Modeling of short-distance running

Maria Camille Theresa Jose Capiral

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MODELING OF SHORT DISTANCE RUNNING

by

Maria Camille Theresa Jose Capiral

Bachelor of Arts
University of Southern California
1992

A Thesis submitted in partial fulfillment
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ABSTRACT

Modeling of Short Distance Running

by

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In this thesis, we construct a model of human performance in short distance foot races. First, we describe the Hill-Keller model of competitive running, based on the solution to an optimal control problem, and then focus on that part of the model dealing with short distance races. Our task is to estimate two underlying physiological parameters that characterize the runner's performance. Second, we customize this sub-model by linearizing Keller's analytical solution for short distance races, we then apply a high quality linear least squares estimation based on the Singular Value Decomposition (SVD) in order to estimate the two physiological parameters. Finally, we apply this computational model to real world data, first on a 1987 World Track record, and more extensively, on larger data sets consisting of split times of high school student athletes.
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CHAPTER ONE

INTRODUCTION

Mathematical sciences constitute an integral part of our everyday lives, although their practical nature and application to the real world often elude us. Due to advances during the past two decades in computing technology, and the resulting use of mathematical modeling, there has been a considerable increase in the use of quantitative techniques within biology, physiology, and other sciences. Such activities have aided in confirming to scientists and engineers that math is a vitally relevant discipline very much needed not only in the search for solutions to abstract problems, but also to better understand physical phenomena.

Mathematical models serve as means through which a small piece of the real world is captured within the scope of the relationships comprising a formal mathematical system. With purpose in mind, they are based on a choice of what to observe and what to ignore necessarily discarding aspects of the real situations deemed irrelevant. For this reason, we must consider certain characteristics of mathematical models such as descriptive realism, precision, robustness, generality, and fruitfulness in order to determine how good they are.

The realm of mathematical modeling necessarily entails the application of numerical methods to computing. Numerical methods enable us to utilize computers as a tool. What was once unsolvable by the human mind is now attainable through the use of...
computer technology with the aid of numerical approximation techniques. This characteristic aspect of modern modeling is illustrated in this thesis. In particular, we describe an important numerical technique for obtaining a high-resolution least squares fit.

First, we analyze the Hill-Keller model for competitive running, which emerged from the contributions of A.V Hill [3] and J.B. Keller [5]. Hill was one of the early pioneers to construct a simple model of running based on the Equation of Motion. Keller expanded Hill's model to produce one based on optimal control theory that resulted in a theoretical relationship between the shortest time $T$ in which a given distance $D$ can be run in terms of four physiological parameters. These physiological parameters were then estimated via non-linear least squares fitting of empirical data to Keller's resulting analytical solution. The theory indicates that a runner should run at maximum acceleration for all races of distance less than a critical distance $D_c \approx 290m$. For races of longer distances, the runner should run at maximum acceleration for a short burst to $t_1$ seconds, then constant speed for the major portion of the race to $t_2$ seconds, until the last one or two seconds to the final optimal time $T$, during which there is a slight deceleration. Although the empirically fitted model predicted world records to a remarkable $\pm 3\%$, in practice, competitors illustrate a positive "kick" as opposed to a loss of energy, thus pointing to possible inadequacies in the model.

Despite its weaknesses, evidence found in the literature has proven the Hill-Keller model to be very fruitful. The ideas behind the model have generated more research in the modeling of running which take into consideration some of the physiological and mechanical effects Keller did not. One example is given by W.G. and J.K. Pritchard's [9]
theory of the effects of wind resistance on the runner. This theory resulted in the accurate evaluation of the much-questioned F. Griffith-Joyner record in the 1988 U.S. Olympic Trials and suggested that clothing design might improve running performances. Also, Professor W. Woodside's [10] reformulation of Keller's problem to a single variable optimization one, (discarding the limiting physical assumptions needed in the first), yielded a more generalized solution. Finally, F. Perronnet and G. Thibault [7] expanded on the use of world running records to develop a model that considered contributions of anaerobic and aerobic metabolism and its correlation to the runner's energy output in correspondence to the duration of the race.

In our case, we will focus on a piece of the Hill-Keller model, namely Keller's theoretical solution involving only short races which he utilized to estimate two of the four physiological factors under consideration. Our problem is to assign these two physiological factors to certain subjects by reducing Keller's solution to a linear approximation requiring a solution to a least squares estimation. We then solve our problem by customizing an appropriate numerical method based on linear algebra, and applying it to computing to perform the necessary computations.

Numerical algorithms involved in such computations rely heavily on matrix factorizations, one of the more important of these is our method of choice, the Singular Value Decomposition (SVD) which decomposes a matrix into well-conditioned orthogonal matrices, preventing the magnification of small perturbations. It yields the best approximation in the least squares sense of a high dimensional matrix by a lower one, giving a particularly useful way of determining the rank of a matrix and providing the most numerically stable solution for our estimation problem.
In Chapter Two, we describe the process Keller used in the formulation, mathematical derivation, and evaluation of the Hill-Keller model. In Chapter Three, we approximate linearly part of his solution and solve the least squares estimation problem by customizing the SVD to our case, an $m \times 2$ overdetermined system of linear equations. Finally, Chapter Four consists of the application of our model to two extreme cases, using running data generated by Olympic athletes and amateur high school track students, thus resulting in the estimation of our desired parameters.
CHAPTER TWO

THE HILL-KELLER MODEL

In the 1920s a British biologist by the name of Archibald V. Hill [3] developed what appears to be the first recorded mathematical model of running. Specifically, Hill's model of sprinting was based on Newton's Second Law of Motion:

$$\frac{dv}{dt} + \frac{v}{\tau} = f(t),$$

where $v(t)$ is the velocity of the runner, $f(t)$ is his/her propulsive force per unit mass, and $\tau$ is a constant related to a resistive force. This model considered only two physiological constants, the maximum propulsive force the runner can exert and a resistive force related to internal losses associated with the runner's action.

Over 40 years later, a mathematician from Stanford University, Professor Joseph B. Keller [5] recognized that Hill had opened the door to mathematical modeling of competitive running. Keller enlarged Hill's basic model and considered an underlying optimal control problem. He then used the calculus of variations to solve the optimal control problem. The Hill-Keller model provides an analyzable and simple theory of running in which the four underlying physiological parameters characterizing the runner can be estimated from finish times in short and long distance races [7].

The four physiological constants that characterize the runner are as follows:

$F$: maximum force a runner can exert per unit mass; $ms^{-2}(kg)^{-1}$

$\tau$: constant used in defining the resistive force opposing the runner; $s$
\( E_0 \): initial amount of energy stored in the runner’s body at the start of the race; \( J(kg)^{-1} \)

\( \sigma \): rate at which energy is supplied per unit mass by oxygen metabolism; \( Js^{-1}(kg)^{-1} \)

The constants \( E_0 \) and \( \sigma \) are measures respectively of the anaerobic and aerobic energy.

The three functions that are used in the model are:

\( v(t) \): velocity at time \( t \); \( ms^{-1} \)

\( f(t) \): propulsive force per unit mass at time \( t \); \( ms^{-2}(kg)^{-1} \)

\( E(t) \): energy left per unit mass at time \( t \); \( J(kg)^{-1} \)

Recall that a joule \( J \) is the amount of energy used when 1 kg moves 1 m when subjected to one Newton force, the unit of force that produces an acceleration of one \( ms^{-2} \) on one kg. In order to measure energy we often use calories instead of Joules and utilize the conversion factor

\[ 1 \text{ calorie} = 4.1840 \text{ J}. \]

A calorie is the amount of heat needed to raise the temperature of 1 kg of water \( 1^\circ C \) and it is used as the unit for measuring the energy produced by food when oxidized by the body. For each of the quantities described above, we have indicated the unit of measure. It is understood that these units will be used throughout the remainder of this thesis and for this reason we will no longer make explicit reference to them.

The model assumes ideal conditions in which diverse factors such as non-rectilinear motion and wind do not play a role. In addition to ideal running conditions, we assume that the resistance to running at a velocity \( v(t) \) is proportional to that velocity, specifically, we assume that the resistive force at time \( t \) is assumed to be proportional to the velocity \( v(t) \) with constant of proportionality \( \frac{1}{r} \). Although experts have questioned
the accuracy of this assumption, it does provide a good approximation and it yields tractable mathematics. We also assume that energy is used at the rate of work $f(t)v(t)$, and that the propulsive force is zero, $f(t) = 0$, for every $t$ beyond the first positive root of $E(t)$, where $E(t)$ is the energy equivalent of the available oxygen per unit mass at time $t$ [4].

The three physiological parameters, $F$, $\tau$, and $\sigma$ and the three functions, $v(t), f(t)$ and $E(t)$ are related by two ordinary differential equations:

\begin{align*}
\text{Force Equation} & \quad \frac{dv}{dt} + \frac{v}{\tau} = f(t) \\
\text{Energy Equation} & \quad \frac{dE}{dt} = \sigma - fv
\end{align*}

The fourth physiological parameter $E_0$ provides the initial value, $E(0) = E_0$, for the Energy Equation, whereas $v(0) = 0$ is the initial value for the Force Equation. The runner is in control of the propulsive force $f(t)$. If one of the two functions, $f(t)$ or $v(t)$, is known then the other is determined mathematically by the Force Equation. When both $f(t)$ and $v(t)$ are known then $E(t)$ is obtained from the Energy Equation.

The optimal control problem considered by Keller can be formulated as follows. Given the four physiological parameters, $\tau, F, \sigma, E_0$ of a runner and given a distance $D$, find the velocity $v(t)$ satisfying the conditions:

\begin{align*}
\frac{dv}{dt} + \frac{v}{\tau} = f(t) & \quad \begin{cases} v(0) = 0 \\ f(t) \leq F \end{cases} \\
\frac{dE}{dt} = \sigma - fv & \quad \begin{cases} E(0) = E_0 \\ E(t) \geq 0 \end{cases}
\end{align*}

such that $T$ defined by

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is minimized. Hence the problem is to find an optimal velocity that minimizes the runner's time for a given distance [5].

The Dual Problem

As it stands, the above problem is very difficult to solve, and for that reason, Keller ingeniously reformulated it. The reformulation yields an equivalent and easier problem, which Keller then proceeded to solve. The equivalence of the two problems is justified by the property that for an optimal velocity there exists a one-to-one correspondence between the best finish time $T$ and the distance $D$ of a race. Therefore, instead of minimizing $T$ for a given $D$, we can maximize $D$ for a given $T$. Thus, the problem is that of finding the possible farthest distance one can run within a certain time rather than finding the fastest time one can run a certain distance. Specifically, the equivalent problem is formulated as follows: Given the four physiological constants $r, F, \sigma, E_r$, and a positive time $T$, find $v(t)$ for $t \in [0, T]$ such that the distance

$$D = \int_0^T v(t) dt$$

is maximized [5]

Estimation of the Physiological Constants

In the next section, we outline Keller's method of solution. First however, we point out that Keller chose to determine the four physiological constants associated with his model by fitting the finish times of 22 races taken from the 1972 World Records to
his theoretical curve (see Table 1). In doing so, Keller illustrated surprising good agreement between his theory and the real world of competitive running.

