Pre-service secondary mathematics teachers' attitudes about the history of mathematics

Kelly Marie Sullivan
University of Nevada, Las Vegas

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PRE-SERVICE SECONDARY MATHEMATICS

TEACHERS’ ATTITUDES ABOUT THE

HISTORY OF MATHEMATICS

by

Kelly Marie Sullivan

Bachelor of Arts
California State University, Fresno
1985

A thesis submitted in partial fulfillment
of the requirements for the

Master of Science Degree
Department of Curriculum and Instruction
College of Education

Graduate College
University of Nevada, Las Vegas
May 2000

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The Thesis prepared by

Kelly Marie Sullivan

Entitled

Pre-Service Secondary Mathematics Teachers' Attitudes About the History of Mathematics

is approved in partial fulfillment of the requirements for the degree of

Master of Science in Curriculum and Instruction

Examination Committee Chair

Dean of the Graduate College

Examination Committee Member

Examination Committee Member

Graduate College Faculty Representative
ABSTRACT

Pre-Service Secondary Mathematics Teachers’ Attitudes About the History of Mathematics

by

Kelly Marie Sullivan

Dr. Juli K. Dixon, Examination Committee Chair
Assistant Professor of Curriculum and Instruction
University of Nevada, Las Vegas

This study was conducted to show that using historical material to teach mathematical content to pre-service secondary mathematics teachers could improve their attitudes about incorporating the history of mathematics into the mathematics classroom. A historical module from the Mathematical Association of America was taught to an undergraduate class of pre-service secondary mathematics teachers. Their attitudes regarding the use of the history of mathematics in the mathematics classroom were then compared to the attitudes of students in another undergraduate class, taught by the same researcher, whose instruction did not incorporate history. Results from t-tests indicate that there was a positive change in attitude amongst the students in the experimental group. This study did not, however, verify a distinction in achievement between the two groups.
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CHAPTER 1

INTRODUCTION

The International Commission on Mathematical Instruction (ICMI) identifies issues or topics they deem to be of particular contemporary significance to mathematics education and initiates specific studies on these themes. At the eighth International Congress on Mathematical Education (ICME) held in Seville, Spain in 1996, the ICMI chose "The Role of the History of Mathematics in the Teaching and Learning of Mathematics" as the theme for the next ICMI Study (Fauvel & van Maanen, 1997). The report on this study will be given at the ninth ICME meeting in Japan in the year 2000. The emphasis placed on this topic of study by the ICMI reflects the emergence of interest within the academic community in incorporating mathematics history as an integral part of the teaching of mathematics.

A strong, growing interest in this academic approach has been building over the years. In several recent publications, promoters of the use of history in the mathematics curriculum have developed a substantial argument outlining the benefits of integrating mathematics history into the classroom (Calinger, 1996; Swetz, et. al., 1995). Fauvel (1991) suggests that the use of the history of mathematics in teaching has the following outcomes: helps to increase motivation for learning; makes mathematics less frightening; allows students to derive comfort from knowing that they are not the only ones with
problems; gives mathematics a human face; and changes students' perceptions of mathematics.

The mathematical experiences of prospective teachers should challenge familiar attitudes, and foster fresh attitudes, about the nature of mathematics. Philippou and Christou (1998) contend that a teacher-preparation program based on the history of mathematics is effective in improving pre-service teachers' attitudes about mathematics.

The Curriculum and Evaluation Standards for School Mathematics (1989a) of the National Council of Teachers of Mathematics (NCTM) calls for reform in the approach to teaching and learning mathematics. Among the recommendations, there are three essential items that should be examined across all grade levels: communicating, connecting, and valuing mathematics. The history of mathematics can be used in the classroom to help reach these goals. Students can communicate about historical facts orally or in writing. They can connect mathematics to various cultures as well as to other intellectual developments in science, philosophy, and religion. Further, history can substantially add to students' value of mathematics learned from the past and in the present.

Over the decades, the benefits of the use of the history of mathematics in the classroom have been noted. In 1969, the NCTM considered the subject to be of such importance as to devote a yearbook to issues related to the history of mathematics in mathematics education. Because of continued demand for information on this subject, the council published an updated edition twenty years later (NCTM, 1989b).

Among those who advocate the use of history in the mathematics classroom, the manner in which to introduce it has been debated. One approach to using history to enrich
mathematics instruction and to improve the learning of mathematics is to have students solve some of the problems that interested early mathematicians (Swetz, 1989). Such problems offer case studies which students can relate to contemporary classroom discussions. According to Swetz, the problems transport the reader back to the age when the problems were posed and illustrate the mathematical concerns of the period.


Can we instill into ‘inhuman’ mathematics more humanity by convincing the learner that mathematics has been conceived by men, or wouldn’t it be a shorter way, a stronger proof, to have some mathematics they are really concerned with recreated by the students themselves? (p. 30)

Statement of the Problem

A search revealed little research-based literature on the role of the history of mathematics in the teaching and learning of mathematics. One can find articles describing ideas and methods for using the history of mathematics in the mathematics classroom, but actual data regarding the effectiveness of such methods appear to be scarce. There are several forces presently working to fill this gap.

The NCTM has issued a call for manuscripts for a focus issue of the Mathematics Teacher on the history of mathematics and mathematics education to be published in the fall of 2000 (NCTM, 1999, p. 239). However, the articles in this journal are usually practitioner-driven. Earlier this decade, an issue of For the Learning of Mathematics was dedicated to the use of the history of mathematics in education (Fauvel, 1991), but these articles also targeted the practicing teacher.
The Mathematical Association of America (MAA) has received a National Science Foundation Award to support a Historical Modules Project. They are presently field-testing twelve self-contained modules. These field tests are being conducted over the 1999-2000 and 2000-2001 school years (NCTM, 1998). A statistical study will be performed at the end of the field tests using a feedback instrument including a questionnaire that will use Likert scales. The research that comes from the MAA project should significantly add to the base of knowledge regarding the integration of the history of mathematics in the mathematics classroom.

Purpose of the Study

This research study sought to identify whether an attitude change occurred in pre-service secondary mathematics teachers who completed one of the MAA historical modules. As well, this study proposed to show whether these pre-service teachers had developed a deeper conceptual understanding of the topic.

The independent variable for this study was the method of instruction for a unit on geometric proof. The experimental group was taught from a historical perspective using the MAA module, and the control group was taught without allusion to history. The dependent variables were survey scores and achievement scores. A pre-/post-survey instrument was used to collect data on attitudes toward the history of mathematics, and proofs were scored for achievement data.

The specific research questions to which this study sought answers were:
1. Is there a change in attitude of pre-service secondary mathematics teachers toward the integration of the history of mathematics into the mathematics classroom after being taught using a historical module?

2. Is there a higher level of mathematical content achievement on a unit of study among pre-service secondary mathematics teachers exposed to a historical module than among those not exposed to the historical module?
CHAPTER 2

REVIEW OF RELATED LITERATURE

This literature review is divided into three sections. The first looks at the evaluations of several techniques for introducing history into the mathematics classroom. The second tells of how history may be related to learning, and the third investigates pre-service teachers' beliefs and influences.

History in the Mathematics Classroom

Jardine (1997) described a study conducted in an undergraduate mathematics class. Each student chose a historical topic related to the syllabus by lesson number. During the specified lesson, the student researcher made a three-to-five minute oral presentation and submitted a one-page essay on the historical figure or concept related to that day's subject matter. Feedback from a qualitative survey instrument verified that engaging students in the study of the history of mathematics enhanced the learning experience for them. Their appreciation of mathematics as a worthy subject of study was increased.

