
10^{-7}	$9.96e-5$	6	$3.16e-5$	7	$4.82e-6$	7	$1.43e-6$	7	$4.21e-7$	8
10^{-8}	$7.18e-5$	7	$4.11e-5$	7	$8.21e-6$	7	$2.87e-6$	8	$4.94e-7$	8
10^{-9}	$6.64e-5$	7	$2.44e-5$	7	$8.32e-6$	8	$1.61e-6$	8	$4.87e-7$	8

From now on, we shall take $\epsilon = 10^{-6}$. In Table 2.2, we display the average CPU times in seconds obtained by running our program ten times. For each run, early exercise prices $B_m^* = Ke^{b_m}$ ($m = 1, 2, \dots, M$) and today's option prices, Deltas, Rhos and Vegas at $S_j = Ke^{x_j - L + b_M}$ ($j = 0, 1, \dots, N$) are computed and written into two files. The option prices, Deltas, Rhos, Thetas, and Vegas for Table 2.5 – Table 2.7 are also computed and written into a file. The maximum numbers of iterations for the Newton's method (numbers inside parentheses) are also presented in this table. The results show that our front-fixing finite element method is very fast. We also display the length L of our computational intervals in Table 2.2. As implied by formula

(2.2.12), L is an increasing function of interest rate and volatility and a decreasing function of dividend rate.

Table 2.2. CPU times and lengths of computational intervals

σ	r	q	L	$M = 512$	$M = 1024$	$M = 2048$	$M = 4096$
0.1	0.05	0.05	0.81	0.45 (5)	1.75 (5)	6.31 (5)	26.0 (5)
		0.1	0.63	0.33 (4)	1.24 (4)	4.59 (4)	17.7 (4)
	0.1	0.05	1.41	0.68 (3)	2.51 (3)	9.39 (3)	38.2 (2)
		0.1	0.72	0.42 (5)	1.55 (5)	5.61 (5)	22.1 (5)
0.2	0.05	0.05	1.61	0.88 (6)	3.34 (6)	15.7 (6)	69.7 (6)
		0.1	1.28	0.72 (5)	2.53 (5)	10.6 (5)	48.6 (6)
	0.1	0.05	2.03	1.11 (3)	3.64 (3)	15.0 (3)	65.7 (3)
		0.1	1.43	0.80 (6)	2.95 (6)	12.9 (6)	59.2 (6)
0.3	0.05	0.05	2.39	1.32 (6)	5.63 (7)	26.1 (7)	115 (7)
		0.1	1.99	1.09 (6)	4.34 (6)	19.3 (7)	84.5 (7)
	0.1	0.05	2.70	1.43 (3)	5.62 (3)	22.0 (3)	93.6 (3)
		0.1	2.13	1.17 (6)	4.79 (7)	21.7 (7)	92.9 (7)

Next, we investigate the convergence of our method. Here we use the approximations of option prices and early exercise prices computed by our program with $M = 10240$ as “exact values”. For each given volatility ($\sigma = 0.1, 0.2, 0.3$) and a given number of steps in time (M), let $E(\sigma, M)$ be the averages of four maximum absolute errors at S_j ($j = 0, 1, \dots, N$) when $r, d = 0.05, 0.01$. In Figure 2.1, we depict $C(\sigma, M) = M^2 E(\sigma, M)$ against M . We observe that $C(M)$ is bounded and thus can conclude that the rate of convergence is of order 2. However, the bounds $C(\sigma, M)$ are very large for Rho and Vega. We also observe that $C(0.1, M)$ is larger than $C(0.2, M)$ and $C(0.3, M)$, which can be explained by the fact that the corresponding γ in (2.3.2) for $\sigma = 0.2, 0.3$ is 4 and 9 times of the one for $\sigma = 0.1$. In other

words, the constant for the coercivity of the bilinear form is smaller when $\sigma = 0.1$.

Figure 2.1. Graphs of $M^2E(\sigma, M)$

In Table 2.3 and Table 2.4, we display the maximum absolute errors for early exercise prices computed at the nodes in intervals $[0, 0.005]$ and $[0.005, 1]$, respectively. We also observe that the proposed method converges with about order 2 when t_m is large enough, say, 0.005. The errors for early exercise prices are large near the option expiration date due to the singularity of B^* at $t = T$. The graphs of early exercise prices near the option expiration date are depicted in Figure 2.2 – Figure 2.4 when $\sigma = 0.2$, which show that the singularity can only be captured by using very small time step size near the option expiration date. Indeed, we are in the same situation as

approximating the function $\sqrt{t}$ on $[0, 1]$ by using its piecewise linear interpolations.

We should point out that the relative errors are very small since early exercise prices are not less than 100 or 200.

Table 2.3. MAEs for early exercise prices on the interval $[0, 0.005]$

σ	r	q	$M = 512$	$M = 1024$	$M = 2048$	$M = 4096$
0.1	0.05	0.05	0.4461	0.2478	0.1253	0.0519
		0.1	0.1991	0.1142	0.0632	0.0306
	0.1	0.05	0.3402	0.2305	0.1391	0.0723
		0.1	0.4154	0.2342	0.1216	0.0516
0.2	0.05	0.05	0.5080	0.2696	0.1279	0.0512
		0.1	0.2650	0.1469	0.0788	0.0362
	0.1	0.05	0.5796	0.3649	0.2292	0.1440
		0.1	0.4745	0.2598	0.1263	0.0504
0.3	0.05	0.05	0.5499	0.2959	0.1461	0.0612
		0.1	0.3051	0.1671	0.0855	0.0387
	0.1	0.05	0.7554	0.4711	0.2862	0.1493
		0.1	0.5126	0.2754	0.1365	0.0554

Table 2.4. MAEs for early exercise prices on the interval $[0.005, 1]$

σ	r	q	$M = 512$	$M = 1024$	$M = 2048$	$M = 4096$
0.1	0.05	0.05	0.1404	0.0352	0.0086	0.0019
		0.1	0.0164	0.0040	0.0010	0.0002
	0.1	0.05	0.2319	0.0429	0.0093	0.0020
		0.1	0.0989	0.0245	0.0060	0.0013
0.2	0.05	0.05	0.1065	0.0272	0.0067	0.0015
		0.1	0.0152	0.0038	0.0009	0.0002
	0.1	0.05	0.1871	0.0366	0.0087	0.0024
		0.1	0.0695	0.0177	0.0043	0.0009
0.3	0.05	0.05	0.1007	0.0260	0.0064	0.0014
		0.1	0.0161	0.0040	0.0010	0.0002
	0.1	0.05	0.1597	0.0342	0.0081	0.0017
		0.1	0.0638	0.0164	0.0040	0.0009

Figure 2.2. Early exercise prices near the expiration date: $r = d = 0.05$

Figure 2.3. Early exercise prices near the expiration date: $r = 0.05$, $d = 0.1$

Figure 2.4. Early exercise prices near the expiration date: $r = 0.1$, $d = 0.05$

Figure 2.5. Early exercise prices near the expiration date: $r = d = 0.1$

Now, let us study the accuracy of our front-fixing method with $M = 512$ when the “exact” values are computed by the binomial method with 20000 time steps. As mentioned before, the semi-analytic approximations are flexible to compute option price, Deltas (hedge ratios) and Thetas at any points once the whole early exercise boundary is available. For simplicity, we use Matlab to implement the semi-analytic approximations. In Table 2.5 – Table 2.7, we display the maximum absolute errors (MAE) at points $S_j = K_1 + j$ for $j = 0, 1, \dots, K_2 - K_1$ in the intervals $[K_1, K_2]$ which are specified in Table 2.5. The interval $[K_1, K_2]$ is determined so that the option price is less than 0.0001 when $S \leq K_1$ and is equal to the payoff when $S \geq K_2$. Here FFEM and SEMI represent the option prices computed by (2.3.9) and the semi-analytic approximation formula (2.5.4). Since the binomial method converges quite slowly and Theta changes rapidly near the early exercise prices, the errors for Thetas are calculated only at the points away from the early exercise prices by \$1, \$2 and \$4 for $\sigma = 0.1, 0.2, 0.3$, respectively. For example, when $\sigma = 0.2, r = d = 0.05, S = \141 , we obtained $\Theta = 0.04376423, 0.04234630, 0.03899506$, and 0.04094303 by

the binomial method with 20000, 40000, 80000, and 100000 time steps, respectively, and $\Theta = 0.04078806$ by our method with $M = 512$. The results in these tables show that our method is very accurate. In particular, the semi-analytic approximation formula provides the same precision as the direct evaluation of option prices from finite element solutions, and it also gives very accurate approximations to Deltas and Thetas.

Table 2.5. Comparison with the binomial method: option prices

σ	Method	$r = d = 0.05$	$r = 0.05, d = 0.1$	$r = 0.1, d = 0.05$	$r = d = 0.1$
0.1	FFEM	$9.514e - 5$	$9.374e - 5$	$1.549e - 4$	$8.395e - 5$
	SEMI	$9.781e - 5$	$8.681e - 5$	$6.081e - 5$	$1.136e - 4$
		[65, 120]	[65, 110]	[60, 212]	[65, 120]
0.2	FFEM	$1.159e - 4$	$7.874e - 5$	$1.154e - 4$	$1.011e - 4$
	SEMI	$1.093e - 4$	$1.035e - 4$	$9.510e - 5$	$9.284e - 4$
		[42, 142]	[43, 123]	[34, 224]	[45, 135]
0.3	FFEM	$1.514e - 4$	$1.118e - 4$	$1.455e - 4$	$1.325e - 4$
	SEMI	$1.461e - 4$	$1.019e - 4$	$1.416e - 4$	$1.355e - 4$
		[25, 170]	[27, 142]	[25, 242]	[26, 158]

Table 2.6. Comparison with the binomial method: Deltas

σ	Method	$r = d = 0.05$	$r = 0.05, d = 0.1$	$r = 0.1, d = 0.05$	$r = d = 0.1$
0.1	FFEM	$2.036e - 4$	$1.190e - 5$	$2.153e - 5$	$1.027e - 4$
	SEMI	$2.036e - 4$	$3.650e - 5$	$1.169e - 5$	$1.232e - 4$
0.2	FFEM	$5.695e - 6$	$2.208e - 4$	$1.044e - 5$	$6.429e - 6$
	SEMI	$6.418e - 6$	$2.203e - 4$	$9.472e - 6$	$6.562e - 6$
0.3	FFEM	$9.268e - 6$	$1.560e - 5$	$8.377e - 6$	$2.625e - 4$
	SEMI	$9.363e - 6$	$1.606e - 5$	$8.034e - 6$	$2.627e - 4$

Table 2.7. Comparison with the binomial method: Thetas

σ	Method	$r = d = 0.05$	$r = 0.05, d = 0.1$	$r = 0.1, d = 0.05$	$r = d = 0.1$
0.1	FFEM	$8.000e - 5$	$1.631e - 4$	$1.576e - 4$	$4.898e - 4$
	SEMI	$8.195e - 5$	$1.502e - 4$	$1.064e - 4$	$4.805e - 4$
0.2	FFEM	$2.259e - 4$	$1.088e - 3$	$4.695e - 4$	$4.736e - 4$
	SEMI	$2.263e - 4$	$1.101e - 3$	$4.663e - 4$	$4.822e - 4$
0.3	FFEM	$1.052e - 3$	$6.549e - 4$	$5.385e - 4$	$1.617e - 3$
	SEMI	$1.053e - 3$	$6.542e - 4$	$5.446e - 4$	$1.614e - 3$

We present the graphs of option prices, Deltas and Thetas when $\sigma = 0.2$ in Figure 2.6 – Figure 2.8 on which the early exercise boundaries are also depicted. Here we have used the semi-analytic formulas to compute all values while the approximations of the early exercise boundaries were computed by our method with 512 time steps. As mentioned in Remark 2.5 we observe that Theta has a singularity when S is equal to the strike price $K = 100$ and the time to the option expiration date is zero. We also see that Deltas change sharply around the early exercise boundaries when $r \geq d$ or around the line $S = K$ when $r < d$. This observation can be explained by the fact that the payoff (the initial value) is not differentiable at $S = K$.

We also depict the graphs of Rhos and Vegas against stock price at the expiration date in Figure 2.9 when $\sigma = 0.2$ and $r = 0.1$. The Rhos and Vegas are computed by solving (2.5.1)–(2.5.2). We observe that Rhos and Vegas of the American options are very close to the European ones when dividend rate d is small and stock prices are not very large. It is as expected since American and European options have the same value when $d = 0$. We are surprised that the Vega may have two local maximum

Figure 2.6. Graphs of option prices: $\sigma = 0.2$

(see Figure 2.9 (a)). To make sure that the graphs are right, we have checked the values of the Vega at $S = 50, 100, 150, 250, 500$ by using central divided difference $(c(\sigma = 0.20001) - c(\sigma = 0.19999))/0.00002$ to approximate $\frac{\partial c}{\partial \sigma}$. The option prices for $\sigma = 0.19999$ and $\sigma = 0.20001$ are computed by the semi-analytic formulas with $M = 1024$ (FEMDD) and the binomial method with 20000 time steps (BINDD). The values of estimated Vegas are displayed in Table 2.8, which shows that our front-fixing method (FEM) gives correct values of Vegas.

Figure 2.7. Graphs of hedge ratios: $\sigma = 0.2$

Figure 2.9. Graphs of Rhos and Vegas: $\sigma = 0.2$ and $r = 0.1$

Figure 2.8. Graphs of Thetas: $\sigma = 0.2$

Table 2.8. Vegas: $\sigma = 0.2, r = 0.1, d = 0.02$

S	FEM	FEMDD	BINDD	European
50	0.2406	0.2406	0.2354	0.2406
100	34.5094	34.5094	34.5090	34.5094
150	2.4061	2.4061	2.3959	2.4061
250	0.0027	0.0027	0.0027	0.0002
500	1.9091	1.9092	1.9091	0.0000

Finally, we consider the American put option with the same parameters in Table 1A of [59]: $K = 100, \sigma = 0.3, r = 0.1, d = 0.0$. In Table 2.9 – Table 2.11, we display the option prices, Deltas, and Thetas, respectively, at 12 stock prices together with the maximum of absolute errors (MAE) and the root of mean squared errors

(RMSE). The errors in the first row and the second row are computed using the “exact” values by the binomial method with 20000 time steps and our method with 10240 time steps, respectively. Here B100, B512, B1024, B6400, and B20000 represent the binomial method with 100, 512, 1024, 6400, and 20000 time steps, respectively. Similar notations are used for the front-fixing finite element method (F100, F512 and F10240) and the semi-analytic approximations (S100). The fifth columns in Table 2.9 and Table 2.10 contain the data from Table 1A of [59] for the front-fixing finite difference method with 100 time steps. We also present the Rhos and Vegas for the put and their errors in Table 2.12 while the “exact” values are computed by our method with 10240 time steps. The results show that our method is much more accurate than the other methods. Again, we notice from Table 2.11 and Table 2.12 that the order of convergence is 2 for our method.