**TABLE 1: KELLER'S CONSTANTS**

<table>
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<tr>
<th>Parameter</th>
<th>Value</th>
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<tr>
<td>$\tau$</td>
<td>0.892 s</td>
</tr>
<tr>
<td>$F$</td>
<td>$12.2 \text{ ms}^{-2} (\text{kg})^{-1}$</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>$41.55 \text{ Js}^{-1} (\text{kg})^{-1}$ or $993 \text{ calories s}^{-1} (\text{kg})^{-1}$</td>
</tr>
<tr>
<td>$E_0$</td>
<td>$2,405.8 \text{ J} (\text{kg})^{-1}$ or $575 \text{ calories} (\text{kg})^{-1}$</td>
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Due to the nature of the solution, Keller partitioned the data into two subsets, sprints and long distance races. In each case a non-linear least squares estimation was necessary. For the first subset, Keller considered the results from eight short dashes ranging from 50 yd to 200 m. He estimated the two constants $F$ and $\tau$ to yield a least squares fit of his calculated time to the given record time. Similarly, he used the second subset of data consisting of the remaining fourteen long distance races (from 400 m to 10,000 m) in order to estimate $\sigma$ and $E_0$. For both cases Keller [5] utilized a computer to carry out the minimization of the sum of the squares of the relative errors,

$$\sum (T_{\text{record}} - T_{\text{calculated}})^2 / T_{\text{record}}^2 .$$

In Chapters Three and Four, we estimate the two constants $F$ and $\tau$. We consider exclusively the short races, using an excellent linear approximation to Keller's solution for that case, followed by a high-resolution linear least squares fit based on the Singular Value Decomposition (SVD), thus getting virtual agreement with Keller's results.
Solution Outline

In the reformulation of the original problem Keller conveniently eliminates \( f(t) \) and \( E(t) \) by expressing them in terms of \( v(t) \). Thus, the problem is given a distance \( D \)

find \( v(t) \) subject to the constraints

\[
\begin{align*}
(2.2) & \quad v(0) = 0, \\
(2.3) & \quad \frac{dv}{dt} + \frac{1}{\tau} v \leq F,
\end{align*}
\]

and

\[
E_0 + \sigma t - \frac{v^2(t)}{2} - \frac{1}{\tau} \int_0^t v^2(s) ds \geq 0
\]

such that \( T \) defined in (2.1) is minimized. In this section we outline the solution to the equivalent problem of maximizing the distance \( D \). At \( t = 0 \), we have \( v(0) = 0 \); regardless of the value of \( f(t) \), thus the initial rate of work is \( f(t)v(t) = 0 \). Therefore, \( f(t) \) may take on its largest value without any energy consumption, allowing us to set

\[ f(0) = F, \]

and assume that the propulsive force is equal to the maximum propulsive force during a small initial interval of time \([0, t_i]\), where \( t_i \) is to be determined. This assumption is clearly unnecessary if \( t_i = 0 \). After using the assumption in (2.3) we obtain

\[
\frac{dv}{dt} + \frac{1}{\tau} v = F, \quad 0 \leq t \leq t_i,
\]

which has the solution

\[ v(t) = F \tau (1 - e^{-\tau t}), \quad 0 \leq t \leq t_i, \]

when (2.2) is satisfied. Utilizing the final constraint and the previous solution we obtain

\[ E_0 + \sigma t - F^2 \tau^2 \left( \frac{t}{\tau} + e^{-\tau t} - 1 \right) \geq 0. \]
In addition, we also assume that

\[ \sigma < F^2 \tau. \]

Otherwise, if \( \sigma \geq F^2 \tau \) then,

\[ E_0 + \sigma t - F^2 \tau^2 \left( \frac{t}{\tau} + e^{-\tau \xi} - 1 \right) \geq 0, \quad 0 \leq t \leq t_1 \]

is satisfied for all \( t \geq 0 \). Hence, the solution to the force equation with \( f(t) = F \),

\[ v(t) = F \tau (1 - e^{-\tau \xi}) \]

is the optimal velocity for all \( t \). This is unrealistic so we disregard this case[5].

Keller's resulting solution involves three cases. The first case optimizes the running strategy for sprints when the given optimal time \( T \) is below or equivalent to a critical time, \( T_c \). \( T_c \) is the unique positive root of

\[ E_0 + \sigma t - F^2 \tau^2 \left( \frac{t}{\tau} + e^{-\tau \xi} - 1 \right) = 0, \]

where the corresponding critical distance, \( D_c \) is mathematically determined using

\[ D_c = F \tau^2 \left( T_c / \tau + e^{-\tau / \tau} - 1 \right). \]

The second case describes what happens when \( T \) is just above \( T_c \), but below or equivalent to a time \( T^* \) which is fairly close to \( T_c \). To determine \( T^* \) we evaluate

\[ T^* = T_c \Delta T \]

where,

\[ \Delta T = \tau \ln 2 - \frac{T}{2} \ln \left[ 1 - \sigma F^{-2} \tau^{-1} \left( 1 - e^{-T / \tau} \right)^{-2} \right]. \]

The final case describes the running strategy for long distance runs, which are the times that are greater than \( T^* \). Thus, the three cases are as follows

I. Short Distances, \( T \leq T_c \).

II. Long Distances Close to the Critical Time, \( T_c < T \leq T^* \).
III. Long Distances, $T > T^*$. 

In each case we describe the five quantities, $v(t), f(t), E(t), d(t)$ and $D$. Note that 

$$d(t) = \int_0^t v(s) ds,$$

and 

$$D = d(T).$$

Upon checking Keller’s calculations a typographical error was revealed in his determination of the values of $d(t)$ and $D$ for the last two cases. It is merely a sign error but definitely worth noting (see Appendix I).

Case 1: Short Distances $T \leq T_c$

$$v(t) = F\tau(1 - e^{-t/\tau})$$

$$f(t) = F$$

$$E(t) = E_0 + \sigma t - F^2 \tau^2 \left( t / \tau + e^{-t/\tau} - 1 \right)$$

$$d(t) = F\tau^2 \left[ \frac{t}{\tau} + e^{-t/\tau} - 1 \right] \quad 0 \leq t \leq T_c$$

$$D = F\tau^2 \left[ \frac{T}{\tau} + e^{-T/\tau} - 1 \right]$$

This tells us for short races, the proposed optimal running strategy is for the runner to apply maximum propulsive force, $f(t) = F$, throughout the duration of the race. The model indicates for short distances a runner will not run out of energy if he/she runs to his/her capacity during the entire race, which seems realistic and quite logical.
Case 2: Long Distances Close to the Critical Time \( T_c < T \leq T^* \)

\[
v(t) = \begin{cases} 
  F \tau (1-e^{-\tau t}) & 0 \leq t \leq T_c \\
  \frac{\sqrt{c^2 - \sigma t}e^{\frac{c^2}{2}}}{\sigma \tau + (c^2 - \sigma t)e^{\frac{c^2}{2}}} & T_c < t \leq T 
\end{cases}
\]

where \( c^2 = v^2(T_c) \)

\[
f(t) = \begin{cases} 
  \frac{F}{\sigma(v(t))} & 0 \leq t \leq T_c \\
  \frac{\sigma}{v(t)} & T_c < t \leq T
\end{cases}
\]

\[
E(t) = \begin{cases} 
  E_0 + \sigma t - F^2 \tau^2\left( t/\tau + e^{-\tau t} - 1 \right) & 0 \leq t \leq T_c \\
  0 & T_c < t \leq T
\end{cases}
\]

\[
d(t) = F\tau^2 \left[ \frac{T_c}{\tau} + e^{-\tau t} - 1 \right] + \tau(\sigma t) \left[ \tanh^{-1}\left\{ 1 + \left( \frac{F^2 \tau}{\sigma} (1 - e^{-\tau t})^2 - 1 \right) e^{-2\tau(t-T_c)} \right\} \right]^{1/2}
\]

\[
-\tanh^{-1}\left( \frac{\tau}{\sigma} \right)^{1/2} F(1-e^{-\tau t}) + \left( \frac{\tau}{\sigma} \right)^{1/2} F(1-e^{-\tau t})
\]

for \( T_c \leq t \leq T \).

\[
D = F\tau^2 \left[ \frac{T_c}{\tau} + e^{-\tau t} - 1 \right] + \tau(\sigma t) \left[ \tanh^{-1}\left\{ 1 + \left( \frac{F^2 \tau}{\sigma} (1 - e^{-\tau t})^2 - 1 \right) e^{-2\tau(t-T_c)} \right\} \right]^{1/2}
\]

\[
-\tanh^{-1}\left( \frac{\tau}{\sigma} \right)^{1/2} F(1-e^{-\tau t}) + \left( \frac{\tau}{\sigma} \right)^{1/2} F(1-e^{-\tau t})
\]

for \( T_c \leq t \leq T^* \).

In this situation, the distance the runner has to run is slightly farther than a sprint.

So naturally he/she will apply maximum force almost throughout the entire race as we saw in case 1. The runner runs to his/her fullest capacity, \( f(t) = F \) during a time interval \([0, T_c] \). Now from the critical time to the time the runner finishes the race \([T_c, T]\), the runner has no energy i.e. \( E(t) = 0 \), but yet has enough momentum to carry him/her to the
finish line. The duration of this 'negative kick' is quite small and the idea is similar to that of a car running out of gas, but continues to move forward.

Case 3: Long Distances $T > T^*$

In this case we must compute $t_1, t_2, \text{ and } \lambda$, by finding the roots of three nonlinear equations:

(i) $F\left(1 - e^{-h/r}\right) = \frac{1}{\lambda}$

(ii) $\lambda = 2\left(\sigma - \frac{r}{\lambda^2}\right)^{-1}\left[\sigma r + \left(\frac{r^2}{\lambda^2} - \sigma r\right)e^{2(t_2 - t)/r}\right]^{1/2} - \frac{r}{\lambda}$

or

$\lambda = \sqrt{\frac{r}{\sigma}}\sqrt{1 - 4e^{-2(t_2 - t)/r}}$

(iii) $E_0 + \sigma t_2 - \frac{r^2}{2\lambda^2} - F^2 r \left(-\frac{3r}{2} + t_1 + 2te^{-h/r} - \frac{r}{2}e^{-2h/r}\right) - \frac{r}{\lambda^2}(t_2 - t_1) = 0$

$v(t) = \begin{cases} 
F r (1 - e^{-\nu t}) & 0 \leq t \leq t_1 \\
\frac{r}{\lambda} & t_1 < t \leq t_2 \\
\sqrt{\sigma r + (c^2 + \sigma r)e^{2(t_2 - t)/r}} & t_2 < t \leq T
\end{cases}$

where $c^2 = v^2(t_2) = \frac{r^2}{\lambda^2}$.

