Questions and conjectures about multinomial coefficients

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QUESTIONS AND CONJECTURES ABOUT

MULTINOMIAL COEFFICIENTS

by

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A thesis submitted in partial fulfillment
of the requirements for the

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is approved in partial fulfillment of the requirements for the degree of

Master of Science in Pure Mathematics

Examination Committee Chair

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The purpose of this thesis is to try to answer some of the questions in Dr. Bachman’s paper "On Divisibility Properties of Certain Multinomial Coefficients". First we let \{a_i\} be any sequence (finite or infinite) of positive integers such that \(\sum \frac{1}{a_i} \leq 1\). It is clear that \(\frac{n!}{(a_1)! (a_2)! (a_3)! \ldots} \) is an integer because it is a multiple of a certain multinomial coefficient. We let \(f_a(n) = \frac{n!}{L(n) [\frac{a_1}{2}] [\frac{a_2}{3}] [\frac{a_3}{4}] \ldots} \) where \(L(n) = lcm(1,2,3,\ldots,n)\). It is easy to show that \(f_a(n)\) is integer-valued. In particular, we would like to study the sequence \(a_1 = b_1 = 2\) and \(a_{k+1} = b_{k+1} = \prod_{i=1}^{k} b_i + 1\). The first goal of my thesis was to prove the following conjecture by computer for all \(m\) up to 100.

**Conjecture 1** For every positive integer \(m\) there exists a number \(n_0\) such that \(m\) divides \(f(n)\) for all \(n > n_0\) where \(f(n) = \frac{n!}{L(n) [\frac{a_1}{2}] [\frac{a_2}{3}] [\frac{a_3}{4}] \ldots} \).

I did this by using Theorem 1 of Dr. Bachman's paper.
**Theorem 1**  
$p^r || f(n)$ if and only if there are exactly $v$ pairs of integers $(k,l), k,l \geq 1$ such that \( \frac{R_k(\lfloor \frac{n}{p} \rfloor)}{B_k} < \frac{R_{k+1}(\lfloor \frac{n}{p} \rfloor)}{B_{k+1}} \) with $R_k(m)$ defined as $m \equiv R_k(m)$ mod $B_k$ and $0 < R_k(m) \leq B_k$ where $B_k = b_{k+1} - 1$.

The second part of my thesis is concerned with attacking Conjecture 1 as it was written in Dr. Bachman's paper. Before we can restate Conjecture 1 we need to define the base $p$ expansion of a positive integer. We write $n_j = a_0p^j + a_1p^{j-1} + \ldots + a_j$ where $0 \leq a_i \leq p - 1$. Now we restate Conjecture 1 as Conjecture 2.

**Conjecture 2**  
Let $\{n_j\}$ be defined above. Then there exist infinitely many integers $j$ for which the inequality \( \frac{R_k(n_j)}{B_k} < \frac{R_{k+1}(n_j)}{B_{k+1}} \) holds for some integer $k = k(j)$.

In my thesis, I will give proofs of Conjecture 1 for special $\{n_j\}$. I discovered these proofs together with Dr. Bachman and Theorem 1 will be used in all the proofs.
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CHAPTER 1
INTRODUCTION AND STATEMENT OF RESULTS

Let \( \{a_i\} \) be any sequence (finite or infinite) of positive integers such that

\[
\sum_i \frac{1}{a_i} \leq 1. \tag{1}
\]

We know that for any natural number \( n \) the quantity

\[
\frac{n!}{\left\lfloor \frac{n}{a_1} \right\rfloor! \left\lfloor \frac{n}{a_2} \right\rfloor! \left\lfloor \frac{n}{a_3} \right\rfloor! \cdots} \tag{2}
\]

is a positive integer where \( \left\lfloor x \right\rfloor \) is the greatest integer function. This is the case because \( \left\lfloor \frac{n}{a_i} \right\rfloor \) will eventually become 0 and (2) becomes a multiple of a multinomial coefficient.

We will now assume that the sequence in \( a_i \) is infinite. Because the quantity in (2) is an integer, we would like to know what divides it and if it is possible to construct a sequence that makes each term of (2) as small as possible.

We can show that the numbers (2) can not be very small because we will show \( L(n) \) divides (2). Where \( L(n) \) is the least common multiple of the numbers 1,2,3,...,n.

Let us now define

\[
f_a(n) = \frac{n!}{L(n)\left\lfloor \frac{n}{a_1} \right\rfloor! \left\lfloor \frac{n}{a_2} \right\rfloor! \left\lfloor \frac{n}{a_3} \right\rfloor! \cdots}. \tag{3}
\]

With this definition we have the following theorem.