Some history of mathematics materials were developed by R. Grunseit for classroom usage under a grant from the Schools Commission Innovations Programme in Australia. These materials presented the history of mathematics to students aged 13 to 15 in the following five ways: as articles relating historical stories which teachers can use as
a basis for lesson preparation; as one-act plays which can be used on stage or in the classroom; as tape-slide presentations; as videotapes; and as biographies of great mathematicians together with drawn portraits. From this range of materials, a play about Thales and an article related to the history of conic sections were selected for intensive study (Fraser & Koop, 1978, 1981; Koop & Fraser, 1978).

A questionnaire survey was used to gauge the opinions of mathematics teachers about these materials (Fraser & Koop, 1978). The survey revealed that, in general, teachers responded favorably to both the play and the article; but opinions about the play tended to be more favorable than opinions about the article. Teachers also considered that the play would be useful for integrating subjects like mathematics, English, and history. However, teachers expressed concern that using a mathematical play in their teaching could involve excessive amounts of time and would require skills not possessed by the average mathematics teacher. The authors suggested that this would not be the case if the play were used in the classroom rather than as a full-scale production. A disappointing finding from the survey was that, despite the fact that teachers expressed favorable opinions about numerous aspects of the material, it was reported that a sizable proportion of teachers responded that they would not use the material in their own teaching.

A group of five experts - a historian of mathematics, two mathematics educators, and two drama critics - provided their views about the Thales play, the conics article, or both (Koop & Fraser, 1978). In general, the historian of mathematics and the mathematics educators expressed favorable opinions about the materials, although opinions about the play were more favorable than opinions about the article. The drama critics recognized that the play possessed positive dramatic qualities.
Fraser and Koop (1981) investigated the changes in affective and cognitive outcomes experienced by a sample of students using the play. It was found that students underwent significant positive changes on both affective and cognitive outcomes during the time of using the Thales play, although the play was more effective in promoting improvement in historical knowledge than mathematics achievement.

The technique of using items from the history of mathematics was effective as a means of promoting positive attitudes toward mathematics for college algebra students (McBride, 1977). The inclusion of human interest in the subject, through mention of sometimes-humorous activities and beliefs of famous mathematicians, was correlated with the development of favorable attitudes. It was reported that students perceived increased enthusiasm on the part of the teachers who used the materials.

Relevance of History to Education

Harper (1987) attempted to address the theory that ontogeny recapitulates phylogeny. That is, he studied whether the historical evolution of algebraic ideas might parallel an individual's lifetime development of algebra. According to Harper, algebra has passed through three identifiable stages: rhetorical, syncopated (Diophantine), and symbolic (Vietan). The rhetorical stage encompasses the period of time when all arguments were written in longhand and no symbols were available to represent unknown quantities. Syncopated algebra is exemplified by the algebraist's use of letters for unknown quantities; and, in symbolic algebra, the algebraist uses letters for given quantities as well. The results from his study of secondary school students indicate a possible alignment between algebraic evolution and conceptual development. The
students advanced from the rhetorical to the Diophantine to the Vietan solution types as they progressed through the school years. This finding strengthens the argument for teaching mathematics, or at least algebra, in the sequence in which it was developed historically.

History is a natural means for promoting the opinion that mathematical definitions are a logical necessity. Arcavi, Bruckheimer, and Ben-Zvi (1987) assessed pre- and in-service teachers’ previous knowledge, conceptions, and misconceptions of irrational numbers. They concluded that it is desirable for teacher-education materials to include the search for a formal definition of irrational number, thereby motivating the necessity of such definitions. The study of the history of irrational numbers would allow teachers an appropriate account of how mathematics is actually accomplished by mathematicians.

Grattan-Guinness (1978) states that “it is an integral part of one’s understanding to be aware of the source of mathematics, but unfortunately it is not necessary for one’s mere knowledge of it (p. 278).” He argues that mathematics educators should strive for student understanding and that the history of mathematics can be used as a great motivator for understanding mathematics.

Pre-Service Teachers’ Beliefs

In the evaluation of a teacher-preparation program based on the history of mathematics, support was found for the hypothesis that the program would be effective in improving prospective teachers’ attitudes toward mathematics (Philippou & Christou, 1998). The program consisted of content and method courses that followed a historic evolutionary process, beginning with pre-Hellenic mathematics; continuing with Greek,
Hindu, Moslem, medieval and enlightenment mathematics; and concluding with topics from calculus, geometry, algebra, set theory, logic, and Boolean algebra. The results of the study affirmed that pre-service teachers bring misconceptions and negative attitudes towards mathematics to teacher education. The history of mathematics proved to be quite potent in changing these attitudes.

There are other factors that influence the attitudes of pre-service teachers. Quinn (1997) found that a mathematics methods course that uses manipulatives, technology, and cooperative learning has an effect on the attitudes of pre-service teachers. While the pre-service elementary teachers' attitudes towards mathematics improved significantly upon completion of the course, the increase in attitude toward mathematics of pre-service secondary mathematics teachers was not significant. He suggests that the pre-service elementary teachers' attitudes were less favorable than the pre-service secondary teachers' attitudes from the beginning and, therefore, had further room for improvement.

Although coursework has been shown to influence pre-service mathematics teachers and their attitudes towards mathematics, findings from Frykholm (1996) indicate that cooperating teachers have the most significant influence on the philosophies and practices of student teachers.

Summary

Several techniques for introducing history into the mathematics classroom have been shown to be effective in increasing motivation on the part of the student and improving attitude on the part of the teacher. In fact, the historical development of a concept may actually serve as a significant tool in the full understanding of the concept.
CHAPTER 3

METHOD

In order to answer the research questions stated in chapter 1, the following null hypotheses were tested:

1. Pre-service secondary mathematics teachers will show no significant change in attitude toward the integration of the history of mathematics into the mathematics classroom due to a historical module instruction.

2. Pre-service secondary mathematics teachers will show no significant difference in achievement due to a historical module instruction.

Participants

The students of two classes of pre-service secondary mathematics teachers at an urban university located in the American Southwest were asked to participate in this study. An upper-division mathematics class from the Department of Mathematical Sciences in the College of Sciences acted as the experimental group, and a mathematics methods class from the Department of Curriculum and Instruction in the College of Education served as the control group. The mathematics class is described in the university catalog as “Probability and Combinatorics for Teachers”, but the professor taught it this particular semester as “History of Mathematics”. The mathematics methods
class is described as “Teaching Secondary Mathematics”. All students in both classes were pre-service secondary mathematics teachers.

There were nineteen students in the mathematics class and thirteen students in the mathematics methods class. Two students were enrolled in both classes, so the data collected from them were not used. One student in the mathematics methods class opted out of the study at the end of the data collection. Therefore, the data analysis involved the data collected from seventeen participants in the experimental group and ten participants in the control group.

All participants in the control group were in their final year of undergraduate education and therefore near the end of their preparation. The majority of the students in the experimental group, conversely, were in the second year of their programs.

Materials

An MAA historical module (See APPENDIX V) was used in this research study. It consisted of classroom instruction material that included (a) a teacher’s guide, (b) the historical and cultural background for the topic, (c) biographical information about the mathematicians who contributed to the topic and information about the cultural context, (d) detailed lesson plans, (e) copy masters, (f) exercises for students, (g) suggestions for student projects, (h) a bibliography, and (i) questions to be answered by the students. The topic was “Geometry and Proof”.

To test the first hypothesis, an attitude scale (See APPENDIX IV) was created. The scale was piloted and refined with the help of the students in a graduate class from the Department of Curriculum and Instruction in the College of Education. The class is
described in the university catalog as "Seminar in Instructional and Curricular Studies" and is intended for students nearing the end of their graduate studies. This particular section was dedicated to issues in mathematics education.