Table 2.9. Prices and errors for an American put

S	F10240	B20000	B100	FFF	F100	S100
77	23.0133	23.0133	23.0000	23.0128	23.0133	23.0137
78	22.0631	22.0631	22.0567	22.0621	22.0631	22.0637
79	21.1488	21.1488	21.1442	21.1469	21.1487	21.1493
80	20.2689	20.2689	20.2576	20.2662	20.2688	20.2695
85	16.3455	16.3455	16.3412	16.3396	16.3454	16.3462
90	13.1207	13.1207	13.1208	13.1124	13.1205	13.1215
95	10.4830	10.4830	10.4799	10.4733	10.4828	10.4838
100	8.3377	8.3376	8.3255	8.3277	8.3373	8.3384
105	6.6031	6.6032	6.6108	6.5936	6.6026	6.6038
110	5.2087	5.2088	5.2250	5.2004	5.2082	5.2094
115	4.0941	4.0941	4.1034	4.0872	4.0935	4.0947
120	3.2077	3.2076	3.1964	3.2023	3.2070	3.2082
MAE	—	—	$1.6e-2$	$9.9e-3$	$6.5e-4$	$8.1e-4$
	—	$8.6e-5$	$1.6e-2$	$1.0e-2$	$6.9e-4$	$7.8e-4$
RMSE	—	—	$9.5e-3$	$6.7e-3$	$3.7e-4$	$6.4e-4$
	—	$4.2e-5$	$9.5e-3$	$6.7e-3$	$3.6e-4$	$6.4e-4$

Table 2.10. Deltas and errors for an American put

S	F10240	B20000	B100	FFF	F100	S100
77	-0.9684	-0.9685	-0.9619	-0.9718	-0.9685	-0.9684
78	-0.9320	-0.9320	-0.9342	-0.9353	-0.9320	-0.9320
79	-0.8969	-0.8969	-0.8987	-0.9001	-0.8969	-0.8969
80	-0.8631	-0.8631	-0.8634	-0.8661	-0.8631	-0.8630
85	-0.7108	-0.7108	-0.7107	-0.7133	-0.7108	-0.7108
90	-0.5828	-0.5828	-0.5832	-0.5848	-0.5829	-0.5828
95	-0.4754	-0.4754	-0.4761	-0.4769	-0.4754	-0.4754
100	-0.3855	-0.3855	-0.3860	-0.3866	-0.3855	-0.3855
105	-0.3107	-0.3107	-0.3110	-0.3116	-0.3107	-0.3107
110	-0.2490	-0.2490	-0.2493	-0.2497	-0.2491	-0.2490
115	-0.1985	-0.1985	-0.1988	-0.1990	-0.1985	-0.1985
120	-0.1575	-0.1575	-0.1574	-0.1578	-0.1575	-0.1575
MAE	—	—	$6.6e-3$	$3.3e-3$	$1.9e-5$	$4.9e-5$
	—	$1.1e-5$	$6.6e-3$	$3.3e-3$	$2.1e-5$	$3.9e-5$
RMSE	—	—	$2.1e-3$	$2.2e-3$	$1.3e-5$	$2.6e-5$
	—	$4.8-6$	$2.1e-3$	$2.2e-3$	$1.6e-5$	$2.2e-5$

Table 2.11. Thetas and errors for an American put

S	F10240	B20000	B512	B1024	B6400	F512	S512
77	0.1437	0.1424	0.1744	0.1300	0.1492	0.1436	0.1436
78	0.3130	0.3124	0.3438	0.3182	0.3065	0.3129	0.3129
79	0.4791	0.4795	0.5060	0.4930	0.4823	0.4791	0.4791
80	0.6417	0.6425	0.6602	0.6383	0.6438	0.6417	0.6417
85	1.3846	1.3846	1.3864	1.3807	1.3848	1.3846	1.3846
90	1.9825	1.9826	1.9811	1.9801	1.9826	1.9825	1.9825
95	2.4165	2.4166	2.4166	2.4157	2.4164	2.4165	2.4165
10	2.6880	2.6881	2.6895	2.6884	2.6881	2.6880	2.6880
105	2.8130	2.8130	2.8112	2.8111	2.8127	2.8130	2.8130
110	2.8158	2.8158	2.8125	2.8160	2.8156	2.8158	2.8158
115	2.7237	2.7239	2.7222	2.7224	2.7235	2.7237	2.7237
120	2.5638	2.5639	2.5636	2.5630	2.5637	2.5638	2.5638
MAE	—	—	0.0320	0.0135	0.0068	0.0012	0.0012
	—	0.0012	0.0308	0.0139	0.0065	$2.3e-5$	$2.8e-5$
RMSE	—	—	0.0159	0.0059	0.0027	0.0005	0.0005
	—	0.0005	0.0158	0.0061	0.0027	$1.4e-5$	$2.1e-5$

Table 2.12. Rhos, Vegas and their errors for an American put

Greek	Rho			Vega		
S	F10240	F512	F1024	F10240	F512	F1024
77	2.6080	2.6082	2.6080	3.1120	3.1122	3.1121
78	5.4411	5.4413	5.4412	6.5698	6.5699	6.5698
79	7.9873	7.9875	7.9873	9.7593	9.7595	9.7594
80	10.2677	10.2679	10.2678	12.6969	12.6970	12.6969
85	18.3354	18.3356	18.3354	24.0918	24.0917	24.0918
90	22.2306	22.2308	22.2307	31.0463	31.0462	31.0463
95	23.3906	23.3908	23.3907	34.6907	34.6905	34.6906
100	22.8286	22.8288	22.8287	35.8971	35.8968	35.8970
105	21.2528	21.2530	21.2528	35.3579	35.3576	35.3578
110	19.1494	19.1495	19.1494	33.6261	33.6258	33.6260
115	16.8429	16.8430	16.8430	31.1384	31.1381	31.1383
120	14.5416	14.5417	14.5417	28.2322	28.2319	28.2321
MAE	—	$2.3e - 4$	$5.7e - 5$	—	$3.1e - 4$	$7.9e - 5$
RMSE	—	$1.9e - 4$	$4.6e - 5$	—	$2.2e - 4$	$5.7e - 5$

CHAPTER 3

AMERICAN OPTIONS ON ZERO-COUPON BONDS

3.1 Introduction

In this chapter we study the front-fixing finite element method for American put option problems on zero-coupon bonds. We begin by giving a description of the interest rate process and the free boundary problem in Section 3.2. In Section 3.3 we consider a front-fixing finite element method for the free boundary value problem and analyze its stability with an appropriate assumption about the approximate free boundaries near the option expiration date. In Section 3.4 we discuss the implementation of the front-fixing finite element method in detail and give the algorithm used to solve the resulting nonlinear systems numerically. We then present numerical results and comparisons of our method with other methods in the last section.

3.2 The interest rate process

Consider a class of one-factor models of the short interest rate process:

$$r(t) = \zeta(X(t)), \quad dX(t) = (\phi(t) - \psi(t)X(t))dt + \sigma(t)dW(t), \quad (3.2.1)$$

where $\zeta(x)$ is an invertible function on $(-\infty, +\infty)$, $\phi(t)$, $\psi(t)$ and $\sigma(t)$ are some known functions of t , and $W(t)$ is a standard Brownian motion under the risk-neutral measure. For $\zeta(x) = x$ and $\zeta(x) = e^x$, we have the popular Hull-White model ([31]) and Black and Karasinski model ([12]), respectively.

Let $x = \eta(r)$ be the inverse function of $r = \zeta(x)$. Assume that $\zeta(x)$ is twice continuously differentiable. By using Ito's formula, we can obtain the stochastic

differential equation for the interest rate process $r(t)$:

$$dr(t) = a(r(t), t)dt + b(r(t), t)dW(t), \quad (3.2.2)$$

where

$$a(r, t) = \zeta'(\eta(r))(\phi(t) - \psi(t)\eta(r)) + \frac{1}{2}\sigma(t)^2\zeta''(\eta(r)), \quad b(r, t) = \sigma(t)\zeta'(\eta(r)).$$

Then we have the following fundamental partial differential equation for the rational price $V(r, t)$ of an interest rate derivative at time t ([10], [53]):

$$V_t + \frac{1}{2}b(r, t)^2V_{rr} + a(r, t)V_r - rV = 0. \quad (3.2.3)$$

Since $\zeta(x)$ is invertible, we can rewrite the above PDE into the PDE for $\tilde{V}(x, t) = V(\zeta(x), t)$:

$$\tilde{V}_t + \frac{1}{2}\sigma(t)^2\tilde{V}_{xx} + (\phi(t) - \psi(t)x)\tilde{V}_x - \zeta(x)\tilde{V} = 0.$$

Now let us consider an American put option on a T^* -maturity zero-coupon bond. The option expiration date is T ($< T^*$) and its exercise price is K . Since the option can be exercised at any time up to its expiration date, there is a critical interest rate $r^*(t)$ which is referred to as the early exercise interest rate. Denote the option price by $p(r, t)$. Let $x^*(t) = \eta(r^*(t))$ and $\tilde{p}(x, t) = p(\zeta(x), t)$. According to the above argument, we can show that $\tilde{p}(x, t)$ and $x^*(t)$ solve the following free boundary problem:

$$\tilde{p}_t + \frac{1}{2}\sigma(t)^2\tilde{p}_{xx} + (\phi(t) - \psi(t)x)\tilde{p}_x - \zeta(x)\tilde{p} = 0, \quad -\infty < x < x^*(t), \quad 0 \leq t \leq T, \quad (3.2.4)$$

$$\tilde{p}(x^*(t), t) = g(x^*(t), t), \quad 0 \leq t \leq T, \quad (3.2.5)$$

$$\tilde{p}_x(x^*(t), t) = g_x(x^*(t), t), \quad 0 \leq t \leq T, \quad (3.2.6)$$

$$\tilde{p}(x, T) = g(x, T), \quad -\infty < x < \infty, \quad (3.2.7)$$

where $g(x, t) = \max(K - \tilde{P}(x, t; T^*), 0)$ is the payoff of the put option and $\tilde{P}(x, t; T^*)$ is the bond price when $r = \zeta(x)$ at time t .

3.3 A front-fixing finite element method

Let L be a positive number large enough such that

$$g(x, t) = 0, \quad \tilde{p}(x, t) \leq \epsilon, \quad \forall x \leq x^*(t) - L,$$

where ϵ is a given error tolerance. Then we can truncate the free boundary problem (3.2.4)–(3.2.7) into the following problem:

$$\tilde{p}_t + \frac{1}{2}\sigma(t)^2\tilde{p}_{xx} + (\phi(t) - \psi(t)x)\tilde{p}_x - \zeta(x)\tilde{p} = 0, \quad \begin{array}{l} x^*(t) - L < x < x^*(t), \\ 0 \leq t \leq T, \end{array} \quad (3.3.1)$$

$$\tilde{p}(x^*(t) - L, t) = g(x^*(t) - L, t), \quad 0 \leq t \leq T, \quad (3.3.2)$$

$$\tilde{p}(x^*(t), t) = g(x^*(t), t), \quad 0 \leq t \leq T, \quad (3.3.3)$$

$$\tilde{p}_x(x^*(t), t) = g_x(x^*(t), t), \quad 0 \leq t \leq T, \quad (3.3.4)$$

$$\tilde{p}(x, T) = g(x, T), \quad -\infty < x < \infty. \quad (3.3.5)$$

Consider the variable transforms:

$$\tau = T - t, \quad y = x - x^*(T - \tau) + L,$$

$$u(x, \tau) = e^{-\beta\tau}\tilde{p}(x, T - \tau), \quad \varphi(\tau) = x^*(T - \tau),$$

where β is a positive constant. Then we have the following nonlinear problem for φ and u :

$$u_\tau - \gamma(\tau)u_{yy} + c(y, \tau, \varphi, \varphi')u_y + d(y, \tau; \varphi)u = 0, \quad 0 \leq \tau \leq T, \quad 0 < y < L, \quad (3.3.6)$$

$$u(0, \tau) = f(0, \tau; \varphi), \quad 0 \leq \tau \leq T, \quad (3.3.7)$$

$$u(L, \tau) = f(L, \tau; \varphi), \quad 0 \leq \tau \leq T, \quad (3.3.8)$$

$$u_y(L, \tau) = f_y(L, \tau; \varphi), \quad 0 \leq \tau \leq T, \quad (3.3.9)$$

$$u(y, 0) = u_0(y), \quad 0 \leq y \leq L, \quad (3.3.10)$$

where

$$\gamma(\tau) = \frac{1}{2}\sigma(T - \tau)^2,$$

$$c(y, \tau; \varphi, \varphi') = \psi(T - \tau)(y + \varphi(\tau) - L) - \phi(T - \tau) - \varphi'(\tau),$$

$$d(y, \tau; \varphi) = \zeta(y + \varphi(\tau) - L) + \beta,$$

$$f(y, \tau; \varphi) = e^{-\beta\tau}g(y + \varphi(\tau) - L, T - \tau),$$

$$v_0(y) = g(y + \varphi(0) - L, T).$$

Notice that we have two boundary conditions at $y = L$. The Neumann boundary condition will be integrated into the variational problem and the Dirichlet boundary condition will be used as a nonlinear equation for $\varphi(\tau)$. Define the bilinear form $\mathcal{A}$ as follows:

$$\mathcal{A}(v, w; \tau, \varphi, \varphi') = \gamma(\tau)(v_y, w_y) + (c(y, \tau; \varphi, \varphi')v_y, w) + (d(y, \tau; \varphi)v, w),$$

where $(\cdot, \cdot)$ denotes the inner product of $L^2(\Omega)$, the space of square integrable functions on $\Omega = (0, L)$. Let $H_E^1(\Omega)$ be the closure of $\{v \in C^\infty(\overline{\Omega}) : v(0) = 0\}$ in the usual Sobolev space $H^1(\Omega)$, and let $H^{-1}(\Omega)$ be the dual space of $H_E^1(\Omega)$ ([1], [43]). The variational form for problem (3.3.6)–(3.3.10) is: Find $u \in L^2(0, T; H_E^1(\Omega))$ and $\varphi \in$

$C([0, T]) \cap C^1((0, T])$ such that $u_\tau \in L^2(0, T; H^{-1}(\Omega))$, $u(0) = u_0$, and

$$(u_\tau, w) + \mathcal{A}(u, w; \tau, \varphi, \varphi') = G(\tau; \varphi)w(L), \quad \forall w \in H_E^1(\Omega), \quad \text{a.e. } 0 \leq \tau \leq T, \quad (3.3.11)$$

$$u(L, \tau) = f(L, \tau; \varphi), \quad 0 \leq \tau \leq T, \quad (3.3.12)$$

where $G(\tau; \varphi) = -\gamma(\tau)f_y(L, \tau; \varphi)$.

Let $\Delta_\tau : 0 = \tau_0 < \tau_1 < \dots < \tau_M = T$ and $\Delta_y : 0 = y_0 < y_1 < \dots < y_N = L$ be partitions of $[0, T]$ and $[0, L]$, respectively, where M and N are positive integers. Let V_h be the piecewise linear element subspace of $H_E^1(\Omega)$ with respect to the partition Δ_y where $h = \max_{1 \leq j \leq N} (y_j - y_{j-1})$. Denote the natural basis functions of V_h by $\omega_1, \omega_2, \dots, \omega_N$, i.e., $\omega_j \in V_h$ such that $\omega_j(y_i) = \delta_{ij}$ for $j = 1, 2, \dots, N$ and $i = 0, 1, \dots, N$, where δ_{ij} is the Kronecker delta.