$f(t) = \begin{cases} 
F & 0 \leq t \leq t_1 \\
\frac{1}{\lambda} & t_1 < t \leq t_2 \\
\frac{\sigma}{v(t)} & t_2 < t \leq T
\end{cases}$
\[
E(t) = \begin{cases} 
E_0 + \sigma \tau - F^2 \tau^2 \left( \frac{t}{\tau} + e^{-t/\tau} - 1 \right) & 0 \leq t \leq t_1 \\
\left( \sigma - \frac{\tau}{\lambda^2} \right) t + e & t_1 < t \leq t_2 \\
0 & t_2 < t \leq T
\end{cases}
\]

where \( e = \left( \frac{\tau}{\lambda^2} - \sigma \right) t_2 = E(t_1) + \left( \frac{\tau}{\lambda^2} - \sigma \right) t_1 \)

\[
d(t) = F \tau^2 \left[ \frac{t}{\tau} + e^{-t/\tau} - 2 \right] + \frac{\tau}{\lambda} (t_2 - t_1) + \tau (\sigma \tau)^{1/2} \left[ \tanh^{-1} \left\{ 1 + \left( \frac{\tau}{\lambda^2 \sigma} - 1 \right) e^{-2/\tau (t_2 - t_1)} \right\} \right]^{1/2} \\
- \left\{ 1 + \left( \frac{\tau}{\lambda^2 \sigma} - 1 \right) e^{-2/\tau (t_2 - t_1)} \right\}^{1/2} - \tanh^{-1} \left\{ \frac{\tau}{\lambda^2 \sigma} \right\}^{1/2} + \left\{ \frac{\tau}{\lambda^2 \sigma} \right\}^{1/2} \right] \text{ for } t_2 < t \leq T
\]

\[
D = F \tau^2 \left[ \frac{t}{\tau} + e^{-t/\tau} - 2 \right] + \frac{\tau}{\lambda} (t_2 - t_1) + (\sigma \tau)^{1/2} \left[ \tan^{-1} \left\{ 1 + \left( \frac{\tau}{\lambda^2 \sigma} - 1 \right) e^{-2/\tau (T - t_1)} \right\} \right]^{1/2} \\
- \left\{ 1 + \left( \frac{\tau}{\lambda^2 \sigma} - 1 \right) e^{-2/\tau (T - t_1)} \right\}^{1/2} - \tan^{-1} \left\{ \frac{\tau}{\lambda^2 \sigma} \right\}^{1/2} + \left\{ \frac{\tau}{\lambda^2 \sigma} \right\}^{1/2} \right].
\]

In case 3, there are three parts to the optimal strategy. Over the short initial time interval \([0, t_1]\), the runner applies maximum force \( f(t) = F \). During the longer interval \([t_1, t_2]\), the runner applies a smaller propulsive force \( f(t) = \frac{1}{\lambda} \), which is constant where his/her rate of energy decreases linearly. At the last stage, like in the last stage in case 2, the runner has no energy over the short interval \([t_2, T]\). Again in the latter part, the runner possesses just enough momentum to enable him/her to finish the race.

**Pros and Cons of the Hill-Keller Model**

Although the Hill-Keller Model does provide a good representation of the 1972 World Track Records from 50 yards to 10,000 meters (see Appendix II [5]), it has several
weaknesses. The greatest flaw is in the optimal strategy for long distance races, i.e. distances $D$ larger than the critical distance $D_c$ (which depends on the four physiological parameters). The exact model indicates that the optimal time $T$ is obtained with a slowing down in the last couple of seconds where

$$E(t)=0.$$ 

As we discussed earlier in the solution outline for cases 2 and 3, this phenomenon can be understood by comparing it to driving a car a fixed distance in the shortest time with limited amount of fuel. Prior to the end, the energy level of the vehicle is zero, yet it has built up enough momentum to move it to the finish line, mathematically yielding the best finish time $T$.

Contrary to the theory, runners often finish with a “positive kick” rather than this “negative kick”. This discrepancy suggests that either, athletes are not performing to their fullest potential or that the Hill-Keller model is inadequate. Perhaps this discrepancy is related to a runner’s goal, which influences his/her strategies. Competitors are concerned with beating their opponents as opposed to achieving the shortest time. It would be both interesting and valuable to investigate if runners practicing the optimal strategy perform better in beating their competitors. Comparing the results when using the optimal strategy with the observed record performances may determine the true optimal strategy.

Although the predicted times came within ±3% of the record times, the empirical fitting of the Hill-Keller model used a single set of physiological constants for all the distances. This should be a point of concern for us since the record holder at 100 m is not likely to be able to match the record holder at 10 km and vice versa. By taking the world
records to estimate the parameters, we are assuming that a single human being is capable of winning all the races. We know that this is not realistic. The physical makeup of sprinters and long distance runners are distinct. Generally, sprinters tend to be muscular while long distance runners tend to be thin. We expect that these individuals undoubtedly must have different physiological constants from one another.

Because it omits various mechanical and physiological effects, some physiologists argue that the model is too simple. It does not take into consideration the importance of the up-and-down motions of the runner’s limbs nor does it consider the non-rectilinear motion on oval tracks. Perhaps the latter effect can be accounted for by adding extra distance when the runner makes a wide turn. The model also fails to represent the accumulation and the process for removal of waste products or the transfer to the use of less efficient fuels nor the depletion of fuels that utilize the least oxygen. It would be of interest to construct a better theory incorporating some of these effects and to adapt it to competitive running as well as other races such as ice-skating, cycling, and swimming [5].

Many external factors such as wind and altitude were ignored in order to assume an “ideal” running situation. It is well known that sprinters run faster at high altitudes than at sea level. The Hill-Keller model only looked at four physiological factors. If one tried to consider all the possible factors that play a role in running it is unlikely that one would get an elegant mathematical solution. The model is good for a first order approximation. It is a simple theory and it is analyzable. The model is attractive because it gives us a meaningful relationship between mathematics and physiology. Since the values between the theoretical and the actual record times agree quite well, we know that
this model is indeed meaningful. Keller had opened the doors for further research in mathematical physiology [9].
CHAPTER THREE

OPTIMAL SHORT RACES, LINEAR APPROXIMATION,
AND THE SINGULAR VALUE DECOMPOSITION

In this chapter we focus on Keller’s solution for short races. For this case, the resulting optimal velocity depends only on the two physiological factors $F$ and $\tau$. We will linearize Keller’s solution and then use a linear least squares regression based on the Singular Value Decomposition (SVD) in order to estimate the two desired parameters. We have specifically chosen the SVD, since our problem can be numerically sensitive, and we want to obtain the greatest accuracy possible.

Short Races

Referring back to the running model, short races are those in which the distances are less than the critical distance $D_c$, where $D_c$ depends on all four physiological constants. For such races, for any $t \in [0, T]$, the Hill-Keller theory yields

$$ v(t) = F \tau (1 - e^{-t/\tau}), $$

and

$$(3.1) \quad d(t) = F \tau^2 \left( \frac{t}{\tau} + e^{-t/\tau} - 1 \right),$$

where the maximum velocity possible is

$$(3.2) \quad v^* = F \tau.$$
Solving for $t$ in (3.1) we have

$$t = \frac{d}{F\tau} - \tau e^{-\frac{d}{v}} + \tau,$$

or equivalently,

$$t = \frac{1}{F\tau} d + \tau - \varepsilon,$$

where $\varepsilon = \tau e^{-\frac{d}{v}}$ is negligible for $t$ away from the initial time $t_0 = 0$. We can therefore take the linear approximation

$$t = \frac{1}{F\tau} d + \tau = \frac{1}{v} d + \tau,$$

and we can write $t$ as a linear function of $d$,

(3.3) \hspace{1cm} t = a + bd,$$

where

$$a = \tau, \hspace{0.5cm} b = \frac{1}{v} = \frac{1}{F\tau}$$

for $t \in [0, T], d \in [0, D], \text{ and } D < D_c.$

As time increases the exponential term, $e^{-\frac{d}{v}}$, approaches zero. Since we are primarily concerned with the final time it takes to run the race, we can safely disregard the negligible term $\varepsilon$. For time $t$ slightly away from the start time $t=0$, the approximation is excellent. Note that at $d=0$, the approximation yields $t = \tau$ instead of $t=0$. In essence, the approximation assumes that the runner has a "flying start" rather than an accelerative start from rest. However, the actual accelerative start quickly levels to the maximum velocity $v^*$. (In the plot of Figure 1, this is represented by the slope $b = \frac{1}{v^*}$ of the straight line graph of $t$ as a function of $d$.)
FIGURE 1: OPTIMAL SHORT RACES AND THE APPROXIMATION LINE

The expression in (3.3) lends itself for linear least squares estimation of the two parameters $a$ and $b$, from which we can then get estimates

$$t = a, F = \frac{1}{ab}.$$ 

Since the estimation problem can be ill conditioned, as mentioned earlier, we have decided to use the SVD. For this reason, at this point, we give a summary of basic properties of the SVD and of the technique involving its use in our linear least squares estimation.

The Singular Value Decomposition

The SVD is a matrix factorization that involves orthogonal matrices. Recall that $Q \in \mathbb{R}^{m \times m}$ is said to be orthogonal if $Q^T Q = I$. From the computational point of view orthogonal matrices are very effective because their condition number is one. The
condition number is a quantifier that indicates to us how sensitive our problem is to be to perturbations (see Appendix IV) [8]. In numerical computation, small perturbations caused by round off errors can lead to major output errors and since the SVD involves well-conditioned orthogonal matrices, we are assured that these perturbations will not be magnified. Another useful characteristic of orthogonal matrices is that they preserve the matrix 2-norm (see Table 2) [2].

TABLE 2: NORM-INvariant PROPERTIES OF ORTHOGONAL MATRICES

<table>
<thead>
<tr>
<th>Property</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>I.</td>
<td>[ \langle Q_1 x, Q_2 y \rangle = \langle x, y \rangle ]</td>
</tr>
<tr>
<td>II.</td>
<td>[ |Q_1|_2 = 1 ]</td>
</tr>
<tr>
<td>III.</td>
<td>[ |Q_1 A Q_2|_2 = |A|_2 ]</td>
</tr>
<tr>
<td>IV.</td>
<td>[ |Q_1 A Q_2|_F = |A|_F ]</td>
</tr>
</tbody>
</table>

Property I indicates that angles and Euclidean norms are invariant under orthogonal transformations. Observe also, (the condition number) \[ \kappa(Q_1) = \|Q_1\|_2 \cdot \|Q_1^t\|_2 = 1. \]

The Singular Value Decomposition Theorem: Let \( A \) be a real \( m \times n \) matrix, then

\[
A_{m \times n} = U_{m \times m} \sum_{m \times n} V_{n \times n}^t .
\]

\( U, V \) are orthogonal matrices and

\[
\Sigma = \begin{pmatrix} \sum_{i} & 0 \\
0 & 0 \end{pmatrix}_{m \times n},
\]

where \( \sum_{i} \) is a non-singular diagonal matrix.
We have provided a proof of the SVD in Appendix III and Figure 2 illustrates the SVD.

**FIGURE 2: DIAGRAM OF THE SINGULAR VALUE DECOMPOSITION**

The SVD indicates that the orthogonal basis \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \) in the domain space \( \mathbb{R}^n \) is mapped onto an orthogonal basis \( \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m \) in the range space \( \mathbb{R}^m \) with the relationship

\[
A \mathbf{v}_j = \sigma_j \mathbf{u}_j , \quad j = 1, \ldots, r .
\]

Observe that the diagonal entries of \( \Sigma \) are non-negative and can be arranged in non-increasing order. Thus

\[
\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_p),
\]

where \( p = \min\{m, n\} \),

and \( \sigma_1 \geq \sigma_2 \geq \cdots \sigma_p \geq 0 \).