The second hypothesis was tested using five questions from the culminating activity at the end of the treatment. A six-point rubric (See APPENDIX VI) was created to score the proofs from this activity, and three practicing teachers were recruited to help in scoring so as to minimize subjectivity.

Procedure

Shortly after the beginning of the semester in which this research was conducted, participants in each group were asked to complete the attitude scale. The professor of the mathematics class administered the survey to the experimental group within the first two weeks of the semester, and the researcher administered the survey to the control group a few weeks later. One week of class-time was allotted to the researcher for instruction. For the experimental group, this meant three classes of fifty minutes each; and for the control group, one class of 150 minutes, of which half was dedicated to this study. At the end of the week, the attitude scale was again administered so that pre- and post-scores could be compared. Also, at the end of instruction, a culminating activity was completed so that a comparison of group achievement could be made.

Experimental Group Instruction

On the first day of instruction, the participants were given copies of the student edition from the MAA module entitled Let Us Pretend: It is 380 B.C.E. and you live in Athens. After an introduction to Plato's Academy, the participants were paired and asked
to complete a worksheet involving relative measurements of segments. As homework, they were asked to complete a worksheet on commensurability of segments. This was to prepare them for the discussion of non-commensurable numbers in the second class session.

The second class consisted of reading *Excerpts from Plato's Meno*, which was found in the student edition. The purpose of this reading was twofold. Firstly, the dialogue among Socrates and the slave boy leads to an incommensurable pair of numbers and how they may be drawn. Secondly, Socrates asserts that learning is a process of discovering knowledge.

On the third and final day of instruction, the participants were asked to construct a sequence of arguments that would lead to the conviction that the side and diagonal of a square are incommensurable. From this culminating activity, the students' proofs of the first five theorems were used as data for comparison with the control group.

**Control Group Instruction**

Instruction for the control group included proving statements leading to the hypothesis that the square root of two is irrational. The arguments were of the same nature of those in the experimental group, but they were not imbedded in the context of history. The same five proofs, with a slight change in wording to conceal the historical link, were collected as data for comparison with the experimental group. Because not all participants completed all of the proofs, the first five theorems (See APPENDIX VI) were chosen for purpose of comparison.
CHAPTER 4

RESULTS

This study sought answers to the following research questions:

1. Is there a change in attitude of pre-service secondary mathematics teachers toward the integration of the history of mathematics into the mathematics classroom after being taught using a historical module?

2. Is there a higher level of mathematical content achievement on a unit of study among pre-service secondary mathematics teachers exposed to a historical module than among those not exposed to the historical module?

First Research Question

To address the first question, a nine-item, Likert-type attitude survey was administered at the beginning of the study, and the same survey was again given at the end of the study. The nine items were:

1. The history of mathematics is a useful tool for learning mathematics.

2. Mathematics lessons should be taught from a historical perspective.

3. The history of mathematics should be taught only as enrichment to the mathematics curriculum.

4. Student learning of school mathematics is increased with teacher use of historical anecdotes.
5. The history of mathematics increases student enjoyment of mathematics.

6. The history of mathematics can be incorporated into the curriculum without increasing student or teacher workload.

7. Mathematics is a static subject without its history.

8. Including the history of mathematics hinders mathematics teaching.

9. I plan to integrate the history of mathematics into the mathematics courses that I will teach.

Responses to the third and eighth items were re-coded so that a score of 5 reflected the most positive response, and a score of 1 reflected the most negative response, on every item. The results from the pre- and post surveys can be found in Table 1. The means and standard deviations are listed for each of the nine items in the pre- and post-surveys, categorized by control group or experimental group.
Table 1

Mean Responses and Standard Deviations on Pre- and Post-Survey Items

<table>
<thead>
<tr>
<th>Item</th>
<th>Control group</th>
<th>Experimental group</th>
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<tr>
<td></td>
<td>Pre-survey M</td>
<td>SD</td>
</tr>
<tr>
<td>1</td>
<td>3.20</td>
<td>1.14</td>
</tr>
<tr>
<td>2</td>
<td>2.00</td>
<td>0.82</td>
</tr>
<tr>
<td>3</td>
<td>2.80</td>
<td>1.40</td>
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<td>4</td>
<td>2.30</td>
<td>1.06</td>
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<td>5</td>
<td>2.00</td>
<td>1.05</td>
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<td>6</td>
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<td>1.58</td>
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<tr>
<td>7</td>
<td>2.00</td>
<td>1.25</td>
</tr>
<tr>
<td>8</td>
<td>4.30</td>
<td>0.95</td>
</tr>
<tr>
<td>9</td>
<td>3.10</td>
<td>0.88</td>
</tr>
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</table>

A reliability test was run for both the pre-survey and the post-survey. The reliability coefficient for the pre-survey was found to be $\alpha = .56$, and that for the post-survey was $\alpha = .80$.

To answer the first research question, a mean and standard deviation were found for each group on the pre-survey and the post-survey (see Table 2).
Table 2

<table>
<thead>
<tr>
<th>Group</th>
<th>M</th>
<th>SD</th>
<th>M</th>
<th>SD</th>
<th>M</th>
<th>SD</th>
<th>N</th>
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<tbody>
<tr>
<td>Control</td>
<td>2.69</td>
<td>0.47</td>
<td>2.86</td>
<td>0.47</td>
<td>0.17</td>
<td>0.24</td>
<td>10</td>
</tr>
<tr>
<td>Experimental</td>
<td>3.08</td>
<td>0.47</td>
<td>3.48</td>
<td>0.64</td>
<td>0.40</td>
<td>0.57</td>
<td>17</td>
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</table>

An independent t-test for equality of means was utilized to compare the pre-survey results between groups, \( t(19.15) = -2.09, p < .05 \), and the post-survey results between groups, \( t(23.53) = -2.87, p < .01 \). A significant difference was found between the two groups both before the intervention and after the intervention.

A dependent t-test for paired samples compared pre- and post-survey results within each group. For the control group, \( t(9) = -2.24, p < .001 \), and for the experimental group, \( t(16) = -2.87, p < .05 \). Scores on the post-survey were significantly higher than the scores on the pre-survey for both the control group and the experimental group. Both groups responded more favorably to the post-survey than to the pre-survey as is indicated by the increase in mean scores for each group from the pre-survey to the post-survey.

An independent t-test for equality of means found that, although the positive change in the experimental group was greater than that in the control group, the difference was not found to be statistically significant \( t(23.12) = -1.47, p > 0.05 \).

These results indicate that there was a positive change in attitude of pre-service secondary mathematics teachers toward the integration of the history of mathematics into the mathematics classroom after being taught using a historical module.
Second Research Question

The results of this study did not suggest a significant distinction between the two groups as regards mathematical content achievement.

To answer the second research question, five proofs from the culminating activity were graded, by rubric, and compared by group. The possible scores on each proof were 1, 2, 3, 4, 5, and 6, with 6 representing an exceptional proof. The mean scores and deviations can be found in Table 3.