Recall that $\varphi(0)$ is the solution of the following equation for x :

$$\tilde{P}(x, T; T^*) = K.$$

Let

$$u_h^0 = \sum_{j=1}^N u_0(y_j) \omega_j(y), \quad \varphi_0 = \varphi(0).$$

The finite element approximation to the variational problem (3.3.11)–(3.3.12) is: For

$m = 1, 2, \dots, N$, find $u_h^m \in V_h$ and $\varphi_m > 0$ such that

$$(\delta_\tau u_h^m, w) + \mathcal{A}_m \left(u_h^{m-\frac{1}{2}}, w \right) = G_m w(L), \quad \forall w \in V_h \quad (3.3.13)$$

$$u_h^m(L) = f(L, \tau_m; \varphi_m), \quad (3.3.14)$$

where

$$\begin{aligned}\mathcal{A}_m(u, w) &= \mathcal{A}\left(u, w; \tau_{m-\frac{1}{2}}, \varphi_m, \delta_\tau \varphi_m\right), \quad G_m = G\left(\tau_{m-\frac{1}{2}}; \varphi_{m-\frac{1}{2}}\right), \\ u_h^{m-\frac{1}{2}} &= \frac{u_h^m + u_h^{m-1}}{2}, \quad \tau_{m-\frac{1}{2}} = \frac{\tau_m + \tau_{m-1}}{2}, \quad \varphi_{m-\frac{1}{2}} = \frac{\varphi_m + \varphi_{m-1}}{2}. \\ \delta_\tau u_h^m &= \frac{u_h^m - u_h^{m-1}}{k_m}, \quad \delta_\tau \varphi_m = \frac{\varphi_m - \varphi_{m-1}}{k_m}, \quad k_m = \tau_m - \tau_{m-1}.\end{aligned}$$

Here we only considered the the Crank-Nicolson scheme in time.

Concerning the stability of the finite element approximations, we assume that the system (3.3.13)–(3.3.14) has a unique solution and that

$$|\delta_\tau \varphi_m| \leq C \tau_m^{\nu-1}, \quad (3.3.15)$$

where $0 < \nu < 1$ and C is a generic positive constant. The above inequality has been verified by numerical tests, but has not been mathematically proven.

Theorem 3.1. *Under assumption (3.3.15) the system (3.3.13)–(3.3.14) is absolutely stable, i.e., we have*

$$\max_{m=1}^M \|u_h^m\| + 2\bar{\gamma} k_m \sum_{m=1}^M \|u_{hy}^{m-\frac{1}{2}}\|^2 \leq \|u_h^0\|^2 + C,$$

where $\bar{\gamma} = \min_{0 \leq \tau \leq T} \gamma(\tau)$.

Proof. Substituting w by $u_h^{m-\frac{1}{2}}$ in (3.3.13), we get

$$\left(\delta_\tau u_h^m, u_h^{m-\frac{1}{2}}\right) + \mathcal{A}_m\left(u_h^{m-\frac{1}{2}}, u_h^{m-\frac{1}{2}}\right) = G_m u_h^{m-\frac{1}{2}}(L). \quad (3.3.16)$$

Let

$$\begin{aligned}\gamma_m &= \gamma\left(T - \tau_{m-\frac{1}{2}}\right), \quad \psi_m = \psi\left(T - \tau_{m-\frac{1}{2}}\right), \\ c_m(y) &= c\left(y, \tau_{m-\frac{1}{2}}; \varphi_m, \delta_\tau \varphi_m\right), \quad d_m(y) = d\left(y, \tau_{m-\frac{1}{2}}; \varphi_m\right).\end{aligned}$$

Then we have

$$\begin{aligned}
\mathcal{A}_m \left(u_h^{m-\frac{1}{2}}, u_h^{m-\frac{1}{2}} \right) &= \gamma_m \left\| u_{hy}^{m-\frac{1}{2}} \right\|^2 + \left(c_m(y) u_{hy}^{m-\frac{1}{2}} + d_m(y) u_h^{m-\frac{1}{2}}, u_h^{m-\frac{1}{2}} \right) \\
&= \gamma_m \left\| u_{hy}^{m-\frac{1}{2}} \right\|^2 + \frac{1}{2} c_m(L) \left| u_h^{m-\frac{1}{2}}(L) \right|^2 - \frac{\psi_m}{2} \left\| u_h^{m-\frac{1}{2}} \right\|^2 + \left(d_m(y) u_h^{m-\frac{1}{2}}, u_h^{m-\frac{1}{2}} \right) \\
&\geq \gamma_m \left\| u_{hy}^{m-\frac{1}{2}} \right\|^2 + \frac{1}{2} c_m(L) \left| u_h^{m-\frac{1}{2}}(L) \right|^2 + \left(\beta + \min_{0 \leq y \leq L} \zeta \left(y + \varphi_{m-\frac{1}{2}} - L \right) - \frac{\psi_m}{2} \right) \left\| u_h^{m-\frac{1}{2}} \right\|^2.
\end{aligned}$$

Thus, when β is sufficiently large, we can get

$$\mathcal{A}_m \left(u_h^{m-\frac{1}{2}}, u_h^{m-\frac{1}{2}} \right) \geq \bar{\gamma} \left\| u_{hy}^{m-\frac{1}{2}} \right\|^2 + \frac{1}{2} c_m(L) \left| u_h^{m-\frac{1}{2}}(L) \right|^2.$$

Hence it follows from (3.3.16) that

$$\left\| u_h^m \right\|^2 - \left\| u_h^{m-1} \right\|^2 + 2\bar{\gamma} k_m \left\| u_{hy}^{m-\frac{1}{2}} \right\|^2 + k_m c_m(L) \left| u_h^{m-\frac{1}{2}}(L) \right|^2 \leq 2k_m G_m u_h^{m-\frac{1}{2}}(L).$$

The assumption (3.3.15) implies that the sequence $\{\varphi_m\}$ is bounded. Thus, the sequences $\{G_m\}$ and $\{u_h^m(L)\} = \{f(L, \tau_m; \varphi_m)\}$ are also bounded. Therefore, we have

$$\left\| u_h^m \right\|^2 - \left\| u_h^{m-1} \right\|^2 + 2\bar{\gamma} k_m \left\| u_{hy}^{m-\frac{1}{2}} \right\|^2 \leq C k_m (1 + \tau_m^{\nu-1}).$$

By summation, we obtain

$$\max_{m=1}^M \left\| u_h^m \right\|^2 + 2\bar{\gamma} k_m \sum_{m=1}^M \left\| u_{hy}^{m-\frac{1}{2}} \right\|^2 \leq \left\| u_h^0 \right\|^2 + C \sum_{m=1}^M k_m (1 + \tau_m^{\nu-1}),$$

which leads to the stability estimate (3.1). $\square$

3.4 Evaluation of bond prices and implementation

In this section we discuss the implementation of our FFEM in detail. Since there are no known analytic formulas for the model other than the Hull-White model (including the Vasicek model), we first consider how to compute zero-coupon bond prices

and their derivatives with respect to r , for example, for the Black-Karasinski model and the model (1.3.5). Then we consider how to solve the nonlinear system (3.3.13)–(3.3.14) and give an algorithm to implement our method.

As discussed in §3.2 we can see that $\tilde{P}(x, t; T^*)$ is the solution of the final value problem:

$$\begin{aligned} \tilde{P}_t + \frac{1}{2}\sigma(t)^2\tilde{P}_{xx} + (\phi(t) - \psi(t)x)\tilde{P}_x - \zeta(x)\tilde{P} &= 0, \quad -\infty < x < \infty, \quad 0 \leq t < T^*, \\ \tilde{P}(x, T^*; T^*) &= 1, \quad -\infty < x < \infty. \end{aligned}$$

In order to solve this problem on a bounded domain, we consider the transforms:

$$\tau = T^* - t, \quad y = \zeta_0(x) \equiv \frac{e^x}{1 + e^x}, \quad v(y, \tau) = \tilde{P}(\zeta_0^{-1}(y), T^* - \tau; T^*).$$

Then we have the following initial value problem for $v(y, \tau)$ on the interval $\Omega = (0, 1)$:

$$v_\tau - c_0(y, \tau)v_{yy} + c_1(y, \tau)v_y + c_2(y, \tau)v = 0, \quad 0 < y < 1, \quad 0 < \tau \leq T^*, \quad (3.4.1)$$

$$v(y, T^*) = 1, \quad 0 < y < 1, \quad (3.4.2)$$

where

$$\begin{aligned} c_0(y, \tau) &= \frac{1}{2}\sigma(T^* - \tau)^2 y^2 (1 - y)^2, \\ c_1(y, \tau) &= y(1 - y) \left(\psi(T^* - \tau)\zeta_0^{-1}(y) - \phi(T^* - \tau) - \frac{1}{2}\sigma(T^* - \tau)^2(1 - 2y) \right), \\ c_2(y) &= \zeta(\zeta_0^{-1}(y)). \end{aligned}$$

Notice that $y = r$ for the model (1.3.5). The new function $v(y, \tau)$ is simply the bond price at time $T^* - \tau$ for this model.

Since $c_0(0, t) = c_0(1, t) = 0$, we essentially do not require any boundary conditions at $y = 0$ and $y = 1$. For the Black-Karasinski model, we have $c_2(y) = y/(1 - y)$ which

is singular at $y = 1$. However, we can show that $\tilde{P}(x, t; T^*) \rightarrow 0$ and $\tilde{P}_x(x, t; T^*) \rightarrow 0$ as $x \rightarrow \infty$ for this model. Thus $V = \{w \in H^1(\Omega) : w(1) = 0\}$ is the natural choice of the space for the weak form of the corresponding problem. Since $c_2(y) = y$ for the model (1.3.5) is bounded, we simply set $V = H^1(\Omega)$. Define the bilinear form $\mathcal{B}$ by

$$\mathcal{B}(\tau; v, w) = (c_0(y, \tau)v_y, w_y) + (c_4(y, \tau)v_y, w) + (c_2(y)v, w),$$

where

$$c_4(y, \tau) = y(1 - y) \left(\psi(T^* - \tau)\zeta_0^{-1}(y) - \phi(T^* - \tau) + \frac{1}{2}\sigma(T^* - \tau)^2(1 - 2y) \right).$$

The variational problem for the initial value problem (3.4.1)–(3.4.2) is: Find $v \in L^2(0, T^*; V)$ such that $v_\tau \in L^2(0, T^*; V')$, $v(0) \equiv 1$, and

$$(v_\tau, w) + \mathcal{B}(\tau, v, w) = 0, \quad \forall w \in V, \quad \text{a.e.}, \quad 0 < \tau \leq T^*. \quad (3.4.3)$$

Furthermore, we can form the following variational problem for $q(y, \tau) = \tilde{P}_x(\zeta_0^{-1}(y), T^* - \tau; T^*)$: Find $q \in L^2(0, T^*; V)$ such that $q_\tau \in L^2(0, T^*; V')$, $q(0) = 0$, and

$$(q_\tau, w) + \mathcal{B}(\tau; q, w) = F(\tau; w), \quad \forall w \in V, \quad \text{a.e.}, \quad 0 < \tau \leq T^*, \quad (3.4.4)$$

where

$$F(\tau; w) = -(\zeta'_0(\zeta_0^{-1}(y))v, w) - \psi(T^* - \tau)(y(1 - y)v_y, w).$$

The above two variational problems can be solved by the same finite element method simultaneously, e.g., using a piecewise linear element space.

Now let us consider how to solve the nonlinear system (3.3.13)–(3.3.14). Write

$$v_h^m(y) = \sum_{j=1}^N v_j^m \omega_j(y).$$

We can rewrite (3.3.13)–(3.3.14) into the following matrix form:

$$\left(A + \frac{1}{2}k_m B_m\right) V^m = \left(A - \frac{1}{2}k_m B_m\right) V^{m-1} + k_m F^m, \quad (3.4.5)$$

$$v_N^m = f(L, \tau_m; \varphi_m), \quad (3.4.6)$$

where

$$A = (\omega_j, \omega_i)_{N \times N}, \quad B_m = (\mathcal{A}_m(\omega_j, \omega_i))_{N \times N},$$

$$V^m = (v_1^m, \dots, v_N^m), \quad F^m = (0, \dots, G_m).$$

For saving computational time, we divide B_m into

$$B_m = B_m^{(1)} + \varphi_m B_m^{(2)},$$

where $B_m^{(1)}$ and $B_m^{(2)}$ are independent of φ_m . Notice that (3.4.5) defines an implicit vector function V^m of φ_m . We can consider (3.4.6) as a nonlinear equation of φ_m which can be rewritten as:

$$H(\varphi_m) = e^{\beta \tau_m} v_N^m - K + P(\varphi_m, T - \tau_m; T^*) = 0,$$

where we have used the fact that $P(\varphi_m, T - \tau_m; T^*) \leq K$. In order to solve this equation by Newton's method, we need $\dot{v}_N^m = \frac{\partial v_N^m}{\partial \varphi_m}$. Differentiating (3.4.5) with respect to φ_m , we get the linear system for the derivative $\dot{V}^m = \frac{\partial V^m}{\partial \varphi_m}$:

$$\left(A + \frac{1}{2}k_m B_m\right) \dot{V}^m = k_m \frac{\partial F^m}{\partial \varphi_m} - \frac{1}{2}k_m B_m^{(2)} V^{m-\frac{1}{2}}. \quad (3.4.7)$$

Systems (3.4.5) and (3.4.7) have the same tridiagonal coefficient matrix and thus can be solved simultaneously by using the Thomas algorithm.

To sum up, for a given tolerance ϵ , our front-fixing finite element method is implemented as follows:

For $m = 1, 2, \dots, M$, do

1. Compute $B_m^{(1)}$ and $B_m^{(2)}$.
2. Let $\varphi_m^{(0)} = \varphi_{m-1}$. For $j = 1, 2, \dots$
 - 2.1. Solve (3.4.3) and (3.4.4) by the same finite element method.
 - 2.2. Build the systems (3.4.5) and (3.4.7).
 - 2.3. Solve the systems (3.4.5) and (3.4.7) by using the Thomas algorithm.
 - 2.4. Compute $\varphi_m^{(j)}$ by

$$\varphi_m^{(j)} = \varphi_m^{(j-1)} - \frac{H\left(\varphi_m^{(j-1)}\right)}{H'\left(\varphi_m^{(j-1)}\right)}.$$

2.5. If

$$\left| \varphi_m^{(j-1)} - \varphi_m^{(j)} \right| < \epsilon,$$

then let $\varphi_m = \varphi_m^{(j)}$ and terminate the j -loop. Otherwise, go to 2.1.

4. Solve system (3.4.5) for a better approximation of V^m .

End do

We conclude this section with some remarks on the partitions Δ_τ and Δ_y . As in [28], we shall use variable step sizes in time by setting

$$\tau_m = \frac{Tm^2}{M^2}, \quad m = 0, 1, \dots, M.$$

In this way, we have relatively more steps near the option expiration date in order to capture the singularity of the early exercise interest rates. For example, we have more than 3% of steps in the interval $[0, 0.001]$ for one-year options. We shall use uniform partitions in the spatial variable y such that the mesh size h in y is almost the same as $\min_{1 \leq m \leq M} \sqrt{k_m} = \sqrt{T}/M$. We just simply set $\beta = 0$ since our tests show that the value of β does not affect the accuracy of computations.

3.5 Numerical results

In this section, we shall examine our front-fixing finite element method (FFEM) numerically and compare it with the usual finite element method (FEM) in [60]. Our programs were written in C++ and run on a computer with an Intel Core 2 CPUs of 3.0 GHz. Here we only consider the Vasicek model and the Hull-White model for simplicity. For the Vasicek model, the values of the parameters are given in Table 3.1 in which $\theta = \phi/\psi$ is the long-term expected interest rate. For the Hull-White model, we assume that the initial term structure is determined by the two-factor CIR model as in [31] and [61]:

$$r(t) = x_1(t) + x_2(t),$$

$$dx_i = \kappa_i(\theta_i - x_i)dt + \sigma_i\sqrt{x_i}dW_i(t), \quad i = 1, 2,$$

where $W_1(t)$ and $W_2(t)$ are two independent standard Brownian motion under the risk-neutral probability. The parameters are specified in Table 3.2. We are referred to [31, 60] for how to calibrate the Hull-White model.