The number of non-zero diagonal entries of \( \Sigma \) is equal to the rank of \( A \). The SVD provides the most efficient computational method for the determination of the rank of a matrix. In Table 3, we have summarized other properties of the SVD [2] [6].
### TABLE 3: OTHER PROPERTIES OF THE SVD

<table>
<thead>
<tr>
<th>Relationship with Eigen Decomposition</th>
</tr>
</thead>
<tbody>
<tr>
<td>I. $\sigma_1, \sigma_2, \cdots, \sigma_r$ are the non-zero eigenvalues of both $A'A \in \mathbb{R}^{n \times n}$ and $AA' \in \mathbb{R}^{m \times m}$.</td>
</tr>
<tr>
<td>II. The right singular vectors $v_1, v_2, \cdots, v_n$ are the eigenvectors of $A'A$.</td>
</tr>
<tr>
<td>III. The left singular vectors $u_1, u_2, \cdots, u_m$ are the eigenvectors of $AA'$.</td>
</tr>
<tr>
<td>IV. If $A$ is symmetric (i.e. $A = A'$) and $\lambda_1, \cdots, \lambda_m$ are its eigenvalues then $</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Orthonormal Bases</th>
</tr>
</thead>
<tbody>
<tr>
<td>V. Rank of $A$: $\mathcal{R}(A) = \text{span}(u_1, \cdots, u_r)$.</td>
</tr>
<tr>
<td>VI. Null space of $A$: $\mathcal{W}(A) = \text{span}(v_{r+1}, \cdots, v_n)$.</td>
</tr>
<tr>
<td>VII. Rank of $A'$: $\mathcal{R}(A') = \text{span}(v_1, \cdots, v_r)$.</td>
</tr>
<tr>
<td>VIII. Null space of $A'$: $\mathcal{W}(A') = \text{span}(u_{r+1}, \cdots, u_n)$.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Norms</th>
</tr>
</thead>
<tbody>
<tr>
<td>IX. $|A|_F = (\sigma_1^2 + \cdots + \sigma_r^2)^{1/2}$ (Frobenius norm).</td>
</tr>
<tr>
<td>X. $|A|_2 = \sigma_1$ where $\sigma_1$ is the largest singular value of $A$.</td>
</tr>
</tbody>
</table>

Properties relating the SVD to eigenvalues, whose proofs are given in Appendix V, were used in the design of our special purpose linear least squares subroutine (see Appendix VII).

Application of the SVD to the Least Squares Problem

The least squares problem under consideration can be formulated as follows:

Given an overdetermined system, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, $m > n$,

$$Ax = b,$$
find $x \in \mathbb{R}^n$ such that

$$\|Ax - b\|_2$$

is minimized.

By the SVD we know that $A = U\sum V^t$. Using the norm-invariant properties of orthogonal matrices, we get

$$\|Ax - b\|_2 = \|U\sum V^tx - b\|_2 = \|U(\sum V^tx - U^tb)\|_2 = \|\sum V^tx - U^tb\|_2.$$ 

By setting $c = U^tb$, we need to find $y = V^tx \in \mathbb{R}^n$ such that

$$\|\sum y - c\|_2$$

is minimized.

This is easier to do since

$$\sum_{i=1}^{n}(\sigma_i y_i - c_i)^2 \text{ with singular values } \sigma_i$$

and we compute

$$y_i = \begin{cases} c_i/\sigma_i & \sigma_i > \varepsilon \\ \text{arbitrary} & \sigma_i \leq \varepsilon \end{cases}, 1 \leq i \leq n.$$ 

Thus, the desired least squares solution is

$$x = Vy.$$ 

Observe that if $A$ has full rank then, $x$ is unique. Otherwise, there are infinitely many least squares solutions. Also, the complete vector $c$ does not need to be computed. Only the columns of $U$ that correspond to the non-zero singular values are needed, thus less computational operations are required. The SVD provides a high resolution least
squares solution to an $m \times n$ overdetermined system of linear equations. In the next section, we illustrate the customization of the SVD to our least squares estimation problem.

The Least Squares Solution for an $m \times 2$ Matrix

For the least squares estimation problem of the two physiological parameters $F$ and $\tau$, the corresponding overdetermined linear system of equations has dimension $m \times 2$, where $m$ is the number of data points available for fitting, namely, $m$ observations $(s_1, t_1), (s_2, t_2), \ldots, (s_m, t_m)$, consisting of split or finish times in short races and where at least two distances $s_i, s_j$ are distinct.

We present a slightly more general set up and allow for weighted linear least squares (though it turned out that this added capability was not needed in our work). Consider a general $m \times 2$ overdetermined system,

$$ar_i + bs_i = t_i, \quad i = 1, \ldots, m,$$

from which in matrix notation, we obtain

$$\begin{bmatrix} r_1 & s_1 \\ r_2 & s_2 \\ \vdots \\ r_m & s_m \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_m \end{bmatrix},$$

or,

$$Ax = b^*,$$

where
Evaluating $A^*A$, we get

$$A^*A = \begin{bmatrix} r_1 & r_2 & \cdots & r_m \\ r_2 & s_1 & & s_m \end{bmatrix} \begin{bmatrix} r_1 & s_1 \\ r_2 & s_2 \\ \vdots & \vdots \\ r_m & s_m \end{bmatrix} = \begin{bmatrix} \sum r^2 & \sum rs \\ \sum rs & \sum s^2 \end{bmatrix}_{2 \times 2},$$

a $2 \times 2$ matrix. Next, we find its eigenvalues $\lambda_k$'s as the two roots of the quadratic equation,

$$\det(A^*A - \lambda I) = \left| \begin{array}{cc} \sum r^2 - \lambda & \sum rs \\ \sum rs & \sum s^2 - \lambda \end{array} \right| = 0.$$

We obtain the singular values $\sigma_1$ and $\sigma_2$ by ordering $\sqrt{\lambda_k}$ in decreasing order:

$$\sigma_k^2 = \lambda_k = \frac{1}{2} \left( \sum rs + \sum ss \right) \pm \sqrt{c},$$

where

$$c = \left( \sum rs - \sum ss \right)^2 + 4 \left( \sum rs \right)^2.$$

Next we orthonormalize the corresponding eigenvectors $V_1$ and $V_2$, and obtain

$$V = [V_1, V_2] \in \mathbb{R}^{2 \times 2}.$$

Let

$$V_1 = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},$$

where
\[ v_1 \left( \sum r^2 - \lambda \right) + v_2 \left( \sum rs \right) = 0. \]

Set \( v_2 = c \) and solve for \( v_1 \) choosing \( c \) of unit length.

Thus,

\[ V_2 = \begin{bmatrix} -v_2 \\ v_1 \end{bmatrix}. \]

Define \( u_i = \frac{1}{\sqrt{\sigma_i}} Av_i, \ i = (1, \ldots, m) \). Hence,

\[ U = [u_1, u_2, \ldots, u_m] \in \mathbb{R}^{m \times m}. \]

Finally, we obtain

\[ U \sum V' = A_{m \times 2}. \]

We only need to solve the quadratic

\[ \|Ax - b^*\|_2 = \|\sum Z - U'b^*\|_2, \]

where \( Z = [z_1, z_2] \),

\[ z_1 = \frac{U'_ib^*}{\sigma_i}, \ z_2 = \frac{U'_ib^*}{\sigma_2}, \]

To find \( U_1 \) and \( U_2 \) we use \( U_i = \frac{AV_i}{\sigma_i}, \ i = 1,2 \).

For \( U_1 \) we get

\[ U_1 = \frac{1}{\sigma_1} \begin{bmatrix} r_1 & s_1 \\ \vdots & \vdots \\ r_m & s_m \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \frac{1}{\sigma_1} \begin{bmatrix} r_1v_1 + s_1v_2 \\ r_2v_1 + s_2v_2 \\ \vdots \\ r_mv_1 + s_mv_2 \end{bmatrix}, \]

thus,
\[ z_1 = \frac{1}{\sigma_1^2} \sum_{i=1}^{m} (r_i v_1 + s_i v_2) t_i. \]

Similarly for \( z_2 \)

\[ z_2 = \frac{1}{\sigma_2^2} \sum_{i=1}^{m} (-r_i v_2 + s_i v_1) t_i. \]

Hence,

\[
\begin{bmatrix}
  a \\
  b
\end{bmatrix} = [V_1, V_2] \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.
\]

Now if \( r_i = 1 \) for all \( i = 1, \ldots, n \) then the problem is equivalent to finding the regression line \( t = a + bs \) through the \( m \) data points \( (s_i, t_i) \). This is precisely our problem.

We want design equations

\[ a + bs_i = t_i, \quad 1 \leq i \leq m \]

to estimate \( a \) and \( b \). We obtain our desired parameters by letting \( r = a \) and \( F = \frac{1}{ab}. \)

To validate our model, we took the same 1972 World record times used by Keller to estimate the physiological parameters \( F \) and \( r \). Our estimates are

\[ F = 13.705, \quad r = 0.784, \]

with respective differences to Keller's estimates by 1.505 ms\(^{-2}\) and 0.108s, thus showing agreement within the context of the model. These slight differences may be due to the fact that Keller used different criteria to estimate the constants. He used the exact optimal solution and minimized the sum of the squares of the relative errors. Since we are not looking at long distance races, we used our linear approximation and minimized the sum of the squares of the deviations.
CHAPTER FOUR

APPLICATION OF OUR MODEL

In this chapter we estimate the two physiological parameters $F$ and $\tau$ used to characterize short races. We apply the model described in Chapter Three, using split times taken from short races run by both Olympic athletes and amateur high school students.

1987 World Championship Data

The data for the first estimations of the physiological constants were taken from the controversial World Championships in Rome (1987). Ben Johnson's use of steroids stripped him of the world record banning him from running still to this day [1]. Unlike Keller, who utilized the finish times produced by 8 different runners, we took two sets of data each produced by a different runner. The 100m split times run by Ben Johnson and Carl Lewis provided two ideal sets of data points for our program. The 10m interval times are illustrated in Table 4 followed by the estimation results $F$ and $\tau$ in Table 5.
TABLE 4: ROME 1987 100M SPLIT TIMES IN SECONDS

<table>
<thead>
<tr>
<th>Meters</th>
<th>Johnson</th>
<th>Lewis</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>1.84</td>
<td>1.94</td>
</tr>
<tr>
<td>20</td>
<td>2.86</td>
<td>2.96</td>
</tr>
<tr>
<td>30</td>
<td>3.8</td>
<td>3.91</td>
</tr>
<tr>
<td>40</td>
<td>4.67</td>
<td>4.78</td>
</tr>
<tr>
<td>50</td>
<td>5.53</td>
<td>5.64</td>
</tr>
<tr>
<td>60</td>
<td>6.38</td>
<td>6.5</td>
</tr>
<tr>
<td>70</td>
<td>7.23</td>
<td>7.36</td>
</tr>
<tr>
<td>80</td>
<td>8.1</td>
<td>8.22</td>
</tr>
<tr>
<td>90</td>
<td>8.96</td>
<td>9.07</td>
</tr>
<tr>
<td>100</td>
<td>9.83</td>
<td>9.93</td>
</tr>
</tbody>
</table>

TABLE 5: ESTIMATION RESULTS FROM THE 1987 WORLD CHAMPIONSHIP DATA

<table>
<thead>
<tr>
<th>Constant</th>
<th>Johnson</th>
<th>Lewis</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F$</td>
<td>10.383</td>
<td>9.468</td>
</tr>
<tr>
<td>$\tau$</td>
<td>1.099</td>
<td>1.203</td>
</tr>
</tbody>
</table>

When comparing the results for the two runners, we observe that Johnson ran the 100m faster than Lewis by a mere 0.10 second and that his estimated maximum propulsive force $F$ is $10.838 \text{ ms}^{-2}(\text{kg})^{-1}$ versus $9.468 \text{ ms}^{-2}(\text{kg})^{-1}$ for Lewis. This estimation appears to represent Johnson's higher muscular strength. Note also Johnson's estimated resistive constant $\tau$ is lower than Lewis', although the two parameters are quite close.
Student Data

To further investigate the estimation procedure for the two physiological parameters $F$ and $\tau$, we took data consisting of split times of high school students who ran short distances for us. The track athletes of La Canada High School from the San Fernando Valley in Southern California ran 100m and 200m sprints. Two females and 4 males participated in the data collection. Since most of these athletes were long distance runners, we were curious to see how they would perform on sprints. The physical characteristics of the students are described in Table 6.

