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Entitled

Average Cayley Genus for Cayley Maps with Dihedral Groups

is approved in partial fulfillment of the requirements for the degree of

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ABSTRACT

Average Cayley Genus for Cayley Maps with Dihedral Groups

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Let $\Gamma$ be a finite group and let $\Delta$ be a generating set for $\Gamma$. A Cayley map associated with $\Gamma$ and $\Delta$ is an oriented 2-cell embedding of the Cayley graph $G_{\Delta}(\Gamma)$ such that the rotation of arcs emanating from each vertex is determined by a unique cyclic permutation of generators and their inverses. A formula for the average Cayley genus is known for the dihedral group with generating set consisting of all the reflections. However, the known formula involves sums of certain coefficients of a generating function and its format does not specifically indicate the Cayley genus distribution. We determine a simplified formula for this average Cayley genus as well as provide improved understanding of the Cayley genus distribution.
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ACKNOWLEDGMENTS

I would like to take this opportunity to thank Dr. Michelle Schultz for all her help, support and patience in being my advisor, a mentor and a friend. I would also like to thank Dr. Gennady Bachman, Dr. Douglas Burke and Dr. William Wells for their time and help. Finally, I would like to thank my family, especially my parents JoAnn and Manny, for without their support and care I wouldn’t have made it this far.
A graph is a finite nonempty set of objects called vertices together with a set of unordered pairs of distinct vertices called edges. The number of vertices is the order of the graph and the number of edges is the size of the graph. A question that arises is, is it possible to draw (or embed) a graph in the plane so that none of its edges cross? These types of graphs are called planar graphs. Through stereographic projection, embedding a graph in the plane is equivalent to embedding that same graph on the sphere, which we denote by \( S_0 \). However, there are many graphs that cannot be embedded on the sphere so that none of its edges cross. We define a surface to be a closed orientable 2-manifold, which can be thought of as a sphere with handles (or equivalently, holes). The number of handles (or holes) is the genus of the surface. For an integer \( k \geq 0 \), let \( S_k \) denote the surface of genus \( k \).

We now formalize the concept of embedding. A graph \( G \) embeds on a surface \( S_k \) if there exists a homeomorphism of \( G \) as a finite 1-complex in \( \mathbb{R}^3 \) with a subspace of \( S_k \). If every region of an embedding of \( G \) on \( S_k \) is homeomorphic to an open disk, then \( G \) is 2-cell embedded on \( S_k \) and we write \( G \prec S_k \). In this work, we require all embeddings to be 2-cell. Figure 1A shows an example of a graph that is drawn on the torus. The left figure is not 2-cell embedded on the torus since a closed curve can be drawn in the outer
region as shown that cannot be contracted to a point, whereas the right figure is the same graph redrawn so that it is now 2-cell embedded on the torus.

![Graphs](image)

Figure 1.A Example of Non 2-Cell Embedding and 2-Cell Embedding.

What we would like to do is to be able to tell which surfaces we are able to 2-cell embed a graph on. To find which surface a graph is embedded on, we often make use of Euler’s identity.

**Theorem 1.1** If $G$ is a connected graph with order $p$ and size $q$ that is 2-cell embedded on $S_k$ with $r$ regions, then

$$p - q + r = 2 - 2k,$$

where the value $2 - 2k$ is called the Euler characteristic of $S_k$.

Let $G$ be a connected graph. Then the genus $\gamma(G)$ of $G$ is the minimum non-negative integer $k$ such that $G$ embeds on $S_k$. A rotation embedding scheme $\rho$ is a collection of cyclic permutations $\rho_v : N(v) \to N(v)$, one for each $v \in V(G)$, where $V(G)$ is the set of vertices of $G$ and $N(v)$ denotes the neighborhood of $v$. It is well
known (see Edmonds [4]) that the 2-cell embeddings of a connected graph $G$ are in one-to-one correspondence with the rotation schemes of $G$.

**Theorem 1.2** Let $G$ be a connected graph with $V(G) = \{1, 2, \ldots, p\}$. If $G \triangleleft S_k$, then this 2-cell embedding uniquely determines a rotation scheme

$$\mathcal{G} = \{\rho_i : N(i) \rightarrow N(i) | 1 \leq i \leq p\}.$$  

Conversely, such a rotation scheme uniquely determines a 2-cell embedding of $G$ on some surface.

It is not difficult to count the number of distinct rotation schemes for a given labeled connected graph $G$. Indeed for a connected graph $G$ with $V(G) = \{1, 2, \ldots, p\}$, there are \(\prod_{i=1}^{p} (\deg(i) - 1)!\) such schemes and due to Theorem 1.2, there are then \(\prod_{i=1}^{p} (\deg(i) - 1)!\) many labeled 2-cell embeddings of $G$. Since there are a finite number of 2-cell embeddings of a given graph, there is a surface of maximum genus on which the graph can be 2-cell embedded. We now define the maximum genus $\gamma_M(G)$ of a graph $G$ to be the maximum integer $k$ such that $G \triangleleft S_k$. So now we have a minimum value and a maximum value, but what of the intermediate values? Duke [3] established the intermediate value theorem for genus distribution, typically referred to as Duke's Theorem.
**Theorem 1.3** There is a 2-cell embedding of a connected graph $G$ on $S_k$ if and only if

$$\gamma(G) \leq k \leq \gamma_M(G).$$

One of the major areas of research has been the study of all 2-cell embeddings of a labeled connected graph $G$ and, in particular, the enumeration of the embeddings of $G$ on a given surface and the determination of the average genus of $G$.

For example, we shall look at the three non-labeled 2-cell embeddings of the complete graph $K_4$ (See Figure 1.B). So what we would like to do now is to enumerate the labeled embeddings. Figure 1.B.a is $K_4$ drawn on the plane and it has two labeled embeddings. Figure 1.B.b and 1.B.c are $K_4$ drawn on the torus (using the planar representation of a torus). Labeling these embeddings, figure 1.B.b has eight labeled embeddings and figure 1.B.c has six labeled embeddings. So there is a total of sixteen labeled 2-cell embeddings with two on $S_0$ and fourteen on $S_1$. Hence, the average genus for $K_4$ is

$$\frac{2(0) + 14(1)}{16} = \frac{14}{16}.$$ 

In general, it is a difficult problem to determine the average genus for classes of graphs.

![Figure 1.B](https://example.com/figure1b.png)

Figure 1.B The Three Embeddings of $K_4$, (a) on the Plane and (b) and (c) on the Torus.
If \( \varphi \) is a rotation scheme for \( G \), then the ordered pair \((G, \varphi)\) is called a map and we say that the genus \( g(G, \varphi) \) of the map \((G, \varphi)\) is \( k \) if \( \varphi \) determines a 2-cell embedding of \( G \) on \( S_k \). Thus, \( \gamma(G) = \min_{\varphi} g(G, \varphi) \) and \( \gamma(\mathcal{M})(G) = \max_{\varphi} g(G, \varphi) \).

The embeddings having non-trivial symmetries are often represented by smaller graph embeddings, called voltage graph embeddings. For our purposes, we will concentrate on the index one voltage graphs (those with exactly one vertex). The reason to focus on these particular voltage graphs is that our primary interest is in studying certain embeddings of Cayley graphs and these are precisely the coverings of index one voltage graphs.

Let \( \Gamma \) be a finite group and \( \Delta \) be a generating set for \( \Gamma \) such that the identity \( e \not\in \Delta \).

Also let \( \Delta^{-1} = \{ \delta^{-1} \mid \delta \in \Delta \} \) and \( \Delta^* = \Delta \cup \Delta^{-1} \). Furthermore, let \( \Delta \) be chosen so that if \( \delta \in \Delta \cap \Delta^{-1} \), then \( \delta^2 = e \). That is, if \( \delta \) is chosen as a generator, then \( \delta^{-1} \) is not chosen, unless \( \delta^2 = e \) (\( \delta \) is its own inverse). The Cayley graph \( G_{\Delta}(\Gamma) \) is that graph whose vertex set is \( \Gamma \) and edge set is \( \{ \{x, x\delta \} \mid x \in \Gamma, \delta \in \Delta^* \} \). After defining a Cayley graph, we can now define a Cayley map. Let \( \Gamma \) be a group and \( \Delta \) be a generating set for \( \Gamma \) with the usual properties, i.e. \( e \not\in \Delta \) and if \( \delta \in \Delta \cap \Delta^{-1} \), then \( \delta^2 = e \) (whenever we consider a generating set, we will always require that it satisfy these conditions). For a cyclic permutation \( \rho: \Delta^* \to \Delta^* \), the Cayley map \((\Gamma, \Delta, \rho)\) is the map \((G_{\Delta}(\Gamma), \varphi)\) where \( \varphi = \{ \rho_x \mid x \in \Gamma \} \) is the rotation scheme for \( G_{\Delta}(\Gamma) \) such that \( \rho_x(y) = x\rho(x^{-1}y) \) for each \( x \in \Gamma \) and each \( y \in N(x) \) (see figure 1.C). In other words, a Cayley map is a 2-cell
embedding of a Cayley graph in which each vertex rotation $\rho_x$ is determined by the same cyclic ordering of the elements of $\Delta^*$.

As said earlier, Cayley maps are coverings of index one voltage graphs. Let $K$ be a pseudograph of order 1. So $K$ consists of one vertex and a finite number of loops. If $e$ is a directed loop of $K$ then we denote the reverse direction of $e$ by $e^{-1}$. Let $K^*$ be the set of all the directed loops of $K$ and their inverses, that is, $K^* = \{ e, e^{-1} | e \in E(K) \}$. Now let $\Gamma$ be a group and let $\phi: K^* \rightarrow \Gamma$ be a one-to-one function satisfying $\phi(e^{-1}) = (\phi(e))^{-1}$ for every $e \in K^*$. Then the ordered triple $(K, \Gamma, \phi)$ is an example of a voltage graph. More specifically, $(K, \Gamma, \phi)$ is an index one voltage graph since it has order 1. The index of a voltage graph is simply the order of the pseudograph used. When the group $\Gamma$ and function $\phi$ are unambiguous, we say simply that $K$ is the voltage graph, where now the group and function are understood.

We now describe the covering of a voltage graph. Let $\Gamma$ be a group with generating set $\Delta$ and let $\Delta^* = \Delta \cup \Delta^{-1}$. Let $(K, \Gamma, \phi)$ be an index one voltage graph such that
\( \phi(E(K)) = \Delta \). That is, \( \phi \) assigns a generator to each directed loop of \( K \). Now, let 
\((K, \Gamma, \phi)\) be 2-cell embedded on a surface. The cyclic ordering of the labeled arcs emanating from the one vertex gives a cyclic permutation \( \rho: \Delta^* \rightarrow \Delta^* \). This voltage graph embedding determines a 2-cell embedding of the Cayley graph \( G_\Delta(\Gamma) \) in some surface and this embedding of \( G_\Delta(\Gamma) \) is the Cayley map \((\Gamma, \Delta, \rho)\). The Cayley graph \( G_\Delta(\Gamma) \) is called the covering graph of the voltage graph \((K, \Gamma, \phi)\) and the embedding of \((K, \Gamma, \phi)\) is said to lift to the embedding of \( G_\Delta(\Gamma) \). We say the embedding of \( G_\Delta(\Gamma) \) is the covering embedding of the voltage graph embedding. However, in voltage graphs, a problem arises when generators of order 2 are used. A directed loop of order two produces a pair of multiple edges in the covering. To avoid this, in voltage graphs, a generator of order 2 is represented by a half-edge.