**Theorem 1.** \( f_a(n) \) is integer valued for all natural numbers \( n \)
When studying the function $f_\alpha(n)$, one may ask the following questions that if answered would give us more information about the divisibility properties of $f_\alpha(n)$.

**Question 1.** Is there a sequence $\{a_i^*\}$ which gives rise to an "optimal" function $f_{a^*}(n)$?

We will call $f_{a^*}(n)$ optimal if $m|f_{a^*}(n)$ implies $m|f_\alpha(n)$ for all $n \geq n_{a,m}$.

**Question 2.** Is it true that for any sequence $\{a_i\}$ and any $m$ we have $m|f_\alpha(n)$, for all $n \geq n_{a,m}$?

If we use a greedy approach to this problem, we will obtain the sequence

$$2, 3, 7, 43, 1807, 3263443...$$

At each step, we let $a_i$ be the largest number so that (1) will continue to be satisfied. We will call this special sequence $a_i$. The sequence $\{a_i\}$ occurs in various contexts and was first systematically studied by J. Sylvester [6]. In general, the $b_i$ are defined as follows

$$b_{k+1} = \prod_{i=1}^{k} b_i + 1, \quad \text{for} \quad k \geq 1$$

Other properties that the $b_i$ satisfy are

$$b_k = b_{k-1}^2 - b_{k-1} + 1$$

$$1 - \sum_{i=1}^{k} \frac{1}{b_i} = \prod_{i=1}^{k} \frac{1}{b_i}$$

$$\sum_{i=1}^{\infty} \frac{1}{b_i} = 1.$$  

In the case that $\{a_i\} = \{b_i\}$, we will use $f(n)$ to represent (3). Thus we have

$$f(n) = \frac{n!}{L(n)\left[\frac{n}{2}\right]!\left[\frac{n}{3}\right]!\left[\frac{n}{4}\right]!...}$$
Another property of the \( b_i \) is that if \( a_i \) is any sequence that satisfies (1) we have

\[
\sum_{i \leq k} \frac{1}{a_i} \leq \sum_{i \leq k} \frac{1}{b_i}
\]

for all \( k \geq 1 \), and equality occurs if and only if \( \{a_i\} = \{b_i\} \). This property was proved by Curtis [2]. This property of \( b_i \) leads one to believe that \( f(n) \) is optimal in the class of functions \( f_a(n) \).

The function \( f \) was first introduced by G. Myerson in [3]. He was the first to prove Theorem 1 in the case \( \{a_i\} = \{b_i\} \) and he asked several questions about the divisibility properties of \( f \). In a joint paper with W. Sander [4], \( f \) was studied in more detail and the following conjecture was made.

**Conjecture 1.** For every positive integer \( m \) there exists a number \( n_0 \) such that \( m \) divides \( f(n) \) for all \( n \geq n_0 \).

Myerson and Sander established the validity of this claim for \( m \leq 8 \). In Chapter 3, I will explain the calculations I did to prove this for \( m \leq 100 \).

**Theorem 2.** There is \( n_0 \) such that \( m|f(n) \) for all \( 2 \leq m \leq 100 \) and all \( n \geq n_0 \).

To explain my work we will need some more definitions and notation. These definitions were created by Dr. Bachman to restate the Myerson and Sander conjecture and prove a theorem that tells us when \( p|f(n) \). We define

\[
B_k = \prod_{i=1}^{k} b_i = b_{k+1} - 1.
\]  

(6)

With this definition it is clear that \( B_{k+1} = B_k^2 + B_k \). Let us also define

\[
m \equiv R_k(m) \pmod{B_k} \quad \text{with} \quad 0 < R_k(m) \leq B_k.
\]  

(7)
We will use the standard notation, that \( p^{\nu} \parallel n \) means that \( p^{\nu} \) is the largest power that divides \( n \). Now we have the following theorem that was proved by Dr. Bachman[1].

**Theorem 3.** \( p^{\nu} \parallel f(n) \) if and only if there are exactly \( \nu \) pairs of integers \((k,l)\), \( k, l \geq 1 \), such that

\[
\frac{R_k([n/p^l])}{B_k} < \frac{R_{k+1}([n/p^l])}{B_{k+1}}. \tag{8}
\]

Using Theorem 3 we see that questions about the divisibility properties of \( f \) are really questions about the distribution of the sequence \([n/p^l]\) when reduced modulo \( B_k \) for \( k \geq 1 \). This leads us to a interesting reformulation of this problem. If we let

\[
n_j = a_0 p^j + a_1 p^{j-1} + \ldots + a_j, \tag{9}
\]

with \( 1 \leq a_0 \leq p - 1 \) and \( 0 \leq a_i \leq p - 1 \), for \( i \geq 1 \). It is clear that \([n_j/p^j] = n_{j-1}\) for \( j \geq l \), and hence, by Theorem 3, if there exists a sequence \( \{n_j\} \) for which the sequence \( \{R_k(n_j)/B_k\} \) is non-increasing for every \( j \), then Conjecture 1 with \( m = p \) fails. Therefore we have the following conjecture by Dr. Bachman, which is of interest in its own right.