<table>
<thead>
<tr>
<th>TABLE 6: PHYSICAL CHARACTERISTICS OF STUDENTS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Females</td>
</tr>
<tr>
<td>Name</td>
</tr>
<tr>
<td>------------------</td>
</tr>
<tr>
<td>Amy Foss</td>
</tr>
<tr>
<td>Reema Khetan</td>
</tr>
<tr>
<td>Males</td>
</tr>
<tr>
<td>Michael Cane</td>
</tr>
<tr>
<td>Tony Hanes</td>
</tr>
<tr>
<td>Matt Moore</td>
</tr>
<tr>
<td>Andy Smith</td>
</tr>
</tbody>
</table>

We used the following procedure to obtain the split times. For the 100m, we marked each 10m interval with orange traffic cones. As the student ran the distance four people in a vehicle were used to collect the data, two timekeepers operating the stopwatches, one record keeper and a driver. The vehicle traveled on a track parallel to the runner so the two timekeepers could convey to the recorder the time the student reached each 10m interval marked. For greater accuracy the timekeepers alternated in
obtaining and conveying the 10 split times. Similarly, we collected the split times for the 200m, but 20m intervals were marked instead of the previous 10m.

Since errors in measurement are inevitable, we decided to have the runners run the distances twice and take the average of the two. In order to try to obtain the best records, the runners were allowed to rest in-between runs and the data collection was conducted in the morning, before the students ate breakfast. Fortunately, the weather was ideal and the runners were not subjected to adverse conditions. Appendix VI contains the tables of the runners' 100m and 200m split times.

After collecting our data, using the average split times, we applied our computational model to estimate the constants $F$ and $\tau$. We ran the program twice for each student and once for their coach, with each try consisting of 10 data points. Table 7 below illustrates our estimation results $F$ and $\tau$ for each runner.

<table>
<thead>
<tr>
<th>TABLE 7: ESTIMATION RESULTS FROM STUDENT DATA</th>
</tr>
</thead>
<tbody>
<tr>
<td>Meters</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>100m</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>200m</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>

When comparing the results in the above table, as we had anticipated, the estimations varied widely from runner to runner. Observing the values for the constant $F$, we note the estimations for the maximum propulsive force ranged from 6.683.
\[ ms^{-2}(kg)^{-1} (\text{Matt 100m}) \text{ to } 16.174 \, ms^{-2}(kg)^{-1} (\text{Reema 200m}) \] where the corresponding estimations for the resistive constant \( r \) are 1.058s and 0.333s respectively. Perhaps these variations are due to the runners' different levels of physical stamina. It is interesting to see that the outlying higher and lower values for both constants were estimated for Reema (100m and 200m) and Matt (100m) the only two weighing under 100lbs. Physically Matt and Reema portray the typical lanky long distance runner so it seems logical our estimations resulted with these extreme values. Reema was the slowest runner of the two females while Matt was the slowest runner of the young males. On the other hand, 160lbs. Andy was the fastest of the group beating the times for both the 100m and 200m. Moreover, when comparing the student data results with the World Championship data results, we see that Mike’s 100m estimation for \( F \) was within Lewis estimation by 0.016 \( ms^{-2}(kg)^{-1} \).

Looking at Table 7 and analyzing the two pairs of estimations for all five students individually, we see that the estimation results taken from the 100m and 200m are considerably close for three out of the five students. Table 8 illustrates these differences.

**Table 8: Individual Differences in Estimations**

|       | \( |F_{100m} - F_{200m}| \) | \( |r_{100m} - r_{200m}| \) |
|-------|-----------------|-----------------|
| Amy   | 0.182           | 0.019           |
| Reema | 0.895           | 0.035           |
| Mike  | 2.726           | 0.168           |
| Matt  | 6.492           | 0.549           |
| Andy  | 0.214           | 0.061           |

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We found the two estimations of the maximum propulsive force individually for Amy differed by $0.182 \text{ms}^{-2}(\text{kg})$, $0.214 \text{ms}^{-2}(\text{kg})$ for Andy, and $0.895 \text{ms}^{-2}(\text{kg})$ for Reema with respective differences in the resistive constant $\tau$ of $0.019s$, $0.061s$, and $0.035s$. This closeness in the estimations of the individual parameters may lead into further research that these parameters are specific to each person and may depend on other physiological factors such as muscle development, stamina and lung capacity. Perhaps we need to take Keller's ideas a step further and develop mathematical models of running which include more individual characteristics.

Conclusion

By using the ideas behind the Hill-Keller model, we have constructed a model which merged mathematical theory, the use of an effective numerical method based on the SVD, (interesting from both the theoretical and applied point of view), and physiology. The Hill-Keller model, which was one of the pioneering models that mathematically investigated running records, contributed to the emerging field of mathematical physiology which is currently undergoing increasing in-depth research. To illustrate, in 1998 the Association of American Publishers bestowed J. Keener's and J. Sneyed's book "Mathematical Physiology" the best new title in mathematics [4]. On a personal level, the work leading to this thesis turned out to be rewarding, especially in the response we received from the high school athletes. We were surprised, but pleased, to see that these students showed great interest in our investigation and wanted feedback from our model.
APPENDIX I

THE DERIVATION OF KELLER'S SOLUTION

The following is the derivation of Keller's solution outlined earlier in Chapter Two. In each of the three cases we describe both the distance at time \( t \) and the distance at the optimal time \( T \), i.e \( d(t) \) and \( D(T) \).

The Three Cases of the Hill-Keller Model

Case 1: \( T \leq T_c \) (Short Races)

Find \( d(t) = \int_0^t v(s) \, ds \) where \( v(s) = F \tau (1 - e^{-st}) \, ds \)

\[
d(t) = \int_0^t F \tau (1 - e^{-st}) \, ds
\]

\[
= F \tau \left[ s \left( e^{-st} \right) \bigg|_0^t - \int_0^t -e^{-st} \, ds \right]
\]

\[
= F \tau \left[ s - \frac{e^{-st}}{s} \bigg|_0^t \right]
\]

\[
= F \tau \left[ t + \tau e^{-st} - 0 - \tau e^{-0} \right]
\]

\[
= F \tau \left[ t + \tau e^{-st} - \tau \right]
\]

\[
d(t) = F \tau^2 \left[ t/\tau + e^{-st} - 1 \right], \quad 0 \leq t \leq T_c.
\]

Calculating \( D \) we get
\[ D = d(T) = F \tau^2 \left[ \frac{T}{\tau + e^{-T/\tau}} - 1 \right], \quad T \leq T_e. \]

For Case 2 and 3 we can determine \( \nu(s) \) similarly by just altering some constants.

Consider

\[ \nu(s) = \left\{ \sigma \tau + \left( \frac{\tau^2}{\lambda^2} - \sigma \tau \right) e^{-2/\tau(x-t_s)} \right\}^{1/2} \]

Let \( b = \frac{c^2}{\sigma \tau} \) where

\[ c = \begin{cases} 
F \tau (1 - e^{-T_e/\tau}) & \text{for case 2} \\
\frac{\tau}{\lambda} & \text{for case 3}
\end{cases} \]

Hence,

\[ \nu(s) = (\sigma \tau)^{1/2} \left\{ 1 + (b - 1) e^{-2/\tau(x-t_s)} \right\}^{1/2}. \]

Now let

\[(A1.1) \quad \Psi(s) = 1 + (b - 1) e^{-2/\tau(x-t_s)}. \]

Thus we obtain

\[ \nu(s) = (\sigma \tau)^{1/2} \left\{ \Psi(s) \right\}^{1/2}. \]

Next we claim

\[ \int \nu(s) ds = \tau (\sigma \tau)^{1/2} \left\{ \tanh^{-1} \left[ \Psi(s) \right]^{1/2} - \left[ \Psi(s) \right]^{1/2} \right\} \equiv V(s). \]

We check the claim and let

\[ \phi(s) = \tanh^{-1} \left[ \Psi(s) \right]^{1/2} - \left[ \Psi(s) \right]^{1/2}, \]

\[ \frac{dV}{ds} = \tau (\sigma \tau)^{1/2} \frac{d\phi}{ds}. \]

Thus,

\[ \frac{d\phi}{ds} = \frac{d}{ds} \left[ \tanh^{-1} \left[ \Psi(s) \right]^{1/2} - \left[ \Psi(s) \right]^{1/2} \right] \]

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\[ \frac{d\phi}{ds} = \frac{d}{ds} \tanh^{-1} \left[ \Psi(s) \right]^{1/2} - \frac{d}{ds} \left[ \Psi(s) \right]^{1/2}. \]

Hence

\[ \frac{d}{ds} \tanh^{-1} \left[ \Psi(s) \right]^{1/2} = \frac{\frac{d}{ds} \left[ \Psi(s) \right]^{1/2}}{1 - \Psi(s)} \]

\[ \frac{d}{ds} \left[ \Psi(s) \right]^{1/2} = \frac{1}{2} \left[ \Psi(s) \right]^{-1/2} \Psi'(s) \]

\[ \frac{d\phi}{ds} = \frac{\frac{1}{2} \left[ \Psi(s) \right]^{-1/2} \Psi'(s) - \frac{1}{2} \left[ \Psi(s) \right]^{-1/2} \Psi''(s)}{1 - \Psi(s)} \]

\[ \frac{d\phi}{ds} = \frac{\frac{1}{2} \Psi(s)^{1/2} \Psi'(s)}{1 - \Psi(s)}. \]

Recall

\[ \Psi(s) = 1 + (b - 1)e^{-2\tau(s-s_1)}, \]

Hence,

\[ \Psi'(s) = -2\tau(b - 1)e^{-2\tau(s-s_1)}, \]

and

\[ -\Psi(s) = (1 - b)e^{-2\tau(s-s_1)}. \]

Therefore,

\[ \frac{d\nu}{ds} = \sigma \tau ^{1/2} \left\{ 1 + (b - 1)e^{-2\tau(s-s_1)} \right\}^{1/2} - \frac{2}{\tau} (b - 1)^{2 - 2\tau(s-s_1)} \]

\[ \frac{d\nu}{ds} = \sigma \tau ^{1/2} \left\{ 1 + (b - 1)e^{-2\tau(s-s_1)} \right\}^{1/2} = \nu(s). \]

Case 3: \( T \geq T^* \) (Long distances)
Find $d(t) = \int_0^t F\tau(1-e^{-\tau t}) + \int_{t_1}^t \frac{\tau}{\lambda} + \int_{t_2}^t v(t) dt$.

