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THE DICHOTOMY IN THE DETERMINACY OF CERTAIN TWO-PERSON INFINITE GAMES WITH MOVES FROM $\{0,1\}$

by

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ABSTRACT

The Dichotomy in the Determinacy of Certain Two-person Infinite Games with Moves from {0,1}

by

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We investigate certain well-known games from the field of set theory; namely, certain two-person games of perfect information with small complexity and with small infinite length. We consider games with moves from the natural numbers and games with moves from $\{0,1\}$. We show that the determinacy of open games with length $\omega \cdot n$ and with moves from $\{0,1\}$ is true regardless of the existence of large cardinals for $n \geq 2$. We show that this is not true, however, for some more complex games: For $k \geq 3$ and $n \geq 2$, the determinacy of Π_k^0 games with length $\omega \cdot n$ and with moves from $\{0,1\}$ is equivalent to the determinacy of Π_k^0 games with length $\omega \cdot n$ and with moves from ω, which in turn requires the existence of large cardinals. We also examine the question of whether for classes Γ properly between Σ_1^0 and Π_3^0, large cardinals are required for the determinacy of Γ games with length $\omega \cdot n$ and with moves from $\{0,1\}$ for $n \geq 2$.

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CHAPTER 1

PRELIMINARIES AND NOTATION

Our research questions arise from the field of determinacy and set theory. We are interested in the determinacy of certain games, namely, certain two-person infinite games of perfect information. Before discussing determinacy, we first review two-person infinite games of perfect information by describing a play in such a game, the length of the game, and a payoff set for the game.

A two-person infinite game of perfect information is infinite as there are infinitely many moves, which can be denoted by $f(0), f(1), f(2), \ldots$, that are played by two players. It is standard that ω denotes the set of natural numbers: $\{0, 1, 2, 3, \ldots\}$. A play of a two-person infinite game of perfect information with length ω can be thought of as a function f from ω into X, that is, $f \in X^{\omega}$. This type of game is said to have length ω due to the ω-many moves. The function f is defined by the two players, player I and player II, alternately choosing an element from the set X (to play as moves). In this thesis, we are interested in games in which the moves are from the natural numbers or from $\{0,1\}$. Player I chooses $f(0) \in X$ first and then player II chooses $f(1) \in X$. This continues with player I choosing an element from X for $f(2k)$, the $2k^{th}$ move, and player II choosing an element from X for $f(2k+1)$, the $2k+1^{st}$ move (assuming that

$f(0)$ is referred to as the 0^{th} move). Associated with this game is a payoff set A, which is a collection of functions from ω into X, that is, $A \subseteq X^{\omega}$. Player I wins the game $G(A)$ if the resulting play f is in the payoff set A; that is, player I wins the game $G(A)$ if $f \in A$. Player II wins the game $G(A)$ if the resulting play f is not in the payoff set A; that is, player II wins the game $G(A)$ if $f \notin A$. It is standard to identify the game $G(A)$ with its payoff set A, so as to refer to $G(A)$ as "the game A".

In this thesis, we are interested in games with length $\omega \cdot n$. We shall now describe a two-person infinite game of perfect information *with length* $\omega \cdot n$ by describing a play in such a game, the length of the game, and a payoff set for the game. A play of a two-person infinite game of perfect information with length $\omega \cdot n$ can be thought of as a function $h = f^1 {}^\frown f^2 {}^\frown \ldots {}^\frown f^n$ from $\omega \cdot n$ into X, that is, $h = f^1 {}^\frown f^2 {}^\frown \ldots {}^\frown f^n \in X^{\omega \cdot n}$. The game is said to have length $\omega \cdot n$ due to the $\omega \cdot n-$many moves. The function f^1 is defined first by the two players with player I and player II alternately choosing an element from the set X (to play as moves) so that $f^1(i) = h(i)$ for all $i \in \omega$. Once f^1 is completely defined, f^2 is similarly defined next so that $f^2(i) = h(\omega + i)$ for all $i \in \omega$. The remaining f^k, $3 \le k \le n$, are similarly defined by $f^k(i) = h(\omega \cdot (k-1) + i)$ for all $i \in \omega$, producing the play $f^1 {}^\frown f^2 {}^\frown \ldots {}^\frown f^n \in X^{\omega \cdot n}$. Although non-standard, we shall refer to f^k as the k^{th} round in the play $f^1 {}^\frown f^2 {}^\frown \ldots {}^\frown f^n$ for $1 \le k \le n$. Note that when $n = 2$ and the length of the game is $\omega \cdot 2$, we shall denote a play from the game as $f {}^\frown g$ instead of $f^1 {}^\frown f^2$. Associated with this game is a payoff set A, which is a collection of functions

from $\omega \cdot n$ into X, that is, $A \subseteq X^{\omega \cdot n}$. Player I wins the game A if the resulting play $f^1 \frown f^2 \frown \ldots \frown f^n \in A$ and player II wins the game A if the resulting play $f^1 \frown f^2 \frown \ldots \frown f^n \notin A$.

The game is said to be of perfect information since at any point in the game, each player has full knowledge of all the previous moves in the game. This fact is important when describing strategies and winning strategies for a given player. σ is a strategy for player I if σ defines each move for player I in terms of the previous moves in the game. Similarly, τ is a strategy for player II if τ defines each move for player II in terms of the previous moves in the game. For $f \in X^{\omega}$, we define $f \upharpoonright n$ and $\overline{f}(n)$ to be the first n terms of the play f, that is, $f \upharpoonright n = \overline{f}(n) = (f(0), f(1), \ldots, f(n-1))$. A play $f^1 \frown f^2 \frown \ldots \frown f^n \in X^{\omega \cdot n}$ is according to a strategy σ for player I has the obvious meaning: For each k, $f^k(2i) = \sigma\left(f^0 \frown f^1 \frown \ldots \frown f^{k-1} \frown \overline{f^k}(2i)\right)$ for all $i \in \omega$. A strategy σ is a winning strategy (abbreviated w.s.) for player I if for all $f^1 \frown f^2 \frown \ldots \frown f^n \in X^{\omega \cdot n}$ such that $f^1 \frown f^2 \frown \ldots \frown f^n$ is according to σ, $f^1 \frown f^2 \frown \ldots \frown f^n \in A$, so that player I wins the game A. A play $f^1 \frown f^2 \frown \ldots \frown f^n \in X^{\omega \cdot n}$ is according to a strategy τ for player II has the obvious meaning: For each k, $f^k(2i+1) = \tau\left(f^0 \frown f^1 \frown \ldots \frown f^{k-1} \frown \overline{f^k}(2i+1)\right)$ for all $i \in \omega$. A strategy τ is a w.s. for player II if for all $f^1 \frown f^2 \frown \ldots \frown f^n \in X^{\omega \cdot n}$ such that $f^1 \frown f^2 \frown \ldots \frown f^n$ is according to τ, $f^1 \frown f^2 \frown \ldots \frown f^n \notin A$, so that player II wins the game A. A game is said to be determined if either player I or player II has a w.s. for the game.

We will now describe an open game, which is a two-person infinite game of perfect information in which the payoff set is an open set. To do this, we will define an

open set and describe the winning conditions for an open game. A set $A \subseteq X^{\omega\cdot n}$ is said to be open if and only if there exists a collection $\left\{\left(\overline{\mathrm{u}_i^1}, \overline{\mathrm{u}_i^2}, \ldots, \overline{\mathrm{u}_i^n}\right) \middle| i \in I\right\}$ of n-tuples of finite sequences from which A can be computed in the following sense: $f^1{}^\frown f^2{}^\frown \ldots {}^\frown f^n \in A$ if and only if there exists $i \in I$ such that f^j extends the finite sequence $\overline{\mathrm{u}_i^j}$ for each j such that $1 \leq j \leq n$. The set $\left\{\left(\overline{\mathrm{u}_i^1}, \overline{\mathrm{u}_i^2}, \ldots, \overline{\mathrm{u}_i^n}\right) \middle| i \in I\right\}$ is said to generate the set $A \subseteq X^{\omega\cdot n}$. Therefore, a set $A \subseteq X^{\omega\cdot n}$ is open if and only if there exists $\left\{\left(\overline{\mathrm{u}_i^1}, \overline{\mathrm{u}_i^2}, \ldots, \overline{\mathrm{u}_i^n}\right) \middle| i \in I\right\}$ that generates A. Notice that player I wins the open game A generated by $\left\{\left(\overline{\mathrm{u}_i^1}, \overline{\mathrm{u}_i^2}, \ldots, \overline{\mathrm{u}_i^n}\right) \middle| i \in I\right\}$ if and only if the play $f^1{}^\frown f^2{}^\frown \ldots {}^\frown f^n$ is such that for some fixed $i \in I$, $\overline{\mathrm{u}_i^j} \subseteq f^j$ for all j such that $1 \leq j \leq n$. Player II wins the open game A generated by $\left\{\left(\overline{\mathrm{u}_i^1}, \overline{\mathrm{u}_i^2}, \ldots, \overline{\mathrm{u}_i^n}\right) \middle| i \in I\right\}$ if and only if the play $f^1{}^\frown f^2{}^\frown \ldots {}^\frown f^n$ is such that for all $i \in I$, $\overline{\mathrm{u}_i^j} \not\subset f^j$ for some $j = j(i)$ that depends on i where $1 \leq j \leq n$.

Gale and Stewart [GS] showed that all open games with length ω are determined (see Theorem 1.10 below). Many other interesting results are well-known about the determinacy of games with length ω and also about games with countable length. Excellent references for material on determinacy beyond this thesis include: Chapter 6 of *Descriptive Set Theory* by Yiannis N. Moschovakis [Mo], Chapter 6 of *The Higher Infinite* by Akihiro Kanamori [Ka], *Classical Descriptive Set Theory* by Alexander S. Kechris [Ke], Donald A. Martin's upcoming book on determinacy [Ma3], and the paper "Long Games" by John R. Steel [St].

As indicated before, we are interested in the determinacy of certain games of length $\omega \cdot n$ in this thesis. We will first generalize the Gale-Stewart result. Dr. Douglas Burke noticed the following generalization as a graduate student of Donald A. Martin at UCLA: Open games with length $\omega \cdot 2$ and moves from $\{0,1\}$ are determined [Bu1]. At the start of our research project, Dr. Burke provided us with a sketch of his proof. In Chapter 2, we provide a detailed proof of this result. In Chapter 3, we extend this result and show that open games with length $\omega \cdot n$ and with moves from $\{0,1\}$ are determined for $n \geq 2$ and note that this result is also true for the same games with moves from any finite set X. This generalization is interesting on its own, but possibly surprising as it can not be done for open games with moves from ω.

In Chapter 4, we examine games in which the payoff sets are Borel sets. Martin showed Borel determinacy for games with length ω [Ma1, Ma2, Ma3]. We now recall the definition of the Borel hierarchy.