**Conjecture 2.** Let \( n_j \) be defined in (9). Then for infinitely many \( j \) the inequality

\[
\frac{R_k(n_j)}{B_k} < \frac{R_{k+1}(n_j)}{B_{k+1}}. \tag{10}
\]

holds for some integer \( k = k(j) \).

**Theorem 4.** Conjecture 1 is true if and only if conjecture 2 is true.

We will prove Theorem 4 in Chapter 4. The existence of a sequence \( \{n_j\} \) with the monotonicity property required to defeat Conjecture 2 appears to be rather
unlikely. We showed that the conjecture is true for a special class of sequences. These sequences $n_j$ have the property that their associated $a_i$ sequences are eventually periodic. We will call any sequence $n_j$ eventually periodic if there exist $i_0$ and $\kappa$ such that for all $i \geq i_0$ we have $a_{i+\kappa} = a_i$. The new way of stating Conjecture 1 seems to be the right approach to this problem. Using these new ideas we were able to prove the following theorem, which we will prove in Chapter 5.

**Theorem 5.** Let $n_j$ be any eventually periodic sequence defined in (9). Then for infinitely many $j$ the inequality

$$\frac{R_k(n_j)}{B_k} < \frac{R_{k+1}(n_j)}{B_{k+1}}.$$  \hspace{1cm} (11)

holds for some integer $k = k(j)$.

This result nicely complements our work on Theorem 2. Theorem 5 seems to dash any hopes of finding a simple counterexample to Conjecture 2 or equivalently Conjecture 1. We also show in the proof of Theorem 5 that (11) holds infinitely often for most $k$. Theorems 2 and 4 and this last statement lend support to the conjectures.
CHAPTER 2
PROOF OF THEOREM 1

We will use the well known fact that if

\[ \nu = \sum_{j = \frac{\log(n)}{\log(p)}}^{\lfloor \log(n) \rfloor} \left\lfloor \frac{n}{p^j} \right\rfloor \]  \hspace{1cm} (1)\]

then we have \( p^{\nu} \parallel n! \). A proof of this fact can be found on page 182 of Niven [5].

Now we will use this fact to calculate the power of \( p \) that divides the numerator and denominator of \( f_a(n) \). If the power for the numerator is greater or equal to that of the denominator, then our theorem is true. The power dividing the numerator is given by (1). The power that divides \( \left\lfloor \frac{n}{a_i} \right\rfloor \) is

\[ \sum_{j = \frac{\log(n)}{log(p)}}^{\lfloor \log(n) \rfloor} \left\lfloor \frac{n}{a_i p^j} \right\rfloor \]  \hspace{1cm} (2)\]

by (1). It is clear that

\[ \frac{\log(n)}{\log(p)} = \sum_{j = \frac{\log(n)}{\log(p)}}^{\lfloor \log(n) \rfloor} 1 \]  \hspace{1cm} (3)\]

is the power of \( p \) that divides \( L(n) \) because this is the largest power of \( p \) that is less than \( n \). The power of \( p \) that divides \( f_a(n) \) is

\[ \sum_{j = \frac{\log(n)}{\log(p)}}^{\lfloor \log(n) \rfloor} \left( \left\lfloor \frac{n}{p^j} \right\rfloor - \sum_{i=1}^{\infty} \left\lfloor \frac{n}{a_i p^j} \right\rfloor - 1 \right) \]  \hspace{1cm} (4)\]

by (1), (2), and (3). But this is equal to

\[ \sum_{j = \frac{\log(n)}{\log(p)}}^{\lfloor \log(n) \rfloor} \left( \left\lfloor \frac{n}{p^j} \right\rfloor - \sum_{i=1}^{\infty} \left\lfloor \frac{n}{a_i p^j} \right\rfloor - 1 \right) \]  \hspace{1cm} (5)\]