A voltage graph can be used to find the number of regions and the structure of the regions in the covering embedding. A walk \( w \) in an index one voltage graph \((K, \Gamma, \phi)\) is a sequence \( f_1, f_2, \ldots, f_m \) of elements of \( K^* \). Define \( \phi(w) = \prod_{i=1}^{m} \phi(f_i) \). Thus, \( \phi(w) \) is an element of \( \Gamma \). Now let \( R \) be a region of a 2-cell embedding of \((K, \Gamma, \phi)\), and let \( w: f_1, f_2, \ldots, f_m \) denote the walk in \( K \) that consists of the oriented boundary of \( R \). Then \( \phi(w) \) is called the boundary element of \( R \) and \( \text{ord}(\phi(w)) \) is the order of the boundary element of \( R \). This leads to a result by Gross and Alpert [5].
Theorem 1.4  If $R$ is a $k$-gon in a 2-cell embedding of an index one voltage graph $(K, \Gamma, \phi)$ and $s$ is the order of the boundary element of $R$, then $R$ lifts to $\frac{|\Gamma|}{s}$ $ks$-gons in the covering of $(K, \Gamma, \phi)$.

This result holds for general voltage graphs, but again, we are specializing to the case of the index one voltage graphs.

If a given Cayley map $(\Gamma, \Delta, \rho)$ determines a 2-cell embedding of $G_\Delta(\Gamma)$ in $S_k$, then $k$ is called the genus $g(\Gamma, \Delta, \rho)$ of the Cayley map $(\Gamma, \Delta, \rho)$. The Cayley genus $\gamma(\Gamma, \Delta)$ is defined as $\gamma(\Gamma, \Delta) = \min_{\rho} g(\Gamma, \Delta, \rho)$ and the maximum Cayley genus $\gamma_M(\Gamma, \Delta)$ is defined by $\gamma_M(\Gamma, \Delta) = \max_{\rho} g(\Gamma, \Delta, \rho)$. Since each Cayley map $(\Gamma, \Delta, \rho)$ is a 2-cell embedding of the Cayley graph $G_\Delta(\Gamma)$, we obtain the inequality $\gamma(G_\Delta(\Gamma)) \leq \gamma(\Gamma, \Delta) \leq \gamma_M(\Gamma, \Delta) \leq \gamma_M(G_\Delta(\Gamma))$. The value that we concentrate on is the average Cayley genus. The average Cayley genus $\overline{\gamma}(\Gamma, \Delta)$ is the average of the genera of all Cayley maps for some group $\Gamma$ with fixed generating set $\Delta$.

So far, we have been using $\Gamma$ as an arbitrary group. What we shall do now is specialize to a particular group. We focus our attention on the dihedral group $D_m$ of order $2m$, where $m \geq 3$. The presentation we use is $D_m = \langle x, y \mid x^m = y^2 = (xy)^2 = e \rangle$. Thus, $D_m = \{ e, x, x^2, \ldots, x^{m-1}, y, xy, x^2y, \ldots, x^{m-1}y \}$ where $e, x, x^2, \ldots, x^{m-1}$ represent the rotations and $y, xy, x^2y, \ldots, x^{m-1}y$ are the reflections. Let
\[ \Delta = \{ y, xy, x^2 y, \ldots, x^{m-1} y \}. \] The Cayley graph \( G_\Delta(D_m) \) is \( K_{m,m} \), the complete bipartite graph. Since \( (x^k y)^2 = e \) for all \( k \in \mathbb{Z} \), we get that \( yx^k = x^{m-k} y \). When \( m \) is odd, the genus for the Cayley maps \( (D_m, \Delta, \rho) \) is constant, as was shown in [9].

**Theorem 1.5** Let \( m = 2n + 1 \geq 3 \). If \( \Delta \) is the generating set consisting of the reflections of the dihedral group \( D_m \), then \( g(D_m, \Delta, \rho) = n(2n-1) \) for every cyclic permutation \( \rho : \Delta \to \Delta \). Therefore the average Cayley genus \( \overline{\gamma}(D_m, \Delta, \rho) = n(2n-1) \).

So each Cayley map \( (D_{2n+1}, \Delta, \rho) \) embeds \( K_{2n+1, 2n+1} \) on a surface of genus \( n(2n-1) \). So the Cayley genus and maximum Cayley genus are equal for this group and generating set, meaning \( \gamma(D_{2n+1}, \Delta) = \gamma_M(D_{2n+1}, \Delta) = n(2n-1) \). As a result of the genus being constant when \( m \) is an odd integer, we will concentrate on \( m \) being an even integer. So \( D_m = D_{2n} \) for \( n \geq 2 \). Again, we take \( \Delta \) as the generating set consisting of all the reflections, that is, \( \Delta = \{ y, xy, x^2 y, \ldots, x^{2n-1} y \} \) and so the Cayley graph is \( G_\Delta(D_{2n}) = K_{2n, 2n} \).

Next, we need to introduce generating functions. A generating function, as mentioned in Brualdi [2], can be thought of as an object, which through manipulation, allows us to count a number of possibilities for a problem. Also, Brualdi indicates that generating functions can be thought of as Taylor series (power series expansions) of infinitely differentiable functions. By finding the function and the Taylor series, the coefficients give the solution. If we let \( h_0, h_1, h_2, \ldots, h_n, \ldots \) be an infinite sequence of
numbers, then its generating function is the infinite series
\[ g(x) = h_0 + h_1 x + h_2 x^2 + \ldots + h_n x^n + \ldots. \]

So, for an example of a generating function, the binomial coefficients in the
expansion of \((a + b)^m \) are \( \binom{m}{0}, \binom{m}{1}, \binom{m}{2}, \ldots, \binom{m}{m} \). The generating function for
these coefficients is \( g_m(x) = \binom{m}{0} + \binom{m}{1} x + \binom{m}{2} x^2 + \ldots + \binom{m}{m} x^m \), or \( g_m(x) = (1+x)^m \) by
the binomial theorem.

Finally, we need to introduce the idea of partitions of integers (see Andrews [1], for
element). A partition of a positive integer \( n \) is a finite non-increasing sequence of
positive integers \( \lambda_1, \lambda_2, \ldots, \lambda_r \) such that \( \sum_{i=1}^{r} \lambda_i = n \). The numbers \( \lambda_i \), for \( 1 \leq i \leq r \) are
called the parts of the partition. So as an example, let’s consider the partitions of 6. The
only partition of 6 with six parts is \( 1+1+1+1+1+1 \). The only partition of 6 with five parts
is given by \( 2+1+1+1+1 \). There are two partitions of 6 with four parts, \( 3+1+1+1 \) and
\( 2+2+1+1 \). The three partitions of 6 into three parts are \( 4+1+1, 3+2+1 \) and \( 2+2+2 \). We
can partition 6 into two parts three ways, namely \( 5+1, 4+2 \) and \( 3+3 \). Finally, 6 is the only
way to partition 6 into one part. Partitions can also be restricted. For example, the
number of parts can be restricted as well as the size of each part. For our work with
Cayley maps for dihedral groups, it will be useful to know the number of partitions of
each of the integers \( \binom{n}{2}, \binom{n}{2}+1, \binom{n}{2}+2, \ldots, 3 \binom{n}{2} \) into exactly \( n-1 \) unequal parts
with no part greater than \( 2n-1 \).
Let $k$ and $j$ be positive integers with $k \leq j$. Then from Riordan [8], we have a generating function $u_j(t, k)$ for the number of partitions having $k$ unequal parts with no part greater than $j$. That generating function is as follows:

$$u_j(t, k) = \begin{cases} \binom{k+1}{2} \frac{(1-t)(1-t^2)(1-t^{j-1})...(1-t^{j-k+1})}{(1-t^2)(1-t^3)...(1-t^k)} & \text{if } k < j \\ \binom{j+1}{2} & \text{if } k = j \end{cases}$$

In Riordan, this generating function appears as an exercise. To see where it comes from, recall what we are trying to do. We wish to enumerate the number of partitions with $k$ unequal parts, no part greater than $j$. The function for this restricted partition is given by $G_j(t, a) = (1 + at)(1 + at^2)...(1 + at^j)$. Since we want unequal parts, the general factor is $1 + at^i$. The term $at^i$ in this general factor is the indicator for one occurrence of $i$. Since we want no part greater than $j$, our last factor is $1 + at^j$. So the exponent of $a$ keeps track of how many parts are being used. Thus, $G_j(t, a) = (1 + at)(1 + at^2)...(1 + at^j) = \sum_{k=0}^{j} u_j(t, k) a^k$, where $u_j(t, k)$ is the generating function we are looking for. We define $u_j(t, 0) = 1$, the empty partition. Start by considering

$$G_j(t, a t) = (1 + at^2)(1 + at^3)...(1 + at^j)$$

so that

$$(1 + at)G_j(t, at) = G_j(t, a)(1 + at^{j+1}),$$

or,

$$(1 + at)\sum_{k=0}^{j} u_j(t, k) t^k a^k = (1 + at^{j+1})\sum_{k=0}^{j} u_j(t, k) a^k,$$

or by distributing,
\[
\sum_{k=0}^{i} u_j(t, k) t^k + \sum_{k=0}^{i} u_j(t, k) t^{k+1} = \sum_{k=0}^{i} u_j(t, k) t^k + \sum_{k=0}^{i} u_j(t, k) t^{k+1} = \sum_{k=0}^{i} \left( t^k - t^{k+1} \right) u_j(t, k) t^k ,
\]

so that
\[
\sum_{k=0}^{i} \left( t^k - 1 \right) u_j(t, k) t^k = \sum_{k=0}^{i} \left( t^{k+1} - t^k \right) u_j(t, k) t^{k+1} = \sum_{k=0}^{i+1} \left( t^{k+1} - t^k \right) u_j(t, k-1) t^k ,
\]

where the last equality holds by changing the index. Hence, by comparing coefficients,
\[
\left( t^k - 1 \right) u_j(t, k) = \left( t^{k+1} - t^k \right) u_j(t, k-1) \text{ for each } k = 1, 2, \ldots, j .
\]

Being true for some \( k \ (2 \leq k \leq j) \), we have the same equation true for \( k - 1 \) and so
\[
\left( t^{k-1} - 1 \right) u_j(t, k-1) = \left( t^{k+1} - t^k \right) u_j(t, k-2) \text{ and thus }
\]
\[
\left( t^{k-1} - 1 \right) \left( t^k - 1 \right) u_j(t, k) = \left( t^{k+1} - t^k \right) \left( t^{k+1} - t^{k-1} \right) u_j(t, k-2).
\]