Table 3

<table>
<thead>
<tr>
<th>Proof</th>
<th>Control group</th>
<th>Experimental group</th>
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<tbody>
<tr>
<td></td>
<td>M</td>
<td>SD</td>
</tr>
<tr>
<td>1</td>
<td>4.20</td>
<td>1.62</td>
</tr>
<tr>
<td>2</td>
<td>4.80</td>
<td>0.92</td>
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<tr>
<td>3</td>
<td>4.40</td>
<td>0.84</td>
</tr>
<tr>
<td>4</td>
<td>3.00</td>
<td>1.25</td>
</tr>
<tr>
<td>5</td>
<td>2.80</td>
<td>1.03</td>
</tr>
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The collapsed mean and standard deviation for each group were used to perform an independent t-test for equality of means between groups. These numbers can be found in Table 4. The control group scored higher than the experimental group, but the difference was not found to be statistically significant ($t(22) = -1.53, p < .20$).
Table 4

Group Means and Standard Deviations on the Five Proofs

<table>
<thead>
<tr>
<th>Group</th>
<th>M</th>
<th>SD</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>Control</td>
<td>3.84</td>
<td>0.96</td>
<td>10</td>
</tr>
<tr>
<td>Experimental</td>
<td>3.23</td>
<td>0.97</td>
<td>14</td>
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</tbody>
</table>

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CHAPTER 5

DISCUSSION

The results of this research support those of Philippou and Christou (1998) in their indication that the integration of the history of mathematics into mathematics instruction improves the attitudes of pre-service secondary teachers. Though the attitudes of both groups improved over the course of the treatment, those participants who were exposed to the historical module showed a significantly greater increase. The mean score for the control group at post-survey, while greater than that at pre-survey, still had not reached the level of the experimental group at pre-survey. Although the pre-survey was given near the beginning of the semester in which the study was conducted, this result may be attributed to the fact that the participants in the experimental group were in an environment that was rich with history, and those in the control group were not.

While the attitude survey was found to be reasonably reliable as a post-survey, the pre-survey reliability was lacking. A possible justification for the poor reliability coefficient found for the pre-survey is that the participants were not as comfortable with the study at the beginning as they were at the end. By the end of the treatment, the participants were quite aware of the researcher and the reasons for her research. They may have taken the post-survey more seriously than they had the pre-survey. Perhaps more fine-tuning of the instrument would be advised for a follow-up study.
This research did not show that the participants who were exposed to the historical module reached a higher level of achievement. Because the instrument was used only at the end of the intervention, a measure of increase in achievement is not available. The control group scored higher than the experimental group, but the difference was not found to be statistically significant. Although none of the participants had had a history of mathematics class prior to the semester in which this study was conducted, all participants had had training in mathematics and mathematics education; but the participants in the control group had undoubtedly had more. Every participant in the control group was within two semesters of graduation, while those in the experimental group were years from graduation. In any case, the treatment of the historical module did not seem to affect a higher level of achievement among the pre-service teachers.

Informal evaluation forms completed by the participants in the experimental group attest to Grattan-Guinness' (1978) argument that the history of mathematics is a great motivator for understanding mathematics. One participant commented, "the historical information created a greater enthusiasm toward the material; the environment was very conducive to learning." Another confirmed Harper's (1987) theory of "following an idea through its progression, from conception to maturity, enhances the lesson being learned." Several participants commented on how they had gained a better appreciation for mathematics, and how the history made the material more relevant.
Limitations of the Study

Certain conditions inherent in this study have been identified. The data for this research were collected from quite small samples of convenience. The participants were not randomly selected. They were placed in each group by way of placement in course of study. The information was collected over a short period of time with no feasible possibility of a follow-up study.

The instruments used to collect data for this study were self-made and not extensively piloted. There was no measure of pre-achievement amongst the participants, which made an assessment of increase in knowledge unattainable.

Implications of the Study

The results of this study bolster the desire of organizations such as the International Commission of Mathematical Instruction, the Mathematical Association of America, and the National Council of Teachers of Mathematics for the integration of the history of mathematics in the mathematics classroom. Indeed, the history of mathematics does have an important role in the teaching and learning of mathematics. Teaching history to pre-service secondary mathematics teachers increases their appreciation for the use of history in the mathematics classroom.

Recommendations for Further Study

It is recommended that a follow-up study be conducted to answer the question regarding in-service secondary mathematics teachers' attitudes about the history of mathematics. Will the appreciation shown by pre-service teachers continue to grow with
the use of history in their eventual classrooms? With what frequency do these teachers use the history of mathematics in their classrooms?

To strengthen the argument for the use of history in pre-service mathematics education programs, it is recommended that an expanded study be conducted with other materials not specific to the MAA module. For example, how would a history of mathematics course requirement enhance the pre-service teacher’s experience? Is there a correlation between the inclusion of the history of mathematics in pre-service instruction and the eventual effectiveness of pre-service teachers?

And, finally, how do historical materials affect the learning of mathematics for high school students? The results from the field tests of the MAA’s Historical Modules Project are anticipated with enthusiasm. Specifically, it is hoped that more data-collection projects will result from the efforts of the MAA.
APPENDICES
Date: Sat, Feb 27, 1999 10:13 PM EDT
From: Histmath
Subj: Re: Historical Modules
To: KMS1574

Dear Kelly Marie:
Let's go with your proposal. In due course, we will get you copies of one or more of the modules. We would like you to test them in some way and provide us with a report of your experience. Hopefully, you might even use this as a project connected with your graduate work. What we want is some research and reaction to the modules. It seems that your input would enrich the testing of the modules.

Best wishes.
Al Buccino
DATE: September 9, 1999

TO: Kelly Marie Sullivan
Department of Curriculum & Instruction
M/S 3005

FROM: Dr. Fred Preston, Chair
Social/Behavioral Sciences Committee

RE: Expedited Review of Human Subject Protocol:
"Integrating the History of Mathematics into the Mathematics Classroom"
OSP #: 311s0999-094x

The protocol for the project referenced above has been reviewed and approved by an expedited review by the Institutional Review Board Social/Behavioral Sciences Committee. This protocol is approved for a period of one year from the date of this notification and work on the project may proceed.

Should the use of human subjects described in this protocol continue beyond a year from the date of this notification, it will be necessary to request an extension.

If you have any questions or require any assistance, please contact Sally Hamilton at 895-1357.

cc: J. Dixon (CI-3005)
OSP File
CONSENT TO PARTICIPATE IN A RESEARCH STUDY

Project Title: The Integration of the History of Mathematics into the Mathematics Classroom

Researcher: Kelly Marie Sullivan
Graduate Student
University of Nevada, Las Vegas
Department of Curriculum and Instruction

Purpose: You are being asked to participate in a research study which will investigate attitude and achievement of pre-service teachers regarding the integration of the history of mathematics in the mathematics classroom.

Procedure: The investigation will involve the teaching of a unit of study. An attitude scale will be administered before and after the teaching of the unit, and an end-of-unit assessment will be administered.

Benefits: This research will add to the body of knowledge regarding the integration of the history of mathematics in the mathematics classroom. Research results will benefit teachers of mathematics and those who design mathematics curriculum.

Conditions: Information collected in this study is confidential, and your name will not be used. The last three digits of your student identification number will be used for identification purposes. The length of time required for the study will not exceed that of the Fall semester ending in December 1999. Your participation in the study is entirely voluntary, and you are free to withdraw your consent at any time. If you have any questions during your association with the research study, before or after its completion, please feel free to ask for further information from the researcher at 270-2416 or kms@nevada.edu. For information regarding the rights of research subjects, please contact the Office of Sponsored Programs at 895-1357.

YOUR SIGNATURE BELOW WILL INDICATE THAT YOU HAVE READ THE INFORMATION PROVIDED ABOVE AND THAT YOU HAVE DECIDED TO VOLUNTEER AS A RESEARCH PARTICIPANT.

______________________________________________  ______________________
Signature of Participant                              Date

______________________________________________  ______________________
Signature of Researcher                              Date
Last three digits of your student identification number __________ Age __________

Major ____________________________________________________________________________________________ Gender __________

First teaching field ________________ Second teaching field ____________________________

Have you taken a class in which the history of mathematics was incorporated:

in pre-college school? ________ in college, before this semester? ________

State in which you attended high school ________ Expected graduation date ________

DIRECTIONS: You may never have had experience with the history of mathematics. However, please evaluate each statement. If you strongly agree, put the number 5 in the blank opposite the statement. If you agree, but not strongly (you have some reservations) then put the number 4 in the blank. If you disagree, but not strongly, then put the number 2 in the blank. If you strongly disagree, then put the number 1 in the blank. If you neither agree nor disagree (you are not certain or you cannot answer) then put the number 3 in the blank. Please answer all parts.