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because \( \lfloor \frac{|z|}{a} \rfloor = \lfloor \frac{z}{a} \rfloor \) for all real \( x \) and all integer \( a \). This fact is also in Niven on page 180. For all \( j \)'s that are being considered in the sum we have \( \lfloor \frac{n}{p^j} \rfloor \geq 1 \) and

\[
\sum_{i=1}^{\infty} \left\lfloor \frac{n}{a_i p^j} \right\rfloor = \sum_{i=1}^{\infty} \left\lfloor \frac{n/p^j}{a_i} \right\rfloor < \sum_{i=1}^{\infty} \left\lfloor \frac{n/p^j}{a_i} \right\rfloor \leq \left\lfloor \frac{n}{p^j} \right\rfloor
\] (6)

by (1) of Chapter 1 and our second fact. Since the quantities on the right and the left of (6) are integers we have

\[
\sum_{i=1}^{\infty} \left\lfloor \frac{n}{a_i p^j} \right\rfloor \leq \left\lfloor \frac{n}{p^j} \right\rfloor - 1
\] (7)

Thus for every \( j \) we have

\[
\left\lfloor \frac{n}{p^j} \right\rfloor - \sum_{i=1}^{\infty} \left\lfloor \frac{n}{a_i p^j} \right\rfloor - 1 \geq 0.
\] (8)

Therefore the theorem follows because the terms of the sum in (5) are all greater than or equal to 0 by (8). \( \Box \)
CHAPTER 3

COMPUTATIONAL RESULTS

In this chapter we will discuss the computations needed to prove Theorem 2. Theorem 2 stated that the conjecture of Myerson and Sander is true for all $m \leq 100$. The algorithm is essentially the algorithm they used to prove the conjecture for $m \leq 8$ in their paper. I emailed Gerald Myerson and he sent his results for $m \leq 16$ before I started my computations. I wanted to check that my new algorithm produced the same results as theirs and it did for all of his data. The only difference between our algorithms is that he uses the function

$$g(x) = \left\lfloor x \right\rfloor - \sum_{i=1}^{\infty} \left\lfloor \frac{x}{b_i} \right\rfloor - 1$$

(1)

to build a tree and I used

$$\frac{R_k(n_j)}{B_k} < \frac{R_{k+1}(n_j)}{B_{k+1}}.$$  

(2)

My algorithm was originally implemented in Mathematica 4.0 on the NSCEE super computer at UNLV. I used Mathematica to prove the conjecture for all $m \leq 37$ and the proof for $m = 37$ required 29 hours of computer time. The rest of the computations were done with C++ and GNU'S MP. Ashley Hatch and John Kilburg helped me translate the Mathematica code into C++. We used GNU'S MP for the large integers that we needed to work with. The C++ version of the algorithm was
substantially faster and the computations were finished on a computer in the Physics department in a couple of days. We will now discuss my algorithm in detail.

Details Of Algorithm For Prime m

Set $B_{max} = 8$


Set $L =$ Empty List

Define $M[a, b] = \begin{cases} 1 & \text{if } b | a \\ \frac{\text{Mod}(a, b)}{b} & \text{otherwise} \end{cases}$

Build First Row

For k = 1 to m

If $M[k, 2] \geq M[k, 6] \geq M[k, 42]$

add k to list $L$

Increment k

Set $len =$ Length of list $L$

Set $sumlen = len$

While $len > 0$ Build Next Row

For i = 1 to len

Look at each list element and calculate children

$q =$ First Element of $L$

Set $n = m \times q$

Set $t \equiv n \pmod{B_{B_{max}}}$
Remove q from list L

For k = 0 to m-1

For j = 1 to Bmax

Set \( r[k, j] = M[t + k, B_j] \)

Increment j

Increment k

M = Empty List

Put all children of q in List M

For k = 0 to m-1

If

\[
    r[k, 1] \geq r[k, 2] \geq r[k, 3] \geq ... \geq r[k, B_{max}] 
\]  

(4)

Add n+k to list M

Set \( L = M \cup L \)

Increment k

L is the new row of tree

Display L

Set len = Length of L

Set sumlen = sumlen + len

End While Loop

Display number of exceptions to conjecture which is m*sumlen+m-1

Display largest exception to conjecture which is m*(largest node)+m-1
Comments on Algorithm

When calculating (4) in the C++ version the computer will check each pair of inequalities from left to right and if one of these is false it will stop calculation and will conclude that the value of the whole statement is false. This is an important savings in execution time because the computer only calculates the needed values of $r[k, j]$ to decide the value of (4). If we are considering a $m$ that is of the form $p^n$, we must count the number of times (2) is satisfied. If this number is less than $\nu$ we add $n + k$ to the list.
CHAPTER 4