Continuing in this manner until we get to \( u_j(t, 0) = 1 \), we find
\[
(t-1)(t^2-1)\ldots(t^{k-1}-1)(t^k-1) u_j(t, k) = \left( t^{j+1} - t^k \right) \left( t^{j+1} - t^{k-1} \right) \ldots \left( t^{j+1} - t^1 \right) u_j(t, 0)
\]
or that
\[
u_j(t, k) = \frac{\left( t^{j+1} - t^k \right) \left( t^{j+1} - t^{k-1} \right) \ldots \left( t^{j+1} - t^1 \right)}{(t-1)(t^2-1)\ldots(t^k-1)} .
\]

By factoring out powers of \( t \) in the numerator and multiplying every factor in the numerator and denominator by \(-1\), we have
\[
u_j(t, k) = \frac{t \cdot t^2 \ldots t^k \left( 1-t^j \right) \left( 1-t^{j-1} \right) \ldots \left( 1-t^{j-k+1} \right)}{(1-t)(1-t^2)\ldots(1-t^k)}
\]
\[
u_j(t, k) = \begin{cases} 
\binom{k+j}{2} \frac{\left( 1-t^j \right) \left( 1-t^{j-1} \right) \ldots \left( 1-t^{j-k+1} \right)}{(1-t)(1-t^2)\ldots(1-t^k)} & \text{if } k < j \\
\binom{j+1}{2} & \text{if } k = j
\end{cases}
\]
We shall use \([t^i]u_j(t, k)\) to denote the coefficient of \(t^i\) in \(u_j(t, k)\). So, \([t^i]u_j(t, k)\) is the number of partitions of the integer \(i\) into \(k\) unequal parts having no part greater than \(j\). So, for us with our interest in Cayley maps for dihedral groups, we, again, want \(n-1\) unequal parts with no part greater than \(2n-1\). So we are interested in \(u_{2n-1}(t, n-1)\).
CHAPTER 2

EXISTING FORMULA

The goal of this thesis is to devise an improved formula for finding the average Cayley genus of Cayley maps with the dihedral group $D_{2n}$ and generating set $\Delta$ consisting of all the reflections of $D_{2n}$. Prior to working on a new formula, we provide a study of the existing formula.

As we saw in Chapter 1, we can use Euler's formula (Theorem 1.1), to determine the genus of a particular Cayley map as long as we know the order of the graph, the size of the graph, and the number of regions in the embedding. The Cayley maps we are interested in are $(D_{2n}, \Delta, \rho)$, where the group chosen is the dihedral group, which has order $4n$, and the generating set $\Delta$ consists of all the reflections. So, $\Delta = \{y, xy, x^2y, \ldots, x^{2n-1}y\}$. Also, let $\rho$ be a cyclic permutation of $\Delta$, say $\rho = (y, x^ky, x^{k+1}y, \ldots, x^{k_{2n-1}}y)$, where $\{k_1, k_2, \ldots, k_{2n-1}\} = \{1, 2, \ldots, 2n-1\}$. The voltage graph whose lift is the Cayley map $(D_{2n}, \Delta, \rho)$ is shown in Figure 2.A.

Recall that the Cayley graph is $G_\Delta(D_{2n}) = K_{2n, 2n}$. For this graph, the order $p$ is the same as the order of the group, so that $p = 4n$. The size $q$ can be found by noticing that the graph being a complete bipartite graph has each vertex in one of the partite sets (there are $2n$ here) adjacent to the $2n$ vertices in the other partite set. Hence, $q = 4n^2$. 

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Using Theorem 1.4 we can find the number of the regions in the Cayley map $(D_{2n}, \Delta, p)$. Since there is only one region in the voltage graph as seen in Figure 2.A, its lift will have $\left| \Gamma \right| = \frac{4n}{s}$ regions, where $s$ is the order of the boundary element of the region that is being used. Thus, using $p = 4n$, $q = 4n^2$ and $r = \frac{4n}{s}$ in Euler's formula and solving for $k$ we get:

$$p - q + r = 2 - 2k,$$

and by substituting,

$$4n - 4n^2 + \frac{4n}{s} = 2 - 2k,$$

and by rearranging,

$$2k = 2 - 4n + 4n^2 - \frac{4n}{s},$$

and solving for $k$,

$$k = 1 + 2n \left[ n - 1 - \frac{1}{s} \right].$$

---

Figure 2.A  Voltage Graph Embedding.
So, to find the genus $k$ given $n$ it remains to find $s$. Again, $s$ is the order of the boundary element. But the problem arising with finding the order for a boundary element is, we start out with $(2n-1)!$ possible cases of boundary elements taking into consideration all the possible cyclic arrangements of the generators. Before we find $(2n-1)!$ orders, we’d like to be able to reduce the number of cases if we can. We start by looking at exactly what a boundary element is. The general boundary element for the one region of our voltage graph is $yx^k yx^{k_2} yx^{k_3} ... yx^{k_{2n-2}} yx^{k_{2n-1}} y$, where $\{k_1, k_2, ..., k_{2n-1}\} = \{1, 2, ..., 2n-1\}$. We group these elements in such a way to see that the boundary element is $(yx^k)(yx^{k_2})...(yx^{k_{2n-2}})(yx^{k_{2n-1}}) y$. So group an $x$ that has an exponent with an odd index of $k$ with the $y$ that comes before that $x$ term. What this allows is to use the properties that $yx^k = x^{-k} y$ and $y^2 = 1$ to see that the boundary element is really just $x^{-k_1 + k_2 - k_3 + k_4 - ... + k_{2n-3} - k_{2n-1}}$. So now we are trying to find $s = \text{ord}(x^{-k_1 + k_2 - k_3 + k_4 - ... + k_{2n-3} - k_{2n-1}})$. We know that $\text{ord}(x) = 2n$ so that $\text{ord}(x^k) = \frac{2n}{\gcd(2n, k)}$.

What we do next is to reorder the exponent of the boundary element to find that we now need to find $\text{ord}(x^{k_2 + k_4 + k_6 + ... + k_{2n-1} - (k_1 + k_3 + k_5 + ... + k_{2n-3})})$. What this means is that we no longer have $(2n-1)!$ cases. All we need to do now is, in the new exponent, choose the $n-1$ values to be added from the $2n-1$ total values. The values that are left will be subtracted, and the order we add the values doesn’t matter since addition is commutative.

So we now have $\binom{2n-1}{n-1}$ distinct choices for

$$\{k_2, k_4, k_6, ..., k_{2n-2}\} \subseteq \{1, 2, 3, ..., 2n-1\}.$$
In the \( \binom{2n-1}{n-1} \) cases, it can happen that \( A = \{k_2, k_4, \ldots, k_{2n-2}\} \) and \( A' = \{k'_2, k'_4, \ldots, k'_{2n-2}\} \) with \( A \neq A' \) but \( \sum_{k \in A} k = \sum_{k' \in A'} k' \). So we can narrow down the cases further by finding all the different sums that are possible. Look at the possible combinations from the set \( \{1, 2, 3, \ldots, 2n-1\} \) that we can sum together. The least possible sum is \( 1 + 2 + 3 + \ldots + (n-2) + (n-1) = \binom{n}{2} \). The next sum that would be possible is given by \( 1 + 2 + 3 + \ldots + (n-2) + n = \binom{n}{2} + 1 \) and the sum after this is \( 1 + 2 + 3 + \ldots + (n-2) + (n+1) = \binom{n}{2} + 2 \) and so on until the final sum. The last sum is \( (n+1) + (n+2) + \ldots + (2n-1) = \binom{2n}{2} - \binom{n+1}{2} = 3 \binom{n}{2} \). So all the possible sums are \( \binom{n}{2}, \binom{n}{2} + 1, \binom{n}{2} + 2, \ldots, 3 \binom{n}{2} \). So this has reduced the number of cases to \( 2 \binom{n}{2} + 1 \).

Also, we have a way to simplify the exponent in \( x^{k_2+k_4+k_6+\ldots+k_{2n-2}-(k_1+k_3+k_5+\ldots+k_{2n-1})} \). In this exponent, let \( S = k_2 + k_4 + k_6 + \ldots + k_{2n-2} \). Then \( k_1 + k_3 + k_5 + \ldots + k_{2n-1} = \binom{2n}{2} - S \). So

\[
x^{k_2+k_4+k_6+\ldots+k_{2n-2}-(k_1+k_3+k_5+\ldots+k_{2n-1})} = x^{S-(\binom{2n}{2}-S)} = x^{2S-(\binom{2n}{2})}.
\]

So the cases at this point are for each \( S = \binom{n}{2}, \binom{n}{2} + 1, \binom{n}{2} + 2, \ldots, 3 \binom{n}{2} \), we must find the order of \( x^{2S-(\binom{2n}{2})} \).

Further, in order to be able to obtain our average Cayley genus, we must also count the number of different cyclic arrangements of generators that produce each such case.
Thus, we want to find how many different sets $A = \{k_2, k_4, \ldots, k_{2n-2}\}$ give the same sum. Each set $A = \{k_2, k_4, \ldots, k_{2n-2}\}$ gives a partition of $\sum_{k \in A} k$ into $n-1$ unequal parts, with no part greater than $2n-1$. A generating function for the number of partitions of an integer with $k$ unequal parts, no part greater than $j$ is given by $u_j(t, k)$ as seen in Chapter 1 and for our situation we are interested in $u_{2n-1}(t, n-1)$.

To reduce further, the same boundary element can occur for different sums. The following theorem states when this will occur.

**Theorem 2.1** Let $A, B \subseteq \{1, 2, 3, \ldots, 2n-1\}$ with $|A| = |B| = n-1$ and suppose

$$S_A = \sum_{a \in A} a \neq \sum_{b \in B} b = S_B.$$ Then $x^{2S_A - \binom{2n}{2}} = x^{2S_B - \binom{2n}{2}}$ if and only if $S_A \equiv S_B \mod n$.