We will calculate each integrand separately.

For the first integrand we get

$$\int_0^t F\tau(1-e^{-\tau t})dt = F\tau^2 \left[ \frac{t}{\tau} + e^{-\tau t} - 1 \right]_0^t$$
$$= F\tau^2 \left[ \frac{t}{\tau} + e^{-\tau t} - 1 \right] - F\tau^2$$
$$= F\tau^2 \left( \frac{1}{\tau} + e^{-\tau t} - 1 - 1 \right)$$
$$= F\tau^2 \left( \frac{1}{\tau} + e^{-\tau t} - 2 \right)$$

Now the second integrand yields

$$\int_{t_1}^{t_2} \frac{\tau}{\lambda} dt = \left. \frac{\tau}{\lambda} t \right|_{t_1}^{t_2} = \frac{\tau}{\lambda} t_2 - \frac{\tau}{\lambda} t_1$$
$$= \frac{\tau}{\lambda} (t_2 - t_1)$$

Finally for the last integrand

$$\int_{t_2}^t v(t) dt = \tau(\sigma t)^{1/2} \left\{ \tanh^{-1}[\Psi(s)]^{1/2} - [\Psi(s)]^{1/2} \right\}$$

recall in (A1.1)

$$\Psi(s) = 1 + (b - 1)e^{-2\sigma(t-t_2)},$$

and for case 3,

$$b = \frac{\tau^2}{\lambda^2 \sigma t} = \frac{\tau}{\lambda^2 \sigma}.$$ 

Hence
\[ \int \nu(t) dt = \tau(\sigma t)^{1/2} \left\{ \tanh^{-1} \left\{ 1 + \left( \frac{\tau}{\lambda^2 \sigma} - 1 \right) e^{-2/\tau(t-t_1)} \right\} \right\}^{1/2} \]

\[ = \tau(\sigma t)^{1/2} \left\{ \tanh^{-1} \left\{ 1 + \left( \frac{\tau}{\lambda^2 \sigma} - 1 \right) e^{-2/\tau(t-t_1)} \right\} \right\}^{1/2} \]

\[ = \tau(\sigma t)^{1/2} \left\{ \tanh^{-1} \left\{ 1 + \left( \frac{\tau}{\lambda^2 \sigma} - 1 \right) e^{-2/\tau(t-t_1)} \right\} \right\}^{1/2} \]

\[ = \tau(\sigma t)^{1/2} \left\{ \tanh^{-1} \left\{ 1 + \left( \frac{\tau}{\lambda^2 \sigma} - 1 \right) e^{-2/\tau(t-t_1)} \right\} \right\}^{1/2} \]

\[ = \tau(\sigma t)^{1/2} \left\{ \tanh^{-1} \left\{ 1 + \left( \frac{\tau}{\lambda^2 \sigma} - 1 \right) e^{-2/\tau(t-t_1)} \right\} \right\}^{1/2} \]

Thus
\[ d(t) = F \tau^2 \left[ \frac{t_1}{\tau} + e^{-\frac{t}{\tau}} - 1 \right] + \frac{\tau}{\lambda} (t_2 - t_1) + \]
\[ \tau(\sigma\tau)^{1/2} \left[ \tanh^{-1} \left\{ 1 + \left( \frac{\tau}{\lambda^2} - 1 \right) e^{-2/\tau(t-t_2)} \right\}^{1/2} - \left\{ 1 + \left( \frac{\tau}{\lambda^2} - 1 \right) e^{-2/\tau(t-t_2)} \right\}^{1/2} \right] \]
\[ - \tanh^{-1} \left\{ \frac{\tau}{\lambda^2} \right\}^{1/2} + \left\{ \frac{\tau}{\lambda^2} \right\}^{1/2} \] for \( t_2 < t \leq T \)

NOTE: Keller's typographical error is a sign error in the last term in the above formula for \( d(t) \).

We easily obtain \( D \) for this case by substituting \( t \) with \( T \) in above equation \( d(t) \).

Case 2: \( T_c < T \leq T^* \), Long distances close to critical time

Case 2 can be formally obtained by taking the formulas of Case 3 with \( t_1 = t_2 = T_c \) and

\[ \lambda = \frac{1}{F(1 - e^{-\frac{t}{\tau}})} \]

We want to find

\[ d(t) = \int_0^{T_c} \left( 1 - e^{-ut} \right) dt + \int_{T_c}^t \left( \sigma \tau + (c^2 - \sigma \tau) e^{-2/\tau(t-t_c)} \right)^{1/2} \]
\[ = F \tau^2 \left\{ \frac{T_c}{\tau} + e^{-\frac{T_c}{\tau}} - 1 \right\} + \int_{T_c}^t v(t) dt. \]

Recall in case 2

\[ b = \frac{F^2 \tau^2 \left( 1 - e^{-\frac{T_c}{\tau}} \right)^2}{\sigma \tau} \]

Hence

\[ \int_{T_c}^t v(t) dt = \tau(\sigma\tau)^{1/2} \left[ \tanh^{-1} \left\{ 1 + (b - 1)e^{-2/\tau(t-t_c)} \right\}^{1/2} \right. \]
\[ - \left\{ 1 + (b - 1)e^{-2/\tau(t-t_c)} \right\}^{1/2} - \tanh^{-1} \left[ b^{1/2} + b^{1/2} \right] \]
Thus

\[ d(t) = F \frac{T_e}{\tau} \left[ \frac{T_e}{\tau} + e^{-\tau_e/\tau} - 1 \right] + \]

\[ \tau(\sigma \tau)^{1/2} \left[ \tanh^{-1} \left\{ 1 + \left( \frac{F^2 \tau}{\sigma} \left( 1 - e^{-\tau_e/\tau} \right)^2 - 1 \right) e^{-2\tau(t-\tau_e)} \right\} \right]^{1/2} \]

\[ - \left\{ 1 + \left( \frac{F^2 \tau}{\sigma} \left( 1 - e^{-\tau_e/\tau} \right)^2 - 1 \right) e^{-2\tau(t-\tau_e)} \right\}^{1/2} \]

\[ - \tanh^{-1} \left( \frac{\tau}{\sigma} \right)^{1/2} F(1 - e^{-\tau_e/\tau}) + \left( \frac{\tau}{\sigma} \right)^{1/2} F(1 - e^{-\tau_e/\tau}) \]

for \( T_e \leq t \leq T \).

Again we can easily obtain \( D \) by substituting \( t \) with \( T \) in the above resulting equation \( d(t) \).
APPENDIX II

KELLER'S ESTIMATES

The two tables contained in this appendix compare Keller's estimates for the short and long distance races with the 1972 World record times.

TABLE A2.1: SHORT DISTANCE RACES

<table>
<thead>
<tr>
<th>Distance, ( D )</th>
<th>Record Time, ( T )</th>
<th>Theoretical Time, ( T_e )</th>
<th>% Error</th>
<th>Average Velocity, ( \frac{D}{T_e} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yards Meters</td>
<td>min:sec:dd</td>
<td>min:sec:dd</td>
<td></td>
<td></td>
</tr>
<tr>
<td>50 45.72</td>
<td>00:05.10</td>
<td>00:05.09</td>
<td>-0.2</td>
<td>8.982318</td>
</tr>
<tr>
<td>54.68 50</td>
<td>00:05.50</td>
<td>00:05.48</td>
<td>-0.4</td>
<td>9.124088</td>
</tr>
<tr>
<td>60 54.864</td>
<td>00:05.90</td>
<td>00:05.93</td>
<td>0.5</td>
<td>9.251939</td>
</tr>
<tr>
<td>65.616 60</td>
<td>00:06.50</td>
<td>00:06.40</td>
<td>-1.5</td>
<td>9.375</td>
</tr>
<tr>
<td>100 91.44</td>
<td>00:09.10</td>
<td>00:09.29</td>
<td>2.1</td>
<td>9.842842</td>
</tr>
<tr>
<td>109.361 100</td>
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<td>00:10.07</td>
<td>1.7</td>
<td>9.930487</td>
</tr>
<tr>
<td>218.722 200</td>
<td>00:19.50</td>
<td>00:19.25</td>
<td>-1.3</td>
<td>10.38691</td>
</tr>
<tr>
<td>220 201.168</td>
<td>00:19.50</td>
<td>00:19.36</td>
<td>-0.7</td>
<td>10.39091</td>
</tr>
<tr>
<td>Distance, $D$</td>
<td>Record Time, $T$</td>
<td>Theoretical Time, $T_c$</td>
<td>% Error</td>
<td>$t_1$</td>
</tr>
<tr>
<td>-------------</td>
<td>-----------------</td>
<td>------------------------</td>
<td>---------</td>
<td>-------</td>
</tr>
<tr>
<td>Yards</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>437.445</td>
<td>400</td>
<td>00:44.50</td>
<td>00:43.27</td>
<td>-2.8</td>
</tr>
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<td>402.336</td>
<td>00:44.90</td>
<td>00:43.62</td>
<td>-2.9</td>
</tr>
<tr>
<td>874.89</td>
<td>800</td>
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<td>01:45.95</td>
<td>1.6</td>
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<tr>
<td>880</td>
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<td>01:44.90</td>
<td>01:46.69</td>
<td>1.7</td>
</tr>
<tr>
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<td>1000</td>
<td>02:16.20</td>
<td>02:18.16</td>
<td>1.4</td>
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<td>1500</td>
<td>03:33.10</td>
<td>03:39.44</td>
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<td>Miles</td>
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<td></td>
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<td>1609.3</td>
<td>03:51.10</td>
<td>03:57.28</td>
<td>2.7</td>
</tr>
<tr>
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<td>2000</td>
<td>04:56.20</td>
<td>05:01.14</td>
<td>1.7</td>
</tr>
<tr>
<td>1.864</td>
<td>3000</td>
<td>07:39.60</td>
<td>07:44.96</td>
<td>1.2</td>
</tr>
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<td>3218.6</td>
<td>08:19.80</td>
<td>08:20.82</td>
<td>0.2</td>
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<td>12:50.40</td>
<td>12:44.89</td>
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<td>5000</td>
<td>13:16.60</td>
<td>13:13.11</td>
<td>-0.4</td>
</tr>
<tr>
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<td>9655.8</td>
<td>26:47.00</td>
<td>25:57.62</td>
<td>-3.1</td>
</tr>
<tr>
<td>6.213</td>
<td>10000</td>
<td>27:39.40</td>
<td>26:54.10</td>
<td>-2.7</td>
</tr>
</tbody>
</table>
APPENDIX III

PROOF OF THE SINGULAR VALUE DECOMPOSITION

Singular Value Decomposition Theorem: Let \( A \) be a real \( m \times n \) matrix, then

\[
A_{m \times n} = U_{m \times m} \sum_{m \times n} V_{n \times n}^t.
\]

\( U, V \) are orthogonal matrices and

\[
\sum = \begin{pmatrix} \sum_{i} & 0 \\ 0 & \sum_{i} \end{pmatrix}_{m \times n}, \text{ where } \sum_{i} \text{ is a nonsingular diagonal matrix.}
\]

Proof: Let \( A \) be any real \( m \times n \) matrix.