PROOF OF THEOREM 4

In this chapter, we will prove that Conjecture 1 is equivalent to Conjecture 2. We will begin by showing that Conjecture 1 implies Conjecture 2. We proceed by assuming that Conjecture 2 is false. It follows that there is a sequence \( \{n_j\} \) such that

\[
\frac{R_k(n_j)}{B_k} < \frac{R_{k+1}(n_j)}{B_{k+1}}
\]  

holds for only a finite number of pairs \((k, j)\). Let \( \nu \) be the number of times (1) is true for this sequence \( \{n_j\} \). Thus for any \( n_j \),

\[
\frac{R_k(\lfloor \frac{n_j}{p^\nu} \rfloor)}{B_k} = \frac{R_k(n_{j-1})}{B_k} < \frac{R_{k+1}(n_{j-1})}{B_{k+1}} = \frac{R_{k+1}(\lfloor \frac{n_j}{p^\nu} \rfloor)}{B_{k+1}}
\]

is true for at most \( \nu \) pairs \((k, l)\). This implies, by Theorem 3, that

\[
p^{\nu+1} \nmid f(n_j),
\]

for every \( j \). Whence Conjecture 1 with \( p^{\nu+1} \) fails. Thus Conjecture 1 implies Conjecture 2.

Now we will prove that Conjecture 2 implies Conjecture 1. We proceed by assuming that Conjecture 1 is false for \( m = p^{\nu} \). This gives us infinitely many \( n \)'s such that \( p^{\nu} \nmid f(n) \). Thus by Theorem 1 we get a set \( \mathcal{N} \) of infinitely many \( n \)'s such that

\[
\frac{R_k(\lfloor \frac{n}{p^{\nu}} \rfloor)}{B_k} < \frac{R_{k+1}(\lfloor \frac{n}{p^{\nu}} \rfloor)}{B_{k+1}}
\]  


is true less than $\nu$ times. Let us now consider the expansion of $n$ in powers of $p$. If we define

$$k = \left\lfloor \frac{\log(n)}{\log(p)} \right\rfloor,$$

then we can expand $n$ in powers of $p$ such that

$$n = a_0 p^k + a_1 p^{k-1} + a_2 p^{k-2} + \ldots + a_{k-1} p + a_k$$  \hspace{1cm} (6)

with $0 \leq a_i < p$. We can partition the set $\mathcal{N}$ into smaller sets $M_i$, $1 \leq i \leq p - 1$, by letting

$$M_i = \{ n \in \mathcal{N} | a_0 = i \}.$$  \hspace{1cm} (7)

Thus by (7) we obtain

$$\mathcal{N} = \bigcup_{i=1}^{p-1} M_i.$$  \hspace{1cm} (8)

We note that one of these $M_i$ in our partition must be infinite. Let $\mathcal{N}_0$ be one of the infinite $M_i$ sets. Thus the elements of $\mathcal{N}_0$ have the same first term of their expansion in powers of $p$. We can partition this set into smaller sets $M_{i,0}$, $0 \leq i \leq p - 1$, by letting

$$M_{i,0} = \{ n \in \mathcal{N}_0 | a_1 = i \}.$$  \hspace{1cm} (9)

Thus by (9) we obtain

$$\mathcal{N}_0 = \bigcup_{i=0}^{p-1} M_{i,0}.$$  \hspace{1cm} (10)
We note that one of these $M_{i,0}$ in our partition must be infinite. Let $\mathcal{N}_1$ be one of the infinite $M_{i,0}$ sets. Thus the elements of $\mathcal{N}_1$ have the same $a_0$ and $a_1$ in their expansion in (6).

We continue this construction by defining

$$M_{i,j} = \{n \in \mathcal{N}_j | a_{j+1} = i\} \quad [0 \leq i \leq p - 1]$$

(11)

and letting $\mathcal{N}_{j+1}$ be one of the infinite $M_{i,j}$. This yields a sequence of sets $\{\mathcal{N}_j\}$, $\mathcal{N}_{j+1} \subset \mathcal{N}_j$, and a sequence of "defining coefficients" $\{a_j\}$, such that all $n \in \mathcal{N}_j$ have expansions (6) with identical coefficients $a_i$, for $0 \leq i \leq j$. Now we can define a sequence $\{n_j\}$, using the $a_i$ sequence above, by

$$n_j = a_0 p^j + a_1 p^{j-1} + a_2 p^{j-2} + ... + a_{j-1} p + a_j.$$ 

(12)