**Proof.** Observe that

$$x^{2S_A - \binom{2n}{2}} = x^{2S_B - \binom{2n}{2}}$$ if and only if

$$2S_A - \binom{2n}{2} \equiv 2S_B - \binom{2n}{2} \mod 2n \quad \text{if and only if} \quad 2S_A - \binom{2n}{2} \equiv 2S_B - \binom{2n}{2} \mod 2n \quad \text{if and only if} \quad S_A \equiv S_B \mod n. \quad \square$$

Theorem 2.1 tells us that we now only need $n$ cases, one for each $i = 0, 1, 2, \ldots, n-1$. Now that the cases have been reduced, using the generating function $u_{2n-1}(t, n-1)$ we can state the existing formula for finding the average Cayley genus from Schultz [9].
Theorem 2.2 Let $n \geq 2$ be an integer and let $a = 0$ if $n$ is even and $a = n$ if $n$ is odd. Then the average Cayley genus $\overline{\gamma}(D_{2n}, \Delta)$, where $\Delta$ is the generating set for $D_{2n}$ consisting of all the reflections, is given by

$$\overline{\gamma}(D_{2n}, \Delta) = \frac{n!(n-1)!}{(2n-1)!} \left[ \sum_{i=0}^{n-1} \left( 2n^2 - 2n + 1 - \gcd(2n, a+2i) \right) \times \sum_{j=0}^{n-1} \left[ t^{i+j} + [i+j] u_{2n-1}(i, n-1) \right] \right]$$

[Note: In Theorem 2.2, we take $\gcd(2n, 0)$ to mean $\gcd(2n, 2n)$ so that $\gcd(2n, 0) = \gcd(2n, 2n) = 2n$.]

Next, we will see this formula in use through an example. For the example, we will consider $n = 12$ and look at the table on the next page.

The numbers in the first column are all the values for $i$ from 0 to $n-1$. The second column is $\gcd(2n, a+2i)$ which is used in the formula. The third column is to find the corresponding genus, $2n^2 - 2n + 1 - \gcd(2n, a+2i)$. The fourth column is using the generating function to find the coefficients needed for each $i$ and then summing them. Then by using the values in the third and fourth columns in the table into the formula we get

$$\overline{\gamma}(D_{24}, \Delta) = \frac{12!11!1!}{23!} \left[ 241(112720) + 263(112632) + 261(112707) + 259(112640) + 257(112710) + 263(112632) + 253(112716) + 263(112632) + 257(112710) + 259(112640) + 261(112707) + 263(112632) \right] = \frac{174631897}{869193}.$$
<table>
<thead>
<tr>
<th>$i$</th>
<th>$\text{gcd}$</th>
<th>genus</th>
<th>Sum of Generating Function Coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>24</td>
<td>241</td>
<td>$1 + 76 + 1109 + 6300 + 18320 + 30554 + 30554 + 18320 + 6300 + 1109 + 76 + 1 = 112720$</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>263</td>
<td>$1 + 98 + 1317 + 7040 + 19496 + 31132 + 29849 + 17125 + 5597 + 921 + 56 = 112632$</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>261</td>
<td>$2 + 129 + 1564 + 7850 + 20696 + 31641 + 29087 + 15968 + 4962 + 766 + 42 = 112707$</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>259</td>
<td>$3 + 165 + 1838 + 8698 + 21863 + 32017 + 28218 + 14812 + 4368 + 628 + 30 = 112640$</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
<td>257</td>
<td>$5 + 212 + 2156 + 9613 + 23034 + 32312 + 27302 + 13703 + 3836 + 515 + 22 = 112710$</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>263</td>
<td>$7 + 266 + 2505 + 10560 + 24152 + 32467 + 26295 + 12608 + 3342 + 415 + 15 = 112632$</td>
</tr>
<tr>
<td>6</td>
<td>12</td>
<td>253</td>
<td>$11 + 336 + 2907 + 11573 + 25261 + 32540 + 25261 + 11573 + 2907 + 336 + 11 = 112716$</td>
</tr>
<tr>
<td>7</td>
<td>2</td>
<td>263</td>
<td>$15 + 415 + 3342 + 12608 + 26295 + 32467 + 24152 + 10560 + 2505 + 266 + 7 = 112632$</td>
</tr>
<tr>
<td>8</td>
<td>8</td>
<td>257</td>
<td>$22 + 515 + 3836 + 13703 + 27302 + 32312 + 23034 + 9613 + 2156 + 212 + 5 = 112710$</td>
</tr>
<tr>
<td>9</td>
<td>6</td>
<td>259</td>
<td>$30 + 628 + 4368 + 14812 + 28218 + 32017 + 21863 + 8698 + 1838 + 165 + 3 = 112640$</td>
</tr>
<tr>
<td>10</td>
<td>4</td>
<td>261</td>
<td>$42 + 766 + 4962 + 15968 + 29087 + 31641 + 20696 + 7850 + 1564 + 129 + 2 = 112707$</td>
</tr>
<tr>
<td>11</td>
<td>2</td>
<td>263</td>
<td>$56 + 921 + 5597 + 17125 + 29849 + 31132 + 19496 + 7040 + 1317 + 98 + 1 = 112632$</td>
</tr>
</tbody>
</table>
As seen in the table there are some problems to address with this formula, specifically, we make two observations. The first is that for different values of \( i \), we have the same greatest common divisor in the second column, the same genus in the third column, and the same coefficient sums in the fourth column. What this suggests is that it may be possible to further reduce the number of cases necessary. Further, by narrowing down the cases, we would consequently arrive at a more accurate representation of the genus distribution. For now, we have the same genus occurring with repetition occasionally. We would like each one to appear only once. Secondly, in the fourth column, we have to go through a lot of tedious work using the generating function to find all the coefficients and then we have to find certain sums of them. We would like another way to determine the sums without having to resort to the generating function.
CHAPTER 3

PREPARATION FOR NEW FORMULA

We begin this chapter by recalling the generating function $u_{2n-1}(t, n-1)$. As seen in the previous chapter, this generating function appears in the existing formula for the average Cayley genus for the Cayley maps we are interested in. From Table 1, the example of $n=12$ using the existing formula, we see how the coefficients from the generating function are listed and then added together for each of the twelve cases. Additional examples of tables are given in Appendix A. These tables show the coefficients from the generating function $u_{2n-1}(t, n-1)$ for each $n = 2, 3, ..., 14$. The column at the right contains the sums of the coefficients in that row. As seen from the example and in the formula, it is these sums in this last column that we are interested in. The rows represent the $n$ cases that are required for the existing formula.

So the tables keep track of the coefficients from the generating function. In particular, for a given $n$, if $u_{2n-1}(t, n-1) = \sum_{k=0}^{n(n-1)} a_k t^{n \choose 2} k^k$, then $a_0, a_1, ..., a_{n-1}$ go into the first column of coefficients, $a_n, a_{n+1}, ..., a_{2n-1}$ go into the second column of coefficients, and so on until $a_{n(n-1)} = 1$ goes into first entry of the last column of coefficients, exhausting the coefficients of $u_{2n-1}(t, n-1)$. Then the coefficients of each row are summed up accordingly in the final column of the table. We let $c_0, c_1, c_2, ..., c_{n-1}$
denote coefficient these sums, where then $c_i$ is the sum of the coefficients from the generating function in the $i$-th row. After reviewing the tables, it is easily noted that the sum in the first row, $c_0$, is different from $c_1$, $c_2$, ..., $c_{n-1}$. These other rows all have $n-1$ coefficients whereas the row $c_0$ has $n$ coefficients. So hopefully we can relate the other row sums in terms of this first row.

The first coefficient of the first row counts the number of partitions of $\binom{n}{2}$ and the other coefficients in this row count the number of partitions of

$$\binom{n}{2} + n, \binom{n}{2} + 2n, \ldots, 3\binom{n}{2}. $$

The coefficients in the first column, starting with the second row, count the partitions of $\binom{n}{2} + 1, \binom{n}{2} + 2, \ldots, \binom{n}{2} + n-1$. Continuing in this manner in the second column, the coefficient that counts the partitions of $\binom{n}{2} + n$ is the first row, and the rest of the coefficients in this second column count the partitions of

$$\binom{n}{2} + n + 1, \binom{n}{2} + n + 2, \ldots, \binom{n}{2} + 2n - 1. $$

We construct the rest of the columns in this same manner. That is, to find the coefficients in the $j$th column for $j = 0, 1, 2, \ldots, n-2$, we use the coefficient in the first row that we have already obtained for that column. Namely, the coefficient in the first row, $j$th column is counting the partitions of $\binom{n}{2} + jn$. Using this entry from the first row, the coefficient in this $i$th row that is in the $j$th column can be represented by
\[ \binom{n}{2} + jn + i \text{ for } i = 0, 1, 2, \ldots, n-1. \] From these tables, we see the importance of this first row, and especially the first row sum \( c_0 \). What we would like to do is to be able to express the other row sums in terms of \( c_0 \).

Now consider again the example of \( n=12 \) and the tables in Appendix A. We would like to know the sum of the coefficients of each row, that is, the entries in the final column of each of the tables, without having to use the generating function to list out each of the individual coefficients in their respective rows. In doing so, we will express each \( c_i \), with \( i = 1, 2, \ldots, n-1 \), in terms of \( c_0 \). Our first step will be to set up a system of \( n-1 \) equations in the variables \( c_1, c_2, \ldots, c_{n-1} \).

Define \( f(t) = \frac{1}{\binom{n}{2}} u_{2n-1}(t, n-1) \). We can rewrite it so that \( f(t) = \sum_{k=0}^{n(n-1)} a_k t^k \), where each \( a_k \) is the entry from the table as defined previously. What we would like to have is a closed formula for

\[
c_0 = a_0 + a_n + a_{2n} + \ldots + a_{n(n-2)} + a_{n(n-1)}
\]

and when \( 1 \leq i \leq n-1 \), a formula for

\[
c_i = a_i + a_{n+i} + a_{2n+i} + \ldots + a_{n(n-2)+i}.
\]

The \( n-1 \) equations will come from considering each expression for \( f(t) \) near the non-trivial \( n \)th roots of unity.

Here we will take a minute to recall some properties from abstract algebra. For a positive integer \( n \), the set of all of the \( n \)th roots of unity forms a multiplicative group that
is cyclic. An \( n \)th root of unity that generates this multiplicative group is called a primitive \( n \)th root of unity.

It is also helpful to recall a few things from complex analysis. First, we define a complex number \( z = a + bi = re^{\theta} = r \cos \theta + ir \sin \theta \). We use complex numbers for their solutions to equations involving roots of unity. The \( n \)th roots of unity are the solutions to the equation \( z^n = 1 \), so they are of the form \( z = e^{2\pi i / n} = \cos \left( \frac{2\pi \ell}{n} \right) + i \sin \left( \frac{2\pi \ell}{n} \right) \), where \( \ell = 0, 1, 2, \ldots, n-1 \). We will use a simpler notation for these solutions by defining \( e^{2\pi i / n} = e^{\ell / n} \).

\[ e^{2\pi i / n} = e^{\ell / n} \] Before preceding any further we mention a few remarks.

**Fact 1** If \( c \in \mathbb{Z} \), then \( e(c) = 1 \). This comes from using the definition of \( e^{\ell / n} \), specifically \( e(c) = e^{2\pi i (c)} = 1 \).

**Fact 2** For \( e(\alpha) \) and \( e(\beta) \) being solutions to \( z^n = 1 \), we have \( e(\alpha + \beta) = e(\alpha) \cdot e(\beta) \).