Definition 1.1 $\Sigma_1^0 = \{A \mid A \text{ is open}\}$

$A \in \Pi_\alpha^0$ if and only if $A^C = A^{\text{Complement}} \in \Sigma_\alpha^0$

$A \in \Sigma_\alpha^0$ if and only if there exists $\{B_n \mid n \in \omega\}$ such that $A = \bigcup_{n \in \omega} B_n$ and

$\forall n \, \exists \alpha(n) < \alpha$ such that $B_n \in \Pi_{\alpha(n)}^0$.

It is well-known that for the spaces $\{0,1\}^{\omega \cdot n}$ and $\omega^{\omega \cdot n}$, the Borel sets are the collection: $\bigcup_{\alpha < \omega_1} \Pi_\alpha^0 = \bigcup_{\alpha < \omega_1} \Sigma_\alpha^0 = \Sigma_{\omega_1 + 1}^0 = \Pi_{\omega_1 + 1}^0$. Most of our work will concern the first few

levels of the Borel hierarchy, so that a general knowledge of the Borel hierarchy is not needed for this thesis.

In Chapter 4, we shall show that the determinacy of Π_3^0 games with length $\omega \cdot 2$ and with moves from $\{0,1\}$ is equivalent to the determinacy of Π_3^0 games with length $\omega \cdot 2$ and with moves from ω. To do this, we use the technique of coding up an integer as a binary sequence. This technique is well-known and is used in Exercise 6A.8 on page 294 of *Descriptive Set Theory* by Yiannis N. Moschovakis [Mo]. In Chapter 5, we extend this result to show that this equivalency holds for the same games with length $\omega \cdot n$ for $n \geq 2$. We also note that it is well-known that the determinacy of Π_3^0 games with length $\omega \cdot n$ and with moves from ω requires the existence of large cardinals for $n \geq 2$. This result indicates that the determinacy of Π_3^0 games with length $\omega \cdot n$ and with moves from $\{0,1\}$ also requires the existence of large cardinals for $n \geq 2$. This is interesting because as shown in Chapter 3, the determinacy of open games with length $\omega \cdot n$ and with moves from $\{0,1\}$ does not require the existence of large cardinals for $n \geq 2$. We also examine the question of whether for classes Γ properly between Σ_1^0 and Π_3^0, large cardinals are required for the determinacy of Γ games with length $\omega \cdot n$ and with moves from $\{0,1\}$ for $n \geq 2$.

We will now introduce the notation used in the paper. Most of the notation used throughout the paper is standard and can be found in *Descriptive Set Theory* by Yiannis N. Moschovakis [Mo].

Definition 1.2 For $f \in X^{\omega}$, define:

$$(f)_{\mathrm{I}} = (f(0), f(2), f(4), \dots)$$

$$(f)_{\mathrm{II}} = (f(1), f(3), f(5), \dots).$$

For $f^{\frown} g \in X^{\omega \cdot 2}$, define:

$$(f, g)_{\mathrm{I}} = (f(0), f(2), f(4), \dots, g(0), g(2), g(4), \dots)$$

$$(f, g)_{\mathrm{II}} = (f(1), f(3), f(5), \dots, g(1), g(3), g(5), \dots).$$

Definition 1.3 We use the notation $\left(\Pi_k^0 \restriction X^{\omega \cdot n}\right)$ for the restriction of Π_k^0 to the space $X^{\omega \cdot n}$. Identifying games with their payoff sets, we shall write $A \in \left(\Pi_k^0 \restriction X^{\omega \cdot n}\right)$ to refer to the game with payoff set $A \in \Pi_k^0$, with length $\omega \cdot n$, and with moves from the set X.

Definition 1.4 We shall write μn to indicate the least n.

Definition 1.5 For $A \subseteq X^{\omega \cdot n}$, we shall write A^C to indicate the complement of A.

Definition 1.6 $A \in E_1 \wedge E_2$ if and only if $A = A_1 \cap A_2$ where $A_1 \in E_1$ and $A_2 \in E_2$. $A \in E_1 \vee E_2$ if and only if $A = A_1 \cup A_2$ where $A_1 \in E_1$ and $A_2 \in E_2$. We do not know of any standard notation for $E_1 \wedge E_2$ and $E_1 \vee E_2$.

We end this chapter with the proof of the Gale-Stewart result since this result is used throughout the paper. We first prove a lemma that will be used in the proof of the Gale-Stewart result.

Lemma 1.7 If player I doesn't have a w.s. for a game $A \subseteq X^\omega$, then $\forall x \, \exists y$ such that player I doesn't have a w.s. for the game $A_{(x,y)} = \left\{ g \,\middle|\, (x,y)^\frown g \in A \right\}$.

Proof. We prove the contrapositive of the lemma. Assume that $\exists x \, \forall y$ player I has a w.s., $s_{(x,y)}$, for $A_{(x,y)}$. We will define a strategy s for player I in the game A.

Definition 1.8 Let $s(\varnothing) = x$. For $\overline{f}(2n)$ that is according to s, define $s\left(\overline{f}(2n)\right)$ to be $s_{(f(0),f(1))}\left(f(2), f(3), \ldots, f(2n-1)\right)$.

Remark 1.9 Note that if $\overline{f}(2n)$ is according to s, then $f(0) = x$ and $\left(f(2), f(3), \ldots, f(2n-1)\right)$ is according to $s_{(f(0),f(1))}$.

If $h \in X^\omega$ is according to s, then $h(0) = x$ and $\left(h(2), h(3), \ldots, h(2n-1)\right)$ is according to player I's w.s., $s_{(h(0),h(1))}$, for $A_{(h(0),h(1))}$ for all n, so that $\left(h(2), h(3), h(4), \ldots\right) \in A_{(h(0),h(1))}$. Therefore by the definition of $A_{(h(0),h(1))}$, $h \in A$, so that s is a w.s. for player I in the game A. ■ (Lemma 1.7)

Theorem 1.10 ([GS]) Every open game is determined.

Proof. Let $A \subseteq X^\omega$ be open. Let $\left\{ \overline{u_i} \,\middle|\, i \in I \right\}$ generate A. If player I has a w.s. for A, then we are done. So assume that player I doesn't have a w.s. for A. Then by Lemma 1.7, we have:

(*) $\forall f(0)\ \exists f(1)$ such that player I doesn't have a w.s. for the game $A_{\bar{f}(2)}$.

We will now define a strategy s for player II in the game A.

Definition 1.11 Define $s(\bar{f}(1)) = f(1)$, the value given by (*). By (*), player I doesn't have a w.s. for $A_{\bar{f}(2)}$, so that by Lemma 1.7, we have:

(**) $\forall f(2)\ \exists f(3)$ such that player I doesn't have a w.s. for the game $A_{\bar{f}(4)}$.

Define $s(\bar{f}(3)) = f(3)$, the value given by (**). More generally, assume that $\bar{f}(2n)$ is according to s and that player I doesn't have a w.s. for $A_{\bar{f}(2n)}$. Then by Lemma 1.7, we have:

(***) $\forall f(2n)\ \exists f(2n+1)$ such that player I doesn't have a w.s. for the game $A_{\bar{f}(2n+2)}$.

Define $s(\bar{f}(2n+1)) = f(2n+1)$, the value given by (***).

Claim 1.12 s is a w.s. for player II in the game A.

Proof. Let f be according to s. Assume towards a contradiction that f is a win for player I. Then $f \in A$, so that $\exists i \in I$ such that $\overline{u_i} \subseteq f$. Pick k so that $\overline{u_i} \subseteq \bar{f}(2k)$. Since $\overline{u_i} \subseteq \bar{f}(2k)$, player I has a w.s. for $A_{\bar{f}(2k)}$ by the definition of $A_{\bar{f}(2k)}$. But since f is according to s, this contradicts that player I doesn't have a w.s. for $A_{\bar{f}(2n)}$ $\forall n$ as stated in (***). Hence, f is a win for player II. Therefore since f is according to s, s is a w.s. for player II in the game A. ■ (Claim 1.12 and Theorem 1.10)

CHAPTER 2

DETERMINACY OF OPEN GAMES IN $\{0,1\}^{\omega\cdot 2}$

The goal of this chapter is to show the determinacy of open games with length $\omega\cdot 2$ and with moves from $\{0,1\}$ (Theorem 2.14). Dr. Douglas Burke noticed this result as a Ph. D. student of Professor Donald A. Martin at UCLA. Our contribution includes providing a detailed write-up of the proof of this result and extending this result as explained in Chapter 3. In building toward the proof, we shall first provide a definition of an open set equivalent to the definition given in Chapter 1. We will show that these definitions are equivalent in Lemma 2.1. Given $A \subseteq \{0,1\}^{\omega\cdot 2}$, we will next define $S(A) = \{f \mid \text{player I has a w.s. for } A_f\} \subseteq \{0,1\}^{\omega}$ where $A_f = \{g \mid f^\frown g \in A\} \subseteq \{0,1\}^{\omega}$. Assuming that A is open, we will show that A_f is open and therefore determined by the Gale-Stewart result. The main lemma, Lemma 2.5, of the proof states that $S(A)$ is open and therefore determined by the Gale-Stewart result. To show that $S(A)$ is open, we will assume for a contradiction that $S(A)$ is not open and use Lemma 2.1 and the Pigeonhole Principle to reach a contradiction. We will use the fact that A_f and $S(A)$ are determined to argue (the conclusion of Theorem 2.14) that A is determined.

Lemma 2.1 The game $S \subseteq X^{\omega}$ is open if and only if $\forall f \in S$, $\exists m_f \in \omega$ such that $\forall \tilde{f}$ with $f \upharpoonright m_f = \tilde{f} \upharpoonright m_f$, we have $\tilde{f} \in S$.

Proof. Assume S is open. Since S is open, there exists $\{\overline{u_i} | i \in I\}$ that generates S. Let $f \in S$. Then $\exists i \in I$ such that $\overline{u_i} \subseteq f$. Let $m_f = \mathrm{lh}(\overline{u_i})$. Fix $\tilde{f}$ such that $f \upharpoonright m_f = \tilde{f} \upharpoonright m_f$. Then $\overline{u_i} = f \upharpoonright \mathrm{lh}(\overline{u_i}) = f \upharpoonright m_f = \tilde{f} \upharpoonright m_f \subseteq \tilde{f}$. Therefore $\overline{u_i} \subseteq \tilde{f}$, so that $\tilde{f} \in S$.

For the other direction, assume:

(i) $\forall f \in S$, $\exists m_f$ such that $\forall \tilde{f}$ with $f \upharpoonright m_f = \tilde{f} \upharpoonright m_f$, we have $\tilde{f} \in S$.

Let $\{\overline{u_i} | i \in I\} = \{f \upharpoonright m_f | f \in S\}$. To argue that S is open, it is enough to show:

Claim 2.2 $f' \in S$ if and only if $\exists i \in I$ such that $\overline{u_i} \subseteq f'$.

Proof. Assume $f' \in S$. Recall that we have $m_{f'}$ as given by our assumption (i). Then by the definition of the $\overline{u_i}$'s, $\exists i \in I$ such that $\overline{u_i} = f' \upharpoonright m_{f'}$. Therefore, $\exists i \in I$ such that $\overline{u_i} \subseteq f'$.