Table 3.1. Parameters for the Vasicek model

Case	σ	ψ	θ	$r(0)$
VAS1	0.06	0.40	0.08	0.08
VAS2	0.10	0.30	0.10	0.10

In the following, one-year American put options written on 5-year and 30-year bonds will be considered. The option exercise prices are chosen to be the same as the current forward bond prices. First, we check the dependency of truncation errors on L .

Table 3.2. Parameters for the TCIR model

Case	σ_1	κ_1	θ_1	$x_1(0)$	σ_2	κ_2	θ_2	$x_2(0)$
TCIR1	0.05	0.2	0.06	0.06	0.10	0.4	0.04	0.04
TCIR2	0.20	0.3	0.06	0.06	0.30	0.4	0.04	0.04

In Table 3.3 and Table 3.4, we display the maximum of the maximum absolute errors (MMAE) for today's option prices and early exercise interest rates (EEIR) for the 4 put options. The maximum absolute errors are computed between the approximate values for $L = 2.0$ and the L as in the tables when the error tolerance for Newton's Method is set to be $1.0e - 10$. The maximum numbers (I) of iterations for Newton's method are also given in these tables. The results suggest that $L = 1.0$ is sufficiently large enough for the eight options. It also shows that Newton's method attains the desired accuracy within at most 7 iterations.

Table 3.3. Dependency of MAEs on L : the Vasicek model

M	100			1000			10000		
L	Price	EEIR	I	Price	EEIR	I	Price	EEIR	I
0.8	$3.3e - 10$	$1.8e - 13$	5	$4.7e - 10$	$3.9e - 11$	6	$4.6e - 09$	$1.5e - 09$	7
1.0	$1.0e - 12$	$1.9e - 13$	5	$4.9e - 11$	$4.8e - 11$	6	$4.6e - 09$	$3.0e - 09$	7
1.2	$4.8e - 13$	$1.8e - 13$	5	$9.1e - 11$	$4.1e - 11$	6	$5.2e - 09$	$1.7e - 09$	7
1.4	$7.0e - 13$	$1.7e - 13$	5	$8.8e - 11$	$3.6e - 11$	6	$4.5e - 09$	$2.6e - 09$	7
1.6	$1.2e - 12$	$1.7e - 13$	5	$4.2e - 11$	$3.4e - 11$	6	$6.4e - 09$	$2.2e - 09$	7

Now we investigate the convergence of our method with $L = 1.0$ and $\epsilon = 1.0e - 8$. We display the the L^2 -norm and H^1 -norm of the error $u_h^M - u_{h/2}^M$ in Figure 3.1 and Figure 3.3. We also display the the L^2 -norm and maximum norm of the error $\varphi_M - \varphi_{2M}$

Table 3.4. Dependency of MAEs on L : the Hull-White model

M	100			1000			10000		
L	Price	EEIR	I	Price	EEIR	I	Price	EEIR	I
0.8	$2.0e-12$	$3.8e-14$	5	$2.1e-11$	$9.8e-12$	5	$1.0e-09$	$1.5e-09$	6
1.0	$1.1e-12$	$4.9e-14$	5	$2.9e-11$	$7.3e-12$	5	$3.8e-10$	$1.1e-09$	6
1.2	$8.7e-13$	$5.5e-14$	5	$2.5e-11$	$1.1e-11$	5	$2.9e-10$	$1.1e-09$	6
1.4	$1.1e-12$	$3.9e-14$	5	$2.2e-11$	$6.4e-12$	5	$1.0e-09$	$1.3e-09$	6
1.6	$7.6e-13$	$3.6e-14$	5	$5.4e-11$	$7.4e-12$	5	$2.3e-10$	$1.1e-09$	6

in Figure 3.2 and Figure 3.4, where φ_M is the piecewise linear interpolation of φ^m ($m = 0, 1, \dots, M$). We can observe that the Crank-Nicolson scheme converges linearly and quadratically in the L^2 -norm and H^1 -norm as expected. The rates of convergence for the early exercise interest rates are more than 1 and 0.5 in the L^2 -norm and maximum norm, respectively.

Figure 3.1. Convergence of option prices: the Vasicek model

Next, we verify the assumption (3.3.15) numerically. To this end, we use a very small step size in time by taking $M = 20000$. In Figures 3.5– 3.6, we depict the graphs of φ_M, φ'_M over the interval $[0, 0.001]$ for which there are about 630 time steps (only

Figure 3.2. Convergence of early exercise interest rates: the Vasicek model

Figure 3.3. Convergence of option prices: the Hull-White model

Figure 3.4. Convergence of early exercise interest rates: the Hull-White model

20 time steps for the uniform partition). The exponential functions in the figures are obtained by fitting the data to the model $a + b * \tau^c$. These figures demonstrate

that the assumption (3.3.15) holds. In particular, we can observe that the sums of the exponents for the fitted exponential functions for φ_M and φ'_M are very close to 1, which also supports the assumption. Furthermore, we can propose a conjecture about the behavior of φ near $\tau = 0$: There are some constants $c_1 > 0$, $c_2 \geq 0$, $\nu > 0$, and $\mu \geq 0$ such that

$$\varphi(\tau) \sim \varphi(0) + \tau^\nu (c_1 - c_2 \log(\tau))^\mu, \quad \tau \rightarrow 0^+. \quad (3.5.1)$$

It seems that we do not have $\nu = \mu = \frac{1}{2}$ as for American options on stocks (see [23]).

Figure 3.5. Early exercise interest rates near the option expiration date: the Vasicek model & Case VAS1

Figure 3.6. Early exercise interest rates near the option expiration date: the Vasicek model & Case VAS2

Finally, we want to compare our front-fixing finite element method (FFEM) with the finite element method (FEM) in [60]. We consider one-year American put options written on bonds with expiration dates 5 years, 10 years, 15 years, and 20 years. The option exercise prices are given as the percentage of the current forward bond price: 87%, 88%, 89%, 90%, and 91%. We display the maximum absolute errors for 20 option prices, hedge ratios ($\partial p / \partial P$), and early exercise interest rates in Table 3.5 – Table 3.10. In these tables, the first and second rows are the maximum absolute errors when the “exact values” are the approximate values computed by FEM and FFEM with 25600 time steps, respectively, for each given number M of time steps. As expected, FFEM provides more accurate approximations, especially for early exercise interest rates.

Figure 3.7. Early exercise interest rates near the option expiration date: the Hull-White model & Case TCIR1

Table 3.5. MAEs for put prices: The Vasicek model

Case	VAS1		VAS2	
	FEM	FFEM	FEM	FFEM
100	$1.27e-2$	$8.63e-3$	$7.19e-3$	$1.81e-3$
	$1.27e-2$	$8.63e-3$	$7.20e-3$	$1.81e-3$
200	$3.42e-3$	$2.12e-3$	$1.49e-3$	$4.25e-4$
	$3.42e-3$	$2.12e-3$	$1.49e-3$	$4.25e-4$
400	$8.05e-4$	$5.25e-4$	$4.05e-4$	$1.01e-4$
	$8.05e-4$	$5.25e-4$	$4.05e-4$	$1.01e-4$
800	$2.19e-4$	$1.30e-4$	$1.06e-4$	$2.40e-5$
	$2.20e-4$	$1.30e-4$	$1.07e-4$	$2.43e-5$
1600	$5.38e-5$	$3.21e-5$	$2.59e-5$	$5.57e-6$
	$5.40e-5$	$3.22e-5$	$2.62e-5$	$5.89e-6$

Figure 3.8. Early exercise interest rates near the option expiration date: the Hull-White model & Case TCIR2

Table 3.6. MAEs for hedge ratios: The Vasicek model

Case	VAS1		VAS2	
	FEM	FFEM	FEM	FFEM
100	$1.39e-3$	$7.34e-4$	$7.17e-4$	$4.69e-5$
	$1.39e-3$	$7.34e-4$	$7.17e-4$	$4.69e-5$
200	$3.04e-4$	$1.74e-4$	$1.79e-4$	$9.85e-6$
	$3.04e-4$	$1.74e-4$	$1.79e-4$	$9.86e-6$
400	$8.48e-5$	$4.38e-5$	$2.81e-5$	$2.27e-6$
	$8.48e-5$	$4.38e-5$	$2.81e-5$	$2.28e-6$
800	$2.28e-5$	$1.11e-5$	$5.70e-6$	$5.06e-7$
	$2.28e-5$	$1.11e-5$	$5.69e-6$	$5.14e-7$
1600	$5.77e-6$	$2.73e-6$	$1.11e-6$	$1.17e-7$
	$5.78e-6$	$2.74e-6$	$1.10e-6$	$1.23e-7$

Table 3.7. MAEs for early exercise interest rates: The Vasicek model

Case	VAS1		VAS2	
M	FEM	FFEM	FEM	FFEM
100	$5.04e-3$	$6.33e-5$	$4.61e-3$	$3.80e-5$
	$5.03e-3$	$4.96e-5$	$4.61e-3$	$2.24e-5$
200	$2.58e-3$	$2.92e-5$	$2.27e-3$	$2.17e-5$
	$2.58e-3$	$1.29e-5$	$2.27e-3$	$5.93e-6$
400	$1.25e-3$	$2.07e-5$	$1.25e-3$	$1.73e-5$
	$1.24e-3$	$3.31e-6$	$1.26e-3$	$1.54e-6$
800	$5.47e-4$	$1.90e-5$	$6.25e-4$	$1.72e-5$
	$5.36e-4$	$8.45e-7$	$6.17e-4$	$3.95e-7$
1600	$3.12e-4$	$1.86e-5$	$3.12e-4$	$1.74e-5$
	$3.14e-4$	$2.14e-7$	$3.04e-4$	$1.01e-7$

Table 3.8. MAEs for put prices: The Hull-White model

Case	TCIR1		TCIR2	
M	FEM	FFEM	FEM	FFEM
100	$1.01e-1$	$8.01e-2$	$9.38e-3$	$5.09e-3$
	$1.01e-1$	$8.01e-2$	$9.38e-3$	$5.09e-3$
200	$2.05e-2$	$1.93e-2$	$2.23e-3$	$1.25e-3$
	$2.05e-2$	$1.93e-2$	$2.23e-3$	$1.25e-3$
400	$5.72e-3$	$4.29e-3$	$5.90e-4$	$3.06e-4$
	$5.73e-3$	$4.29e-3$	$5.90e-4$	$3.07e-4$
800	$1.74e-3$	$1.05e-3$	$1.55e-4$	$7.55e-5$
	$1.74e-3$	$1.06e-3$	$1.55e-4$	$7.57e-5$
1600	$4.26e-4$	$2.62e-4$	$3.96e-5$	$1.85e-5$
	$4.26e-4$	$2.62e-4$	$3.98e-5$	$1.87e-5$

Table 3.9. MAEs for hedge ratios: The Hull-white model

Case	TCIR1		TCIR2	
M	FEM	FFEM	FEM	FFEM
100	$5.04e-2$	$7.87e-2$	$1.22e-3$	$1.72e-4$
	$5.04e-2$	$7.87e-2$	$1.22e-3$	$1.72e-4$
200	$2.14e-2$	$2.61e-2$	$1.74e-4$	$3.35e-5$
	$2.14e-2$	$2.61e-2$	$1.74e-4$	$3.35e-5$
400	$1.91e-3$	$2.43e-3$	$4.63e-5$	$8.86e-6$
	$1.91e-3$	$2.43e-3$	$4.63e-5$	$8.86e-6$
800	$5.82e-4$	$7.48e-4$	$1.05e-5$	$2.01e-6$
	$5.83e-4$	$7.47e-4$	$1.05e-5$	$2.01e-6$
1600	$1.58e-4$	$1.82e-4$	$2.66e-6$	$5.01e-7$
	$1.58e-4$	$1.81e-4$	$2.66e-6$	$5.01e-7$

Table 3.10. MAEs for early exercise interest rates: The Hull-White model

Case	TCIR1		TCIR2	
M	FEM	FFEM	FEM	FFEM
100	$6.17e-3$	$2.97e-5$	$5.16e-3$	$5.49e-5$
	$6.17e-3$	$2.50e-5$	$5.16e-3$	$3.74e-5$
200	$2.77e-3$	$1.46e-5$	$2.38e-3$	$2.76e-5$
	$2.76e-3$	$6.19e-6$	$2.40e-3$	$9.76e-6$
400	$1.17e-3$	$1.75e-5$	$1.17e-3$	$2.05e-5$
	$1.17e-3$	$1.57e-6$	$1.18e-3$	$2.52e-6$
800	$7.03e-4$	$1.82e-5$	$5.86e-4$	$1.91e-5$
	$7.03e-4$	$4.00e-7$	$5.98e-4$	$6.44e-7$
1600	$3.13e-4$	$1.84e-5$	$3.12e-4$	$1.95e-5$
	$3.28e-4$	$1.01e-7$	$3.01e-4$	$1.63e-7$

CHAPTER 4

AMERICAN OPTIONS WITH REGIME-SWITCHING

4.1 Introduction

In Chapter 2 we considered American options on a risky asset, e.g., a stock. We assumed that the price dynamics of the underlying asset depended only on time and a random component. In this chapter we extend that model so that the price of the underlying asset also depends on the state of the economy, which is modeled by a finite state Markov chain $X(t)$.

In Section 4.2 we give a brief statement of the problem. In Section 4.3 we propose our front-fixing finite element method for the free boundary value problem and formulate the variational form of the problem and its finite element approximation. We also establish the stability of the method in this section. In Section 4.4 we discuss implementation of the method in detail and provide the algorithm used to compute the approximate solutions. We conclude the chapter by presenting numerical results and comparisons of our method with the usual finite element method in the last section.

4.2 The regime-switching model

The stock price process $S(t)$ is assumed to follow the stochastic differential equation

$$dS(t) = S(t) (\mu_{X(t)} dt + \sigma_{X(t)} dW(t)), \quad t > 0, \quad (4.2.1)$$

where $W(t)$ is a standard Brownian motion under the risk neutral measure, and μ_{e_i} and σ_{e_i} are constant when the economy is in the i -th state at time t , i.e., when

$X(t) = e_i$. For simplicity, we consider the case that there are only two states which are denoted by $e_1 = (1, 0)'$ and $e_2 = (0, 1)'$.

Consider an American call option with strike price K and expiry date T years. As usual, there is an early exercise price $S_i^c(t)$ such that the option should be exercised when the stock price $S \geq S_i^c(t)$ at time t in the i -th state of economy. Otherwise, the option should be held. Thus $[S_i^c(t), \infty) \times [0, T]$ is known as the exercise region and $(0, S_i^c(t)) \times [0, T]$ is called the continuation region. Denote by $C_i(S, t)$ the call price in the i -th state. Then $C_i(S, t)$ and $S_i^c(t)$ solve the free boundary value problem: for $i = 1, 2$,

$$C_{i,t}(S, t) + \mathcal{A}_i C(S, t) = 0, \quad 0 \leq S < S_i^c(t), \quad 0 < t \leq T, \quad (4.2.2)$$

$$C_i(0, t) = 0, \quad 0 \leq t \leq T, \quad (4.2.3)$$

$$C_i(S, t) = S - K, \quad S \geq S_i^c(t), \quad 0 \leq t \leq T, \quad (4.2.4)$$

$$C_{i,S}(S_i^c(t), t) = 1, \quad 0 < t \leq T, \quad (4.2.5)$$

$$C_i(S, T) = (S - K)^+, \quad S \geq 0, \quad (4.2.6)$$

where $C(S, t) = (C_1(S, t), C_2(S, t))^T$,

$$\mathcal{A}_i C(S, t) = \frac{1}{2} \sigma_i^2 S^2 C_{i,SS}(S, t) + \mu_i S C_{i,S}(S, t) - r_i C_i(S, t) + \langle Q C(S, t), e_i \rangle,$$

$\langle \cdot, \cdot \rangle$ denotes the usual inner product on R^2 , constant r_i is the interest rate in the i -th state of economy, and Q is the rate matrix of $X(t)$.