Claim (1) \( A^t A \) and \( AA^t \) are Hermitian. Since \( A \) is real then \( A^t A \) is real and we need only to show \( A^t A = (A^t A)^t \).

Proof of Claim (1): Let \( a_{ir} \) be the \( i \)th row of \( A^t \) and \( a_{jr} \) be the \( j \)th col. of \( A \) so

\[
a_{ir} \circ a_{jr} = a_{ij}.
\]

Similarly if we let \( a_{jr} \) be the \( j \)th row of \( A^t \) we get\( a_{jr} \circ a_{ir} = a_{ij} \).

For \( a_{jr} \circ a_{ir} = a_{jr} \circ a_{ir} \) we get

\[
a_{ij} = a_{ij} \quad \forall i, j \quad i \neq j
\]

If \( i = j \) then

\[
a_{ii} = a_{jj}
\]

Thus \( A^t A = (A^t A)^t \) i.e. \( A^t A \) is symmetric. Hence \( A^t A \) and \( (A^t A)^t \) are Hermitian.
Since $A'A$ and $AA'$ are Hermitian, then they have a complete set of orthonormal eigenvectors. The columns of $U$ are the eigenvectors of $AA'$ and the columns of $V$ are the eigenvectors of $A'A$. Thus $U_{m \times m}$ and $V_{n \times n}$ are unitary matrices.

By the Spectral Theorem, $A'A$ is orthogonally diagonalizable and has real eigenvalues. So $A'A$ and $AA'$ are positive, semi-definite matrices, whose nonzero eigenvalues are positive and equal. In particular, the positive square roots of these eigenvalues are defined as the singular values of $A$.

$$\sqrt{\lambda_j} = \sigma_j, \quad j = 1, \ldots, n.$$ 

We can now order the singular values in non-increasing order,

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0 \quad \text{where} \quad \sigma_{r+1} = \sigma_{r+2} = \cdots = \sigma_n = 0.$$ 

Next we consider the corresponding orthonormal eigenvectors of $A'A$,

$$v_1, \ldots, v_r, \quad v_{r+1}, \ldots, v_n,$$

and let

$$V = [v_1, v_2, \ldots, v_r, \quad v_{r+1}, \ldots, v_n],$$

form an orthonormal basis of $\mathbb{R}^n$.

We may assume the 1st row cols. of $V$ are eigenvectors associated with the eigenvectors of $A'A$, i.e., $(A'A)v_i = \sigma_i^2 v_i \quad \forall i = 1, 2, \ldots, r$. The $r$ eigenvectors by definition are the right singular vectors of $A$. The remaining $n - r$ columns of $V$ are the eigenvectors of $A'A$ corresponding to its zero eigenvalue. Since columns of $V$ are orthonormal it is unitary.

Now we construct $U$. Define

$$u_i = \left( \frac{1}{\sigma_i} \right) A v_i \quad \forall i = 1, 2, \ldots, r.$$
Consider any two vectors $u_i \neq u_j$ from this set. Then

$$\langle u_i, u_j \rangle = \left(\frac{1}{\sigma_i}\right)\left(\frac{1}{\sigma_j}\right)\langle Av_i, Av_j \rangle = \left(\frac{1}{\sigma_i}\right)\left(\frac{1}{\sigma_j}\right)\langle Av_i, A^t Av_j \rangle = \left(\frac{1}{\sigma_i}\right)\left(\frac{1}{\sigma_j}\right)\langle v_i, \sigma^2 v_j \rangle = \left(\frac{\sigma_i}{\sigma_i}\right)\langle v_i, v_j \rangle = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

For $i \neq j$, the $v_i$'s are orthogonal so $\langle v_i, v_j \rangle = 0$. For $i = j$, we have $\frac{\lambda_i}{\sigma_i^2} = 1$, by definition of the singular values and $\langle v_i, v_j \rangle = 1$ because $v_i$ is unit. Thus the set $\{u_1, u_2, \ldots, u_r\}$ is orthonormal. Let the $r$-orthonormal vectors compose the first $r$ columns of $U$ and the remaining $m - r$ columns of $U$ are orthonormal vectors that form a basis of $\mathbb{R}^m$. Hence,

$$U = [u_1, \ldots, u_r, u_{r+1}, \ldots, u_m].$$

We need to show $U^tAV = \left(\sum_{i=1}^r 0 \begin{smallmatrix} \sigma_i \\ 0 \\ \vdots \\ 0 \end{smallmatrix} \begin{smallmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{smallmatrix} \begin{smallmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{smallmatrix} \right)$ where $\sum_{i=1}^r = \text{diag}(\sigma_1, \ldots, \sigma_r)$. Let

$$S = \left(\sum_{i=1}^r 0 \begin{smallmatrix} \sigma_i \\ 0 \\ \vdots \\ 0 \end{smallmatrix} \begin{smallmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{smallmatrix} \begin{smallmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{smallmatrix} \right), \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0.$$

($S$ is composed of the singular values of $A$ and sufficient zeros to complete the diagonal)
\[ U'AV = U'A[v_1, \ldots, v_r, v_{r+1}, \ldots, v_n] \]
\[ = U'[Av_1, \ldots, Av_r, Av_{r+1}, \ldots, Av_n] \]
\[ = U'[\sigma_1 u_1, \ldots, \sigma_r u_r, 0, \ldots, 0] \]
\[ = [\sigma_1 U' u_1, \ldots, \sigma_r U' u_r, 0, \ldots, 0] \]
\[ = [\sigma_1 e_1, \ldots, \sigma_r e_r, 0, \ldots, 0] = S \]

Therefore

\[ U'AV = \begin{pmatrix} \sum_1 & 0 \\ 0 & 0 \end{pmatrix} \text{ where } \sum_1 = \text{diag}(\sigma_1, \ldots, \sigma_r). \]
APPENDIX IV

THE CONDITION NUMBER OF A MATRIX

The condition number of a matrix \( A \) denoted \( \kappa(A) \) is used to quantify the stability of the matrix. It measures how sensitive a problem is to perturbations. A large condition number increases the number of iterations required and limits accuracy to which a solution is obtained. In particular, the matrix \( A \) is said to be ill-conditioned if \( \kappa(A) \) is too large, in some subjective sense, within the context of the problem at hand and the desired accuracy.

Let us consider the problem

\[
Ax = b
\]

with the corresponding perturbed problem

\[
A\tilde{x} = b + \Delta b
\]

The condition number of \( A \) \( \kappa(A) \), is the largest "amplification factor", \( c \) such that

\[
\frac{\|Ax\|}{\|x\|} = c \frac{\|\Delta b\|}{\|b\|}
\]

and

\[
\frac{\|\Delta x\|}{\|x\|} = c \frac{\|\Delta A\|}{\|A\|} \text{ where } c' = c.
\]
Theorem 1: Given the above conditions, suppose $A$ is invertible and symmetric, i.e. $A^{-1}$ exists and $A^T = A$. Let $\lambda_i \in \mathbb{R}$ such that $0 \leq |\lambda_1| \leq |\lambda_2| \leq \ldots \leq |\lambda_n|$, then

$$\kappa(A) = \frac{|\lambda_n|}{|\lambda_1|},$$

where $|\lambda_n| = \max |\lambda_i|$ and $|\lambda_1| = \min |\lambda_i|$.

Remark 1: From the assumption the perturbed problem is

$$A\tilde{x} = b + \Delta b,$$

so the maximum amplification factor, $\kappa(A) = c$, occurs when $b \neq 0$ is pointing toward the eigenvector $x_n$ corresponding to the largest eigenvalue $\lambda_n$ of $A$ and when the perturbation $\Delta b$ is pointing toward the eigenvector $x_1$, corresponding to the smallest eigenvalue, $\lambda_1$.

Proof: By the diagonalization theorem we can factor $A$ as

$$A = SDS^{-1}$$

where $D = \text{diag}(\lambda_1, \ldots, \lambda_n)$ and the jth column of $S$ is any eigenvector corresponding to the eigenvalue, $\lambda_j$. Then

$$c = A^{-1} = \left[SDS^{-1}\right] = SD^{-1}S^{-1}$$

and

$$D^{-1} = \text{diag}\left(\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}\right)$$

so

$$cS = SD^{-1}$$

where the jth column of $S$ is an eigenvector of $A$ corresponding to $\lambda_j$ (i.e. $S = x_j = \text{e-vector of A corresponding to } \lambda_j$).

Thus we obtain
\[ cx_j = \frac{1}{\lambda_j} x_j \]

Among all possible \( b \neq 0 \) and \( \Delta b \), we want to find:

(i) \( \Delta b \) so that corresponding \( \Delta x \) is amplified as much as possible, i.e. find \( \Delta b \) and largest \( c_1 \) such that \( \|\Delta x\| = c_1 \|\Delta b\| \)

(ii) \( b \) so that corresponding \( x \) is as little amplified as possible, i.e. find \( b \) and smallest \( c_2 \) such that \( \|x\| = c_2 \|b\| \).

For arbitrary \( b \neq 0, \Delta b \), we will then have

\[
\|\Delta x\| \leq c_1 \|\Delta b\| \quad \text{and} \quad \|x\| \geq c_2 \|b\| \Rightarrow \frac{1}{\|x\|} \leq \frac{1}{c_2} \cdot \frac{1}{\|b\|}
\]

Once \( c_1, c_2 \) are found, we can pick \( c = \frac{c_1}{c_2} \).

Let \( x_1, x_2, \ldots, x_n \) be orthonormal eigenvectors and recall that \( A \) is symmetric. The eigenvalues of \( A^{-1} \) are

\[ \frac{1}{\lambda_j} \text{ with } 0 < \frac{1}{|\lambda_n|} \leq \ldots \leq \frac{1}{|\lambda_1|}. \]

For any \( v \in \mathbb{R}^n \),

\[ v = \beta_1 x_1 + \ldots + \beta_n x_n, \]

\[ A^{-1} v = \frac{1}{\lambda_1} \beta_1 x_1 + \frac{1}{\lambda_2} \beta_2 x_2 + \ldots + \frac{1}{\lambda_n} \beta_n x_n, \]

where \( \beta_i \in \mathbb{R} \), \( i = 1, \ldots, n \).

The largest amplification under \( A^{-1} \) occurs when \( v = \beta_1 x_1 \) with amplification \( \frac{1}{|\lambda_1|} \).

Thus
Similarly, the smallest amplification under $A^{-1}$ occurs when $v = \beta_n x_n$ with amplification $\frac{1}{|\lambda_n|}$. Thus

$$\|A^{-1}v\| = \frac{1}{|\lambda_n|} \|v\|.$$ 

Hence $c_1 = \frac{1}{|\lambda_1|}$ and $c_2 = \frac{1}{|\lambda_n|}$.