For the other direction, assume $\exists i \in I$ such that $\overline{u_i} \subseteq f'$. Then $\overline{u_i} = f \upharpoonright m_f$ for some $f \in S$ by the definition of the $\overline{u_i}$'s. Since $\overline{u_i} = f \upharpoonright m_f$ and $\overline{u_i} \subseteq f'$, $f \upharpoonright m_f = f' \upharpoonright m_f$, so that by assumption (i), $f' \in S$. ■ (Claim 2.2 and Lemma 2.1)

We now set up some notation and prove some lemmas in order to eventually show that every open game $A \subseteq \{0,1\}^{\omega \cdot 2}$ is determined.

Definition 2.3 Given $A \subseteq \{0,1\}^{\omega \cdot 2}$, define $S(A) = \{f \mid \text{player I has a w.s. for } A_f\}$ where $A_f = \{g \mid f^\frown g \in A\}$.

Lemma 2.4 If $A \subseteq \{0,1\}^{\omega \cdot 2}$ is open, then $A_f = \{g \mid f^\frown g \in A\} \subseteq \{0,1\}^\omega$ is open and therefore determined (by Gale-Stewart).

Proof. Assume $A \subseteq \{0,1\}^{\omega \cdot 2}$ is open. Then there exists $\{(\overline{u_i}, \overline{v_i}) \mid i \in I\}$ that generates A. We shall show that some collection of the $\overline{v_i}$'s generates $A_f \subseteq \{0,1\}^\omega$ as an open set. Note that the following are equivalent:

(i) $g \in A_f$

(ii) $f^\frown g \in A$

(iii) $\exists i \in I$ such that $\overline{u_i} \subseteq f$ and $\overline{v_i} \subseteq g$

(iv) $\overline{v_i} \subseteq g$ for some $i \in I_f$ where $I_f = \{i \mid \overline{u_i} \subseteq f\}$.

Therefore, $\{\overline{v_i} \mid i \in I_f\}$ generates A_f as an open set. ■ (Lemma 2.4)

We now prove the main technical lemma that we shall use to show that every open game $A \subseteq \{0,1\}^{\omega \cdot 2}$ is determined.

Lemma 2.5 If $A \subseteq \{0,1\}^{\omega \cdot 2}$ is open, then

$$S(A) = \{f \mid \text{player I has a w.s. for } A_f\} \subseteq \{0,1\}^{\omega}$$

is open and therefore determined (by Gale-Stewart).

Proof. Assume for a contradiction that $S(A)$ is not open. Then by Lemma 2.1, there exists $f \in S(A)$ such that $\forall m\, \exists f_m$ such that $f \restriction m = f_m \restriction m$, but $f_m \notin S(A)$. Since $f \in S(A)$, by the definition of $S(A)$, player I has a w.s., σ, for A_f. Since $f_m \notin S(A)$, by the definition of $S(A)$, player I doesn't have a w.s. for A_{f_m}. By Lemma 2.4, A_{f_m} is determined, so that since player I doesn't have a w.s. for A_{f_m}, player II has a w.s., τ_m, for A_{f_m} $\forall m$.

For each m, we shall now define g_m by playing player I's strategy σ against player II's strategy τ_m. That is, we shall define player I's moves in each g_m to be according to σ, player I's w.s. for A_f, and we shall define player II's moves in each g_m to be according to τ_m, player II's w.s. for A_{f_m} $\forall m$.

Definition 2.6 For all k, define:

$$g_m(2k) = \sigma(g_m(0), g_m(1), \ldots, g_m(2k-1))$$

$$g_m(2k+1) = \tau_m(g_m(0), g_m(1), \ldots, g_m(2k)).$$

Remark 2.7 Note that since g_m is according to τ_m and τ_m is a w.s. for player II in the game A_{f_m}, $g_m \notin A_{f_m}$, so that $f_m{}^\frown g_m \notin A$ $\forall m$.

Definition 2.8 Define:

$$W_1 = \begin{cases} \{m \in \omega \mid g_m(1) = 0\} & \text{if } \{m \in \omega \mid g_m(1) = 0\} \text{ is infinite} \\ \{m \in \omega \mid g_m(1) = 1\} & \text{otherwise.} \end{cases}$$

Since $\omega = \{m \in \omega \mid g_m(1) = 0\} \cup \{m \in \omega \mid g_m(1) = 1\}$, at least one (maybe both) of these sets is infinite, so that W_1 is infinite. More generally, we have:

Definition 2.9 Setting $W_{-1} = \omega$, define:

$$W_{2k+1} = \begin{cases} \{m \in W_{2k-1} \mid g_m(2k+1) = 0\} & \text{if } \{m \in W_{2k-1} \mid g_m(2k+1) = 0\} \text{ is infinite} \\ \{m \in W_{2k-1} \mid g_m(2k+1) = 1\} & \text{otherwise.} \end{cases}$$

Note that by the Pigeonhole Principle and by induction, each W_{2k+1} is infinite for all k.

We shall now define g. We shall define player I's moves in g to be according to σ, player I's w.s. for A_f. We shall define player II's moves in g to be a zero or one depending on how each W_{2k+1} was defined, so that g will agree with infinitely many of the g_m's up to any finite point.

Definition 2.10 For fixed k, define:

$$g(2k) = \sigma(g(0), g(1), \ldots, g(2k-1))$$

$$g(2k+1) = \begin{cases} 0 & \text{if } W_{2k+1} = \{m \in W_{2k-1} \mid g_m(2k+1) = 0\} \\ 1 & \text{if } W_{2k+1} = \{m \in W_{2k-1} \mid g_m(2k+1) = 1\}. \end{cases}$$

Claim 2.11 If $m \in W_{2k+1}$, then $g(2k+1) = g_m(2k+1)$.

Proof. Assume $m \in W_{2k+1}$.

Case 1 $W_{2k+1} = \{m \in W_{2k-1} | g_m(2k+1) = 0\}$.

Then by the definition of g, $g(2k+1) = 0$. Since $m \in W_{2k+1}$, $g_m(2k+1) = 0$. Hence, $g(2k+1) = g_m(2k+1)$.

Case 2 $W_{2k+1} = \{m \in W_{2k-1} | g_m(2k+1) = 1\}$.

Then by the definition of g, $g(2k+1) = 1$. Since $m \in W_{2k+1}$, $g_m(2k+1) = 1$. Hence, $g(2k+1) = g_m(2k+1)$. ■ (Claim 2.11)

Claim 2.12 If $m \in W_{2k+1}$, then $g \restriction 2k+2 = g_m \restriction 2k+2$.

Proof. Assume $m \in W_{2k+1}$. Since g and g_m are both according to σ and σ is a w.s. for player I, $g \restriction 2k+2 = g_m \restriction 2k+2$ if player II's moves are the same in $g \restriction 2k+2$ and $g_m \restriction 2k+2$. By the definition of W_{2k+1}, $W_1 \supseteq W_3 \supseteq \ldots \supseteq W_{2k-1} \supseteq W_{2k+1} \supseteq \ldots$, so that $m \in W_{2i+1}$ for i such that $0 \le i \le k$. Therefore by Claim 2.11, $g(2i+1) = g_m(2i+1)$ for i such that $0 \le i \le k$, so that $g \restriction 2k+2 = g_m \restriction 2k+2$. ■ (Claim 2.12)

Recall that we assumed, toward a contradiction, that A is open, but that $S(A)$ is not open. Since A is open, A is generated by some $\{(\overline{u_i}, \overline{v_i}) | i \in I\}$. We will now use Claim 2.12 to reach our contradiction. Recall that g is according to σ and σ is a w.s.

for player I in A_f. Therefore $g \in A_f$, so that $f^\frown g \in A$. Hence, $\exists i \in I$ such that $\overline{u_i} \subseteq f$ and $\overline{v_i} \subseteq g$. Let $2k+1 \geq \mathrm{lh}(\overline{v_i})$. Since W_{2k+1} is infinite, there exists $M \in W_{2k+1}$ such that $M > \mathrm{lh}(\overline{u_i})$. By Claim 2.12, $g \restriction 2k+2 = g_M \restriction 2k+2$, so that since $2k+1 \geq \mathrm{lh}(\overline{v_i})$, $g \restriction \mathrm{lh}(\overline{v_i}) = g_M \restriction \mathrm{lh}(\overline{v_i})$. Therefore since $\overline{v_i} = g \restriction \mathrm{lh}(\overline{v_i}) = g_M \restriction \mathrm{lh}(\overline{v_i})$, $\overline{v_i} \subseteq g_M$. By the definition of f_m (at the beginning of the proof), $f \restriction M = f_M \restriction M$, so that since $M > \mathrm{lh}(\overline{u_i})$, $f \restriction \mathrm{lh}(\overline{u_i}) = f_M \restriction \mathrm{lh}(\overline{u_i})$. Therefore since $\overline{u_i} = f \restriction \mathrm{lh}(\overline{u_i}) = f_M \restriction \mathrm{lh}(\overline{u_i})$, $\overline{u_i} \subseteq f_M$. Hence since $\overline{u_i} \subseteq f_M$ and $\overline{v_i} \subseteq g_M$, $f_M{}^\frown g_M \in A$, contradicting $f_m{}^\frown g_m \notin A$ $\forall m$ as noted in Remark 2.7. Thus, we have shown that $S(A)$ is open. ■ (Lemma 2.5)

We now use the fact that both A_f and $S(A)$ are open and therefore determined by Gale-Stewart to argue that A is determined.

Lemma 2.13 If player I has a w.s. for $S(A)$, then player I has a w.s. for A.

Proof. Assume player I has a w.s., σ, for $S(A)$. If f is according to σ, then $f \in S(A)$, so that by the definition of $S(A)$, player I has a w.s., σ_f, for A_f. If g is according to σ_f, then by the definition of A_f, $f^\frown g \in A$. Hence player I has a w.s. for A, namely, playing according to σ to obtain f and then playing according to σ_f to obtain g. ■ (Lemma 2.13)

Theorem 2.14 ([Bu1]) If $A \subseteq \{0,1\}^{\omega\cdot 2}$ is open, then A is determined.

Proof. If player I has a w.s. for A, then A is determined and we are done. So assume that player I doesn't have a w.s. for A. Therefore by the contrapositive to Lemma 2.13, player I doesn't have a w.s. for $S(A)$. Since $S(A)$ is determined by Lemma 2.5 and player I doesn't have a w.s. for $S(A)$, player II has a w.s., τ, for $S(A)$. Let f be according to τ. Then since τ is a w.s. for player II, $f \notin S(A)$. Since $f \notin S(A)$, by the definition of $S(A)$, player I doesn't have a w.s. for A_f. Since A_f is determined and player I doesn't have a w.s. for A_f, player II has a w.s., τ_f, for A_f. Let g be according to τ_f. Then since τ_f is a w.s. for player II, $g \notin A_f$. Therefore by the definition of A_f, $f^\frown g \notin A$. Thus player II has a w.s. for A, namely, playing according to τ to obtain f and then playing according to τ_f to obtain g. Therefore, either player I or player II has a w.s. for A. Hence, A is determined. ■ (Theorem 2.14)

Thus, we have shown the determinacy of open games with length $\omega\cdot 2$ and with moves from $\{0,1\}$. In the next chapter, we will extend this result for open games with length $\omega\cdot n$ and with moves from $\{0,1\}$ for $n \geq 2$. We will also note that the result is true not only for open games with length $\omega\cdot n$ and with moves from $\{0,1\}$, but also for open games with length $\omega\cdot n$ in which the moves are from any finite set X for $n \geq 2$.