THIS ATTITUDE SCALE IS BEING USED FOR RESEARCH PURPOSES ONLY. YOUR RESPONSES ARE CONFIDENTIAL. DO NOT SIGN THIS FORM.

(Strongly Disagree) 1 2 3 4 5 (Strongly Agree)

The history of mathematics is a useful tool for learning mathematics. ________

Mathematics lessons should be taught from a historical perspective. ________

The history of mathematics should be taught only as enrichment to the mathematics curriculum. ________

Student learning of school mathematics is increased with teacher use of historical anecdotes. ________

The history of mathematics increases student enjoyment of mathematics. ________

The history of mathematics can be incorporated into the curriculum without increasing student or teacher workload. ________

Mathematics is a static subject without its history. ________

Including the history of mathematics hinders mathematics teaching. ________

I plan to integrate the history of mathematics into the mathematics courses that I will teach. ________
Let Us Pretend:

It is 380 B.C.E.

and you live in

Αθήνα. (Athens)

(Student Copy)

Part 1.

Activity 1: Find the location of Athens on a globe or in an atlas.

Activity 2: Construct a time line starting with 500 B.C.E. showing some important historical events that you studied in history class.

Activity 3: Find out what you can about what Athens was like in 380 B.C.E.

Activity 4: You are lucky enough to have been admitted to the only university in the world. Find out all you can about it.

Activity 5: A professor (Theodorus or perhaps even Theoremus – yes, we know the name – of some of the professors who taught math at your University is lecturing and you are taking notes. You are learning Greek mathematics, which may seem a little unusual to you. To help you with these lessons your (American) teacher will teach these lessons with the help of worksheets. To avoid confusion, we will call your Greek teacher Mr. T.

[Students work Lessons 1 & 2 on worksheets distributed by their teacher.]

By now you should know that your university is called the Academy and that the President and founder of the Academy is the great philosopher Plato. Our next major project is to study a part of one of Plato’s works.

Activity 6: Find out what you can of Plato’s life, of his writing and of his philosophy.

Activity 7: Write a short essay explaining why Plato considered mathematics so important.

Activity 8: Find out what you can about Socrates the man and “Socrates”, the character in Plato’s dialogues.

Activity 9: Your teacher will hand you a piece of paper with a square drawn on it. Construct a square whose area is twice that of the square that you have been given. (This is Lesson 3.)
Activity 10: Read Plato's dialogue *Meno*. What is the main thrust of the argument in *Meno*, and what role in this dialogue is served by Socrates' experiment with the slave boy?
Excerpts from Plato's
Meno

MENO: ... But what do you mean when you say that we don't learn anything, but
that we call learning is recollection? Can you teach me that it is so?

SOCR: I have just said that you're a rascal, and now you ask me if I can teach you,
when I say there's no such thing as teaching, only recollection. Evidently you want
to catch me contradicting myself straight away.

MENO: No, honestly, Socrates. I wasn't thinking of that. It was just habit. If you in
any way can make clear to me that what you say is true, please do.

SOCR: It isn't an easy thing, but still I should like to do what I can since you ask
me. I see you have a large number of retainers here. Call one of them, anyone you
like, and I will use him to demonstrate it to you.

MENO: Certainly. [To a slave boy.] Come here.

SOCR: He is a Greek, and speaks our language?


SOCR: Listen carefully then, and see whether it seems to you that he is learning from
me or simply being reminded.

MENO: I will.

SOCR: Now boy, you know that a square is a figure like this?
[Socrates begins to draw figures in the sand at his feet. He points to the square ABCD.]
BOY: Yea.

SOCR: It has all these four sides equal?

BOY: Yea.

SOCR: And these lines which go through the middle of it are also equal? [Socrates draws lines EF and GH and the figure looks like this.]

BOY: Yea.

SOCR: Such a figure could be either larger or smaller, could it not?

BOY: Yea.
SOCR: Now if this side is two feet long, and this side the same, how many feet will the whole be? Put it this way. If it were two feet in this direction and only one in that, must not the whole area be two feet taken once?

---

BOY: Yes.

SOCR: But since it is two feet this way also, does it not become twice two feet?

[Soocrates points to previous figure.]

BOY: Yes.

SOCR: And how many feet is twice two? Work it out and tell me.

BOY: Four.

SOCR: How could one draw another figure double the size of this but similar, that is, with all its sides similar, like this one?

BOY: Yes.

SOCR: How many feet will its area be?

BOY: Eight.
SOCR: Now then, try to tell me how long each of its sides will be. The present figure has a side two feet. [Points to figure.] What will be the side of the double sized one?

BOY: But it will be double, Socrates, obviously.

SOCR: You see, Meno, that I am not teaching him anything, only asking. Now he thinks he knows the side of the eight-foot square.

MENO: Yes.

SOCR: But does he?

MENO: Certainly not.

SOCR: He thinks it is twice the length of the other.

MENO: Yes.

SOCR: Now watch as he recollects things in order – the proper way to re-lect.

You say that the side of double length produces the double sized figure. Like this, I mean. [points to first figure] not long this way and short that. [Points to second figure.] It must be equal on all sides like the first figure, only twice its size, that is, eight feet. Think a moment whether you will expect to get it from doubling the side.

BOY: Yes, I do.

SOCR: Well now, shall we have a line double the length of this (points to AB) if we add another the same length at this end? [Adds BJ to AB producing this figure.]

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BOY: Yes.

SOCR: It is on this line then, according to you, that we shall make the eight foot square by taking four of the same length.

BOY: Yes.

SOCR: Let us draw in four equal lines using the first as a base. Does this not give you what you call the eight-foot base? [Completes the square.]

BOY: Certainly.

SOCR: But does it contain these four squares, each equal to the original four foot one? [Extends segment BC to N and DC to M.]
BOY: Yes.

SOCR: How big is it then, won't it be four times as big?

BOY: Of course.

SOCR: And is four times the same as twice?

BOY: Of course not.

SOCR: So doubling the side has given us, not a double, but a four-fold figure.

BOY: True.

SOCR: And four times four are sixteen, are they not?

BOY: Yes.

SOCR: Then how big is the side of the eight-foot figure? This one has given us four times the original area, hasn't it?

BOY: Yes.

SOCR: And a side half the length gave us a square of four feet?

BOY: Yes.

SOCR: Good. And isn't a square of eight feet double this one [points to smaller square] and half that? [Points to larger square.]

BOY: Yes.

SOCR: Will it not have a side greater than this one and less than that? [Points as before.]

BOY: I think it will.

SOCR: Right. Always answer what you think. Now tell me: Was not this side two feet long and this one four? [Points as before.]

BOY: Yes.
SOCR: Then the side of the eight-foot figure must be longer than two feet but shorter than four?

BOY: It must.

SOCR: Try to say how long you think it is.

BOY: Three feet.

SOCR: If so, shall we add half of this bit and make it three feet? [Marks the point O in the figure so that BO is half of AB and completes the square AOPQ.] Here are two, and this is one, and on this side similarly we have two plus one, and here is the figure you want.

BOY: Yes.

SOCR: If it is three feet this way and three that, will the whole area be three times three feet?

BOY: It looks like it.

SOCR: And that is how many?

BOY: Nine.

SOCR: Whereas the square double our first square had to be how many?

BOY: Eight.

SOCR: But we haven't yet got a square of eight feet, even from a three-foot side?
BOY: No.