We arrive at this fact by again using the definition. We have that \( e(\alpha + \beta) = e^{2\pi i (\alpha + \beta)} = e^{2\pi i \alpha + 2\pi i \beta} = e^{2\pi i \alpha} e^{2\pi i \beta} = e(\alpha) \cdot e(\beta) \).

**Fact 3** Let \( \xi \) be a primitive \( n \)th root of unity, then \( \xi^{n-1} + \xi^{n-2} + \ldots + \xi = -1 \). This comes from using that 1 cannot be a primitive root of unity since 1 does not generate the other roots. So since \( \xi \) is a primitive \( n \)th root of unity, we have \( \xi^n = 1 \). Thus \( \xi^n - 1 = 0 \) and by factoring we have \( (\xi - 1)(\xi^{n-1} + \xi^{n-2} + \ldots + \xi + 1) = 0 \). Since \( \xi \neq 1 \), we get that \( \xi - 1 \neq 0 \) so that \( \xi^{n-1} + \xi^{n-2} + \ldots + \xi + 1 = 0 \). Thus we obtain fact 3. What this fact means is that the sum of all the nontrivial \( n \)th roots of unity is \(-1\).
Previously, it was noted that \( \frac{1}{\binom{n}{2}} u_{2n-1}(t, n-1) = \frac{(1-t^{2n-1})(1-t^{2n-2}) \cdots (1-t^{n+1})}{(1-t^{n-1})(1-t^{n-2}) \cdots (1-t)} \) and 
\[
\sum_{k=0}^{n(n-1)} a_k t^k
\]
were considered equal. This was because we were using them as generating functions. For our purposes now, let \( f(t) = \frac{(1-t^{2n-1})(1-t^{2n-2}) \cdots (1-t^{n+1})}{(1-t^{n-1})(1-t^{n-2}) \cdots (1-t)} \) and 
\[
g(t) = \sum_{k=0}^{n(n-1)} a_k t^k.
\]
Then \( g(t) \) is defined for all complex numbers and \( f(t) \) is defined on \( \mathbb{C} - \mathcal{N} \), where \( \mathcal{N} = \bigcup_{N=1}^{n-1} \left\{ e^{k/N} : 0 \leq k \leq N - 1 \right\} \), i.e. \( \mathcal{N} \) is the set of the \( N \)th roots of unity for \( N = 1, 2, \ldots, n-1 \). Observe that \( f(t) = g(t) \) for all \( t \in \mathbb{C} - \mathcal{N} \). We know \( g(t) \) is a polynomial and so \( g(t) \) is continuous on \( \mathbb{C} \). Thus, \( \lim_{t \to t_0} g(t) = g(t_0) \) for all \( t_0 \in \mathbb{C} \). But since \( f(t) = g(t) \) for all \( t \in \mathbb{C} - \mathcal{N} \) and \( \mathcal{N} \) is a finite set, it follows that \( g(t_0) = \lim_{t \to t_0} g(t) = \lim_{t \to t_0} f(t) \). So, we use \( t_0 = e^{\left( \frac{\ell}{n} \right)} \) for \( \ell = 1, 2, \ldots, n-1 \) to get \( n-1 \) equations.

Now, we evaluate \( g(t) \) at each of the \( n \)th roots of unity \( e^{\left( \frac{\ell}{n} \right)} \) for \( \ell = 1, 2, \ldots, n-1 \).

Substituting into \( g(t) = \sum_{k=0}^{n(n-1)} a_k t^k \), we have that \( g\left(e^{\left( \frac{\ell}{n} \right)}\right) = \sum_{k=0}^{n(n-1)} a_k \left(e^{\left( \frac{\ell}{n} \right)}\right)^k \). By using the identity that \( e^{\left( \frac{\ell}{n} \right)^{r+ns}} = e^{\left( \frac{\ell}{n} \right)^r} \), we get
\[
\sum_{k=0}^{n(n-1)} a_k \left(e^{\left( \frac{\ell}{n} \right)}\right)^k =
\]
\[
\sum_{j=0}^{n-1} a_{nj} \left( e \left( \frac{j}{n} \right) \right)^{n+j} + \sum_{j=0}^{n-2} a_{j+1n} \left( e \left( \frac{j}{n} \right) \right)^{2+j} + \sum_{j=0}^{n-2} a_{2jn} \left( e \left( \frac{j}{n} \right) \right)^{2+j} + \ldots + \sum_{j=0}^{n-2} a_{n-1+jn} \left( e \left( \frac{j}{n} \right) \right)^{n+j} =
\sum_{j=0}^{n-1} a_{nj} + e \left( \frac{\ell}{2} \right) \sum_{j=0}^{n-2} a_{1+jn} + \left( e \left( \frac{\ell}{n} \right) \right)^{2} \sum_{j=0}^{n-2} a_{2+jn} + \ldots + \left( e \left( \frac{\ell}{n} \right) \right)^{n-1} \sum_{j=0}^{n-2} a_{n-1+jn}.
\]

Now by definition, \( c_0 = \sum_{j=0}^{n-1} a_{nj} \) and \( c_i = \sum_{j=0}^{n-2} a_{i+jn} \) for each \( i = 1, 2, \ldots, n-1 \), so we get that

\[
g \left( e \left( \frac{\ell}{n} \right) \right) = \sum_{k=0}^{n(n-1)} a_k \left( e \left( \frac{\ell}{n} \right) \right)^k = c_0 + c_1 e \left( \frac{\ell}{n} \right) + c_2 e \left( \frac{2\ell}{n} \right) + c_3 e \left( \frac{3\ell}{n} \right) + \ldots + c_{n-1} e \left( \frac{(n-1)\ell}{n} \right)
\]

for each \( \ell = 1, 2, \ldots, n-1 \). So we have \( n-1 \) equations in \( n-1 \) variables. Namely, the variables are \( c_1, c_2, \ldots, c_{n-1} \). So we can solve this system of equations by setting up a matrix equation. The matrix equation we get is given by the following:

\[
\begin{bmatrix}
\begin{array}{cccc}
e \left( \frac{1}{n} \right) & e \left( \frac{2}{n} \right) & \ldots & e \left( \frac{n-1}{n} \right) \\
e \left( \frac{2}{n} \right) & e \left( \frac{4}{n} \right) & \ldots & e \left( \frac{2(n-1)}{n} \right) \\
\vdots & \vdots & \ddots & \vdots \\
e \left( \frac{n-1}{n} \right) & e \left( \frac{2(n-1)}{n} \right) & \ldots & e \left( \frac{(n-1)(n-1)}{n} \right)
\end{array}
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2 \\
\vdots \\
c_{n-1}
\end{bmatrix}
= \begin{bmatrix}
\lim_{t \to e \left( \frac{1}{n} \right)} f(t) - c_0 \\
\lim_{t \to e \left( \frac{2}{n} \right)} f(t) - c_0 \\
\vdots \\
\lim_{t \to e \left( \frac{n-1}{n} \right)} f(t) - c_0
\end{bmatrix}.
\]

A comment here is needed about this matrix equation. The coefficient matrix satisfies the conditions for the Vandermonde matrix. This means that the determinant is non-zero, which implies that this equation not only has a solution, but that the solution is unique. We proceed to determine this solution. First, we calculate \( \lim_{t \to e \left( \frac{\ell}{n} \right)} f(t) \).
Theorem 3.1 Let \( \ell \) be an integer such that \( 1 \leq \ell \leq n-1 \) and let \( d = \gcd(n, \ell) \). Then

\[
\lim_{t \to e^{\frac{\ell}{n}}} f(t) = \binom{2d - 1}{d - 1}.
\]

Proof. We will consider two cases.

Case 1. Suppose that \( d = 1 \). In this case, \( e^{\frac{\ell}{n}} \) is a primitive \( n \)th root of unity so that if

\[
\xi = e^{\frac{\ell}{n}},
\]

then \( \xi^n = 1 \). Since, \( f(t) \) is defined at \( \xi \) and is continuous there, we have

\[
\lim_{t \to \xi} f(t) = f(\xi) = \frac{(1-\xi^{2\ell-1})(1-\xi^{2\ell-2})\ldots(1-\xi^{n+1})}{(1-\xi^{\ell})(1-\xi^{\ell-2})\ldots(1-\xi)} = \frac{(1-\xi^{\ell})(1-\xi^{\ell-2})\ldots(1-\xi^{n+1})}{(1-\xi^{\ell})(1-\xi^{\ell-2})\ldots(1-\xi)} = 1.
\]

Also when \( d = 1 \), we have \( \binom{2d - 1}{d - 1} = \binom{1}{0} = 1 \).

Case 2. Suppose that \( d \) satisfies \( 1 < d < n \). Since \( d = \gcd(n, \ell) \), we may write \( n = dN \) and \( \ell = dL \) for some integers \( N \) and \( L \) with \( \gcd(N, L) = 1 \). So \( \lim_{t \to e^{\frac{\ell}{n}}} f(t) = \lim_{t \to \xi} f(t) \). Since \( \gcd(N, L) = 1 \), it follows that \( e^{\frac{L}{N}} \) is a primitive \( N \)th root of unity. So let \( \xi = e^{\frac{L}{N}} \) and then \( \xi^N = 1 \). Evaluating for \( \lim_{t \to \xi} f(t) \), we obtain

\[
\lim_{t \to \xi} f(t) = \lim_{t \to \xi} \frac{(1-t^{2\ell-1})(1-t^{2\ell-2})\ldots(1-t^{n+1})}{(1-t^{\ell})(1-t^{\ell-2})\ldots(1-t)}.
\]

Next, look at the general term

\[
\frac{1-t^{n+i}}{(1-t^i)},
\]

for some \( i \) with \( 1 \leq i \leq n-1 \), in this equation. Observe that if \( N \mid i \), then since
\( n = dN, \ \xi^N = 1, \) and \( \xi' \neq 1 \) we obtain \( \frac{1 - \xi^{n+i}}{1 - \xi'} = \frac{1 - \xi^{dN}}{1 - \xi'} = 1. \) Using this, we have

\[
\lim_{t \to \xi} f(t) = \lim_{t \to \xi} \left( \prod_{N_i = 1}^{N} \frac{1 - t^{n+i}}{1 - t'} \right) = \lim_{t \to \xi} \left( \prod_{N_i = 1}^{N} \frac{1 - t^{n+i}}{1 - t'} \right) = \lim_{t \to \xi} \left( \prod_{1 \leq n \leq n-1} \frac{1 - t^{n+i}}{1 - t'} \right)
\]

\[
\lim_{t \to \xi} \left( \frac{1 - t^{(2d-1)N}}{1 - t^{(d-1)N}} \right) = \lim_{t \to \xi} \left( \frac{1 - t^{(d-j)N}}{1 - t^{jN}} \right) = \lim_{t \to \xi} \left( \frac{1}{1 - t^{N}} \right) = \lim_{t \to \xi} \left( \frac{1 + t^{N} + t^{2N} + \cdots + t^{(d-j)N}}{1 + t^{N} + t^{2N} + \cdots + t^{(j-1)N}} \right) = \frac{d + 1}{d}
\]

\[
\frac{(d+1)(d+2) \cdots (2d-1)}{1 \cdot 2 \cdot 3 \cdots (d-1)} = \frac{(2d-1)!}{d!(d-1)!} = \frac{(2d-1)}{d-1}. \quad \Box
\]