CHAPTER 3

DETERMINACY OF OPEN GAMES IN $X^{\omega\cdot n}$ WHERE X IS FINITE

The goal of this chapter is to extend the results of Chapter 2 and show the determinacy of open games with length $\omega\cdot n$ and with moves from a finite set X for $n\geq 2$ (Theorem 3.18). This chapter parallels Chapter 2 as most of the preliminary definitions and lemmas correspond to those in Chapter 2. As in Chapter 2, we shall first state and/or prove a number of definitions and lemmas that will be used to build toward the proof. Note that in Chapter 2, we used $f^\frown g$ to denote a play in $\{0,1\}^{\omega\cdot 2}$. In this chapter, we will use $f^1{}^\frown f^2{}^\frown\ldots{}^\frown f^n$ to denote a play in $X^{\omega\cdot n}$ for $n\geq 2$.

Lemma 3.1 The game $S\subseteq X^{\omega\cdot n}$ is open if and only if $\forall\overline{f}=f^1{}^\frown f^2{}^\frown\ldots{}^\frown f^n\in S$, $\exists m=m_{\overline{f}}$ such that $\forall\overline{g}=g^1{}^\frown g^2{}^\frown\ldots{}^\frown g^n$ with $f^j\restriction m=g^j\restriction m$ for all j such that $1\leq j\leq n$, we have $g^1{}^\frown g^2{}^\frown\ldots{}^\frown g^n\in S$.

Proof. Assume S is open. Since S is open, there exists $\left\{\left(\overline{u_i^1},\overline{u_i^2},\ldots,\overline{u_i^n}\right)\middle| i\in I\right\}$ that generates S. Let $\overline{f}=f^1{}^\frown f^2{}^\frown\ldots{}^\frown f^n\in S$. Then $\exists i\in I$ such that $\overline{u_i^j}\subseteq f^j$ for all j, $1\leq j\leq n$. Let $m=m_{\overline{f}}=\max\left\{\mathrm{lh}\left(\overline{u_i^j}\right)\middle|1\leq j\leq n\right\}$. Fix $\overline{g}=g^1{}^\frown g^2{}^\frown\ldots{}^\frown g^n$ such that

$f^j \restriction m = g^j \restriction m$ for all j, $1 \le j \le n$. Then $\overline{u_i^j} = f^j \restriction \mathrm{lh}\left(\overline{u_i^j}\right) \subseteq f^j \restriction m = g^j \restriction m$ for all j, $1 \le j \le n$. Therefore $\overline{u_i^j} \subseteq g^j$ for all j such that $1 \le j \le n$, so that $g^1 {}^\frown g^2 {}^\frown \ldots {}^\frown g^n \in S$.

For the other direction, assume:

(i) $\forall f^1 {}^\frown f^2 {}^\frown \ldots {}^\frown f^n \in S$, $\exists m = m_{\bar{f}}$ such that $\forall \bar{g} = g^1 {}^\frown g^2 {}^\frown \ldots {}^\frown g^n$ with $f^j \restriction m = g^j \restriction m$ for all j such that $1 \le j \le n$, we have $g^1 {}^\frown g^2 {}^\frown \ldots {}^\frown g^n \in S$.

Let $\left\{\left(\overline{u_i^1}, \overline{u_i^2}, \ldots, \overline{u_i^n}\right) \middle| i \in I\right\} = \left\{\left(f^1 \restriction m, f^2 \restriction m, \ldots, f^n \restriction m\right) \middle| f^1 {}^\frown f^2 {}^\frown \ldots {}^\frown f^n \in S\right\}$. To argue that S is open, it is enough to show:

Claim 3.2 $h^1 {}^\frown h^2 {}^\frown \ldots {}^\frown h^n \in S$ if and only if $\exists i \in I$ such that $\overline{u_i^j} \subseteq h^j$ for all j such that $1 \le j \le n$.

Proof. Assume $h^1 {}^\frown h^2 {}^\frown \ldots {}^\frown h^n \in S$. Recall that we have $m = m_{\bar{h}}$ as given by our assumption (i). Then by the definition of the $\overline{u_i^j}$'s, $\exists i \in I$ such that $\left(\overline{u_i^1}, \overline{u_i^2}, \ldots, \overline{u_i^n}\right) = \left(h^1 \restriction m, h^2 \restriction m, \ldots, h^n \restriction m\right)$. Therefore, $\exists i \in I$ such that $\overline{u_i^j} \subseteq h^j$ for all j such that $1 \le j \le n$.

For the other direction, assume $\exists i \in I$ such that $\overline{u_i^j} \subseteq h^j$ for all j such that $1 \le j \le n$. By the definition of the $\overline{u_i^j}$'s, $\left(\overline{u_i^1}, \overline{u_i^2}, \ldots, \overline{u_i^n}\right) = \left(f^1 \restriction m, f^2 \restriction m, \ldots, f^n \restriction m\right)$ for some $f^1 {}^\frown f^2 {}^\frown \ldots {}^\frown f^n \in S$. Since $\overline{u_i^j} = f^j \restriction m$ for $1 \le j \le n$ and $\overline{u_i^j} \subseteq h^j$ for $1 \le j \le n$,

$f^j \upharpoonright m = h^j \upharpoonright m$ for all j such that $1 \le j \le n$, so that by assumption (i), $h^1{}^\frown h^2{}^\frown \dots {}^\frown h^n \in S$.

■ (Claim 3.2 and Lemma 3.1)

We now set up some notation and prove some lemmas in order to eventually show that every open game $A \subseteq X^{\omega \cdot n}$ where X is finite is determined for $n \ge 2$. To simplify notation, we will let $\overline{f} = f^1{}^\frown f^2{}^\frown \dots {}^\frown f^n$ and $\overline{f_m} = f_m^1{}^\frown f_m^2{}^\frown \dots {}^\frown f_m^n$.

Definition 3.3 Given $A \subseteq X^{\omega(n+1)}$, define

$$S(A) = \left\{ f^1{}^\frown f^2{}^\frown \dots {}^\frown f^n \,\middle|\, \text{player I has a w.s. for } A_{\overline{f}} \right\}$$

where $A_{\overline{f}} = \left\{ f^{n+1} \,\middle|\, f^1{}^\frown f^2{}^\frown \dots {}^\frown f^{n+1} \in A \right\}$.

Lemma 3.4 If $A \subseteq X^{\omega(n+1)}$ is open, then $A_{\overline{f}} = \left\{ f^{n+1} \,\middle|\, f^1{}^\frown f^2{}^\frown \dots {}^\frown f^{n+1} \in A \right\} \subseteq X^\omega$ is open and therefore determined (by Gale-Stewart).

Proof. Assume $A \subseteq X^{\omega(n+1)}$ is open. Then there exists $\left\{ \left(\overline{u_i^1}, \overline{u_i^2}, \dots, \overline{u_i^{n+1}} \right) \middle| i \in I \right\}$ that generates A. We shall show that some collection of the $\overline{u_i^{n+1}}$'s generates $A_{\overline{f}} \subseteq X^\omega$ as an open set. Note that the following are equivalent:

(i) $f^{n+1} \in A_{\overline{f}}$

(ii) $f^1{}^\frown f^2{}^\frown \dots {}^\frown f^{n+1} \in A$

(iii) $\exists i \in I$ such that $\overline{u_i^j} \subseteq f^j$ for all j such that $1 \le j \le n+1$

(iv) $\overline{\mathrm{u}_i^{n+1}} \subseteq f^{n+1}$ for some $i \in I_{\vec{f}}$ where

$I_{\vec{f}} = \left\{ i \middle| \overline{\mathrm{u}_i^j} \subseteq f^j \text{ for all } j \text{ such that } 1 \le j \le n \right\}$.

Therefore, $\left\{ \overline{\mathrm{u}_i^{n+1}} \middle| i \in I_{\vec{f}} \right\}$ generates $A_{\vec{f}}$ as an open set. ■ (Lemma 3.4)

Remark 3.5 For our applications of Lemma 3.4, we shall consider X finite and in particular $X = \{0,1\}$. Of interest is for what sets A are the sets $A_{\vec{f}}$ still determined, though for our applications we shall only need A to be open.

We now prove the main technical lemma which we shall use to show that every open game $A \subseteq X^{\omega n}$ where X is finite is determined for $n \ge 2$.

Lemma 3.6 If $A \subseteq \{0,1\}^{\omega(n+1)}$ is open, then

$$S(A) = \left\{ f^1 {}^\frown f^2 {}^\frown \ldots {}^\frown f^n \middle| \text{player I has a w.s. for } A_{\vec{f}} \right\} \subseteq \{0,1\}^{\omega \cdot n}$$

is open.

Proof. Assume for a contradiction that $S(A)$ is not open. Then by Lemma 3.1, there exists $f^1 {}^\frown f^2 {}^\frown \ldots {}^\frown f^n \in S(A)$ such that $\forall m \, \exists f_m^1 {}^\frown f_m^2 {}^\frown \ldots {}^\frown f_m^n$ such that $f^j \restriction m = f_m^j \restriction m$ for all j such that $1 \le j \le n$, but $f_m^1 {}^\frown f_m^2 {}^\frown \ldots {}^\frown f_m^n \notin S(A)$. Since $f^1 {}^\frown f^2 {}^\frown \ldots {}^\frown f^n \in S(A)$, by the definition of $S(A)$, player I has a w.s., σ, for $A_{\vec{f}}$. Since $f_m^1 {}^\frown f_m^2 {}^\frown \ldots {}^\frown f_m^n \notin S(A)$, by the definition of $S(A)$, player I doesn't have a w.s. for $A_{\vec{f}_m}$. By lemma 3.4, $A_{\vec{f}_m}$ is

determined, so that since player I doesn't have a w.s. for $A_{\overline{f_m}}$, player II has a w.s., τ_m, for $A_{\overline{f_m}}$ $\forall m$.

For each m, we shall now define f_m^{n+1} similar to how g_m was defined in Chapter 2 by playing player I's strategy σ against player II's strategy τ_m. That is, we shall define player I's moves in each f_m^{n+1} to be according to σ, player I's w.s. for $A_{\overline{f}}$, and we shall define player II's moves in each f_m^{n+1} to be according to τ_m, player II's w.s. for $A_{\overline{f_m}}$ $\forall m$.