SOCR: Then what length will give it? Try to tell us exactly. If you don’t want to count it up, just show us on the diagram.

BOY: It’s no use, Socrates, I just don’t know.

SOCR: Observe, Meno, the stage he has reached on the path of recollection. At the beginning he did not know the side of a square of eight feet. Nor indeed, does he know it now, but then he thought he knew it and answered boldly, as was appropriate — he felt no perplexity — Now, however, he does feel perplexed. Not only does he not know the answer; he doesn’t even think he knows.

MENO: Quite true.

SOCR: Isn’t he in a better position now in relation to what he didn’t know?

MENO: I admit that too.

SOCR: So in perplexing him and numbing him like the stingray, have we done him any harm?

MENO: I think not.

SOCR: In fact we have helped him to some extent toward finding out the right answer, for now not only is he ignorant of it but he will be quite glad to look for it. Up to now, he thought he could speak well and fluently, on many occasions and before large audiences, on the subject of a square double the size of a given square, maintaining that it must have a side of double the length.

MENO: No doubt.

SOCR: Do you suppose then that he would have attempted to look for, or learn, what he thought he knew, though he did not, before he was thrown into perplexity, became aware of his ignorance, and felt a desire to know?

MENO: No.

SOCR: Then the numbing process was good for him?

MENO: I agree.

SOCR: Now notice what, starting from this state of perplexity, he will discover by seeking the truth, in company with me, though I simply ask him questions without teaching him. Be ready to catch me if I give him any instruction or explanation instead of simply interrogating him on his own opinions.

[Socrates erases the previous figure and starts again.]
Tell me, boy, is this not our square of four feet? You understand?

BOY: Yes.

SOCR: Now we can add another, equal to it, like this?

BOY: Yes.
SOCR: And a third here, equal to each of the others?

BOY: Yes.

SOCR: And we can fill in this one in the corner?
BOY: Yes.
SOCR: Then we have four equal squares?
BOY: Yes.

SOCR: And how many times the first square is the whole?
BOY: Four times.

SOCR: And we ant double the size. You remember?
BOY: Yes.

SOCR: Now do these lines, going from corner to corner, cut each of these squares in half? [Draws the diagonals.]
BOY: Yes.

SOCR: And these are four equal lines, enclosing this area? [Points to square BDNE.]

BOY: They are.

SOCR: Now think, how big is this area? [Points to square BDNE.]

BOY: I don’t understand.

SOCR: Here are four squares. Has not each line cut off the inner half of each of them?

BOY: Yes.

SOCR: And how many halves are there in this figure? [Points to BDNE.]

BOY: Four.

SOCR: And how many in this one? [Points to ABCD.]

BOY: Two.

SOCR: And what is the relation of four to two?

BOY: Double.

SOCR: How big is this figure then?

BOY: Eight feet.

SOCR: On what base?

BOY: This one. [Points to the segment BD.]

SOCR: The one that goes from corner to corner of the square of four feet?

BOY: Yes.

SOCR: The technical name for it is “diagonal”; so if we use that name, it is your personal opinion that the square on the diagonal of the original square is double its area.

BOY: That is so, Socrates.
SOCR: What do you think, Meno? Has he answered with any opinions that were not his own?

MENO: No, they were all his.

SOCR: Yet he did not know, as we agreed a few minutes ago.

MENO: True.

SOCR: So a man who does not know has in himself true opinions on a subject without having knowledge.

MENO: It would appear so.

SOCR: At present, these opinions, being newly aroused, have a dreamlike quality. But if the same questions are put to him on many occasions and in different ways you can see that in the end he will have a knowledge on the subject as accurate as anybody's.

MENO: Probably.

SOCR: This knowledge will not come from teaching but from questioning. He will recover it himself.

MENO: Yes.

SOCR: And the spontaneous recovery of knowledge that is in him is recollection, isn't it?

MENO: Yes.

++++++++++++++++++++++++++
Activity 11: Perform the Play.

Activity 12: The following are questions that will be discussed in class. Record your opinions in a journal.

Q. 1. Did Socrates convince you that you don’t learn — that your soul remembers? If not, what could Socrates have done to convince you of this?

Q. 2. What, if any, conclusions can you draw about the nature of Greek society in 380 B.C.E. from watching this play?

Q. 3. Why, do you think, did Socrates choose to have the slave boy “recall” mathematical rather than other facts?

Q. 4. What do you think of Socrates as a teacher? How would you like it if your teacher treated you or a fellow student the way Socrates treated the slave?

Q. 5. Did Socrates convince you that if you are given a square S with diameter d, then the square constructed on d has twice the area of S?

Q. 6. Are you now convinced that if you wanted to construct a square with twice the area of a given square you would construct it on the diagonal of the given square?

Activity 13: Try to explain to someone how to construct a square with twice the area of a given square and help that person understand why your method works.

Activity 14: Look for other instances of logical reasoning in Memo.

Activity 15: Read the Declaration of Independence and look for instances of logical reasoning.

Activity 16: Examine other important historical documents looking for instances of logical reasoning.

**BREAK**

*We reflect about what we have done and describe what still needs to be done.*

We hope that by now you can distinguish a proof from an argument that purports to be a proof but isn’t. Proofs need not be cumbersome and formal as they appear in some geometry books — a line of argument — considered a proof if it convinces the listener. Different listeners have different standards of rigor. Our point is that sufficient for the occasion is the proof that convinces you.

An equally important role that proofs play is explanation. You cannot be convinced of the truth of a statement unless you understand it, and if a statement is difficult enough, its proof must not only convince, it must also explain. Activity 13 is intended to convince you of this.

**Part 2.**

You will now construct a sequence of arguments that ought to convince anyone that the side and diagonal of a square are incommensurable. The proofs which we will help you construct are similar to those which, according to Knorr (1975) and Becker (1936) were used by the Pythagoreans to establish some of the
most basic properties of whole numbers. They assert that refinements of these arguments can be found in Euclid's famous *Elements*, and that subsequent developments of these lines of thought led to the Greeks' recognition of the fact that certain pairs of line segments are incommensurable.

Before you continue working on this module, you need to be aware of certain historical facts.

Today, beginning the second millennium, once we have agreed on a unit of length like the meter, we think of the length of a line segment as a number. One line segment may be 10 meters long, the length of another may be $\frac{37}{142}$ meters and the length of the diagonal of a square whose side is one meter long is $\sqrt{2}$ meters long. These lengths, 10, $\frac{37}{142}$, and $\sqrt{2}$ are all numbers. We think of the length of a line segment as a number that is attached to that segment to help us visualize how big it is.

It took humanity many years to arrive at this insight. In 380 B.C.E., people did not distinguish between a line segment and its length. It was the line segment, not its length that was thought of as a quantity. The Greeks used the word number exclusively to denote what are called today natural numbers or positive integers. Thus a line segment could not have a number attached to it to denote its length because what we call today the length of a line segment depends on the unit of measurement that is chosen. "Thus," an ancient Greek would say, "whether or not one can attach a number to a line segment depends on the unit of length that is the scale, that is selected. Since, from a theoretical point of view, one unit of length is as good as another, some segments will have length and others won't. Moreover, if you change your scales, segments that used to have lengths may no longer have them while segments that did not used to have lengths now do. Line segments, not their lengths, are quantities, and numbers and quantities are entirely different things."