We now observe that $n_j = \lfloor \frac{n}{p^l} \rfloor$ for some $n \in \mathcal{N}$ and appropriate value of $l$. It follows by (4) that there are less than $\nu$ values of $j$ such that (1) holds. Therefore Conjecture 2 fails for $\{n_j\}$ and Conjecture 2 implies Conjecture 1.
CHAPTER 5

PROOF OF THEOREM 5

Let $\kappa$ be a period of the sequence $\{a_i\}_{i=0}^\infty$. Fix $i_0$ so that for all $i \geq i_0$ we have

$$a_{i+\kappa} = a_i,$$

and set

$$F = a_0 p^{i_0 - 1} + a_1 p^{i_0 - 2} + \cdots + a_{i_0 - 1} \quad (1)$$

and

$$G = a_{i_0} p^{\kappa - 1} + a_{i_0 + 1} p^{\kappa - 2} + \cdots + a_{i_0 + \kappa - 1}. \quad (2)$$

Also set

$$P = p^\kappa \quad (3)$$

and let $\{N_j\}_{j=0}^\infty$ be the subsequence of $\{n_j\}_{j=0}^\infty$ defined by

$$N_j = n_{\kappa j + i_0 - 1} = p^\kappa j (a_0 p^{i_0 - 1} + \cdots + a_{i_0 - 1}) + p^\kappa (j-1) (a_{i_0} p^{\kappa - 1} + \cdots + a_{i_0 + \kappa - 1})$$

$$+ p^\kappa (j-2) (a_{i_0} p^{\kappa - 1} + \cdots + a_{i_0 + \kappa - 1}) + \cdots + (a_{i_0} p^{\kappa - 1} + \cdots + a_{i_0 + \kappa - 1}),$$

so that

$$N_j = FP^j + G \frac{p^j - 1}{P - 1}. \quad (4)$$
To prove the theorem we will show that there are infinitely many pairs of integers $j$ and $k$ such that the inequality

$$\frac{R_k(N_j)}{B_k} < \frac{R_{k+1}(N_j)}{B_{k+1}}$$

holds.

Although our argument is somewhat technical, the basic idea behind it is very simple. For the convenience of the reader, we will now illustrate the idea by giving a proof of (5) under certain additional assumptions. More precisely, let us now consider the case where $a_0 = 1$, $a_i = 0$, for all $i \geq 1$, and $p$ is coprime with all $B_k$. In this case we can take $\kappa = 1$ and $i_0 = 1$, and the quantities defined in (1)-(4) become $F = 1$, $G = 0$, $P = p$, and $N_j = n_j = p^j$. Now, given any $k$ there is $j_0$ such that

$$p^{j_0} \equiv 1 \pmod{B_k}. \quad (6)$$

We fix $j_0$ and, if necessary, increase $k$ so that in addition to (6) we also have $p^{j_0} \not\equiv 1 \pmod{B_{k+1}}$. In view of the identity

$$B_{k+1} = B_k b_{k+1} = B_k (B_k + 1),$$

by (6) of Chapter 1, and the assumption $(p, B_{k+1}) = 1$, we also see that $p^{j_0} \not\equiv 1 + B_k \pmod{B_{k+1}}$. It follows that

$$p^{j_0} \equiv 1 + tB_k \pmod{B_{k+1}} \quad (8)$$

for some $1 < t < b_{k+1}$. But (6)-(8) yield the inequality

$$\frac{R_{k+1}(p^{j_0})}{B_{k+1}} = \frac{1 + tB_k}{B_{k+1}} > \frac{1 + B_k}{B_k(B_k + 1)} = \frac{R_k(p^{j_0})}{B_k},$$

as desired.
Let us now return to the general case. By the definition of $b_k$ (4) of Chapter 1 we see that

$$\langle b_k, b_l \rangle = 1$$

(9)

for $k \neq l$. Therefore either $(p, b_k) = 1$ for all $k$ or there is exactly one index $k_0$ such that $p \mid b_{k_0}$. In the latter case let $\tau$ be such that $p^\tau \| b_{k_0}$, while in the former case set $\tau = 0$ and $k_0 = 1$. With these conventions we see that in either case we have

$$p^\tau \| B_k,$$

(10)

for $k \geq k_0$, by the definition of $B_k$ (6) of Chapter 1. Analogously, there is $k_1 \geq k_0$ such that for all $k \geq k_1$

$$(P - 1, b_k) = 1.$$ 

(11)

It follows from (11) that the congruence

$$(P - 1)V \equiv 1 \pmod{b_{k+1}}$$

(12)

is solvable for every $k \geq k_1$ and we let $V = V_k$ be any fixed solution of (12). We also fix an integer $\nu_0$ satisfying $\nu_0 \kappa \geq \tau$, so that

$$P_0 = P^{\nu_0} \geq p^\tau.$$

(13)

With these definitions in place we are ready to state our principal lemma.