**Corollary 3.2** \( \lim_{t \to e} f(t) = \lim_{t \to e} f(t) \) if and only if \( \gcd(n, \ell_1) = \gcd(n, \ell_2). \)

Proof. Let \( d = \gcd(n, \ell_1) \) and let \( d' = \gcd(n, \ell_2). \) Then \( \lim_{t \to e} f(t) = \left( \frac{2d-1}{d-1} \right) \) and

\[
\lim_{t \to e} f(t) = \left( \frac{2d'-1}{d'-1} \right). \quad \text{Thus,} \quad \lim_{t \to e} f(t) = \lim_{t \to e} f(t) \quad \text{if and only if} \quad d = d', \quad \text{that is, if}
\]

and only if \( \gcd(n, \ell_1) = \gcd(n, \ell_2). \) \( \Box \)
In order to solve the matrix equation (\(\ast\)), it will be useful to have the following notation for certain column vectors. Also, if \(\vec{x}\) is a column vector, then we write \(\vec{x}^T\) for the transpose of \(\vec{x}\), meaning \(\vec{x}^T\) is \(\vec{x}\) written as a row vector. Let \(n\) be a positive integer and let \(d\) be a divisor of \(n\) with \(n = dN\). Then define the \((n-1)\times 1\) column vectors as follows using the transpose notation:

\[
\vec{c} = (c_1, c_2, \ldots, c_{n-1})^T,
\]
\[
\vec{u} = (1, 1, \ldots, 1)^T,
\]
\[
\vec{v}_d = (v_1, v_2, \ldots, v_{n-1})^T \quad \text{where} \quad v_k = \begin{cases} 1 & \text{if} \quad d = \gcd(k, n) \\ 0 & \text{if} \quad d \neq \gcd(k, n) \end{cases},
\]
\[
\vec{u}_d = (u_1, u_2, \ldots, u_{n-1})^T \quad \text{where} \quad u_k = \begin{cases} 0 & \text{if} \quad N \mid k \\ 1 & \text{if} \quad N \nmid k \end{cases}.
\]

Then observe that

\[
\vec{u} - \vec{u}_d = (1-u_1, 1-u_2, \ldots, 1-u_{n-1})^T \quad \text{where} \quad 1-u_k = \begin{cases} 1 & \text{if} \quad N \mid k \\ 0 & \text{if} \quad N \nmid k \end{cases}.
\]

Also let \(A\) be the coefficient matrix of the equation \((\ast)\) and \(\vec{b}\) be the solution vector of \((\ast)\). Thus observe that if \(D = \{d : d \mid n, 1 < d < n\}\), then

\[
\vec{b} = (1-c_0)\vec{u} + \sum_{d \in D} \left( \frac{2d-1}{d-1} - 1 \right) \vec{v}_d.
\]

Using these vectors, the equation \((\ast)\) is \(A\vec{c} = \vec{b}\). We proceed to present several lemmas that will be useful in solving this equation.
Lemma 3.3 Using $A$ and $\tilde{u}$ as defined previously, $A\tilde{u} = (-1, -1, ..., -1)^T$.

Proof. Let $A\tilde{u} = (k_1, k_2, ..., k_{n-1})^T$ so that $k_j$ is the sum of the $j$th row of the matrix $A$, that is, $k_j = \sum_{t=1}^{n-1} e\left(\frac{j \ell}{n}\right)$ for each $j$ where $1 \leq j \leq n-1$. If $\gcd(j, n) = 1$, then by Fact 3 we have $\sum_{t=1}^{n-1} e\left(\frac{j \ell}{n}\right) = -1$. So suppose that the $\gcd(j, n) = d$, where $d \neq 1$. Then we may write $j = Jd$ and $n = Nd$ for integers $J$ and $N$ with $\gcd(J, N) = 1$. By grouping terms as follows,

$$
\sum_{t=1}^{n-1} e\left(\frac{j \ell}{n}\right) = \sum_{t=1}^{n-1} e\left(\frac{dJ \ell}{dN}\right) = \sum_{t=1}^{n-1} e\left(\frac{J \ell}{N}\right) = \left[ e\left(\frac{J}{N}\right) + e\left(\frac{2J}{N}\right) + ... + e\left(\frac{(N-1)J}{N}\right) + e\left(\frac{NJ}{N}\right)\right] +
$$

$$
\left[ e\left(\frac{(N+1)J}{N}\right) + e\left(\frac{(N+2)J}{N}\right) + ... + e\left(\frac{(2N-1)J}{N}\right) + e\left(\frac{2NJ}{N}\right)\right] + ... +
$$

$$
\left[ e\left(\frac{(d-1)NJ}{N}\right) + e\left(\frac{[(d-1)N+1]J}{N}\right)\right] + ... + e\left(\frac{dNJ}{N}\right),
$$

we see that we have added all $N$ $N$th roots of unity $d-1$ times and then have added the $N-1$ nontrivial $N$th roots of unity once. Thus, using Fact 3, we obtain

$$
\sum_{t=1}^{n-1} e\left(\frac{j \ell}{n}\right) = (d-1) \left[ \sum_{t=1}^{N} e\left(\frac{J \ell}{N}\right) \right] + \sum_{t=1}^{N-1} e\left(\frac{J \ell}{N}\right) = (d-1) \cdot (0) + (-1) = -1. \quad \Box
$$

So Lemma 3.3 gives us that $A\left[(c_0-1)\tilde{u}\right] = (1-c_0)\tilde{u}$, which is the first term in our solution vector $\tilde{b}$. Later we define values $U_N$ for each $N \in D$ such that
\[ A \left( \sum_{N \in D} U_N \bar{u}_N \right) = \sum_{d \in D} \left( \frac{2d-1}{d-1} - 1 \right) \bar{v}_d, \] the second part of our solution vector. However, it is useful to first provide a few more helpful observations.

**Lemma 3.4** Let \( N \in D \) and \( n = dN \). Then \( A \bar{u}_N = -N(\bar{u} - \bar{u}_d) \).

**Proof.** Notice that \( \bar{u}_N = (u_1, u_2, ..., u_{n-1})^T \), where \( u_k = \begin{cases} 0 & \text{if } d \mid k \\ 1 & \text{if } d \nmid k \end{cases} \). Then

\[
A \bar{u}_N = (i_1, i_2, ..., i_{n-1}), \quad \text{where } i_k = \sum_{\substack{1 \leq s \leq n-1 \\ d \mid t}} e \left( \frac{k\ell}{n} \right).
\]

Since the sum of the entries in an entire row of \( A \) is \(-1\), we have that

\[
-1 - \sum_{L=1}^{N-1} e \left( \frac{kL}{N} \right) = -1 - \sum_{L=1}^{N-1} e \left( \frac{kL}{n} \right) = -1 - \sum_{L=1}^{N-1} e \left( \frac{kL}{n} \right) = -1 - \sum_{L=1}^{N-1} e \left( \frac{kL}{n} \right).
\]

If \( N \mid k \), then \( k = NK \) for some integer \( K \) so that by using fact 1 \( A \bar{u}_N = -1 - \sum_{L=1}^{N-1} e \left( \frac{kL}{n} \right) = -1 - \sum_{L=1}^{N-1} e \left( kL \right) = -1 - (N-1) = -N \). If \( N \nmid k \), then \( k = NK + r \) for some integers \( K \) and \( r \) with \( 1 \leq r \leq N-1 \) so that using facts 1 and 3, we have \( A \bar{u}_N = -1 - \sum_{L=1}^{N-1} e \left( \frac{(NK+r)L}{n} \right) = -1 - \sum_{L=1}^{N-1} e \left( \frac{NL}{n} \right) e \left( \frac{rL}{n} \right) = -1 - \sum_{L=1}^{N-1} e \left( \frac{rL}{n} \right) = -1 - (1-1) = 0. \)

So \( A \bar{u}_N = \begin{cases} -N & \text{if } N \mid k \\ 0 & \text{if } N \nmid k \end{cases} \). Thus, since \( \bar{u} - \bar{u}_d = (1-u_1, 1-u_2, ..., 1-u_{n-1})^T \), where

\[
1-u_k = \begin{cases} 1 & \text{if } N \mid k \\ 0 & \text{if } N \nmid k \end{cases},
\]

it follows that \( A \bar{u}_N = -N(\bar{u} - \bar{u}_d) \). \( \Box \)
Lemma 3.5  Let $D = \{d : d \mid n, 1 < d < n\}$ and let $N \in D$. Then $\vec{u} - \vec{u}_{n/N} = \sum_{d \mid N} \vec{v}_d$.

Proof. Previously, it was noted that $\vec{u} - \vec{u}_{n/N} = (i_1, i_2, \ldots, i_{n-1})^T$ where

$$i_k = \begin{cases} 1 & \text{if } N \mid k \\ 0 & \text{if } N \not\mid k \end{cases}$$

Let $\mathcal{K} = \{d \in D : N \mid d\} = \{\ell_1, \ell_2, \ldots, \ell_a\}$. So, $\mathcal{K}$ is the set of positive proper divisors of $n$ that are also multiples of $N$. (For example, when $n=12$ and $N=2$ we have $D = \{2, 3, 4, 6\}$ and $\mathcal{K} = \{2, 4, 6\}$.) So $\sum_{d \mid N} \vec{v}_d = \sum_{\mathcal{K}} \vec{v}_d = \vec{v}_{\ell_1} + \vec{v}_{\ell_2} + \ldots + \vec{v}_{\ell_a} = (v_1, v_2, \ldots, v_{n-1})^T$ where, for each $k = 1, 2, \ldots, n-1$, we have

$$v_k = \begin{cases} 1 & \text{if } \gcd(k, n) \in \mathcal{K} \\ 0 & \text{if } \gcd(k, n) \not\in \mathcal{K} \end{cases}$$

We verify that $v_k = i_k$ for each $k = 1, 2, \ldots, n-1$. If $\gcd(k, n) \in \mathcal{K}$, then $k = \ell_j K$ for some $\ell_j \in \mathcal{K}$ and some integer $K$. Since $N \mid \ell_j$, we have $\ell_j = N L_j$ for some integer $L_j$, so that $k = NL_j K$. So $N \mid k$ in this case. On the other hand, if $\gcd(k, n) = g \not\in \mathcal{K}$, then $N \not\mid g$ by the defining property of $\mathcal{K}$. Since $N \mid n$, we know $N \not\mid k$ for otherwise we would have $N \mid g$, a contradiction. Thus, $v_k = i_k$ for each $k = 1, 2, \ldots, n-1$. □

What these two lemmas allow us to do is to show the relationship between the $\vec{u}_N$ vectors, where $N \in D$, as a sum of vectors $\vec{v}_d$, for $d \in D$ as well. This will prove useful in finding the second part of the solution vector $\vec{b}$.