Which brings us to the troublesome question regarding the nature of fractions. The Greeks did not have fractions. They had ratios instead. A ratio was neither a number nor a quantity, a ratio was a concept which enabled people to compare the size of one number with another number, or one quantity with another quantity. What we call today the fraction, that is the rational number, $\frac{3}{4}$ did not exist as far as the Greeks were concerned. They had instead the ratio $3:4$ that helped them conceptualize the relative sizes of the numbers 3 and 4. They were not allowed to divide 3 by 4. Today we divide 3 by 4 and obtain as answer the number $\frac{3}{4}$. I like to think of the ratio $3:4$ as the gain on the Cheshire cat: the last thing you see as the numbers 3 and 4, upon division, disappear and the number $\frac{3}{4}$ takes their place.

Now two quantities could also have a ratio, one to the other. If the two quantities were commensurable, their ratio was just like the ratio of two numbers. If the two quantities were not commensurable, this was no the case. The Greeks had an algorithm to help them determine whether or not the ratio of a pair of numbers was equal to the ratio of another pair. Thus they could determine whether or not the ratio of two commensurable quantities was equal to the ratio of another pair of commensurable quantities. It required an elaborate theory to determine how one should go about deciding when the ratios of two arbitrary pairs of quantities are equal, or when the ratio of a pair of quantities was as the ratio of a pair of numbers. We no longer need, nor do we study this theory. It became unnecessary when people realized that numbers, ratios and quantities can all be subsumed under the concept of the real number system.

But this does not mean that in creating this theory the ancient Greeks wasted their time. Without the insight they gained into the nature of commensurability, or better yet, incommensurability, we would not be able to distinguish between rational and irrational numbers, we would not have been able to create the concept of the real number system and much of modern mathematics would not exist today.

We are now ready to begin investigating the reasons and implications of the statement that the side and diagonal of a square are incommensurable. To be able to do that we need certain facts about the natural numbers. We will present the facts to you in the manner in which, Knorr (1975) suggests the Greeks discovered them. Our task will be to provide a translation into modern algebra of the work of the ancient Pythagoreans.
First of all, from now on we will use the word number in the Greek sense: natural number, positive whole number, positive integer. Whenever we use the word "number" in this way, we shall print it in boldface, thus: number.

According to Knorr, a number was represented by a set of pebbles arranged in a row, thus:

```
    ***  
```

The number seven.

This trend of thought led the Greeks to think of a single dot as a unit and to define a number as that which is obtained when units are added to each other. If we are given some number n today we might say: "Please buy n oranges at the grocery store." Mr. T would say: "Please buy as many oranges as there are units in this number."

**Definition 1:** An even number is one that can be partitioned into two equal parts. It will be represented thus:

```
    *** | ***  
```

The number six.

**Definition 2:** An odd number is that which cannot be separated into two equal parts. It will be represented thus:

```
    *** | ***  
```

The number seven.

**Definition 3:** The product of two numbers is the number obtained when the first is added to itself as many times as there are units in the second.

**Theorem 1:** The sum of an arbitrary amount of even numbers is an even number.

**Greek Proof:** Arrange the numbers as follows:

```
    *** | ***  
```

```
    *** | ***  
```

```
    *** | ***  
```

Now create a new number by arranging all the dots on the left of the vertical lines on the left of a new vertical line and all the dots on the right of the vertical lines on the right of the new vertical line. It is clear that the new number is even.

**Activity 17:** Create a "modern" proof of this theorem. Is there anything you find unsatisfactory about the Greek "proof"?

**Theorem 2:** The sum of two odd numbers is even.

**Greek proof:** Arrange the two numbers as follows:

```
    *** | ***  
```

Now create a new number by drawing a vertical line. Put all the dots on the left of the bracketed dots on the left of that vertical line and all the dots to the right of the bracketed dots to the right of
the vertical line. Finally put the top bracketed dot on the left of the new vertical line and the bottom bracketed dot on the right of that line. Clearly the number thus created is even.

Activity 18: Create a "modern" proof of this theorem. Is there anything you find unsatisfactory about the Greek "proof"?

Theorem 3: The sum of an even and an odd number is odd.

Greek proof: Arrange the given numbers as follows:

```
   *   * |*   *   *
   *   * |*   *   *
```

Now create a new number as follows: Draw a vertical line. Put all the dots that are on the left of the bracket dot in the first number and all the dots that are on the left of the vertical line in the second number on the left of the new vertical line. Put all the dots that are on the right of the bracketed dot in the first number and all the dots that are on the right of the vertical line in the second number on the right of the new vertical line. Now replace the new vertical line by the bracketed dot of the first number. Clearly the number so created is odd.

Activity 19: Create a "modern" proof of this theorem. Is there anything you find unsatisfactory about the Greek "proof"?

Theorem 3: The sum of an even quantity of odd numbers is even.

Activity 20: Create a "Greek proof" and a "modern" proof of this theorem. Is there anything you find unsatisfactory about the Greek "proof"?

Theorem 4: The sum of an odd amount of odd numbers is odd.

Activity 21: Create a "Greek proof" and a "modern" proof of this theorem. Is there anything you find unsatisfactory about the Greek "proof"?

Theorem 5: The product of an even number and any number is even. The product of two odd numbers is odd.

Activity 22: Create a proof that would be acceptable to Mr. T as well as to your American teacher.

Definition 4: A square number is the number obtained when some number is multiplied by itself. A square number can be represented like this:

```
   * * * * *
   * * * * *
   * * * * *
   * * * * *
   * * * * *
```

The square number 25.
Definition 5: If $X$ is a square number, the number that was multiplied by itself to obtain is called the side of $X$. (Today we call the side of $X$ the square root of $X$.)

Theorem 6: The square of an even number is even. The square of an odd number is odd.

Activity 23: Convince Mr. T and your American teacher.

Theorem 7: If a square number is even, its side is even. If a square number is odd, its side is odd.

Activity 24: Convince Mr. T and your American teacher.

Theorem 8: Any even square number is divisible into four equal parts.

Outline of Greek Proof:

```
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  * * | * * | * *
  * * | * * | * *
  * * | * * | * *
```

Activity 25: Complete the “Greek proof” and create a “modern” proof of this theorem. Is there anything you find unsatisfactory about the “Greek proof”?

Theorem 9: Any odd square number, when diminished by a unit becomes divisible into four equal parts.