**Lemma 1:** For infinitely many integers $k \geq k_1$ there are integers $\nu_1 = \nu_1(k)$ and $T = T_k$ such that the congruence

$$N_{\nu_0 + \nu_1 m} \equiv X + YV(1 - T^m)B_k \pmod{B_{k+1}}$$

(14)
holds for all \( m \geq 1 \), with

\[
X = FP_0 + G\frac{P_0 - 1}{P - 1}, \quad Y = X(P - 1) + G,
\]

and \( V \) given by (12). Furthermore, we have

\[(T, b_{k+1}) = 1 \text{ and } T \not\equiv 1 \pmod{b_{k+1}}. \quad (16)\]

Recall that our aim is to establish the inequality (5). Observe that, by (15), (1)-(3), and (13), the quantities \( X \) and \( Y \) are absolute. Hence, assuming that \( k \) in Lemma 1 is sufficiently large, as we may, and writing \( j(m) = \nu_0 + \nu_1m \) we get \( R_k(N_{j(m)}) = X \), by (14) and (7). To evaluate \( R_{k+1}(N_{j(m)}) \) we let \( c_m, m \geq 1 \), be the integer satisfying

\[
c_m \equiv YV(1 - T^m) \pmod{b_{k+1}}, \quad (17)
\]

with \( 0 \leq c_m < b_{k+1} \). Thus, (14), and (7) yield \( R_{k+1}(N_{j(m)}) = X + c_mB_k \). Therefore, for every \( m \) such that \( X < c_m < b_{k+1} \) we obtain, by (7),

\[
\frac{R_{k+1}(N_{j(m)})}{B_{k+1}} = \frac{X + c_mB_k}{B_{k+1}} > \frac{X(1 + B_k)}{B_k(1 + B_k)} = \frac{R_K(N_{j(m)})}{B_k},
\]

as desired. Whence (5) follows from Lemma 1 and Lemma 2 below.

**Lemma 2.** For all sufficiently large \( k \) occurring in Lemma 1 there are infinitely many indices \( m \) such that the quantity \( c_m \) defined by (17) satisfies \( X < c_m < b_{k+1} \).

We remark that Lemmas 1 and 2 show that, in fact, there are infinitely many \( k \) such that (5), and consequently (11) of Chapter 1, hold with \( k \) fixed for infinitely many \( j \). Thus it only remains to prove these lemmas.
Proof of Lemma 2. Let $k$ occurring in Lemma 1 be fixed. Since $c_m$ is periodic modulo $b_{k+1}$, it suffices to show that there is at least one $m$ such that $X < c_m < b_{k+1}$.

To see this we assume that

$$0 \leq c_m \leq X \quad (18)$$

and show that this leads to a contradiction.

Recall that the quantities $P - 1$, $X$, and $Y$ are absolute. Hence, by (9), we may assume that $k$ is so large that

$$([2(P - 1)(X + 1)^2Y]!, b_{k+1}) = 1. \quad (19)$$

Let $h$ be the order of $T$ modulo $b_{k+1}$, which is well defined by (16). It readily follows from (17), (12), and (19) that if $c_m = c_l$, then $h \mid l - m$. Hence, by (18), we have

$$h \leq X + 1. \quad (20)$$

Note that $(P - 1)c_m \equiv Y(1 - T^m) \pmod{b_{k+1}}$, by (17) and (12). This yields

$$\left(1 - T\right)\left(Yh - (P - 1) \sum_{m=1}^{h-1} c_m\right) \equiv Y(1 - T^h)$$

$$\equiv 0 \pmod{b_{k+1}}. \quad (21)$$

But, by (18) and (20), we have

$$|Yh - (P - 1) \sum_{m=1}^{h-1} c_m| < 2(P - 1)(X + 1)^2 Y.$$
Hence, by (19), the second factor on the left-hand side of (21) is coprime with $b_{k+1}$, unless it is equal to 0. Whence (21) and (16) yield

$$Yh - (P - 1) \sum_{m=1}^{h-1} c_m = 0. \quad (22)$$

Now, on the one hand (18) implies

$$\sum_{m=1}^{h-1} c_m < Xh,$$

on the other hand (22), (15), and (2) give

$$\sum_{m=1}^{h-1} c_m \geq Xh.$$

This shows that (18) is false and concludes the proof of the lemma.