Similarly, for the put option we have for $i = 1, 2$,

$$P_{i,t}(S, t) + \mathcal{A}_i P(S, t) = 0, \quad S > S_i^p(t), \quad 0 < t \leq T, \quad (4.2.7)$$

$$P_i(0, t) = 0, \quad 0 \leq t \leq T, \quad (4.2.8)$$

$$P_i(S, t) = K - S, \quad 0 < S \leq S_i^p(t), \quad 0 \leq t \leq T, \quad (4.2.9)$$

$$P_{i,S}(S(t), t) = -1, \quad 0 < t \leq T, \quad (4.2.10)$$

$$P_i(S, T) = (K - S)^+, \quad S \geq 0. \quad (4.2.11)$$

4.3 A front-fixing finite element method

Consider the variable transforms

$$S = Ke^x, \quad C_i(S, T - t) = Ke^{-\alpha_i x - \beta_i t} u_i(x, t), \quad S_i^c(T - t) = Ke^{\phi_i(t)}, \quad i = 1, 2,$$

where α_i and β_i are some constants. Write

$$Q = \begin{pmatrix} -q_1 & q_1 \\ q_2 & -q_2 \end{pmatrix},$$

where q_1, q_2 are positive constants. Then the free boundary problem (4.2.2)–(4.2.6) for the American call option becomes: for $i = 1, 2$ (where t is now the time until expiration),

$$u_{i,t} - \gamma_i u_{i,xx} + \nu_i u_{i,x} + \kappa_i u_i - \xi_i u_{3-i} = 0, \quad x < \phi_i(t), \quad 0 < t \leq T, \quad (4.3.1)$$

$$u_i(x, t) = f_i(x, t), \quad x \geq \phi_i(t), \quad 0 \leq t \leq T, \quad (4.3.2)$$

$$u_{i,x}(\phi_i(t), t) = f_{i,x}(\phi_i(t), t), \quad 0 \leq t \leq T, \quad (4.3.3)$$

$$u_i(x, 0) = f_i(x, 0), \quad x \in (-\infty, +\infty), \quad (4.3.4)$$

where

$$\gamma_i = \frac{1}{2} \sigma_i^2, \quad \nu_i = \gamma_i(1 + 2\alpha_i) - \mu_i, \quad \kappa_i = q_i + r_i + \alpha_i \mu_i - \gamma_i \alpha_i(1 + \alpha_i) - \beta_i,$$

$$f_i(x, t) = e^{\alpha_i x + \beta_i t} (e^x - 1)^+, \quad \xi_i(x, t) = q_i e^{(\alpha_i - \alpha_{3-i})x + (\beta_i - \beta_{3-i})t}.$$

For a given tolerance ϵ , we can choose a positive number L which is large enough such that

$$|u_i(x, t)| \leq \epsilon, \quad x \leq \phi_i(t) - L, \quad 0 \leq t \leq T.$$

Let $\psi_i(t) = \phi_i(t) - L$. Then we can truncate the above free boundary problem into the following variable domain problem: for $i = 1, 2$,

$$u_{i,t} - \gamma_i u_{i,xx} + \nu_i u_{i,x} + \kappa_i u_i - \xi_i u_{3-i} = 0, \quad \psi_i(t) < x < \phi_i(t), \quad 0 \leq t < T, \quad (4.3.5)$$

$$u_i(x, t) = 0, \quad x \leq \psi_i(t), \quad 0 \leq t \leq T, \quad (4.3.6)$$

$$u_i(x, t) = f_i(x, t), \quad x \geq \phi_i(t), \quad 0 \leq t \leq T, \quad (4.3.7)$$

$$u_{i,x}(\phi_i(t), t) = f_{i,x}(\phi_i(t), t), \quad 0 \leq t \leq T, \quad (4.3.8)$$

$$u_i(x, 0) = f_i(x, 0), \quad -\infty < x < \infty. \quad (4.3.9)$$

Let

$$x = y_i + \psi_i(t), \quad w_i(y_i, t) = u_i(y_i + \psi_i(t), t).$$

Then equation (4.3.5) becomes

$$w_{i,t} - \gamma_i w_{i,y_i y_i} + (\nu_i - \phi_i'(t)) w_{i,y_i} + \kappa_i w_i - \xi_i(y_i + \psi_i(t), t) u_{3-i}(y_i + \psi_i(t), t) = 0$$

for $0 < y_i < L$, $0 < t \leq T$. Notice that

$$u_{3-i}(y_i + \psi_i(t), t) = w_{3-i}(y_i + \psi_i(t) - \psi_{3-i}(t), t) = w_{3-i}(y_i + \phi_i(t) - \phi_{3-i}(t), t),$$

and let

$$\delta(t) = \phi_1(t) - \phi_2(t), \quad \eta_i(y, t, \phi_i) = \xi_i(y + \phi_i - L, t).$$

Then we have

$$w_{i,t} - \gamma_i w_{i,y_i y_i} + (\nu_i - \phi_i'(t)) w_{i,y_i} + \kappa_i w_i - \eta_i(y_i, t, \phi_i(t)) w_{3-i}(y_i + (-1)^{3-i} \delta(t), t) = 0$$

for $0 < y_i < L$, $0 < t \leq T$. Equations (4.3.6)–(4.3.9) can be transformed in the same fashion. After changing y_i into y , we obtain the following nonlinear problem: for $i = 1, 2$,

$$w_{i,t} + \mathcal{L}_i w_i - \eta_i(y, t, \phi_i(t)) w_{3-i}(y + (-1)^{3-i} \delta(t), t) = 0, \quad 0 < y < L, \quad 0 < t \leq T, \quad (4.3.10)$$

$$w_i(y, t) = 0, \quad y \leq 0, \quad 0 \leq t \leq T, \quad (4.3.11)$$

$$w_i(y, t) = g_i(y, t, \phi_i(t)), \quad y \geq L, \quad 0 \leq t \leq T, \quad (4.3.12)$$

$$w_{i,y}(L, t) = g_{i,y}(L, t, \phi_i(t)), \quad 0 \leq t \leq T, \quad (4.3.13)$$

$$w_i(y, 0) = w_{i0}(y), \quad -\infty < y < \infty, \quad (4.3.14)$$

where

$$\mathcal{L}_i w = -\gamma_i w_{i,yy} + (\nu_i - \phi_i'(t)) w_{i,y} + \kappa_i w_i,$$

$$g_i(y, t, \phi_i) = f_i(y + \phi_i - L, t), \quad w_{i0}(y) = g_i(y, 0, \phi_i(0)).$$

Denote by $(\cdot, \cdot)$ the inner product of $L^2(\Omega)$, the space of square integrable functions on $\Omega = (0, L)$. Let $H_E^1(\Omega)$ be the closure of $\{v \in C^\infty(\overline{\Omega}) : v(0) = 0\}$ in the usual Sobolev space $H^1(\Omega)$ ([1], [43]), and let $H^{-1}(\Omega)$ be the dual space of $H_E^1(\Omega)$.

For $v \in H_E^1(\Omega)$, we have by (4.3.11) and (4.3.12)

$$(\eta_i(y, t, \phi_i(t)) w_{3-i}(y + (-1)^{3-i} \delta(t), t), v) = b_i(t, \phi_1, \phi_2; w_{3-i}, v) + c_i(t, \phi_1, \phi_2; v),$$

where

$$b_1(t, \phi_1, \phi_2; w_2, v) = \int_{\max(0, \delta(t))}^{\min(L, L + \delta(t))} q_1 e^{(\alpha_1 - \alpha_2)(z + \phi_2(t) - L) + (\beta_1 - \beta_2)t} w_2(z, t) v(z - \delta(t)) dz,$$

$$\begin{aligned}
c_1(t, \phi_1, \phi_2; v) &= \int_{\phi_2(t)}^{\max(\phi_1(t), \phi_2(t))} q_1 e^{\alpha_1 z + \beta_1 t} (e^z - 1) v(z + L - \phi_1) dz, \\
b_2(t, \phi_1, \phi_2; w_1, v) &= \int_{\max(0, -\delta(t))}^{\min(L, L - \delta(t))} q_2 e^{(\alpha_2 - \alpha_1)(z + \phi_1(t) - L) + (\beta_2 - \beta_1)t} w_1(z, t) v(z + \delta(t)) dz, \\
c_2(t, \phi_1, \phi_2; v) &= \int_{\min(\phi_1(t), \phi_2(t))}^{\phi_2(t)} q_2 e^{\alpha_2 z + \beta_2 t} (e^z - 1) v(z + L - \phi_2) dz.
\end{aligned}$$

Define the bilinear form for $\mathcal{L}_i$,

$$a_i(\phi_i'(t); w, v) = \gamma_i(w_y, v_y) + (\nu_i - \phi_i'(t))(w_y, v) + \kappa_i(w, v).$$

Then we formulate the variational form for the nonlinear problem (4.3.10)–(4.3.14):

Find $w_i \in L^2(0, T; H_E^1(\Omega))$ and $\phi_i \in C([0, T]) \cap C^1((0, T])$ such that $w_{i,t} \in L^2(0, T; H^{-1}(\Omega))$,

$w_i(0) = w_{i0}$, and for a.e. $0 \leq t \leq T$

$$(w_{i,t}, v) + a_i(\phi_i'(t); w_i, v) - b_i(t, \phi_1, \phi_2; w_{3-i}, v) = F_i(t, \phi_1, \phi_2; v), \quad \forall v \in H_E^1(\Omega), \quad (4.3.15)$$

$$w_i(L, t) = g_i(L, t, \phi_i(t)), \quad (4.3.16)$$

where

$$F_i(t, \phi_1, \phi_2; v) = c_i(t, \phi_1, \phi_2; v) + \gamma_i g_{i,y}(L, t, \phi_i(t)) v(L).$$

Let $\Delta_t : 0 = t_0 < t_1 < \dots < t_M = T$ and $\Delta_y : 0 = y_0 < y_1 < \dots < y_N = L$

be partitions of $[0, T]$ and $[0, L]$, respectively, where M and N are positive integers.

Let V_h be the piecewise linear element subspace of $H_E^1(\Omega)$ with respect to partition

Δ_y , where $h = \max_{1 \leq j \leq N} (y_j - y_{j-1})$. Denote the basis functions of V_h by $\omega_1, \omega_2, \dots, \omega_N$,

i.e., $\omega_j \in V_h$ and $\omega_j(y_i) = \delta_{ij}$ for $j = 1, 2, \dots, N$ and $i = 0, 1, \dots, N$, where δ_{ij} is the

Kronecker delta. Let

$$w_{i,h}^0 = \sum_{j=1}^N w_{i0}(y_j) \omega_j(y), \quad \phi_i^0 = \phi_i(0) = \log(\max(r_i/d_i, 1)),$$

where $d_i = r_i - \mu_i$ is regarded as the dividend rate. The finite element approximation to the variational problem (4.3.15)–(4.3.16) is: For $m = 1, 2, \dots, M$, find $w_{i,h}^m \in V_h$ and $\phi_i^m > 0$ such that

$$(\delta_t w_{i,h}^m, v) + A_i^{m-\theta}(w_{1,h}, w_{2,h}; v) = F_i(t_{m-\theta}, \phi_1^{m-\theta}, \phi_2^{m-\theta}; v), \quad \forall v \in V_h, \quad (4.3.17)$$

$$w_{i,h}^m(L) = g_i(L, t_m, \phi_i^m), \quad (4.3.18)$$

where $\theta \in [0, 1]$ and

$$\begin{aligned} A_i^{m-\theta}(w_{1,h}, w_{2,h}; v) &= a_i(\delta_t \phi_i^m; w_{i,h}^{m-\theta}, v) - b_i(t_{m-\theta}, \phi_1^{m-\theta}, \phi_2^{m-\theta}; w_{3-i,h}^{m-\theta}, v), \\ \delta_t w_{i,h}^m &= \frac{w_{i,h}^m - w_{i,h}^{m-1}}{k_m}, \quad \delta_t \phi_i^m = \frac{\phi_i^m - \phi_i^{m-1}}{k_m}, \quad k_m = t_m - t_{m-1}, \\ w_{i,h}^{m-\theta} &= (1 - \theta)w_{i,h}^m + \theta w_{i,h}^{m-1}, \quad \phi_i^{m-\theta} = (1 - \theta)\phi_i^m + \theta \phi_i^{m-1}. \end{aligned}$$

When $\theta = 0, \frac{1}{2}$, we have the backward Euler and Crank-Nicolson methods, respectively.

Now we would like to consider the stability of the finite element approximations. As in Chapter 2, we assume that the system (4.3.17)–(4.3.18) has a unique solution and that

$$|\delta_t \phi_i^m| \leq C t_m^{\varsigma_i-1}, \quad (4.3.19)$$

where $0 < \varsigma_i < 1$ for $i = 1, 2$, and C is a generic positive constant. The above inequality has been verified by numerical tests. This implies that

$$|\phi_i^m| \leq \Phi, m = 0, 1, \dots, M \quad i = 1, 2, \quad (4.3.20)$$

where Φ is a positive constant.