Definition 3.7 For all k, define:

$$f_m^{n+1}(2k) = \sigma\left(f_m^{n+1}(0), f_m^{n+1}(1), \ldots, f_m^{n+1}(2k-1)\right)$$

$$f_m^{n+1}(2k+1) = \tau_m\left(f_m^{n+1}(0), f_m^{n+1}(1), \ldots, f_m^{n+1}(2k)\right).$$

Remark 3.8 Note that since f_m^{n+1} is according to τ_m and τ_m is a w.s. for player II in the game $A_{\overline{f_m}}$, $f_m^{n+1} \notin A_{\overline{f_m}}$, so that $f_m^1 {}^\frown f_m^2 {}^\frown \ldots {}^\frown f_m^{n+1} \notin A$ $\forall m$.

Definition 3.9 Define:

$$W_1 = \begin{cases} \{m \in \omega \mid f_m^{n+1}(1) = 0\} & \text{if } \{m \in \omega \mid f_m^{n+1}(1) = 0\} \text{ is infinite} \\ \{m \in \omega \mid f_m^{n+1}(1) = 1\} & \text{otherwise.} \end{cases}$$

Since $\omega = \{m \in \omega \mid f_m^{n+1}(1) = 0\} \cup \{m \in \omega \mid f_m^{n+1}(1) = 1\}$, at least one (maybe both) of these sets is infinite, so that W_1 is infinite. More generally, we have:

Definition 3.10 Setting $W_{-1} = \omega$, define:

$$W_{2k+1} = \begin{cases} \{m \in W_{2k-1} \mid f_m^{n+1}(2k+1) = 0\} & \text{if } \{m \in W_{2k-1} \mid f_m^{n+1}(2k+1) = 0\} \text{ is infinite} \\ \{m \in W_{2k-1} \mid f_m^{n+1}(2k+1) = 1\} & \text{otherwise.} \end{cases}$$

Note that by the Pigeonhole Principle and by induction, each W_{2k+1} is infinite for all k.

We shall now define f^{n+1} similar to how g was defined in Chapter 2. We shall define player I's moves in f^{n+1} to be according to σ, player I's w.s. for $A_{\bar{f}}$. We shall define player II's moves in f^{n+1} to be a zero or one depending on how each W_{2k+1} was defined, so that f^{n+1} will agree with infinitely many of the f_m^{n+1}'s up to any finite point.

Definition 3.11 For fixed k, define:

$$f^{n+1}(2k) = \sigma\left(f^{n+1}(0), f^{n+1}(1), \ldots, f^{n+1}(2k-1)\right)$$

$$f^{n+1}(2k+1) = \begin{cases} 0 & \text{if } W_{2k+1} = \{m \in W_{2k-1} \mid f_m^{n+1}(2k+1) = 0\} \\ 1 & \text{if } W_{2k+1} = \{m \in W_{2k-1} \mid f_m^{n+1}(2k+1) = 1\}. \end{cases}$$

Claim 3.12 If $m \in W_{2k+1}$, then $f^{n+1}(2k+1) = f_m^{n+1}(2k+1)$.

Proof. Assume $m \in W_{2k+1}$.

Case 1 $W_{2k+1} = \{m \in W_{2k-1} | f_m^{n+1}(2k+1) = 0\}$.

Then by the definition of f^{n+1}, $f^{n+1}(2k+1) = 0$. Since $m \in W_{2k+1}$, $f_m^{n+1}(2k+1) = 0$. Hence, $f^{n+1}(2k+1) = f_m^{n+1}(2k+1)$.

Case 2 $W_{2k+1} = \{m \in W_{2k-1} | f_m^{n+1}(2k+1) = 1\}$.

Then by the definition of f^{n+1}, $f^{n+1}(2k+1) = 1$. Since $m \in W_{2k+1}$, $f_m^{n+1}(2k+1) = 1$. Hence, $f^{n+1}(2k+1) = f_m^{n+1}(2k+1)$. ■ (Claim 3.12)

Claim 3.13 If $m \in W_{2k+1}$, then $f^{n+1} \upharpoonright 2k+2 = f_m^{n+1} \upharpoonright 2k+2$.

Proof. Assume $m \in W_{2k+1}$. Since f^{n+1} and f_m^{n+1} are according to σ and σ is a w.s. for player I, $f^{n+1} \upharpoonright 2k+2 = f_m^{n+1} \upharpoonright 2k+2$ if player II's moves are the same in $f^{n+1} \upharpoonright 2k+2$ and $f_m^{n+1} \upharpoonright 2k+2$. By the definition of W_{2k+1}, $W_1 \supseteq W_3 \supseteq \cdots \supseteq W_{2k-1} \supseteq W_{2k+1} \supseteq \cdots$, so that $m \in W_{2i+1}$ for i such that $0 \le i \le k$. Therefore by Claim 3.12, $f^{n+1}(2i+1) = f_m^{n+1}(2i+1)$ for i such that $0 \le i \le k$, so that $f^{n+1} \upharpoonright 2k+2 = f_m^{n+1} \upharpoonright 2k+2$. ■ (Claim 3.13)

Recall that we assumed, toward a contradiction, that A is open, but that $S(A)$ is not open. Since A is open, A is generated by some $\{(\overline{\mathrm{u}_i^1}, \overline{\mathrm{u}_i^2}, \ldots, \overline{\mathrm{u}_i^{n+1}}) | i \in I\}$. We will now use Claim 3.13 to reach our contradiction. Recall that f^{n+1} is according to σ and σ is a w.s. for player I in $A_{\bar{f}}$. Therefore $f^{n+1} \in A_{\bar{f}}$, so that $f^1 {}^\frown f^2 {}^\frown \ldots {}^\frown f^{n+1} \in A$. Hence

$\exists i \in I$ such that $\overline{u_i^j} \subseteq f^j$ for all j such that $1 \le j \le n+1$. Let $2k+1 \ge \mathrm{lh}\left(\overline{u_i^{n+1}}\right)$. Since W_{2k+1} is infinite, there exists $M \in W_{2k+1}$ such that $M > \max\left\{\mathrm{lh}\left(\overline{u_i^j}\right) \middle| 1 \le j \le n\right\}$. By Claim 3.13, $f^{n+1} \restriction 2k+2 = f_M^{n+1} \restriction 2k+2$, so that since $2k+1 \ge \mathrm{lh}\left(\overline{u_i^{n+1}}\right)$, $f^{n+1} \restriction \mathrm{lh}\left(\overline{u_i^{n+1}}\right) = f_M^{n+1} \restriction \mathrm{lh}\left(\overline{u_i^{n+1}}\right)$. Therefore since $\overline{u_i^{n+1}} = f^{n+1} \restriction \mathrm{lh}\left(\overline{u_i^{n+1}}\right) = f_M^{n+1} \restriction \mathrm{lh}\left(\overline{u_i^{n+1}}\right)$, $\overline{u_i^{n+1}} \subseteq f_M^{n+1}$. By the definition of $\overline{f_m} = f_m^1 {}^\frown f_m^2 {}^\frown \ldots {}^\frown f_m^n$ (at the beginning of the proof), $f^j \restriction M = f_M^j \restriction M$ for all j such that $1 \le j \le n$, so that since $M > \max\left\{\mathrm{lh}\left(\overline{u_i^j}\right) \middle| 1 \le j \le n\right\}$, $f^j \restriction \mathrm{lh}\left(\overline{u_i^j}\right) = f_M^j \restriction \mathrm{lh}\left(\overline{u_i^j}\right)$ for all j such that $1 \le j \le n$. Therefore since $\overline{u_i^j} = f^j \restriction \mathrm{lh}\left(\overline{u_i^j}\right) = f_M^j \restriction \mathrm{lh}\left(\overline{u_i^j}\right)$ for $1 \le j \le n$, $\overline{u_i^j} \subseteq f_M^j$ for all j such that $1 \le j \le n$. Recall that $\overline{u_i^{n+1}} \subseteq f_M^{n+1}$, so that since $\overline{u_i^{n+1}} \subseteq f_M^{n+1}$ and $\overline{u_i^j} \subseteq f_M^j$ for all j such that $1 \le j \le n$, $f_M^1 {}^\frown f_M^2 {}^\frown \ldots {}^\frown f_M^{n+1} \in A$, contradicting $f_m^1 {}^\frown f_m^2 {}^\frown \ldots {}^\frown f_m^{n+1} \notin A$ $\forall m$ as noted in Remark 3.4. Thus, we have shown that $S(A)$ is open. ■ (Lemma 3.6)

Lemma 3.14 If $A \subseteq X^{\omega(n+1)}$ is open for finite X, then

$$S(A) = \left\{ f^1 {}^\frown f^2 {}^\frown \ldots {}^\frown f^n \middle| \text{player I has a w.s. for } A_{\overline{f}} \right\}$$

is open.

Remark 3.15 Note that if X is finite, then the proof of Lemma 3.6 goes through to show that if $A \subseteq X^{\omega \cdot (n+1)}$ is open, then $S(A)$ is open. Immediately following the definition of W_{2k+1}, we used the Pigeonhole Principle to see that each W_{2k+1} is infinite, assuming $X = \{0,1\}$. This point is still true if X is finite.

Remark 3.16 ([Bu2]) Dr. Burke noted that for infinite X, Lemma 3.14 is false. Pick open $A_n \subseteq \omega^\omega$ such that the Π_2^0 set $\bigcap_{n=0}^{\infty} A_n$ is not open. Then $A = \left\{ f^\frown g \,\middle|\, f \in A_{g(1)} \right\} \subseteq \omega^{\omega \cdot 2}$ is open, but $S(A) = \bigcap_{n=0}^{\infty} A_n$, so that $S(A)$ is not open.

We now use the fact that $A_{\vec{f}}$ is open and therefore determined and that $S(A)$ is open to argue (the conclusion of Theorem 3.18) that A is determined. We first prove a lemma that will be used in the proof of Theorem 3.18.

Lemma 3.17 Let X be finite and $A \subseteq X^{\omega \cdot (n+1)}$. If player I has a w.s. for $S(A)$, then player I has a w.s. for A.

Proof. Assume player I has a w.s., σ, for $S(A)$. If $f^{1\frown} f^{2\frown} \ldots ^\frown f^n$ is according to σ, then $f^{1\frown} f^{2\frown} \ldots ^\frown f^n \in S(A)$, so that by the definition of $S(A)$, player I has a w.s., $\sigma_{\vec{f}}$, for $A_{\vec{f}}$. If f^{n+1} is according to $\sigma_{\vec{f}}$, then by the definition of $A_{\vec{f}}$, $f^{1\frown} f^{2\frown} \ldots ^\frown f^{n+1} \in A$. Hence player I would have a w.s. for A, namely, playing according to σ to obtain $f^{1\frown} f^{2\frown} \ldots ^\frown f^n$ and then playing according to $\sigma_{\vec{f}}$ to obtain f^{n+1}. ∎ (Lemma 3.17)

Theorem 3.18 If X is finite and $A \subseteq X^{\omega\cdot(n+1)}$ is open, then A is determined.