Therefore

$$c = \frac{c_1}{c_2} = \frac{|\lambda_n|}{|\lambda_1|}.$$ 

**Remark 2:** From the above we can define the condition number of a square matrix $A$ to be

$$\kappa(A) = \begin{cases} \sqrt{\frac{\lambda_{\text{max}}(A^T A)}{\lambda_{\text{min}}(A^T A)}} & \text{if } A \text{ is nonsingular} \\ \infty & \text{if } A \text{ is singular} \end{cases}.$$ 

**Remark 3:** In the worst case scenario $b$ is proportional to a "maximal" eigenvector and $\Delta b$ is proportional to a "minimal" eigenvector. In which case we have

$$\frac{\|\Delta x\|}{\|x\|} = \kappa(A) \frac{\|\Delta b\|}{\|b\|}.$$ 

Otherwise, in general we have

$$\frac{\|\Delta x\|}{\|x\|} \leq \kappa(A) \frac{\|\Delta b\|}{\|b\|}.$$ 

**Remark 4:** In our case, by using the SVD we have a relationship between the eigenvalues of $A^T A$ and the singular values of $A$ (i.e. $\sqrt{\lambda_i} = \sigma_i$).
In particular, after factoring the matrix $A$, we can easily calculate the condition number as the ratio comparing the largest singular value of $A$ to the smallest. That is

$$
\kappa(A) = \frac{\sigma_1}{\sigma_n}.
$$
APPENDIX V

THE SVD AND EIGENVALUE DECOMPOSITION

In this appendix we state and prove the Singular Value Decomposition properties used in the design of the subroutine listed in Appendix VII.

Suppose

\[ A = U \Sigma V^* \]

where \( U, V \) are orthogonal matrices, \( \Sigma = \begin{pmatrix} \sum_1 & 0 \\ 0 & 0 \end{pmatrix} \), where \( \sum_1 = \text{diag}(\sigma_1, \ldots, \sigma_r) \).

\( \sigma_1 \geq \cdots \geq \sigma_r > 0 \).

Claim 1: \( \sigma_1^2, \ldots, \sigma_r^2 \) are nonzero eigenvalues of both \( A^t A \) and \( A A^t \).

Proof: Recall \( A^t A \) and \( A A^t \) are Hermitian positive semidefinite (see Claim 1 of proof of SVD Appendix I). \( (A^t A)^{1/2} \) exists and it’s eigenvalues are non-negative square roots of eigenvalues of \( A^t A \). Hence \( \sigma_i^2 \) is a true eigenvalue of \( A^t A \) and \( A A^t \) for every \( i = 1, \ldots, r \).

Claims 2 & 3: Right \( s \)-vectors \( v_1, \ldots, v_n \) are the eigenvectors of \( A^t A \) and left \( s \)-vectors \( u_1, \ldots, u_n \) are the eigenvectors of \( A A^t \).

Proof: Recall \( \sigma \) is a singular value of \( A \) repeated \( n(m) \) times. So there exists \( n \) linearly independent eigenvectors \( \{v_1, v_2, \ldots, v_n\} \) of \( A^t A \) corresponding to \( \sigma^2 \) and there exists \( m \) linearly independent eigenvectors \( \{u_1, \ldots, u_m\} \) of \( A A^t \) corresponding to the eigenvalue \( \sigma^2 \).
of $AA'$. These vectors by definition are the right singular vectors of $A'A$ and left singular vectors of $AA'$.

Claim 4: If $A = A'$ has eigenvalues $\lambda_1, \ldots, \lambda_2$ then the singular values of $A$ are $|\lambda_j|$.

Proof: Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $A$ for all $x_j \neq 0 (j = 1, \ldots, n)$.

Then

$$Ax_j = \lambda_j x_j$$

$$A'Ax_j = \lambda_j A'x_j$$

$$A'Ax_j = \lambda_j Ax_j$$

$$A'Ax_j = \lambda_j (\lambda_j x_j)$$

$$A'Ax_j = \lambda_j^2 x_j (j = 1, \ldots, n)$$

Thus $\lambda_j^2$ are eigenvalues of $A'A$, so the singular values of $A$ are $\sqrt{\lambda_j^2} = |\lambda_j| (j = 1, \ldots, n)$. 

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APPENDIX VI

STUDENTS SPLIT TIMES

The following tables in this appendix illustrate the 100m and 200m split times of the student athletes who participated in our data collection.

TABLE A6.1: FEMALES 100M SPLIT TIMES

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APPENDIX VII

FORTRAN 77 PROGRAM

PARAMETER (mmax=100, nt=0)
IMPLICIT DOUBLE PRECISION (a-h,o-z)
DIMENSION r(mmax), s(mmax), t(mmax)

c Given m points (s(i), t(i)) where s=d=distance (m) and t=time (sec),
c during one or several optimal sprints, estimate the physiological
c parameters F and tau using regression fitting via our customized
c SVD routine applied on the (good approximate) line t=a+b*s where
c a=tau and b=1/(F*tau).

10 WRITE(*,*) 'm?'
   READ(*,*,END=90) m
   IF (m.LT.2 OR M.GT.MMAX) GO TO 10
   DO I=1,m
       R(I)=1D0
       WRITE(*,'(6H POINT,12)?') I
       READ(*,*) S(I), T(I)
   END DO
   WRITE(*,70) (S(I), T(I), I=1,m)
70 FORMAT(/ 13H DATA POINTS: / ( 5(F5.0,F5.2,4X) ) )
   CALL SVD2(M, R, S, T, NT, A, B, N, CD, S1, S2, A1, B1)

   TAU = A
   F = 1D0/(A*B)
   TAU1 = A1
   F1 = 1D0/(A1*B1)
   WRITE(*,80) F, TAU, F1, TAU1, S1, S2, CD, N
   GO TO 10
80 FORMAT(/ 23H RESULTS OF ESTIMATION:   
   + / 5H F = F7.3, 7X, 5HTAU = F7.3 
   + / 5H F = F7.3, 7X, 5HTAU = F7.3 
   + / 4H S1=E10.3,5X,3H S2=E9.3,5X,3H CD=E8.3,5X,3H NR=I1/)
   + / 5H F = F7.3, 7X, 5HTAU = F7.3 
   + / 4H S1=E10.3,5X,3H S2=E9.3,5X,3H CD=E8.3,5X,3H NR=I1/)
90 CONTINUE
Subroutine SVD2

SUBROUTINE SVD2(m,r,s,t,nt,a,b,nr,cd,s1,s2,a1,b1)
IMPLICIT DOUBLE PRECISION (a-h,o-z)
DIMENSION r(m), s(m), t(m)

c High resolution least squares solution via the SVD of an mx2
overdetermined system of linear equations
a*r(i) + b*s(i) = t(i), 1 le i le m, m ge 2,
in the unknowns a and b and where r(i),s(i),t(i) are 3m given
constants. If every r(i)=1 the problem is equivalent to finding
the regression line t=a+b*s thru the m data points (s(i),t(i)).
The inputs are m,r,s,t,nt where nt prescribes a tolerance used
in getting the numerical rank (see description below). The outputs
are the SVD solution a and b, the numerical rank nr, the condition
c number cd, and singular values s1 and s2 of the design matrix A,
and for comparison purposes, values a1 and b1 of the two unknowns
c computed as the solution of the normal system with matrix A^t*A.

iff(m.lt.2) then
  write(*,*) 'm is too small in SVD2'
  return
endif
c Find Srr,Srs,Sss needed in the SVD computation and the tolerance
tol=10**(c)*||A||_oo used in determining the numerical rank
with c=nt if nt is within the IEEE double precision number 1-15
c of significant digits or with c=15 otherwise.

Srr = 0d0
Srs = 0d0
Sss = 0d0
tol = 0d0
do i=1,m
  Srr = Srr + r(i)*r(i)
  Srs = Srs + r(i)*s(i)
  Sss = Sss + s(i)*s(i)
  tol = dmax1(tol, dabs(r(i)+s(i)) )
dendo
c=nt
tol = 10**(c) * tol
Find the two singular values $s_1$ and $s_2$ in decreasing order of the $m \times 2$ design matrix $A$ and then its numerical rank $nr$. These singular values are square roots of the eigenvalues of $A^*A$.

$$c = (S_{rr}-S_{ss})*(S_{rr}-S_{ss}) + 4d0*S_{rs}*S_{rs}$$
$$s_{sl} = 0.5d0*( (S_{rr} + S_{ss}) + dsqrt(c) )$$
$$s_{s2} = 0.5d0*( (S_{rr} + S_{ss}) - dsqrt(c) )$$
$$s_1 = dsqrt(s_{ss1})$$
$$s_2 = dsqrt(dmax1(0d0,s_{s2}))$$

$$cd=0d0$$
if($s_{1}.le.tol$) then
   $nr=0$
else if($s_{2}.le.tol$) then
   $nr=1$
else
   $nr=2$
   $cd=s_1/s_2$
endif

Find the right singular vectors $V_1=(v_1,v_2)^t$ and $V_2=(-v_2,v_1)^t$, namely, the normalized eigen vectors of the $2 \times 2$ matrix $A^*A$.

$$c = -S_{rs}/(S_{rr}-s_{ss1})$$
$$v_2 = dsqrt( 1d0/(c*c+1d0) )$$
$$v_1 = c*v_2$$

Find the solution $Z=(z_1,z_2)^t$ of the transformed problem where $z_1=U_1^*T/s_1$ and $z_2=U_2^*T/s_2$ using the left singular vectors $U_1=A*V_1/s_1$ and $U_2=A*V_2/s_2$ dependent on the numerical rank $nr$. If $nr$ is 2 the minimum norm solution is used ($z_2=0$ or $z_1=z_2=0$).

$$z_1 = 0d0$$
$$z_2 = 0d0$$
if($nr.ge.1$) then
do $i=1,m$
   $z_1 = z_1 + ( r(i)*v_1+s(i)*v_2 )*t(i)$
   $z_2 = z_2 + (-r(i)*v_2+s(i)*v_1 )*t(i)$
endo

$$z_1 = z_1/s_{ss1}$$
endif
if($nr.eq.2$) $z_2 = z_2/s_{s2}$

Find the desired coefficients as the matrix-vector product $V^*Z$.

$$a = v_1*z_1-v_2*z_2$$
\[ b = v_2 z_1 + v_1 z_2 \]

For comparison purposes, also compute the coefficients as the solution of the 2x2 normal system \( A^t A (a_1, b_1)^t = A^t T \).

\[
\begin{align*}
    \text{Srt} &= 0d0 \\
    \text{Sst} &= 0d0 \\
    \text{do} \ i=1,m \\
    &\quad \text{Srt} = \text{Srt} + r(i) * t(i) \\
    \text{do} \ i=1,m \\
    &\quad \text{Srt} = \text{Srt} + r(i) * t(i) \\
    &\quad \text{Sst} = \text{Sst} + s(i) * t(i) \\
    \text{enddo} \\
    a_1 &= 0d0 \\
    b_1 &= 0d0 \\
    c &= \text{Srt} * \text{Sst} - \text{Srt} * \text{Srs} \\
    \text{if}(c \text{.gt.} 0d0) \text{ then} \\
    &\quad a_1 = (\text{Srt} * \text{Sst} - \text{Sst} * \text{Srs}) / c \\
    &\quad b_1 = (\text{Srt} * \text{Sst} - \text{Srt} * \text{Srs}) / c \\
    \text{endif} \\
    \text{return} \\
    \text{END}
\]
REFERENCES


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