Theorem 4.1. Assume β_1 and β_2 are determined such that $\Lambda^2 - \kappa_1 \kappa_2 \leq 0$, where

$$\Lambda = \max_{x \in (0, L)} \left(q_1 e^{(\alpha_1 - \alpha_2)(x + \phi_2^{m-\theta} - L) + (\beta_1 - \beta_2)t_{m-\theta}} + q_2 e^{(\alpha_2 - \alpha_1)(x + \phi_1^{m-\theta} - L) + (\beta_2 - \beta_1)t_{m-\theta}} \right). \quad (4.3.21)$$

Under assumption (4.3.19) the system (4.3.17)–(4.3.18) is absolutely stable when $\theta =$

$0, \frac{1}{2}$, i.e., we have

$$\sum_{i=1}^2 \max_{m=1}^M \|w_{i,h}^m\|^2 + \sum_{m=1}^M k_m \sum_{i=1}^2 \gamma_i \|w_{i,h,y}^{m-\theta}\|^2 \leq \sum_{i=1}^2 \|w_{i,h}^0\|^2 + C. \quad (4.3.22)$$

Proof. Substituting $w_{i,h}^{m-\theta}$ for v in (4.3.17) and taking the summation over $i = 1, 2$,

we get

$$\begin{aligned} & \sum_{i=1}^2 (\delta_t w_{i,h}^m, w_{i,h}^{m-\theta}) + \sum_{i=1}^2 \gamma_i (w_{i,h,y}^{m-\theta}, w_{i,h,y}^{m-\theta}) + \sum_{i=1}^2 (\nu_i - \delta_t \phi_i^m) (w_{i,h,y}^{m-\theta}, w_{i,h}^{m-\theta}) \\ & + \sum_{i=1}^2 \kappa_i (w_{i,h}^{m-\theta}, w_{i,h}^{m-\theta}) - \sum_{i=1}^2 b_i (t_{m-\theta}, \phi_1^{m-\theta}, \phi_2^{m-\theta}; w_{3-i,h}^{m-\theta}, w_{i,h}^{m-\theta}) \\ & = \sum_{i=1}^2 c_i (t_{m-\theta}, \phi_1^{m-\theta}, \phi_2^{m-\theta}; w_{i,h}^{m-\theta}) + \sum_{i=1}^2 \gamma_i g_{i,y} (L, t_{m-\theta}, \phi_i^{m-\theta}) w_{i,h}^{m-\theta} (L). \end{aligned} \quad (4.3.23)$$

Using the facts that

$$(\delta_t w_{i,h}^m, w_{i,h}^{m-\theta}) = \frac{1}{2k_m} \left(\|w_{i,h}^m\|^2 - \|w_{i,h}^{m-1}\|^2 + (1 - 2\theta) \|w_{i,h}^m - w_{i,h}^{m-1}\|^2 \right),$$

and

$$(w_{i,h,y}^{m-\theta}, w_{i,h}^{m-\theta}) = \frac{1}{2} |w_{i,h}^{m-\theta} (L)|^2,$$

we can write (4.3.23) as

$$\begin{aligned} & \frac{1}{2k_m} \sum_{i=1}^2 \left(\|w_{i,h}^m\|^2 - \|w_{i,h}^{m-1}\|^2 + (1 - 2\theta) \|w_{i,h}^m - w_{i,h}^{m-1}\|^2 \right) + \sum_{i=1}^2 \gamma_i \|w_{i,h,y}^{m-\theta}\|^2 \\ & + \frac{1}{2} \sum_{i=1}^2 (\nu_i - \delta_t \phi_i^m) |w_{i,h}^{m-\theta} (L)|^2 + \sum_{i=1}^2 \kappa_i \|w_{i,h}^{m-\theta}\|^2 - \sum_{i=1}^2 b_i (t_{m-\theta}, \phi_1^{m-\theta}, \phi_2^{m-\theta}; w_{3-i,h}^{m-\theta}, w_{i,h}^{m-\theta}) \\ & = \sum_{i=1}^2 c_i (t_{m-\theta}, \phi_1^{m-\theta}, \phi_2^{m-\theta}; w_{i,h}^{m-\theta}) + \sum_{i=1}^2 \gamma_i g_{i,y} (L, t_{m-\theta}, \phi_i^{m-\theta}) w_{i,h}^{m-\theta} (L). \end{aligned} \quad (4.3.24)$$

Using the Cauchy-Schwarz, Young's, and Friedrich's inequalities, we find that

$$\sum_{i=1}^2 c_i (t_{m-\theta}, \phi_1^{m-\theta}, \phi_2^{m-\theta}; w_{i,h}^{m-\theta}) \leq C + \frac{\gamma_i}{2} \sum_{i=1}^2 \|w_{i,h,y}^{m-\theta}\|^2, \quad (4.3.25)$$

where C is a positive constant. Using (4.3.20), (4.3.21), and the Cauchy-Schwarz inequality, we find that

$$\sum_{i=1}^2 b_i (t_{m-\theta}, \phi_1^{m-\theta}, \phi_2^{m-\theta}; w_{3-i,h}^{m-\theta}, w_{i,h}^{m-\theta}) \leq \Lambda \|w_{2,h}^{m-\theta}\| \|w_{1,h}^{m-\theta}\|. \quad (4.3.26)$$

Using the inequalities (4.3.25) and (4.3.26), equation (4.3.24) becomes

$$\begin{aligned} & \sum_{i=1}^2 \left(\|w_{i,h}^m\|^2 - \|w_{i,h}^{m-1}\|^2 + (1-2\theta) \|w_{i,h}^m - w_{i,h}^{m-1}\|^2 \right) + \\ & k_m \sum_{i=1}^2 \gamma_i \|w_{i,h,y}^{m-\theta}\|^2 + 2k_m \sum_{i=1}^2 \kappa_i \|w_{i,h}^{m-\theta}\|^2 - 4\Lambda k_m \|w_{2,h}^{m-\theta}\| \|w_{1,h}^{m-\theta}\| \\ & \leq k_m C + 2k_m \sum_{i=1}^2 \gamma_i g_{i,y}(L, t_{m-\theta}, \phi_i^{m-\theta}) w_{i,h}^{m-\theta}(L) - k_m \sum_{i=1}^2 (\nu_i - \delta_t \phi_i^m) |w_{i,h}^{m-\theta}(L)|^2. \end{aligned} \quad (4.3.27)$$

The assumption (4.3.19) implies that the sequences $\{\phi_i^m\}$ are bounded. Thus, the sequences $\{g_{i,y}(L, t_m, \phi_i^m)\}$ and $\{w_{i,h}^m(L)\} = \{g_i(L, t_m, \phi_i^m)\}$ are also bounded. Therefore equation (4.3.27) can be written as

$$\sum_{i=1}^2 \left(\|w_{i,h}^m\|^2 - \|w_{i,h}^{m-1}\|^2 \right) + k_m \sum_{i=1}^2 \gamma_i \|w_{i,h,y}^{m-\theta}\|^2 \leq C \frac{k_m}{\epsilon} \sum_{i=1}^2 (1 + t_m^{\zeta_i-1}).$$

Taking the summation over $m = 1, \dots, M$, we obtain the stability estimate (4.3.22). □

4.4 Implementation

In this section we give details of the implementation of our front-fixing finite element method and outline the algorithm used to produce the numerical results.

The matrix form of (4.3.17)–(4.3.18) is

$$A\delta_t W_i^m + (\kappa_i A + \gamma_i B + (\nu_i - \delta_t \phi_i^m)C) W_i^{m-\theta} - D_i^{m-\theta} W_{3-i}^{m-\theta} = F_i^{m-\theta} \quad (4.4.1)$$

$$w_{i,N}^m = g_i(L, t_m, \phi_i^m), \quad (4.4.2)$$

for $i = 1, 2$, $m = 1, \dots, M$, where

$$A = (\omega_j, \omega_\ell)_{N \times N}, \quad B = (\omega'_j, \omega'_\ell)_{N \times N}, \quad C = (\omega'_j, \omega_\ell)_{N \times N},$$

$$D_i^{m-\theta} = (b_i(t_{m-\theta}, \phi_1^{m-\theta}, \phi_2^{m-\theta}, \omega_j, \omega_\ell))_{N \times N}, \quad F_i^{m-\theta} = (F_i(t_{m-\theta}, \phi_1^{m-\theta}, \phi_2^{m-\theta}, \omega_\ell))_{N \times 1}.$$

Let

$$\begin{aligned} A_{i,i} &= A + (1 - \theta)k_m(E_i - \delta_t \phi_i^m C), \quad A_{i,3-i} = -(1 - \theta)k_m D_i^{m-\theta}, \\ G_i^m &= k_m F_i^{m-\theta} + (A - \theta k_m(E_i - \delta_t \phi_i^m C))W_i^{m-1} + \theta k_m D_i^{m-\theta} W_{3-i}^{m-1}, \end{aligned}$$

where $E_i = \kappa_i A + \gamma_i B + \nu_i C$. System (4.4.1) can be rewritten as

$$A_{i,i} W_i^m + A_{i,3-i} W_{3-i}^m = G_i^m, \quad i = 1, 2. \quad (4.4.3)$$

Notice that

$$g_i(L, t_m, \phi_i^m) = f_i(y + \psi_i, t_m) = e^{\alpha_i \phi_i^m + \beta_i t_m} (e^{\phi_i^m} - 1).$$

Equation (4.4.2) becomes

$$Q_i(\phi_1^m, \phi_2^m) \equiv \phi_i^m - \log(1 + e^{-\alpha_i \phi_i^m - \beta_i t_m} w_{i,N}^m) = 0, \quad i = 1, 2. \quad (4.4.4)$$

Here we have regarded $w_{1,N}^m$ and $w_{2,N}^m$ as the implicit functions of ϕ_1^m and ϕ_2^m defined by system (4.4.3). We want to solve this nonlinear system by a Newton's method and

thus need $\frac{\partial w_{i,N}^m}{\partial \phi_j^m}$ for $i, j = 1, 2$ since

$$\begin{aligned}\frac{\partial Q_i}{\partial \phi_i^m} &= 1 - \frac{p_i \left(\frac{\partial w_{i,N}^m}{\partial \phi_i^m} - \alpha_i w_{i,N}^m \right)}{1 + p_i w_{i,N}^m}, \\ \frac{\partial Q_i}{\partial \phi_{3-i}^m} &= -\frac{p_i \frac{\partial w_{i,N}^m}{\partial \phi_{3-i}^m}}{1 + p_i w_{i,N}^m},\end{aligned}$$

where $p_i = e^{-\alpha_i \phi_i^m - \beta_i t_m}$. Differentiating (4.4.1) with respect to ϕ_j^m ($j = 1, 2$) leads to the system for $\frac{\partial W_i^m}{\partial \phi_j^m}$:

$$A_{i,i} \frac{\partial W_i^m}{\partial \phi_j^m} + A_{i,3-i} \frac{\partial W_{3-i}^m}{\partial \phi_j^m} = G_{j,i}^m, \quad i = 1, 2, \quad (4.4.5)$$

where

$$G_{i,i}^m = k_m \frac{\partial F_i^m}{\partial \phi_i^m} + C W_i^{m-\theta} - \frac{\partial A_{i,3-i}}{\partial \phi_i^m} W_{3-i}^{m-\theta}, \quad G_{i,3-i}^m = k_m \frac{\partial F_{3-i}^m}{\partial \phi_i^m} - \frac{\partial A_{3-i,i}}{\partial \phi_i^m} W_i^{m-\theta}$$

for $j = 1, 2$. We should point out that the matrices $D_i^{m-\theta}$ and their derivatives are not tridiagonal, as is usually the case when implementing the finite element method. We have three systems to solve, each with identical coefficient matrices, and different right hand side matrices.

Our algorithm is as follows: For a given tolerance ϵ

For $m = 1, 2, \dots, M$, do

1. Compute free boundary independent matrices.

2. Let $\phi_i^{m(0)} = \phi_i^{m-1}$ for $i = 1, 2$. For $j = 1, 2, \dots$

2.1. Compute free boundary dependent matrices.

2.2. Build the systems (4.4.3) and solve.

2.3. Build the systems (4.4.5) and solve.

2.4. Compute $\phi_i^{m(j)}$ for $i = 1, 2$ using (4.4.4)

2.5. If $\left| \phi_i^{m(j-1)} - \phi_i^{m(j)} \right| < \epsilon$ for $i = 1, 2$, then let $\phi_i^m = \phi_i^{m(j)}$ for $i = 1, 2$ and terminate the j -loop. Otherwise, go to 2.1.

End do

As in previous chapters, we use variable step sizes in time to concentrate more nodes near the singularity at option expiration.

4.5 Numerical results

In this section, we shall examine the results from our front-fixing finite element method and compare it with the usual finite element method. Our programs were written in Matlab and, despite the complete vectorization of the code, the computational times were not impressive. We have tested our method with several combinations of values for α_1 and α_2 . Our numerical experiments show that the accuracy of our method is the the same for all cases, and is also independent of the choice of β . In the following, we only present numerical results for the case with $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$.

Let $P(\Delta t)$ be the transition probability matrix from t to Δt . Write

$$P(\Delta t) = \begin{pmatrix} 1 - P_1 & P_1 \\ P_2 & 1 - P_2 \end{pmatrix},$$

where P_i is the probability in which the economy switches from state i to state $3 - i$, for $i = 1, 2$. Since $P(\Delta t) = e^{Q\Delta t}$, we have by simple calculation

$$q_i = -\frac{P_i \log(1 - P_1 - P_2)}{(P_1 + P_2)\Delta t}, \quad i = 1, 2, \quad (4.5.1)$$

which are used to determine q_1 and q_2 . In the following numerical examples, we assume that the probability for which the economy switches from state 1 to state 2 is 40%, or from state 2 to state 1 is 30%, during a half year period ($P_1 = 0.4$, $P_2 = 0.3$, $\Delta t = 0.5$). Then we have from (4.5.1) that $q_1 = 1.375968919$ and $q_2 = 1.031976689$. If we set $q_1 = q_2 = 0$, then the program can be used to compute option prices and early exercise prices without switching. The computational domain L was set to be the

maximum of the computed domains for each state, as in (2.2.12), with a tolerance of 10^{-10} . The tolerance for Newton's method was set to be 10^{-10} . The following examples are for an American call option with a one year expiration and strike price of \$100.

Let us first look at the rate of convergence which is displayed in figures (4.1)–(4.8). We are displaying graphs of the L^2 norm and H^1 norm of the error $w_{i,h}^M - w_{i,h/2}^M$ with respect to the mesh size for the option prices in both states. We are also displaying the L^2 norm of the error $\phi_i^M - \phi_i^{2M}$, where ϕ_i^M is the piecewise linear interpolation of ϕ^m ($m = 0, 1, \dots, M$), with respect to the number of time steps for the early exercise price (EE price) for both states. We have considered cases with regime-switching volatility and constant interest rate as well as cases with regime-switching interest rate and constant volatility. The graphs display these errors for each dividend rate tested for the given set of parameters. For example, figure (4.7) shows the rates of convergence for the cases when $\sigma_1 = \sigma_2 = 0.4$, $r_1 = 0.08$, $r_2 = 0.04$, and $d_1 = d_2 = 0.02, 0.04, 0.06, 0.08$, and 0.10 . We can see that for the option price, the method is converging quadratically in the L^2 norm and linearly in the H^1 norm. For the early exercise option price, the rate of convergence is more than one in the L^2 norm. These results agree with the results obtained in Chapter 2.

Next, let us look at figures (4.9)–(4.15) where we have displayed the behavior of the early exercise prices near expiration. As before, we have used a very small step size in time by taking $M = 6400$ and have displayed the graphs of ϕ_i^M and $\phi_i^{M'}$ over the interval $[0, 0.01]$. The exponential functions in the figures are obtained by fitting the data to the model $a + b * \tau^c$. Our results verify assumption (4.3.19), and allow us

to make a conjecture about the behavior of the early exercise prices near expiration, similar to (2.4.2) in Chapter 2. That is: there are some constants $c_i > 0$, $\mu_i \geq 0$, and $\nu_i > 0$ such that

$$S_i(T-\tau) \sim K \max\left(1, \frac{r_i}{d}\right) \left(1 + \tau^{\nu_i} \left(\log \frac{c_i}{\tau}\right)^{\mu_i}\right), \quad \text{as } \tau \rightarrow 0^+, \quad \text{for } i = 1, 2, \quad (4.5.2)$$

where τ is the time to expiration. It should be noted, however, that when the interest rate is less than the dividend rate, i.e., $r < d$, our program has some difficulty in achieving this very high accuracy for the free boundary very near to the singularity at expiration. We currently believe this may have something to do with the Matlab software, but until we investigate further we cannot be sure.