Proof. The proof is by induction on n. For $n = 0$, $A \subseteq X^{\omega}$ is determined (by Gale-Stewart). As the inductive hypothesis, assume every open game $A \subseteq X^{\omega\cdot n}$ is determined. If player I has a w.s. for A, then A is determined and we are done. So assume player I doesn't have a w.s. for A. Therefore by the contrapositive to Lemma 3.17, player I doesn't have a w.s. for $S(A)$. By Lemma 3.14, $S(A) \subseteq X^{\omega\cdot n}$ is an open game, so that by the inductive hypothesis, $S(A)$ is determined. Since $S(A)$ is determined and player I doesn't have a w.s. for $S(A)$, player II has a w.s., τ, for $S(A)$. Let $f^1{}^\frown f^2{}^\frown \ldots {}^\frown f^n$ be according to τ. Then since τ is a w.s. for player II, $f^1{}^\frown f^2{}^\frown \ldots {}^\frown f^n \notin S(A)$. Since $f^1{}^\frown f^2{}^\frown \ldots {}^\frown f^n \notin S(A)$, by the definition of $S(A)$, player I doesn't have a w.s. for $A_{\bar{f}}$. Since $A_{\bar{f}}$ is determined and player I doesn't have a w.s. for $A_{\bar{f}}$, player II has a w.s., $\tau_{\bar{f}}$, for $A_{\bar{f}}$. Let f^{n+1} be according to $\tau_{\bar{f}}$. Then since $\tau_{\bar{f}}$ is a w.s. for player II, $f^{n+1} \notin A_{\bar{f}}$. Therefore by the definition of $A_{\bar{f}}$, $f^1{}^\frown f^2{}^\frown \ldots {}^\frown f^{n+1} \notin A$. Thus player II has a w.s. for A, namely, playing according to τ to obtain $f^1{}^\frown f^2{}^\frown \ldots {}^\frown f^n$ and then playing according to $\tau_{\bar{f}}$ to obtain f^{n+1}. Therefore, either player I or player II has a w.s. for A. Hence, A is determined. ■ (Theorem 3.18)

Thus, we have shown the determinacy of open games with length $\omega \cdot n$ and with moves from any finite set X for $n \geq 2$. In the next two chapters, we will see that this result is in some sense optimal as it cannot be extended to slightly more complex games.

CHAPTER 4

THE DETERMINACY OF $\left(\Pi_3^0 \restriction \{0,1\}^{\omega\cdot 2}\right)$ IS EQUIVALENT TO THE DETERMINACY OF $\left(\Pi_3^0 \restriction \omega^{\omega\cdot 2}\right)$

The goal of this chapter is to show that the determinacy of Π_3^0 games with length $\omega \cdot 2$ and with moves from $\{0,1\}$ is equivalent to the determinacy of Π_3^0 games with length $\omega \cdot 2$ and with moves from ω (Theorem 4.13). This is significant because it is well-known that certain large cardinal hypotheses are needed to prove the determinacy of Π_3^0 games with length $\omega \cdot 2$ and with moves from ω (see Theorem 5.4). This result implies that certain large cardinal hypotheses are also needed to prove the determinacy of Π_3^0 games with length $\omega \cdot 2$ and with moves from $\{0,1\}$. However, as shown it Chapter 2, these additional hypotheses are not needed to prove the determinacy of Σ_1^0 games with length $\omega \cdot 2$ and with moves from $\{0,1\}$.

In order to show that the determinacy of $\left(\Pi_3^0 \restriction \{0,1\}^{\omega\cdot 2}\right)$ implies the determinacy of $\left(\Pi_3^0 \restriction \omega^{\omega\cdot 2}\right)$ (Lemma 4.14), we will let $A \in \left(\Pi_3^0 \restriction \omega^{\omega\cdot 2}\right)$ and play an auxiliary game $A^{**} \in \left(\Pi_3^0 \restriction \{0,1\}^{\omega\cdot 2}\right)$. We will code up the move $x \in \omega$ for a given player in the game A into moves in the auxiliary game A^{**} by having the given player play x number of ones

followed by a zero in the appropriate positions. Provided that neither player plays a tail end of ones in the auxiliary game, we will decode moves for a given player in the auxiliary game A^{**} into a move in the game A by counting the number of ones played by the given player before playing a zero in the appropriate positions and then playing the number of ones as the move for the given player in the game A. We will assume that $A^{**} \in \left(\Pi_3^0 \restriction \{0,1\}^{\omega \cdot 2}\right)$ is determined, so that either player I or player II has a w.s. for A^{**}.

If player I has a w.s. for A^{**}, the plays will be coded and decoded as pictured in Figure 4.A. If player II has a w.s. for A^{**}, the plays will be coded and decoded as pictured in Figure 4.B. Figures 4.A and 4.B are meant to help the reader view the coding and decoding that will be done in this chapter. The precise definitions of the auxiliary game A^{**} and the decoding function will appear later in this chapter.

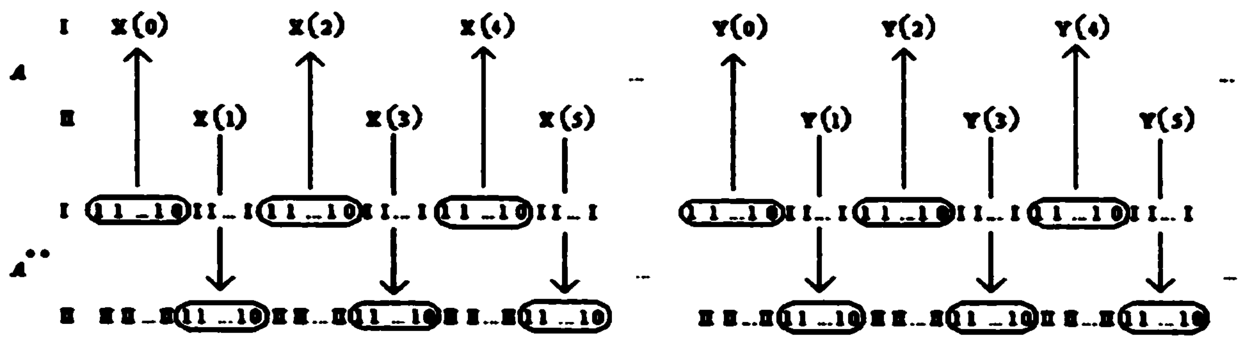

Figure 4.A The coding and decoding of plays if player I has a w.s. for A^{**}.

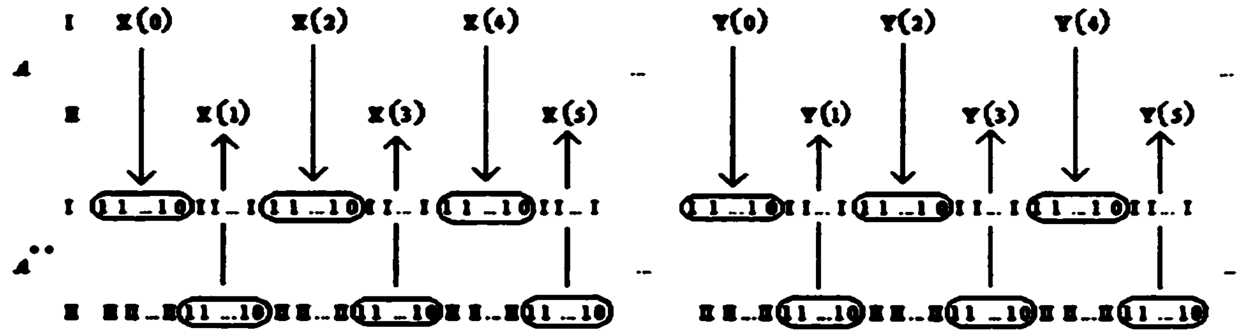

Figure 4.B The coding and decoding of plays if player II has a w.s. for A^{**}.

In building toward the proof of Lemma 4.14, we shall first set up some helpful notation that will enable us to code up a move in the game A into moves in the auxiliary game A^{**} and to decode moves in the auxiliary game A^{**} into a move in the game A. In the next definition, we associate with each binary finite sequence, $\overline{w} \in \{0,1\}^{<\omega}$, a finite sequence $\widehat{w} \in \omega^{<\omega}$. We shall think of the binary sequence $\overline{w}$ as a code for the sequence $\widehat{w}$ and $\hat{}$ as a decoding function. (Note that the $\hat{}$ will change size throughout the paper to nicely accommodate the size of the sequence that is being decoded.) Given a finite binary sequence $\overline{w}$, we shall define $\widehat{w}$ so that the first value of $\widehat{w}$ will be the number of ones played by player I before playing a zero in $\overline{w}$. The remaining values of $\widehat{w}$ will be obtained by alternately counting, starting with player II, the number of ones played by a given player before playing a zero after the last "significant" zero played by the other player in $\overline{w}$. A zero will be "significant" if it affects the value of $\widehat{w}$.

Definition 4.1 Fix $\bar{\mathrm{w}} = \left(\mathrm{w}(0), \mathrm{w}(1), \ldots, \mathrm{w}\left(\mathrm{lh}(\bar{\mathrm{w}})-1\right)\right) \in \{0,1\}^{<\omega}$. Let

$$n_0^{\bar{\mathrm{w}}} = \begin{cases} \mu n \text{ such that } \mathrm{w}(2n)=0 & \text{if } \exists n \text{ such that } \mathrm{w}(2n)=0 \\ \text{undefinded} & \text{otherwise.} \end{cases}$$

$$n_{2k+1}^{\bar{\mathrm{w}}} = \begin{cases} \mu n \geq n_{2k}^{\bar{\mathrm{w}}} \text{ such that } \mathrm{w}(2n+1)=0 & \text{if } \exists n \geq n_{2k}^{\bar{\mathrm{w}}} \text{ such that } \mathrm{w}(2n+1)=0 \\ \text{undefinded} & \text{otherwise.} \end{cases}$$

$$n_{2k+2}^{\bar{\mathrm{w}}} = \begin{cases} \mu n > n_{2k+1}^{\bar{\mathrm{w}}} \text{ such that } \mathrm{w}(2n)=0 & \text{if } \exists n > n_{2k+1}^{\bar{\mathrm{w}}} \text{ such that } \mathrm{w}(2n)=0 \\ \text{undefinded} & \text{otherwise.} \end{cases}$$

Then we define $\widehat{\mathrm{w}}(l)$ whenever $n_l^{\bar{\mathrm{w}}}$ is defined by:

$$\widehat{\mathrm{w}}(0) = n_0^{\bar{\mathrm{w}}}$$

$$\widehat{\mathrm{w}}(2k+1) = n_{2k+1}^{\bar{\mathrm{w}}} - n_{2k}^{\bar{\mathrm{w}}}$$

$$\widehat{\mathrm{w}}(2k+2) = n_{2k+2}^{\bar{\mathrm{w}}} - n_{2k+1}^{\bar{\mathrm{w}}} - 1$$

Remark 4.2 Note that a tail end of ones for either player in $\bar{\mathrm{w}}$ does not affect $\hat{\bar{\mathrm{w}}}$. More generally, $\hat{\bar{\mathrm{w}}} = \hat{\bar{\mathrm{u}}}$ if the following holds: $\bar{\mathrm{w}} = \bar{\mathrm{u}}^\frown \bar{\mathrm{v}}$, the last value of $\bar{\mathrm{u}}$ is a zero, and all the even values of $\bar{\mathrm{v}}$ are ones.