It only remains to prove Lemma 1. It turns out that in order to deduce congruence (14) we first want to do arithmetic modulo $C_k$, $k \geq k_1$, defined by

$$C_k = (P - 1)p^{-r}B_k. \quad (23)$$

Observe that, by (3), we have

$$(P, C_k) = 1, \quad (24)$$

for all $k \geq k_1$. Furthermore, (23) and (7) yield the identities

$$C_{k+1} = b_{k+1}C_k \quad (25)$$

and

$$(P - 1)C_{k+1} = p^rC_k^2 + (P - 1)C_k. \quad (26)$$
Next we observe that, by (24), for every \( k \geq k_1 \) there is \( \nu_1 \) such that \( P^{\nu_1} \equiv 1 \pmod{C_k} \). Furthermore, by keeping \( \nu_1 \) fixed and, if necessary, increasing \( k \), we can arrange it so that we also have \( P^{\nu_1} \not\equiv 1 \pmod{C_{k+1}} \). With such a pair of integers \( k \) and \( \nu_1 \) fixed, we set

\[
Q = P^{\nu_1}
\]

and write, by (25),

\[
Q \equiv 1 + tC_k \pmod{C_{k+1}},
\]

with \( 0 < t < b_{k+1} \). Moreover, fix an integer \( W \) satisfying

\[
p^W \equiv 1 \pmod{b_{k+1}}
\]

and set

\[
T = 1 - (P - 1)Wt.
\]

Triples of integers \( k, \nu_1, \) and \( T \) just specified exist for infinitely many values of \( k \), and we will now show that for each such triple, the assertions of Lemma 1 follow from the following lemma.

**Lemma 3.** For \( m \geq 1 \) we have

\[
Q^m \equiv 1 + p^W(1 - T^m)C_k \pmod{C_{k+1}}.
\]

Proof of Lemma 1. Let a triple of integers \( k, \nu_1, \) and \( T \) defined in the paragraph preceeding Lemma 3 be fixed. Note that, by (4), (13), and (27), the left hand side
of (14) is

\[ N_{\nu_0 + \nu_1 m} = F P_0 Q^m + G \frac{P_0 Q^m - 1}{P - 1}. \]  

(32)

Note also that, by Lemma 3, for every \( m \geq 1 \) there is an integer \( s = s(m) \) such that

\[ Q^m = 1 + p^r V (1 - T^m) C_k + s C_{k+1}. \]

Whence, by (23) and (13), we get

\[ P_0 Q^m \equiv P_0 + P_0 (P - 1) V (1 - T^m) B_k \pmod{B_{k+1}} \]  

(33)

as well as

\[ \frac{P_0 Q^m - 1}{P - 1} \equiv \frac{P_0 - 1}{P - 1} + P_0 V (1 - T^m) B_k \pmod{B_{k+1}}. \]  

(34)

Substituting (33) and (34) into (32) yields (14).

Next, recall that \((Q, C_{k+1}) = 1\), by (27) and (24). Therefore, for some integer \( h \) we have \( Q^h \equiv 1 \pmod{C_{k+1}} \). But, by Lemma 3, (25), and (12), this gives \( T^h \equiv 1 \pmod{b_{k+1}} \), and shows, in particular, that the first assertion in (16) holds. The second assertion in (16) follows simply from (30) and (29), since \( 0 < t < b_{k+1} \).

Thus, the validity of assertions (14) and (16) for the triple \( k, \nu_1, \) and \( T \) has been established. But, as we already noted, there are infinitely many such triples of integers \( k, \nu_1(k), \) and \( T_k \) (corresponding to infinitely many values of \( k \)), and the proof of the lemma is complete.

It now only remains to prove Lemma 3.

Proof Of Lemma 3. We argue by induction. Note that for \( m = 1 \) congruence (31) is equivalent to (28), by (30), (29), and (12).

For the inductive step we write
\[ Q^{m+1} \equiv \{1 + p^r V(1 - T^m)C_k\} \{1 + p^r V(1 - T)C_k\} \]

\[ \equiv 1 + p^r V(2 - T - T^m)C_k + p^r V^2(1 - T)(1 - T^m)C_k^2 \pmod{C_{k+1}}. \]

Combining this with (26), (25), and (12) yields

\[ Q^{m+1} \equiv 1 + p^r V(2 - T - T^m - (1 - T)(1 - T^m))C_k \]

\[ \equiv 1 + p^r V(1 - T^{m+1})C_k \pmod{C_{k+1}}, \]

as desired. The proof of Lemma 1 is now complete.
## APPENDIX

### CALCULATION DATA

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