Finally, we would like to compare our front-fixing finite element method (FFEM) to the usual finite element method (FEM) in [62]. To this end, we computed “exact” values with the usual finite element method with 12,800 time steps and our own method with 6400 time steps. Tables (4.1)–(4.12) display the maximum absolute errors for option prices, early exercise prices (EE price), and hedge ratios (partial derivative of option price with respect to stock price) of both states as compared to the “exact values”. The M column in each table displays the number of time steps used with both methods. The top number in each cell is the maximum absolute error of the method with M time steps compared to the usual finite element method with 12,800 time steps, and the bottom number in each cell is the method with M time steps compared to our method with 6400 time steps. We can see that the front-fixing finite element method provides more accurate results and converges more quickly than the usual finite element method, especially for the early exercise prices. Thus, our

method can obtain more accurate results with fewer time steps, saving computational time.

Figure 4.1. Rates of convergence: $\sigma_1 = \sigma_2 = 0.2$, $r_1 = 0.08$, $r_2 = 0.06$

Figure 4.2. Rates of convergence: $\sigma_1 = \sigma_2 = 0.2$, $r_1 = 0.10$, $r_2 = 0.08$

Figure 4.3. Rates of convergence: $\sigma_1 = 0.3$, $\sigma_2 = 0.2$, $r_1 = r_2 = 0.06$

Figure 4.4. Rates of convergence: $\sigma_1 = 0.3$, $\sigma_2 = 0.2$, $r_1 = r_2 = 0.10$

Figure 4.5. Rates of convergence: $\sigma_1 = 0.4$, $\sigma_2 = 0.3$, $r_1 = r_2 = 0.06$

Figure 4.6. Rates of convergence: $\sigma_1 = 0.4$, $\sigma_2 = 0.3$, $r_1 = r_2 = 0.10$

Figure 4.7. Rates of convergence: $\sigma_1 = \sigma_2 = 0.4$, $r_1 = 0.08$, $r_2 = 0.04$

Figure 4.8. Rates of convergence: $\sigma_1 = \sigma_2 = 0.4$, $r_1 = 0.10$, $r_2 = 0.06$

Figure 4.9. Behavior of EE price: $\sigma_1 = \sigma_2 = 0.4$, $r_1 = 0.10$, $r_2 = 0.06$, $d = 0.04$

Figure 4.10. Behavior of EE price: $\sigma_1 = \sigma_2 = 0.4$, $r_1 = 0.10$, $r_2 = 0.06$, $d = 0.06$

Figure 4.11. Behavior of EE price: $\sigma_1 = \sigma_2 = 0.4$, $r_1 = 0.10$, $r_2 = 0.06$, $d = 0.08$

Figure 4.12. Behavior of EE price: $\sigma_1 = \sigma_2 = 0.4$, $r_1 = 0.10$, $r_2 = 0.06$, $d = 0.10$

Figure 4.13. Behavior of EE Price: $\sigma_1 = \sigma_2 = 0.4$, $r_1 = 0.10$, $r_2 = 0.06$, $d = 0.12$

Figure 4.14. Behavior of EE price: $\sigma_1 = 0.4$, $\sigma_2 = 0.3$, $r_1 = r_2 = 0.06$, $d = 0.06$

Figure 4.15. Behavior of EE price: $\sigma_1 = 0.4$, $\sigma_2 = 0.3$, $r_1 = r_2 = 0.06$, $d = 0.08$

Table 4.1. MAE Price-State I: $\sigma_1 = 0.4$, $\sigma_2 = 0.3$, $r_1 = r_2 = 0.06$

M	$d = 0.04$		$d = 0.06$		$d = 0.08$	
	FEM	FFEM	FEM	FFEM	FEM	FFEM
100	$1.02e-3$	$2.24e-3$	$1.20e-3$	$2.25e-3$	$1.34e-3$	$2.11e-3$
	$1.03e-3$	$2.24e-3$	$1.17e-3$	$2.22e-3$	$1.33e-3$	$2.10e-3$
200	$4.06e-4$	$6.57e-4$	$4.88e-4$	$5.39e-4$	$5.75e-4$	$5.05e-4$
	$4.10e-4$	$6.59e-4$	$4.57e-4$	$5.18e-4$	$5.63e-4$	$4.96e-4$
400	$1.75e-4$	$1.11e-4$	$2.09e-4$	$1.30e-4$	$2.56e-4$	$1.26e-4$
	$1.79e-4$	$1.13e-4$	$1.79e-4$	$1.11e-4$	$2.44e-4$	$1.17e-4$
800	$7.88e-5$	$2.10e-5$	$9.22e-5$	$5.30e-5$	$1.15e-4$	$4.08e-5$
	$8.23e-5$	$1.82e-5$	$6.16e-5$	$2.34e-5$	$1.03e-4$	$2.94e-5$
1600	$3.54e-5$	$2.82e-5$	$4.03e-5$	$5.42e-5$	$5.04e-5$	$2.98e-5$
	$3.89e-5$	$2.54e-5$	$2.98e-5$	$1.90e-5$	$3.84e-5$	$1.29e-5$

Table 4.2. MAE Price-State II: $\sigma_1 = 0.4$, $\sigma_2 = 0.3$, $r_1 = r_2 = 0.06$

M	$d = 0.04$		$d = 0.06$		$d = 0.08$	
	FEM	FFEM	FEM	FFEM	FEM	FFEM
100	$7.28e-4$	$2.60e-3$	$7.55e-4$	$2.47e-3$	$8.62e-4$	$2.31e-3$
	$7.27e-4$	$2.60e-3$	$7.23e-4$	$2.45e-3$	$8.47e-4$	$2.30e-3$
200	$1.70e-4$	$7.60e-4$	$2.22e-4$	$5.92e-4$	$3.14e-4$	$5.53e-4$
	$1.65e-4$	$7.63e-4$	$1.87e-4$	$5.69e-4$	$2.96e-4$	$5.43e-4$
400	$4.25e-5$	$1.30e-4$	$7.09e-5$	$1.50e-4$	$1.22e-4$	$1.41e-4$
	$3.75e-5$	$1.33e-4$	$3.46e-5$	$1.28e-4$	$1.04e-4$	$1.31e-4$
800	$1.06e-5$	$9.53e-5$	$2.39e-5$	$8.30e-5$	$4.87e-5$	$4.86e-5$
	$8.05e-6$	$9.03e-5$	$1.82e-5$	$4.84e-5$	$3.06e-5$	$3.49e-5$
1600	$2.80e-6$	$5.13e-5$	$8.10e-6$	$7.01e-5$	$1.93e-5$	$3.59e-5$
	$2.48e-6$	$4.64e-5$	$3.33e-5$	$3.17e-5$	$2.03e-6$	$1.75e-5$

Table 4.3. Relative MAE EE Price-State I: $\sigma_1 = 0.4$, $\sigma_2 = 0.3$, $r_1 = r_2 = 0.06$

M	$d = 0.04$		$d = 0.06$		$d = 0.08$	
	FEM	FFEM	FEM	FFEM	FEM	FFEM
100	$6.98e-3$	$4.55e-4$	$7.67e-3$	$2.08e-4$	$4.83e-3$	$8.81e-5$
	$7.02e-3$	$4.97e-4$	$7.75e-3$	$2.81e-4$	$4.87e-3$	$1.24e-4$
200	$3.54e-3$	$7.94e-5$	$3.77e-3$	$4.76e-6$	$4.83e-3$	$6.19e-6$
	$3.59e-3$	$1.21e-4$	$3.85e-3$	$6.85e-5$	$4.87e-3$	$2.99e-5$
400	$1.83e-3$	$1.09e-5$	$1.83e-3$	$5.79e-5$	$1.11e-3$	$2.85e-5$
	$1.87e-3$	$3.11e-5$	$1.90e-3$	$1.54e-5$	$1.14e-3$	$7.59e-6$
800	$9.67e-4$	$3.49e-5$	$8.52e-4$	$6.87e-5$	$1.11e-3$	$3.40e-5$
	$1.01e-3$	$7.15e-6$	$9.25e-4$	$4.48e-6$	$1.14e-3$	$2.10e-6$
1600	$5.37e-4$	$3.94e-5$	$3.65e-4$	$7.17e-5$	$6.40e-4$	$3.54e-5$
	$5.79e-4$	$2.67e-6$	$4.38e-4$	$1.57e-6$	$6.77e-4$	$7.29e-7$

Table 4.4. Relative MAE EE Price-State II: $\sigma_1 = 0.4$, $\sigma_2 = 0.3$, $r_1 = r_2 = 0.06$

M	$d = 0.04$		$d = 0.06$		$d = 0.08$	
	FEM	FFEM	FEM	FFEM	FEM	FFEM
100	$3.60e-3$	$4.33e-4$	$7.58e-3$	$2.20e-4$	$6.32e-3$	$7.61e-5$
	$3.66e-3$	$4.94e-4$	$7.64e-3$	$2.76e-4$	$6.36e-3$	$1.12e-4$
200	$3.60e-3$	$6.07e-5$	$3.85e-3$	$1.20e-5$	$2.77e-3$	$8.76e-6$
	$3.66e-3$	$1.22e-4$	$3.90e-3$	$6.74e-5$	$2.81e-3$	$2.75e-5$
400	$1.98e-3$	$2.98e-5$	$1.98e-3$	$3.61e-5$	$9.99e-4$	$2.84e-5$
	$2.04e-3$	$3.10e-5$	$2.04e-3$	$1.94e-5$	$1.04e-3$	$7.80e-6$
800	$1.17e-3$	$4.52e-5$	$1.05e-3$	$4.92e-5$	$9.99e-4$	$3.38e-5$
	$1.23e-3$	$1.56e-5$	$1.11e-3$	$6.26e-6$	$1.04e-3$	$2.44e-6$
1600	$3.56e-4$	$5.58e-5$	$5.83e-4$	$5.32e-5$	$5.55e-4$	$3.54e-5$
	$4.16e-4$	$5.02e-6$	$6.39e-4$	$2.23e-6$	$5.91e-4$	$8.56e-7$

Table 4.5. MAE Hedge ratio-State I: $\sigma_1 = 0.4$, $\sigma_2 = 0.3$, $r_1 = r_2 = 0.06$

M	$d = 0.04$		$d = 0.06$		$d = 0.08$	
	FEM	FFEM	FEM	FFEM	FEM	FFEM
100	$5.40e-4$	$9.47e-5$	$5.41e-4$	$7.70e-5$	$6.00e-4$	$7.63e-5$
	$5.40e-4$	$9.47e-5$	$5.41e-4$	$7.66e-5$	$6.00e-4$	$7.61e-5$
200	$4.13e-4$	$2.58e-5$	$4.14e-4$	$1.91e-5$	$4.14e-4$	$1.86e-5$
	$4.13e-4$	$2.58e-5$	$4.13e-4$	$1.89e-5$	$4.14e-4$	$1.83e-5$
400	$3.70e-4$	$5.39e-6$	$3.70e-4$	$4.60e-6$	$3.70e-4$	$4.62e-6$
	$3.70e-4$	$5.41e-6$	$3.70e-4$	$4.15e-6$	$3.70e-4$	$4.38e-6$
800	$2.22e-5$	$1.74e-6$	$2.22e-5$	$1.75e-6$	$5.80e-5$	$3.47e-6$
	$2.21e-5$	$1.05e-6$	$2.20e-5$	$1.28e-6$	$5.33e-5$	$1.26e-6$
1600	$7.28e-6$	$1.46e-6$	$7.37e-6$	$1.41e-6$	$1.96e-5$	$4.26e-6$
	$7.13e-6$	$4.08e-7$	$8.03e-6$	$5.32e-7$	$2.43e-5$	$4.78e-7$

Table 4.6. MAE Hedge ratio-State II: $\sigma_1 = 0.4$, $\sigma_2 = 0.3$, $r_1 = r_2 = 0.06$

M	$d = 0.04$		$d = 0.06$		$d = 0.08$	
	FEM	FFEM	FEM	FFEM	FEM	FFEM
100	$4.02e-4$	$1.17e-4$	$3.06e-4$	$8.88e-5$	$5.17e-4$	$8.80e-5$
	$4.03e-4$	$1.17e-4$	$3.05e-4$	$8.82e-5$	$5.16e-4$	$8.77e-5$
200	$7.29e-5$	$3.18e-5$	$8.99e-5$	$2.16e-5$	$2.33e-4$	$2.14e-5$
	$7.33e-5$	$3.19e-5$	$8.92e-5$	$2.11e-5$	$2.33e-4$	$2.11e-5$
400	$2.54e-5$	$6.65e-6$	$1.66e-5$	$5.60e-6$	$1.08e-4$	$6.92e-6$
	$2.45e-5$	$6.69e-6$	$1.66e-5$	$4.90e-6$	$1.09e-4$	$5.11e-6$
800	$1.75e-5$	$3.94e-6$	$1.14e-5$	$3.22e-6$	$2.63e-5$	$5.08e-6$
	$1.86e-5$	$3.04e-6$	$1.13e-5$	$2.23e-6$	$2.68e-5$	$1.47e-6$
1600	$4.07e-6$	$2.14e-6$	$4.31e-6$	$2.18e-6$	$1.68e-5$	$4.34e-6$
	$3.65e-6$	$1.17e-6$	$3.15e-6$	$1.05e-6$	$1.63e-5$	$6.92e-7$

Table 4.7. MAE Price-State I: $\sigma_1 = \sigma_2 = 0.4$, $r_1 = 0.08$, $r_2 = 0.04$

M	$d = 0.02$		$d = 0.04$		$d = 0.06$		$d = 0.08$		$d = 0.10$	
	FEM	FFEM	FEM	FFEM	FEM	FFEM	FEM	FFEM	FEM	FFEM
100	$9.75e-4$	$5.81e-3$	$1.11e-3$	$1.86e-3$	$1.23e-3$	$1.16e-3$	$1.35e-3$	$1.89e-3$	$1.44e-3$	$1.89e-3$
	$9.79e-4$	$5.81e-3$	$1.11e-3$	$1.86e-3$	$1.23e-3$	$1.16e-3$	$1.36e-3$	$1.89e-3$	$1.45e-3$	$1.89e-3$
200	$3.95e-4$	$1.44e-3$	$4.48e-4$	$4.97e-4$	$5.19e-4$	$2.56e-4$	$5.79e-4$	$4.53e-4$	$6.24e-4$	$4.54e-4$
	$3.99e-4$	$1.44e-3$	$4.51e-4$	$4.97e-4$	$5.21e-4$	$2.56e-4$	$5.83e-4$	$4.52e-4$	$6.29e-4$	$4.54e-4$
400	$1.72e-4$	$3.60e-4$	$1.93e-4$	$1.19e-4$	$2.28e-4$	$8.49e-5$	$2.57e-4$	$1.10e-4$	$2.80e-4$	$1.10e-4$
	$1.77e-4$	$3.60e-4$	$1.97e-4$	$1.18e-4$	$2.31e-4$	$8.42e-5$	$2.61e-4$	$1.09e-4$	$2.85e-4$	$1.10e-4$
800	$7.81e-5$	$8.98e-5$	$8.60e-5$	$2.84e-5$	$1.02e-4$	$1.59e-5$	$1.14e-4$	$2.64e-5$	$1.26e-4$	$2.62e-5$
	$8.24e-5$	$9.00e-5$	$8.92e-5$	$2.79e-5$	$1.05e-4$	$1.70e-5$	$1.18e-4$	$2.59e-5$	$1.31e-4$	$2.62e-5$
1600	$3.52e-5$	$2.25e-5$	$3.80e-5$	$7.89e-6$	$4.51e-5$	$7.10e-6$	$5.00e-5$	$6.07e-6$	$5.54e-5$	$5.88e-6$
	$3.95e-5$	$2.27e-5$	$4.12e-5$	$7.35e-6$	$4.78e-5$	$6.04e-6$	$5.38e-5$	$5.59e-6$	$6.04e-5$	$5.88e-6$