We will now extend this definition for any $f \in \{0,1\}^{\omega}$, so that $\hat{f} \in \omega^{\omega}$ is the unique sequence that extends all of the $\widehat{\bar{f}(n)}$ where $n \in \omega$. We shall think of this infinite binary sequence as a code for either an infinite or finite sequence of integers depending on whether or not f has a tail end of ones.

Definition 4.3 For any $f \in \{0,1\}^{\omega}$, define $\hat{f} = \left\langle \widehat{f(n)} \middle| n \in \omega \right\rangle$.

For $\bar{u} \in \omega^{<\omega}$, we next define the set $E(\bar{u})$ to be the collection of all finite binary sequences that are coded for $\bar{u}$.

Definition 4.4 Given any $\bar{u} \in \omega^{<\omega}$, define $E(\bar{u}) \subseteq \{0,1\}^{<\omega}$ by: $\bar{w} \in E(\bar{u})$ if and only if $\hat{\bar{w}} = \bar{u}$.

Proposition 4.5 For $f^\frown g \in \{0,1\}^{\omega \cdot 2}$, $\bar{u} \subseteq \hat{f}$ and $\bar{v} \subseteq \hat{g}$ if and only if $\exists \tilde{u} \in E(\bar{u})$ and $\exists \tilde{v} \in E(\bar{v})$ such that $\tilde{u} \subseteq f$ and $\tilde{v} \subseteq g$.

Proof. Assume $\bar{u} \subseteq \hat{f}$ and $\bar{v} \subseteq \hat{g}$. Then $\exists n_{\hat{f}}$ and $\exists n_{\hat{g}}$ such that $\bar{u} = \hat{f} \restriction n_{\hat{f}}$ and $\bar{v} = \hat{g} \restriction n_{\hat{g}}$. Let $m_{n_{\hat{f}}} = \sum_{i < n_{\hat{f}}} \left(2\hat{f}(i) + 1\right)$ and let $m_{n_{\hat{g}}} = \sum_{i < n_{\hat{g}}} \left(2\hat{g}(i) + 1\right)$. Let $\tilde{u} = f \restriction m_{n_{\hat{f}}}$ and let $\tilde{v} = g \restriction m_{n_{\hat{g}}}$. Then $\tilde{u} \subseteq f$ and $\tilde{v} \subseteq g$ are the sequences of shortest length such that $\hat{\tilde{u}} = \bar{u}$ and $\hat{\tilde{v}} = \bar{v}$. Therefore by the definition of $E(\bar{u})$, $\tilde{u} \in E(\bar{u})$ and by the definition of $E(\bar{v})$, $\tilde{v} \in E(\bar{v})$.

For the other direction, assume that $\exists\tilde{\mathrm{u}} \in E(\bar{\mathrm{u}})$ and $\exists\tilde{\mathrm{v}} \in E(\bar{\mathrm{v}})$ such that $\tilde{\mathrm{u}} \subseteq f$ and $\tilde{\mathrm{v}} \subseteq g$. By the definition of $E(\bar{\mathrm{u}})$, $\hat{\tilde{\mathrm{u}}} = \bar{\mathrm{u}}$ and by the definition of $E(\bar{\mathrm{v}})$, $\hat{\tilde{\mathrm{v}}} = \bar{\mathrm{v}}$. Therefore since $\tilde{\mathrm{u}} \subseteq f$ and $\tilde{\mathrm{v}} \subseteq g$, $\bar{\mathrm{u}} = \hat{\tilde{\mathrm{u}}} \subseteq \hat{f}$ and $\bar{\mathrm{v}} = \hat{\tilde{\mathrm{v}}} \subseteq \hat{g}$. ■ (Proposition 4.5)

In this chapter, we shall introduce an auxiliary game whose payoff set includes conditions that penalize a player if the player plays a tail end of ones. We now introduce the open sets from which such conditions will be formally defined.

Definition 4.6 For $f\,\hat{}\,g \in \{0,1\}^{\omega\cdot 2}$, define $f\,\hat{}\,g \in O_m^F$ if and only if there exists $\bar{\mathrm{w}} \in \{0,1\}^{<\omega}$ such that $\bar{\mathrm{w}} \subseteq f$ and $\mathrm{lh}\left(\hat{\bar{\mathrm{w}}}\right) > m$.

Definition 4.7 For $f\,\hat{}\,g \in \{0,1\}^{\omega\cdot 2}$, define $f\,\hat{}\,g \in O_m^G$ if and only if there exists $\bar{\mathrm{w}} \in \{0,1\}^{<\omega}$ such that $\bar{\mathrm{w}} \subseteq g$ and $\mathrm{lh}\left(\hat{\bar{\mathrm{w}}}\right) > m$.

Remark 4.8 Note that O_m^F is an open subset of $\{0,1\}^{\omega\cdot 2}$ since $\left\{(\bar{\mathrm{w}},\varnothing) \middle| \mathrm{lh}\left(\hat{\bar{\mathrm{w}}}\right) > m\right\}$ generates O_m^F and that O_m^G is an open subset of $\{0,1\}^{\omega\cdot 2}$ since $\left\{(\varnothing,\bar{\mathrm{w}}) \middle| \mathrm{lh}\left(\hat{\bar{\mathrm{w}}}\right) > m\right\}$ generates O_m^G.

Remark 4.9 Notice that $f\,\hat{}\,g \in O_m^F$ implies that f codes up at least $m+1$ elements from ω. Therefore $f\,\hat{}\,g \in \bigcap_{m=0}^{\infty} O_m^F$ if and only if f doesn't contain a tail end of ones, so

that f codes up an infinite sequence of elements from ω. Similar comments of course hold for O^G_m and $\bigcap_{m=0}^{\infty} O^G_m$.

Remark 4.10 $f^\frown g \in \bigcup_{m=0}^{\infty}\left(\left(O^F_{2m+1}\right)^C \cap \left(\bigcap_{k\le m} O^F_{2k}\right)\right)$ if and only if the following hold for the play $f^\frown g$:

(i) player II played a tail end of ones in the first round

(ii) if player I did play a tail end of ones in the first round, then player II's tail end started before player I's tail end did.

$f^\frown g \in \left(\bigcap_{m=0}^{\infty} O^F_m\right) \cap \left(\bigcup_{m=0}^{\infty}\left(\left(O^G_{2m+1}\right)^C \cap \left(\bigcap_{k\le m} O^G_{2k}\right)\right)\right)$ if and only if the following hold for the play $f^\frown g$:

(i) neither player played a tail end of ones in the first round

(ii) player II played a tail end of ones in the second round

(iii) if player I did play a tail end of ones in the second round, then player II's tail end started before player I's tail end did.

$f^\frown g \in \bigcap_{m=0}^{\infty}\left(O^F_m \cap O^G_m\right)$ if and only if neither player played a tail end of ones in the first or second round.

In light of Remarks 4.9 and 4.10, we make the following definitions:

Definition 4.11 Define:

$$E_{f\ \mathrm{NTE}} = \bigcap_{m=0}^{\infty} O_m^F$$

f NTE stands for "f has no tail end of ones".

$$E_{f\ \mathrm{TE}}^{\mathrm{II\ before\ I}} = \bigcup_{m=0}^{\infty}\left(\left(O_{2m+1}^F\right)^C \cap \left(\bigcap_{k\le m} O_{2k}^F\right)\right)$$

f TE stands for "f has a tail end of ones" and II before I indicates that "player II started playing a tail end of ones before player I *possibly* started such a tail end".

$$E_{f\ \mathrm{NTE},\ g\ \mathrm{TE}}^{\mathrm{II\ before\ I}} = \left(\bigcap_{m=0}^{\infty} O_m^F\right) \cap \left(\bigcup_{m=0}^{\infty}\left(\left(O_{2m+1}^G\right)^C \cap \left(\bigcap_{k\le m} O_{2k}^G\right)\right)\right)$$

f NTE stands for "f has no tail end of ones", g TE stands for "g has a tail end of ones", and II before I indicates that "player II started playing a tail end of ones in the second round before player I *possibly* started such a tail end".

$$E_{f\ \mathrm{NTE},\ g\ \mathrm{NTE}} = \bigcap_{m=0}^{\infty}\left(O_m^F \cap O_m^G\right)$$

f NTE stands for "f has no tail end of ones" and g NTE stands for "g has no tail end of ones".

Remark 4.12 Note that by the definition of Π_2^0 and Σ_2^0 sets of the Borel hierarchy, $E_{f\ \mathrm{NTE}} \in \Pi_2^0$, $E_{f\ \mathrm{TE}}^{\mathrm{II\ before\ I}} \in \Sigma_2^0$, $E_{f\ \mathrm{NTE},\ g\ \mathrm{TE}}^{\mathrm{II\ before\ I}} \in \Pi_2^0 \wedge \Sigma_2^0$ (i.e. $E_{f\ \mathrm{NTE},\ g\ \mathrm{TE}}^{\mathrm{II\ before\ I}} = E_1 \cap E_2$ where $E_1 \in \Pi_2^0$ and $E_2 \in \Sigma_2^0$), and $E_{f\ \mathrm{NTE},\ g\ \mathrm{NTE}} \in \Pi_2^0$.

We now state the main theorem of this chapter.

Theorem 4.13 The determinacy of $\left(\Pi_3^0 \restriction \{0,1\}^{\omega \cdot 2}\right)$ is equivalent to the determinacy of $\left(\Pi_3^0 \restriction \omega^{\omega \cdot 2}\right)$.

In order to prove this theorem, we shall show each implication separately and state each as a lemma.

Lemma 4.14 The determinacy of $\left(\Pi_3^0 \restriction \{0,1\}^{\omega \cdot 2}\right)$ implies the determinacy of $\left(\Pi_3^0 \restriction \omega^{\omega \cdot 2}\right)$.

Proof. Assume the determinacy of $\left(\Pi_3^0 \restriction \{0,1\}^{\omega \cdot 2}\right)$. Let $A \in \left(\Pi_3^0 \restriction \omega^{\omega \cdot 2}\right)$. We shall first define the set $A^* \subseteq \{0,1\}^{\omega \cdot 2}$ to be the set of all plays $f^\frown g \in \{0,1\}^{\omega \cdot 2}$ in which neither player plays a tail end of ones (both in the first and in the second round) and in which the decoded play $\hat{f}^\frown \hat{g}$ is in A. We shall next define the payoff set for our auxiliary game $A^{**} \subseteq \{0,1\}^{\omega \cdot 2}$. In order to code moves from $\omega^{\omega \cdot 2}$ into moves from $\{0,1\}^{\omega \cdot 2}$ and to decode moves from $\{0,1\}^{\omega \cdot 2}$ into moves from $\omega^{\omega \cdot 2}$, we need neither player to play a tail end of ones in the first and second rounds of our auxiliary game with moves from $\{0,1\}$. That is, we need $f^\frown g \in E_{f\,\mathrm{NTE},\, g\,\mathrm{NTE}}$ by Remark 4.9 and Definition 4.11. We will incorporate this condition into the payoff set for our auxiliary game $A^{**} \subseteq \{0,1\}^{\omega \cdot 2}$, so

that if either player plays a tail end of ones, then the player that starts doing so first will lose. By Remark 4.10 and Definition 4.11, this corresponds to A^{**} being defined so that $E^{\text{II before I}}_{f\ \text{TE}} \subseteq A^{**}$ and $E^{\text{II before I}}_{f\ \text{NTE}, g\ \text{TE}} \subseteq A^{**}$. If neither player plays a tail end of ones in the auxiliary game A^{**}, then player I wins the auxiliary game A^{**} if and only if $\hat{f}^\frown \hat{g} \in A$ and player II wins the auxiliary game A^{**} if and only if $\hat{f}^\frown \hat{g} \notin A$. We now give precise definitions of the payoff sets A^* and A^{**}.