Now we are ready to define the values $U_N$ for $N \in D$ such that
$$A \left( \sum_{N \in D} U_N \bar{u}_N \right) = \sum_{N \in D} \left( \binom{2d-1}{d-1} - 1 \right) \bar{v}_d .$$

We define

$$U_N = \frac{\left( \frac{2N-1}{N-1} \right) - 1 - \sum_{1 < k < N \atop k \mid N} k U_k}{N} .$$

Looking into this definition, we see that it is recursive. What this means is that to find a particular $U_N$, it depends on what values $U_k$ we get for the divisors $k$ of $N$ with $1 < k < N$, if $N$ has any such divisors.

Recall the solution vector $\bar{b}$ and the expression $\left( \binom{2d-1}{d-1} - 1 \right) \bar{v}_d$ that lies within it.

This leads to the following lemma in which we will use the definition of $U_N$.

**Lemma 3.6** For each $d \in D$, the sum $\sum_{N \in D \atop N \not\mid d} NU_N = \left( \binom{2d-1}{d-1} - 1 \right) - 1$.

**Proof.** Observe $\sum_{N \in D \atop N \not\mid d} NU_N = \sum_{N \not\mid d} NU_N + dU_d = \sum_{N \not\mid d} NU_N + \left( \binom{2d-1}{d-1} - 1 \right) - \sum_{1 < N \not\mid d} NU_N = \left( \binom{2d-1}{d-1} - 1 \right) - 1$. □

With the help of the previous four lemmas, we are now prepared to solve equation (*), in other words, we are now ready to solve $A\bar{c} = \bar{b}$ for $\bar{c}$. We do this in the following theorem.
Theorem 3.7 The solution to the matrix equation $A\vec{c} = \vec{b}$ shown as (*) is

$$\vec{c} = (c_0 - 1)\vec{u} - \sum_{N \in D} U_N \vec{u}_N$$

Proof: Consider $A\vec{c} = A\left[(c_0 - 1)\vec{u} - \sum_{N \in D} U_N \vec{u}_N\right]$.

By Lemma 3.3, we have

$$A\vec{c} = (1 - c_0)\vec{u} - \sum_{N \in D} AU_N \vec{u}_N.$$  

Using Lemma 3.4, it follows that

$$A\vec{c} = (1 - c_0)\vec{u} + \sum_{N \in D} NU_N \left(\vec{u} - \vec{u}_{N/N}\right).$$

Applying Lemma 3.5, we obtain

$$A\vec{c} = (1 - c_0)\vec{u} + \sum_{N \in D} NU_N \left(\sum_{d \in D} v_d\right) = \left(1 - c_0\right)\vec{u} + \sum_{d \in D} \left(\sum_{N \in D} NU_N\right) \vec{v}_d.$$

Finally, by Lemma 3.6,

$$A\vec{c} = (1 - c_0)\vec{u} + \sum_{d \in D} \left(\frac{2d - 1}{d - 1}\right) \vec{v}_d = \vec{b}. \quad \square$$

Before going further, we shall introduce the Euler $\phi$-function $\phi(m)$, which denotes the number of positive integers less than $m$ that are relatively prime to $m$ and $\phi(1)$ is defined to be 1. Then, for each $c_i$ where $i \in D$, the number of elements in$
\{c_i : 1 \leq i \leq n-1, \gcd(i, n) = d\}$ for a given $d$ is $\phi\left(\frac{n}{d}\right)$. Thus narrowing down the number of cases even further. Instead of $n$ cases, we now only need one case for each
divisor \( d \) of \( n \) where \( 1 \leq d < n \) as well as the initial case of \( c_0 \). A basic result of number theory (see [7], for example) that will be useful in the following proof is that

\[
\sum_{d|n} \phi\left(\frac{n}{d}\right)\bigg|_{d|n} = n.
\]

For our purposes, we will use that \( \sum_{1 \leq d < n} \phi\left(\frac{n}{d}\right) = n - 1 \). This follows directly since \( \sum_{1 \leq d < n} \phi\left(\frac{n}{d}\right) = \left[ \sum_{d|n} \phi\left(\frac{n}{d}\right) \right] - \phi\left(\frac{n}{n}\right) = n - 1 \). We now use this in the following corollary of Theorem 3.7. Also, we use the notation \( D^* = D \cup \{1\} \).

**Corollary 3.8** Let \( n \geq 2 \) be a positive integer. Let \( u_{2n-1}(t, n-1) = \sum_{k=0}^{n(n-1)} a_{kn} t^k \) be the generating function for partitions of integers with \( n-1 \) unequal parts and no part greater than \( 2n-1 \). Define \( c_0 = \sum_{j=0}^{n-1} a_{nj} \) and for each \( i = 1, 2, \ldots, n-1 \), let \( c_i = \sum_{j=0}^{n-2} a_{ij} \). Then \( c_i = c_j \) if and only if \( \gcd(i, n) = \gcd(j, n) \). Furthermore,

\[
c_0 = \frac{\binom{2n-1}{n-1} + n - 1 + \sum_{k \in D^*} \phi\left(\frac{n}{k}\right) \sum_{N \in D^*} U_N}{n}
\]

and for each \( i \in D^* \), the value

\[
c_i = c_0 - \left[ 1 + \sum_{N \in D^*} U_N \right].
\]
Proof. That $c_i = c_j$ if and only if $\text{gcd}(i, n) = \text{gcd}(j, n)$ follows from Corollary 3.2.

For each $i \in D^*$, the value $c_i = c_0 - \left[ 1 + \sum_{k \in D \atop n \mid kN} U_N \right]$ follows from Theorem 3.7. Finally, to solve for $c_0$, we know that the total $c_0 + \sum_{i=1}^{n-1} c_i = \binom{2n-1}{n-1}$ so that $\sum_{i=1}^{n-1} c_i = \binom{2n-1}{n-1} - c_0$. By the comments preceding this corollary, we have that

$$\binom{2n-1}{n-1} - c_0 = \sum_{i=1}^{n-1} c_i = \sum_{k \in D^*} \phi\left(\frac{n}{k}\right) c_k =$$

$$\sum_{k \in D^*} \left[ \phi\left(\frac{n}{k}\right) (c_0 - 1) - \sum_{N \in D \atop n \mid kN} U_N \right] =$$

$$\sum_{k \in D^*} \left[ \phi\left(\frac{n}{k}\right) (c_0 - 1) \right] - \sum_{k \in D^*} \left[ \phi\left(\frac{n}{k}\right) \sum_{N \in D \atop n \mid kN} U_N \right] =$$

$$(c_0 - 1)(n-1) - \sum_{k \in D^*} \left[ \phi\left(\frac{n}{k}\right) \sum_{N \in D \atop n \mid kN} U_N \right].$$

Hence,

$$\binom{2n-1}{n-1} - c_0 = (c_0 - 1)(n-1) - \sum_{k \in D^*} \left[ \phi\left(\frac{n}{k}\right) \sum_{N \in D \atop n \mid kN} U_N \right]$$

and solving for $c_0$, we have

$$c_0 = \frac{\binom{2n-1}{n-1} + n-1 + \sum_{k \in D^*} \left[ \phi\left(\frac{n}{k}\right) \sum_{N \in D \atop n \mid kN} U_N \right]}{n}. \quad \Box$$

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So now, we have reduced the number of cases from $n$ to the number of divisors of $n$. To show what an improvement this is, from Hardy [6], we have that the number of divisors of $n$ is $d(n) = O(n^\delta)$ for all positive $\delta$. Also, from Corollary 3.8, we are now able to determine the coefficient sums that we have been looking for without having to use the generating function. These were our two main goals in achieving a new, simplified formula for finding the average Cayley genus for dihedral groups.
CHAPTER 4

NEW FORMULA AND SPECIAL CASES

In this chapter, we will devise a new formula for finding the average Cayley genus for
dihedral groups. Recall that we are working with Cayley maps for $D_{2n}$ with
$\Delta = \{ y, xy, x^2y, \ldots, x^{2n-1}y \}$. A particular Cayley map $(D_{2n}, \Delta, \rho)$ is the lift of a
voltage graph that has one region. That region has an associated boundary element and
the genus of the Cayley map in question is $k = 1 + 2n \left[ n-1 \frac{1}{s} \right]$, where $s$ is the order of
this boundary element of the region from the voltage graph embedding. From our work
in Chapter 3, we need only find the order of the boundary elements that correspond to the
terms $c_0$ and $c_d$, where $d | n$ and $1 \leq d < n$. We consider these in two cases.

We first consider $c_0$ and try to find the order of the boundary element of the region
corresponding to $c_0$. From Chapter 2 (page 17), recall that this boundary element is

$$ x^{\binom{n}{2} - \binom{2n}{2}}. $$

Simplifying the exponent, we obtain

$$ 2 \binom{n}{2} - 2n \binom{2n}{2} = \frac{2 \cdot n!}{2!(n-2)!} - \frac{(2n)!}{2!(2n-2)!} = $$

$$ \frac{2n(n-1)}{2} - \frac{2n(2n-1)}{2} = n(n-1) - n(2n-1) = n[(n-1) - (2n-1)] = n[-n] = -n^2. $$

Recall

that the order of $x$ is $2n$, so that the exponent $2 \binom{n}{2} - \binom{2n}{2}$ is $-n^2 \equiv n^2 \mod 2n$. Let
$a \equiv n^2 \mod 2n$ with $a \in \{0, 1, 2, \ldots, 2n-1\}$. Then $a = \begin{cases} 0 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases}$. Now that we know what the boundary element is, we can find the order of it. The $\text{ord}(x^a) = \frac{2n}{\gcd(2n, a)} = \begin{cases} 1 & \text{if } n \text{ is even} \\ 2 & \text{if } n \text{ is odd} \end{cases}$. (Recall that we use $\gcd(2n, 0) = \gcd(2n, 2n) = 2n$.)

Now, we consider $c_d$ and find its boundary element and the order of that boundary element. Again, from page 17, we have that the boundary element is $x^a \equiv 2d \left( \begin{array}{c} 2n \\ 2 \end{array} \right) \mod 2n$. Through similar calculations, we get that the exponent $2 \left( \begin{array}{c} n \\ 2 \end{array} \right) + 2d - \left( \begin{array}{c} 2n \\ 2 \end{array} \right) = -n^2 + 2d$. So now let $b \in \{0, 1, 2, \ldots, 2n-1\}$ such that

$\begin{align*} b &= (-n^2 + 2d) \mod 2n. \end{align*}$

Then $b = \begin{cases} 2d & \text{if } n \text{ is even} \\ n - 2d & \text{if } n \text{ is odd} \end{cases}$. Now that we have the boundary elements for $c_d$, we can find the order of these elements. The $\text{ord}(x^b) = \frac{2n}{\gcd(2n, b)} = \begin{cases} \frac{n}{d} & \text{if } n \text{ is even} \\ \frac{2n}{d} & \text{if } n \text{ is odd} \end{cases}$. Now that we have the orders of the boundary elements, we can find the corresponding genera.