Table 4.8. MAE Price-State II: $\sigma_1 = \sigma_2 = 0.4$, $r_1 = 0.08$, $r_2 = 0.04$

M	$d = 0.02$		$d = 0.04$		$d = 0.06$		$d = 0.08$		$d = 0.10$	
	FEM	FFEM	FEM	FFEM	FEM	FFEM	FEM	FFEM	FEM	FFEM
100	$9.67e-4$	$1.72e-3$	$1.13e-3$	$1.20e-3$	$1.27e-3$	$1.31e-3$	$1.40e-3$	$1.90e-3$	$1.49e-3$	$1.92e-3$
	$9.70e-4$	$1.71e-3$	$1.13e-3$	$1.19e-3$	$1.28e-3$	$1.31e-3$	$1.40e-3$	$1.90e-3$	$1.49e-3$	$1.92e-3$
200	$3.93e-4$	$4.45e-4$	$4.57e-4$	$4.35e-4$	$5.41e-4$	$2.94e-4$	$5.96e-4$	$4.56e-4$	$6.37e-4$	$4.60e-4$
	$3.96e-4$	$4.40e-4$	$4.58e-4$	$4.33e-4$	$5.44e-4$	$2.93e-4$	$6.00e-4$	$4.55e-4$	$6.42e-4$	$4.61e-4$
400	$1.72e-4$	$9.78e-5$	$1.97e-4$	$8.56e-5$	$2.38e-4$	$9.14e-5$	$2.64e-4$	$1.10e-4$	$2.84e-4$	$1.11e-4$
	$1.75e-4$	$1.00e-4$	$1.98e-4$	$8.43e-5$	$2.40e-4$	$9.05e-5$	$2.68e-4$	$1.10e-4$	$2.89e-4$	$1.11e-4$
800	$7.80e-5$	$2.68e-5$	$8.73e-5$	$1.71e-5$	$1.06e-4$	$1.80e-5$	$1.18e-4$	$2.65e-5$	$1.28e-4$	$2.65e-5$
	$8.07e-5$	$2.72e-5$	$8.84e-5$	$1.61e-5$	$1.09e-4$	$1.73e-5$	$1.22e-4$	$2.61e-5$	$1.33e-4$	$2.66e-5$
1600	$3.52e-5$	$7.08e-6$	$3.85e-5$	$7.43e-6$	$4.67e-5$	$5.96e-6$	$5.15e-5$	$6.09e-6$	$5.61e-5$	$5.97e-6$
	$3.79e-5$	$5.81e-6$	$3.95e-5$	$5.87e-6$	$4.93e-5$	$5.45e-6$	$5.55e-5$	$5.69e-6$	$6.12e-5$	$6.03e-6$

Table 4.9. Relative MAE EE price-State I: $\sigma_1 = \sigma_2 = 0.4$, $r_1 = 0.08$, $r_2 = 0.04$

M	$d = 0.02$		$d = 0.04$		$d = 0.06$		$d = 0.08$		$d = 0.10$	
	FEM	FFEM	FEM	FFEM	FEM	FFEM	FEM	FFEM	FEM	FFEM
100	$8.11e-3$	$1.59e-3$	$4.48e-3$	$7.73e-4$	$8.13e-3$	$2.76e-4$	$5.18e-3$	$9.61e-5$	$8.48e-3$	$4.14e-6$
	$8.16e-3$	$1.63e-3$	$4.51e-3$	$8.06e-4$	$8.19e-3$	$3.31e-4$	$5.26e-3$	$1.76e-4$	$8.56e-3$	$8.01e-5$
200	$2.22e-3$	$3.60e-4$	$4.48e-3$	$1.71e-4$	$4.62e-3$	$2.52e-5$	$5.18e-3$	$3.69e-5$	$4.76e-3$	$6.49e-5$
	$2.27e-3$	$4.07e-4$	$4.51e-3$	$2.05e-4$	$4.67e-3$	$8.06e-5$	$5.26e-3$	$4.32e-5$	$4.85e-3$	$1.94e-5$
400	$7.41e-4$	$5.43e-5$	$1.37e-3$	$1.67e-5$	$1.10e-3$	$3.45e-5$	$1.33e-3$	$6.96e-5$	$1.05e-3$	$7.96e-5$
	$7.88e-4$	$1.02e-4$	$1.40e-3$	$5.04e-5$	$1.16e-3$	$2.09e-5$	$1.41e-3$	$1.06e-5$	$1.13e-3$	$4.70e-6$
800	$7.41e-4$	$2.19e-5$	$5.85e-4$	$2.14e-5$	$1.10e-3$	$5.05e-5$	$1.33e-3$	$7.76e-5$	$1.05e-3$	$8.32e-5$
	$7.88e-4$	$2.53e-5$	$6.19e-4$	$1.23e-5$	$1.16e-3$	$4.83e-6$	$1.41e-3$	$2.55e-6$	$1.13e-3$	$1.12e-6$
1600	$3.70e-4$	$4.09e-5$	$1.95e-4$	$3.07e-5$	$2.20e-4$	$5.43e-5$	$3.61e-4$	$7.96e-5$	$5.81e-4$	$8.40e-5$
	$4.18e-4$	$6.23e-6$	$2.29e-4$	$3.04e-6$	$2.75e-4$	$1.06e-6$	$4.42e-4$	$6.02e-7$	$6.65e-4$	$2.78e-7$

Table 4.10. Relative MAE EE price-State II: $\sigma_1 = \sigma_2 = 0.4$, $r_1 = 0.08$, $r_2 = 0.04$

M	$d = 0.02$		$d = 0.04$		$d = 0.06$		$d = 0.08$		$d = 0.10$	
	FEM	FFEM	FEM	FFEM	FEM	FFEM	FEM	FFEM	FEM	FFEM
100	$6.95e-3$	$1.21e-3$	$1.29e-2$	$3.30e-4$	$9.96e-3$	$8.06e-5$	$6.70e-3$	$1.90e-5$	$7.00e-3$	$5.68e-5$
	$7.00e-3$	$1.26e-3$	$1.30e-2$	$4.10e-4$	$1.00e-2$	$1.22e-4$	$6.74e-3$	$5.68e-5$	$7.07e-3$	$1.66e-5$
200	$3.31e-3$	$2.61e-4$	$4.69e-3$	$2.77e-5$	$6.07e-3$	$1.30e-5$	$2.97e-3$	$2.40e-5$	$3.39e-3$	$7.01e-5$
	$3.36e-3$	$3.12e-4$	$4.77e-3$	$1.07e-4$	$6.11e-3$	$2.81e-5$	$3.01e-3$	$1.38e-5$	$3.46e-3$	$3.28e-6$
400	$1.48e-3$	$2.57e-5$	$2.63e-3$	$5.42e-5$	$2.18e-3$	$3.34e-5$	$1.11e-3$	$3.45e-5$	$1.58e-3$	$7.27e-5$
	$1.53e-3$	$7.64e-5$	$2.71e-3$	$2.54e-5$	$2.23e-3$	$7.68e-6$	$1.15e-3$	$3.23e-6$	$1.65e-3$	$7.14e-7$
800	$5.70e-4$	$3.24e-5$	$5.78e-4$	$7.39e-5$	$1.21e-3$	$3.95e-5$	$1.11e-3$	$3.70e-5$	$6.78e-4$	$7.33e-5$
	$6.21e-4$	$1.83e-5$	$6.58e-4$	$5.78e-6$	$1.25e-3$	$1.55e-6$	$1.15e-3$	$7.28e-7$	$7.51e-4$	$1.42e-7$
1600	$5.70e-4$	$4.68e-5$	$5.78e-4$	$7.84e-5$	$7.28e-4$	$4.08e-5$	$6.41e-4$	$3.75e-5$	$2.26e-4$	$7.34e-5$
	$6.21e-4$	$3.92e-6$	$6.58e-4$	$1.22e-6$	$7.69e-4$	$2.64e-7$	$6.79e-4$	$1.95e-7$	$2.99e-4$	$5.36e-8$

Table 4.11. MAE Hedge ratio-State I: $\sigma_1 = \sigma_2 = 0.4$, $r_1 = 0.08$, $r_2 = 0.04$

M	$d = 0.02$		$d = 0.04$		$d = 0.06$		$d = 0.08$		$d = 0.10$	
	FEM	FFEM	FEM	FFEM	FEM	FFEM	FEM	FFEM	FEM	FFEM
100	$5.42e-4$	$1.03e-4$	$5.40e-4$	$8.59e-5$	$5.41e-4$	$5.77e-5$	$5.42e-4$	$6.72e-5$	$5.95e-4$	$6.81e-5$
	$5.42e-4$	$1.03e-4$	$5.41e-4$	$8.57e-5$	$5.41e-4$	$5.78e-5$	$5.42e-4$	$6.72e-5$	$5.95e-4$	$6.81e-5$
200	$4.11e-4$	$2.57e-5$	$4.12e-4$	$2.28e-5$	$4.13e-4$	$1.45e-5$	$4.13e-4$	$1.64e-5$	$4.14e-4$	$1.66e-5$
	$4.10e-4$	$2.58e-5$	$4.12e-4$	$2.25e-5$	$4.13e-4$	$1.43e-5$	$4.13e-4$	$1.64e-5$	$4.14e-4$	$1.66e-5$
400	$3.69e-4$	$6.40e-6$	$3.69e-4$	$5.77e-6$	$3.70e-4$	$4.14e-6$	$3.70e-4$	$4.01e-6$	$3.70e-4$	$4.05e-6$
	$3.69e-4$	$6.43e-6$	$3.69e-4$	$5.54e-6$	$3.70e-4$	$3.92e-6$	$3.70e-4$	$3.99e-6$	$3.70e-4$	$4.05e-6$
800	$2.21e-5$	$1.62e-6$	$2.22e-5$	$1.58e-6$	$2.22e-5$	$1.42e-6$	$2.31e-5$	$1.11e-6$	$2.48e-5$	$1.97e-6$
	$2.20e-5$	$1.60e-6$	$2.21e-5$	$1.35e-6$	$2.22e-5$	$9.37e-7$	$2.22e-5$	$9.64e-7$	$2.43e-5$	$9.79e-7$
1600	$1.61e-6$	$4.55e-7$	$1.70e-6$	$5.70e-7$	$4.47e-6$	$8.38e-7$	$1.06e-5$	$8.87e-7$	$1.91e-5$	$2.34e-6$
	$1.82e-6$	$3.91e-7$	$1.71e-6$	$3.41e-7$	$3.77e-6$	$2.63e-7$	$9.49e-6$	$2.17e-7$	$1.85e-5$	$2.25e-7$

Table 4.12. Hedge ratio-State II: $\sigma_1 = \sigma_2 = 0.4$, $r_1 = 0.08$, $r_2 = 0.04$

M	$d = 0.02$		$d = 0.04$		$d = 0.06$		$d = 0.08$		$d = 0.10$	
	FEM	FFEM	FEM	FFEM	FEM	FFEM	FEM	FFEM	FEM	FFEM
100	$5.42e-4$	$6.84e-5$	$5.40e-4$	$4.78e-5$	$5.42e-4$	$5.28e-5$	$5.62e-4$	$6.71e-5$	$5.43e-4$	$6.80e-5$
	$5.42e-4$	$6.84e-5$	$5.40e-4$	$4.78e-5$	$5.42e-4$	$5.28e-5$	$5.62e-4$	$6.71e-5$	$5.43e-4$	$6.80e-5$
200	$4.11e-4$	$1.42e-5$	$4.13e-4$	$1.54e-5$	$4.13e-4$	$1.27e-5$	$4.14e-4$	$1.63e-5$	$4.14e-4$	$1.65e-5$
	$4.11e-4$	$1.42e-5$	$4.13e-4$	$1.53e-5$	$4.13e-4$	$1.28e-5$	$4.14e-4$	$1.63e-5$	$4.14e-4$	$1.65e-5$
400	$3.69e-4$	$4.57e-6$	$3.70e-4$	$3.29e-6$	$3.70e-4$	$4.32e-6$	$3.70e-4$	$4.00e-6$	$3.70e-4$	$4.04e-6$
	$3.69e-4$	$4.60e-6$	$3.70e-4$	$3.26e-6$	$3.70e-4$	$3.54e-6$	$3.70e-4$	$3.99e-6$	$3.70e-4$	$4.04e-6$
800	$2.21e-5$	$1.15e-6$	$5.67e-5$	$9.81e-7$	$4.81e-5$	$3.34e-6$	$2.99e-5$	$1.38e-6$	$2.22e-5$	$2.55e-6$
	$2.20e-5$	$1.19e-6$	$5.67e-5$	$6.82e-7$	$5.14e-5$	$8.63e-7$	$2.82e-5$	$9.62e-7$	$2.22e-5$	$9.79e-7$
1600	$2.55e-6$	$4.90e-7$	$6.40e-6$	$8.06e-7$	$1.07e-5$	$3.24e-6$	$1.51e-5$	$1.59e-6$	$1.13e-5$	$2.37e-6$
	$2.34e-6$	$2.69e-7$	$6.39e-6$	$2.22e-7$	$1.40e-5$	$2.53e-7$	$1.34e-5$	$2.18e-7$	$1.08e-5$	$2.26e-7$

CHAPTER 5

CONCLUSION

In summary, we have proposed and studied a new front-fixing finite element method for American options on stocks, zero-coupon bonds, and stocks under a regime-switching model. Stability has been established under some appropriate assumptions, and numerical results are presented in all three cases. We have also discussed implementations in detail and provided the algorithms used to solve the nonlinear systems. Our numerical results show that our method with the Crank-Nicolson scheme converges quadratically and linearly in the L^2 and H^1 norms respectively for option prices in all three cases. Together with variable step sizes in time, the front-fixing finite element method produces very fast and accurate approximations of early exercise prices.

In Chapter 2, for the American option on stocks, we present a result concerning the far field estimate of the option prices which we use to formulate an approximate variable domain problem. Extensive numerical results are given where we compare our method to the binomial method and observe that not only is our method very fast in comparison, but it is also very accurate. We also give details and numerical results about the computation of the greek ratios, which are of particular interest to practitioners. In Chapter 3, for the American option on zero-coupon bonds, we explain how our method can be used to simultaneously compute zero-coupon bond prices and early exercise option prices for any model of the short interest rate. We also propose and numerically verify a conjecture about the asymptotic expansion of early exercise

interest rates near option expiration for the Hull-White model. Numerical results are presented, and we observe that our method provides more accurate approximations than the usual finite element method, especially for early exercise interest rates. In Chapter 4, for the American option on stocks under a regime-switching model, we extend the model studied in Chapter 2 so that the price of the underlying asset also depends on the state of the economy, which is modeled by a finite state Markov chain. We compare our method to the usual finite element method and observe that the front-fixing finite element method produces very accurate results, especially for early exercise option prices. Based on our results, we were also able to propose and numerically verify a conjecture about the behavior of the early exercise option price near option expiration, similar to what was done in Chapter 3. We conclude that our method can obtain more accurate results with fewer time steps, saving computational time.

APPENDIX

Date: Fri, 12 Mar 2010 10:59:34 -0500
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To whom it may concern,

My name is Anthony D. Holmes. I am a PhD candidate in the Mathematical Sciences department at the University of Nevada Las Vegas. I am planning to use a paper that my advisor, Hongtao Yang, and I published in one of your journals as a chapter in my dissertation. The article was:

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Pages: 2158-2180

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