Outline of Greek proof:

```
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```

Activity 26: Complete the “Greek proof” and create a “modern” proof of this theorem. Is there anything you find unsatisfactory about the Greek “proof”?

End of Preliminaries.

We are now ready to provide a proof very much like the one that Mr. T might have shown to his students.

Look at this diagram, which is the very one which Socrates showed to the slave boy.
We will show that the segments DB and DH are incommensurable.

Suppose they are not. Then a part of DB, say DX, measures both DB and DH. The ratio $\frac{DH}{DB}$ is then as the number of times DX goes into DH to the number of times that DX goes into DB. Let these numbers be $P$ and $Q$, that is $DH:DB::P:Q$. We may require that these numbers are a small as possible (that will cause them to be in reduced terms) which means that not both of them are even.

[Basic Assumption: $P$ and $Q$ are not both even.]

The squares DBHI and AGFE, are on the sides DB and AC respectively, and so their areas are square numbers. (Mr. T would say: "The squares DBHI and AGFE, are on the sides DB and AC respectively, and thus represent square numbers."

As far as Mr. T is concerned, square are quantities, which, under certain circumstances, may be like numbers.)

Now AGFE is the double of DBHI as Socrates pointed out is clear from the diagram. Thus AGFE represents an even square number and so its side AG represents an even number. But AG is equal (in length) to DH, and so DH represents an even number.

[First result: $P$ is even.]

Since AGFE represents an even square number, it can be divided into four equal parts. But AGCY.. is one such part, hence AGCY represents a number. Its double is the square number DBHI as was also pointed out by Socrates. Hence DBHI represents an even square number, and so its side DB represents an even number. That is: $Q$ is even.
[Second result: Q is even.]

Now compare our Basic Assumption these two results. They contradict each other. This means that the assumption:

"BD and DH are commensurable"

leads to a contradiction and so this assumption must be false. We are thus led inevitably to the conclusion:

The side and diagonal of a square are not commensurable.
Lesson 1.

Note to the teacher:

The purpose of this lesson is to acquaint students with what the Greeks meant by the expression: “The line segment $x$ measures the line segment $y$.”

This expression means: The length of the segment $x$ is $k$ times the length of the segment $y$, where $k$ is a positive integer.

Students will be given three sheets. Sheet 1 will contain several thin rectangles of various lengths and uniform widths. Students will pretend that these rectangles are line segments. Students will cut out these rectangles use them like rulers to “measure”, in the Greek sense, the other line segments. They will record their results on sheet 2. (Line segment 1 measures segments 1, 2, 5 etc.; it does not measure segments 3, 4, 6 etc.) Sheet 3 will be used to do the following exercises:

Guess whether the following statement is true or false:

1. If the segment $a$ measures the segment $b$, then the segment $b$ measures the segment $a$.

Convince me that your guess is correct.

37. (Note to the teacher: This statement is not correct, students should be able to come up with counter examples.)

2. It never happens that segment $a$ measures segment $b$ and at the same time segment $b$ measures segment $a$.

Convince me that your guess is correct.

38. (Note to the teacher: Segments $a$ and $b$ both measure each other if and only if they have the same length.)

3. If segment $a$ measures segment $b$ and segment $b$ measures segment $c$, then segment $a$ measures segment $c$.

Convince me that your guess is correct.

39. (Note to the teacher: The ideal argument would be as follows: Since segment $a$ measures segment $b$, the length of $b$ is $m$ times that of $a$. Since segment $b$ measures segment $c$, the length of $c$ is $n$ times that of $b$. Hence the length of $c$ is $mn$ times that of $a$, and so $a$ measures $c$.)

Question 4: In your own words explain what the Greeks meant when they said: “The segment $a$ measures the segment $b$.”
Worksheet 1.1

The Greeks said: "One line segment "measures" another if the length of the second is a whole number times the length of the second."

Cut out the rectangles marked 1, 2, ..., 10. Think of them as line segments.
**Worksheet 1.2**

Write the numeral 1 in the box in row i and column j if the i-th segment measures the j-th segment. If not, put a 0 in that box.

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Worksheet 1.3

Guess whether the following statement is true or false:

1. If the segment $a$ measures the segment $b$, then the segment $b$ measures the segment $a$.

   Convince me that your guess is correct.

2. It never happens that segment $a$ measures segment $b$ and at the same time segment $b$ measures segment $a$.

   Convince me that your guess is correct.

3. If segment $a$ measures segment $b$ and segment $b$ measures segment $c$ then segment $a$ measures segment $c$.

   Convince me that your guess is correct.

Question 4: In your own words explain what the Greeks meant when they said: "The segment $a$ measures the segment $b."
Lesson 2

Note to the teacher: The purpose of this lesson is to explain to the students what the Greeks meant by the expression: "Two line segments are commensurable."

The classical definitions are as follows:

1. A part of the segment a is a sub-segment b which measures a. (See Lesson 1 for a definition of the expression: "The segment b measures a."")

2. The line segment a is commensurable with segment b if a part of a measures b. This means that if the length of a is one unit, then the segment a can be divided into q equal parts in such a way that p of these parts will cover b. The length of the segment b is therefore p/q units.

3. Students will be given three worksheets. The first will be the same as Worksheet I of Lesson 1. Each line segment turns out to be commensurable with every line segment on that sheet.

4. Students will then fill in the table on worksheet 2 as follows: They will enter in the box in row i column j the length of the j-th segment when the i-th segment is considered to be one unit long.

5. On worksheet 3 students will be asked to guess whether the following statement is true: "If segment a is commensurable with segment b then segment b is commensurable with segment a." They should be able to guess that the answer is yes. They will then be asked to convince their teacher that their guess is correct. (If the length of segment a is commensurable with b then the length of b is p/q units when the length of a is taken as the unit. Hence the length of a is q/p units if the length of b is taken as the unit. Hence b is commensurable with a.)

6. Students should then be told:

1. In view of 5 we will say from now on that the segments a and b are commensurable.

2. In spite of the results shown in your table, there exist pairs of line segments that are not commensurable. Such line segments are called incommensurable."
Worksheet 2.1

Cut out these rectangles marked 1, 2, 3, ..., 10. Think of them as line segments. The Greeks said: "Line segment a is commensurable with line segment b if the segment a can be broken up into equal sub-segments in such a way that a sub-segment of a measures segment b." Compare each pair of line segments and determine if the first is commensurable with the second.
Worksheet 2.2

Suppose that segment a is commensurable with segment b. Then if the length of a is taken as one unit, the length of segment b will be \( p/q \) units where \( p \) and \( q \) are positive whole numbers. (\( q \) will be the number of parts into which segment a must be divided and \( p \) will be the number of these pieces which, laid end to end, will cover segment b.) In the table below, indicate in the box in row \( i \) and column \( j \) the length of the segment \( j \) when segment \( i \) is used as a unit.

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Worksheet 2.3

Guess whether the following statement is true or false:

If segment $a$ is commensurable with segment $b$ then segment $b$ is commensurable with segment $a$.

Convince me that your guess is correct.

The italicized statement is true. In view of this fact Mr. T would say: "Segments $a$ and $b$ are commensurable" instead of "$a$ is commensurable with $b$.

Complete the sentence: "Two segments $a$ and $b$ are commensurable if...

Convince me: "If segments $a$ and $b$ are commensurable and segments $b$ and $c$ are commensurable, then segments $a$ and $c$ are commensurable."

Important Information: For thousands of years people thought that all pairs of line segments were commensurable. Today (380 B.C.E.) Mr. T knows that this isn’t so; there are pairs of line segments that are not commensurable, they are called incommensurable. We’ll explore this phenomenon in more detail in this module.
Lesson 3

Assignment: Draw a square whose area is twice the area of this square.
STUDENT EVALUATION

Name of Activity: ______________________________

1. What did you learn from this activity?

2. How might the activity be changed to make it better?

3. How was this set of activities different than what you have done in previous courses?

4. Suggest 2 or more ways the material could be improved.

5. How did the historical information influence your attitude and ideas about mathematics?
THEOREMS

1. The sum of an arbitrary amount of even natural numbers is even.
2. The sum of two odd natural numbers is even.
3. The sum of an even natural number and an odd natural number is odd.
4. The sum of an even quantity of odd natural numbers is even.
5. The sum of an odd amount of odd natural numbers is odd.

SCORING RUBRIC

Good Responses

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<tr>
<th>6 Exemplary Proof</th>
<th>Gives a complete proof with clear, coherent, unambiguous explanations</th>
</tr>
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<tbody>
<tr>
<td>5 Competent Proof</td>
<td>Gives a complete proof with reasonably clear explanations</td>
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</tbody>
</table>

Adequate Responses

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<tr>
<th>4 Minor Flaws, But Satisfactory Proof</th>
<th>Completes the proof satisfactorily, but the explanation may be muddled</th>
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<tbody>
<tr>
<td>3 Serious Flaws, But Satisfactory Proof</td>
<td>Begins the proof appropriately but may fail to complete or may omit significant parts of the proof</td>
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</table>

Inadequate Responses

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<tr>
<th>2 Unsatisfactory Proof</th>
<th>Proof is not understandable</th>
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<tbody>
<tr>
<td>1 Unable to Effectively Begin Proof</td>
<td>Work does not reflect a proof; no attempt</td>
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</table>
REFERENCES


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