Recall, from the beginning of Chapter 2, the genus was calculated to be $k = 1 + 2n \left[ n - 1 - \frac{1}{s} \right]$. So the genus depends on the order of the boundary element, and this depends on which case, $c_0$ or $c_d$, we have and whether $n$ is even or odd, as seen in finding the orders for the boundary elements.
First, we will find the genus corresponding to $c_0$. The order of the boundary element for the case $c_0$ is given by $\text{ord}(x^n) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 2 & \text{if } n \text{ is odd} \end{cases}$. This gives us that the genus $k$ is given by $k = \begin{cases} 2n^2 - 4n + 1 & \text{if } n \text{ is even} \\ 2n^2 - 3n + 1 & \text{if } n \text{ is odd} \end{cases}$. Similarly, we now find the genus corresponding to $c_d$. The order of the boundary element for the case $c_d$, where $d | n$, from above, is given by $\text{ord}(x^b) = \begin{cases} \frac{n}{d} & \text{if } n \text{ is even} \\ \frac{2n}{d} & \text{if } n \text{ is odd} \end{cases}$. Thus, the genus $k$ for this case is given by $k = \begin{cases} 2n^2 - 2n - 2d + 1 & \text{if } n \text{ is even} \\ 2n^2 - 2n - d + 1 & \text{if } n \text{ is odd} \end{cases}$. Now that we have formulas for each of the Cayley genera, we can now calculate the average Cayley genus.

**Theorem 4.1** Let $n \geq 2$ be an integer. Let $D = \{d : d | n, 1 < d < n\}$ and $D^* = D \cup \{1\}$. For $N \in D$ and $i \in D^*$, let

\[
U_N = \frac{\binom{2N-1}{N-1}^{N-1} - \sum_{1 \leq k < N} k \cdot U_k}{N},
\]

\[
c_0 = \frac{\binom{2n-1}{n-1}^n + n - 1 + \sum_{k \in D^*} \phi\left(\frac{n}{k}\right) \sum_{N \in D, n | N} U_N}{n},
\]

\[
c_i = c_0 - \left[1 + \sum_{N \in D, n | N} U_N\right].
\]
Then the average Cayley genus $\overline{γ}(D_{2n}, \Delta)$ where $D_{2n}$ is the dihedral group and $\Delta$ is the generating set consisting of all the reflections is given by

$$\overline{γ}(D_{2n}, \Delta) = \begin{cases} 
\left(\frac{2n-1}{n-1}\right) \left[ c_0(2n^2 - 4n + 1) + \sum_{d \in D^*} c_d \phi\left(\frac{n}{d}\right)(2n^2 - 2n - 2d + 1) \right] & \text{if } n \text{ is even} \\
\left(\frac{2n-1}{n-1}\right) \left[ c_0(2n^2 - 3n + 1) + \sum_{d \in D^*} c_d \phi\left(\frac{n}{d}\right)(2n^2 - 2n - d + 1) \right] & \text{if } n \text{ is odd.}
\end{cases}$$

What we will do now is to revisit the example of $n=12$. We shall make a new table using the new formula. However, we have to recognize that we have to work out the part with $c_0$ separately from the rest of the divisors $D^* = \{1, 2, 3, 4, 6\}$. With $n$ being even, we use the first equation in the formula and build the following table.

### Table 2  Example of $n=12$ Using New Formula.

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Now, we use the values from the table in the formula for when $n$ is even. We obtain

$$\overline{γ}(D_{2n}, \Delta) = \frac{12!11!}{23!} \left[ 241(112720) + 4 \cdot 263(112632) + 2 \cdot 261(112707) + 2 \cdot 259(112640) + 2 \cdot 257(112710) + 253(112716) \right] = \frac{174631897}{869193}.$$
This being the same average Cayley genus that we obtained from Table 1 in Chapter 2 using the existing formula. Comparing Table 1 with Table 2 that we just constructed for finding the average Cayley genus when \( n = 12 \), we can see the two major problems in the existing formula and the reasons for the new formula. The new formula as seen in Table 2 reduced the number of cases from twelve to six. Also, as can be seen in the final column of both tables, using the new formula, we directly achieve the coefficient sums. So this new formula provides the improvements we were hoping for.

Now, to look at some special cases using this formula. The first special case we will look at is when \( n = p \), where \( p \) is an odd prime. Being that it is an odd prime, the only divisor \( d \) of \( p \) that satisfies \( 1 \leq d < p \) is 1. So for an odd prime integer, there are only two cases, \( c_0 \) and \( c_1 \). Using Corollary 3.8, we have that

\[
c_0 = \frac{\left( \frac{2p-1}{p-1} \right) + p - 1}{p}
\]

and

\[
c_1 = c_0 - 1 = \frac{\left( \frac{2p-1}{p-1} \right) - 1}{p}.
\]

And from Theorem 4.1 when \( n \) is odd, we obtain a formula that depends only on \( p \).

**Corollary 4.2** If \( p \) is an odd prime, then

\[
\overline{\gamma}(D_{2p}, \Delta) = \left( \frac{2p-1}{p-1} \right)^{-1} \left[ \left( \frac{2p-1}{p-1} + p - 1 \right) \left( 2p^2 - 3p + 1 \right) + 2(p-1)^2 \left( \frac{2p-1}{p-1} - 1 \right) \right].
\]
Another special case we will consider is when \( n = p^2 \) where again \( p \) is an odd prime.

So the only divisors of \( p^2 \) satisfying \( 1 \leq d < p^2 \) are 1 and \( p \). So this leads us to three cases, \( c_0 \), \( c_1 \) and \( c_p \). From Corollary 3.8,

\[
c_0 = \frac{\left( \frac{2p^2-1}{p^2-1} \right) + p^2 - 1 + (p-1) \left( \frac{2p-1}{p-1} - 1 \right)}{p^2},
\]

\[
c_1 = c_0 - 1 - U_p = \frac{\left( \frac{2p^2-1}{p^2-1} \right) - \left( \frac{2p-1}{p-1} \right)}{p^2},
\]

and

\[
c_p = c_0 - 1 = \frac{\left( \frac{2p^2-1}{p^2-1} \right) - 1 + (p-1) \left( \frac{2p-1}{p-1} - 1 \right)}{p^2}.
\]

Using Theorem 4.1 when \( n \) is odd, we again obtain a formula that depends only on \( p \).

**Corollary 4.3** If \( p \) is an odd prime, then

\[
\bar{r}(D_{2p^2}, \Delta) = \left( \frac{2p^2-1}{p^2-1} \right)^{-1} \left[ \left( \frac{2p^2-1}{p^2-1} \right) + p^2 - 1 + (p-1) \left( \frac{2p-1}{p-1} - 1 \right) \right] \left( 2p^4 - 3p^2 + 1 \right) +
\]

\[
2p \left( \frac{2p^2-1}{p^2-1} \right) \left( \frac{2p-1}{p-1} \right) +
\]

\[
\left[ \left( \frac{2p^2-1}{p^2-1} \right) - 1 + (p-1) \left( \frac{2p-1}{p-1} - 1 \right) \right] \left( p-1 \right) \left( 2p^4 - 2p^2 - p + 1 \right).
\]
A third special case we will look at is when \( n = 2p \), where again \( p \) is an odd prime. So the divisors of \( 2p \) satisfying \( 1 \leq d < 2p \) are 1, 2 and \( p \). Thus, we have four cases to look at, namely \( c_0, c_1, c_2 \) and \( c_p \). Using Corollary 3.8 to find \( c_0, c_1, c_2 \) and \( c_p \) we get that

\[
c_0 = \frac{(4p-1) + (2p-1) + 4p - 4}{2p},
\]

\[
c_1 = c_0 - 1 - U_2 - U_p = \frac{(4p-1) - (2p-1) - 2}{2p},
\]

\[
c_2 = c_0 - 1 - U_2 = \frac{(4p-1) + (2p-1) - 4}{2p},
\]

and

\[
c_p = c_0 - 1 - U_p = \frac{(4p-1) - (2p-1) + 2p - 2}{2p}.
\]

From Theorem 4.1 when \( n \) is even, we obtain a formula that depends solely on \( p \).
Corollary 4.4 If $n = 2p$, where $p$ is an odd prime, then

$$\overline{\tau}(D_{4p}, \Delta) = (\frac{4p-1}{2p-1})^{-1} \left[ \left( \frac{4p-1}{2p-1} + \frac{(2p-1)}{(p-1)} + 4p-4 \right) \frac{2p}{2p} \right] (8p^2 - 8p + 1) +$$

$$(p-1) \left( \frac{4p-1}{2p-1} - \frac{(2p-1)}{2p} (p-2) \right) (8p^2 - 4p - 1) +$$

$$(p-1) \left( \frac{4p-1}{2p-1} + \frac{(2p-1)}{2p} - 4 \right) \frac{2p}{2p} (8p^2 - 4p - 3) +$$

$$\left( \frac{4p-1}{2p-1} - \frac{(2p-1)}{2p} + 2p - 2 \right) \frac{2p}{2p} (8p^2 - 6p + 1).$$

In conclusion, we now have developed a new formula for finding the average Cayley genus for the dihedral group with generating set consisting of all the reflections. This formula is an improvement in that it allows for fewer cases and enables us to find directly the coefficient sums without having to use the generating function. What sparked our interest in improving this formula can best be seen by looking at the tables in Appendix A. In particular, the prime case seemed the most interesting. For example, look at $n = 13$. What was intriguing the most is what happens in the rows $c_1$, $c_2$, ..., $c_6$. From the generating function, we see six totally different sets of coefficients, but when summed
up in each row, each sums to the same answer. After noticing this pattern for the prime case, we tried to see if there were similar patterns in other cases. That’s when we noticed the patterns in the $p^2$ and $2p$ cases. This inspired us to then try and simplify the formula more. Now this need not be the end for the average Cayley genus for Cayley maps involving the dihedral group. For example we only considered the generating set $\Delta$ consisting of all the reflections of the dihedral group. What would happen if we now looked at the generating set to be a subset of the reflections? Also, knowing that for other groups, finding the average Cayley genus has been difficult to find, can the techniques used in this paper be used for determining the average Cayley genus for Cayley maps using another group other than the dihedral group? These questions, as well as others still await further work.
### APPENDIX A

**GENERATING FUNCTION COEFFICIENTS FOR $n = 2, 3, \ldots, 14$**

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Thesis Title:
Average Cayley Genus for Cayley Maps with Dihedral Groups

Thesis Examination Committee:
Chairperson, Dr. Michelle Schultz, Ph. D.
Committee Member, Dr. Gennady Bachman, Ph. D.
Committee Member, Dr. Douglas Burke, Ph. D.
Graduate Faculty Representative, Dr. William Wells, Ph. D.