Definition 4.15 $A^* = \left\{ f^\frown g \in \{0,1\}^{\omega \cdot 2} \middle| f^\frown g \in E_{f\ \text{NTE}, g\ \text{NTE}} \text{ and } \hat{f}^\frown \hat{g} \in A \right\}$.

Definition 4.16 $A^{**} = E^{\text{II before I}}_{f\ \text{TE}} \cup E^{\text{II before I}}_{f\ \text{NTE}, g\ \text{TE}} \cup A^*$.

We shall now set up some notation for the proof of the next claim. Since $A \in \left(\Pi^0_3 \restriction \omega^{\omega \cdot 2}\right)$, $A^C \in \left(\Sigma^0_3 \restriction \omega^{\omega \cdot 2}\right)$, so that there exist open sets $\widehat{O_{i,j}} \subseteq \omega^{\omega \cdot 2}$ such that $A^C = \bigcup_{i \in \omega} \bigcap_{j \in \omega} \widehat{O_{i,j}}$. (We use the hat notation, $\hat{}$, here to remind the reader that $\widehat{O_{i,j}} \subseteq \omega^{\omega \cdot 2}$ and not as a decoding function.) For $i, j \in \omega$, let $\left\{ \left(\overline{\mathrm{u}^k_{i,j}}, \overline{\mathrm{v}^k_{i,j}} \right) \middle| k \in K \right\}$ generate $\widehat{O_{i,j}}$. Recall the definition of $E(\bar{\mathrm{u}})$ as defined in Definition 4.4. Let $O_{i,j}$ be the open set generated by

$$\left\{ \left(\tilde{\mathrm{u}}, \tilde{\mathrm{v}} \right) \in \{0,1\}^{<\omega} \times \{0,1\}^{<\omega} \middle| \exists k \text{ such that } \tilde{\mathrm{u}} \in E\left(\overline{\mathrm{u}^k_{i,j}} \right) \text{ and } \tilde{\mathrm{v}} \in E\left(\overline{\mathrm{v}^k_{i,j}} \right) \right\}.$$

Define $B = \bigcap_{i \in \omega} \bigcup_{j \in \omega} \left(O_{i,j} \right)^C \in \Pi^0_3$.

Claim 4.17 $A^{*} \in \left(\Pi_3^0 \restriction \{0,1\}^{\omega \cdot 2}\right)$.

Proof. For $f^\frown g \in E_{f\,\mathrm{NTE},\,g\,\mathrm{NTE}}$, the following are equivalent:

(i) $f^\frown g \notin A^{*}$

(ii) $\hat{f}^\frown \hat{g} \notin A$

(iii) $\hat{f}^\frown \hat{g} \in A^C = \bigcup_{i\in\omega}\bigcap_{j\in\omega}\widehat{O_{i,j}}$

(iv) $\exists i\ \forall j\ \hat{f}^\frown \hat{g} \in \widehat{O_{i,j}}$

(v) $\exists i\ \forall j\ \exists k$ such that $\overline{u_{i,j}^k} \subseteq \hat{f}$ and $\overline{v_{i,j}^k} \subseteq \hat{g}$

(vi) $\exists i\ \forall j\ \exists k\ \exists \tilde{u} \in E\left(\overline{u_{i,j}^k}\right)\ \exists \tilde{v} \in E\left(\overline{v_{i,j}^k}\right)$ such that $\tilde{u} \subseteq f$ and $\tilde{v} \subseteq g$

(by Proposition 4.5)

(vii) $\exists i\ \forall j\ \ f^\frown g \in O_{i,j}$

(viii) $f^\frown g \in \bigcup_{i\in\omega}\bigcap_{j\in\omega} O_{i,j}$

(ix) $f^\frown g \notin B$ (recall $B = \bigcap_{i\in\omega}\bigcup_{j\in\omega}\left(O_{i,j}\right)^C$).

Therefore, $A^{*} = E_{f\,\mathrm{NTE},\,g\,\mathrm{NTE}} \cap B$. Recall that $E_{f\,\mathrm{NTE},\,g\,\mathrm{NTE}} \in \Pi_2^0$ (by Remark 4.12) and that $B \in \Pi_3^0$, so that $A^{*} \in \left(\Pi_3^0 \restriction \{0,1\}^{\omega \cdot 2}\right)$. ■ (Claim 4.17)

We now show that the auxiliary game $A^{**} \in \left(\Pi_3^0 \restriction \{0,1\}^{\omega \cdot 2}\right)$ using Remark 4.12 and Claim 4.17.

Claim 4.18 $A^{**} \in \left(\Pi_3^0 \restriction \{0,1\}^{\omega \cdot 2}\right)$.

Proof. Recall that $A^{**} = E_{f\ \mathrm{TE}}^{\mathrm{II\ before\ I}} \cup E_{f\ \mathrm{NTE},g\ \mathrm{TE}}^{\mathrm{II\ before\ I}} \cup A^{*}$, $E_{f\ \mathrm{TE}}^{\mathrm{II\ before\ I}} \in \Sigma_2^0$, $E_{f\ \mathrm{NTE},g\ \mathrm{TE}}^{\mathrm{II\ before\ I}} \in \Pi_2^0 \wedge \Sigma_2^0$, and that $A^{*} \in \Pi_3^0$. Therefore by the definition of Π_2^0, Σ_2^0, and Π_3^0 sets of the Borel hierarchy, $A^{**} \in \left(\Pi_3^0 \restriction \{0,1\}^{\omega \cdot 2}\right)$. ■ (Claim 4.18)

We shall now use the auxiliary game A^{**} to show that A is determined. Since $A^{**} \in \left(\Pi_3^0 \restriction \{0,1\}^{\omega \cdot 2}\right)$ by Claim 4.18, A^{**} is determined by the hypothesis of Lemma 4.14. Therefore, either player I or player II has a w.s., s^{**}, for A^{**}.

Case 1 Player I has a w.s., s^{**}, for A^{**}.

We plan to construct a w.s., s, for player I in the game A. Our definition of s is by induction. We shall define s on sequences of length $2k$ assuming s has been defined on sequences of smaller length. Due to this assumption, we can set up some notation, $f_{\bar{x}}$, for finite sequences, $\bar{x} = (x(0), x(1), \ldots, x(2k\text{-}1)) \in \omega^{<\omega}$, that are according to s.

Definition 4.19 Given $\bar{x} = (x(0), x(1), \ldots, x(2k\text{-}1)) \in \omega^{<\omega}$ that is according to s, define $f_{\bar{x}} \in \{0,1\}^{<\omega}$ by:

(i) $f_{\bar{x}}$ is according to s^{**}

(ii) $\widehat{f_{\bar{x}}}(2i-1) = x(2i-1)$ for $1 \le i \le k$

(iii) set $f_{\bar{x}}(2i-1) = 0$ if $f_{\bar{x}}(2i-1)$ does not affect the value of $\widehat{f_{\bar{x}}}$ (that is, set

$f_{\bar{x}}(2i-1) = 0$ if $\overline{\left(\overline{f_{\bar{x}}}(2i-1)\right)^\frown(0)} = \overline{\left(\overline{f_{\bar{x}}}(2i-1)\right)^\frown(1)}$)

(iv) $f_{\bar{x}}$ is the shortest sequence satisfying (i)-(iii).

Remark 4.20 Notice that by property (iv), the last value of $f_{\bar{x}}$ is a move for player II that is a zero.

We now extend the definition of $f_{\bar{x}}$ and define $f_{\bar{X}}$ for infinite sequences $\bar{X} = (X(0), X(1), X(2), \ldots) \in \omega^\omega$ that are according to s.

Definition 4.21 Given $\bar{X} = (X(0), X(1), X(2), \ldots) \in \omega^\omega$ that is according to s, define $f_{\bar{X}} \in \{0,1\}^\omega$ by: $f_{\bar{X}} = \bigcup_{i=0}^{\infty} f_{\bar{x}_i}$ where $\bar{x}_i = \bar{X} \restriction i$, $i \in \omega$.

We now set up similar notation, $g_{\bar{X},\bar{y}}$, for infinite sequences $\bar{X} = (X(0), X(1), X(2), \ldots) \in \omega^\omega$ and finite sequences $\bar{y} = (y(0), y(1), \ldots, y(2k-1)) \in \omega^{<\omega}$ such that $\bar{X}^\frown\bar{y}$ is according to s.

Definition 4.22 Given $\overline{X} = (X(0), X(1), X(2), \ldots) \in \omega^\omega$ and $\bar{y} = (y(0), y(1), \ldots, y(2k-1)) \in \omega^{<\omega}$ such that $\overline{X}^\frown\bar{y}$ is according to s, define $g_{\overline{X},\bar{y}} \in \{0,1\}^{<\omega}$ by:

(i) $f_{\overline{X}}{}^\frown g_{\overline{X},\bar{y}}$ is according to s^*

(ii) $\widehat{g_{\overline{X},\bar{y}}}(2i-1) = y(2i-1)$ for $1 \le i \le k$

(iii) set $g_{\overline{X},\bar{y}}(2i-1) = 0$ if $g_{\overline{X},\bar{y}}(2i-1)$ does not affect the value of $\widehat{g_{\overline{X},\bar{y}}}$ (that is,

set $g_{\overline{X},\bar{y}}(2i-1) = 0$ if $\overline{\left(\overline{g_{\overline{X},\bar{y}}}(2i-1)\right)^\frown(0)} = \overline{\left(\overline{g_{\overline{X},\bar{y}}}(2i-1)\right)^\frown(1)}$)

(iv) $g_{\overline{X},\bar{y}}$ is the shortest sequence satisfying (i)-(iii).

We now extend the definition of $g_{\overline{X},\bar{y}}$ and define $g_{\overline{X},\overline{Y}}$ for infinite sequences $\overline{X} = (X(0), X(1), X(2), \ldots) \in \omega^\omega$ and $\overline{Y} = (Y(0), Y(1), Y(2), \ldots) \in \omega^\omega$ such that $\overline{X}^\frown\overline{Y}$ is according to s.

Definition 4.23 Given $\overline{X}^\frown\overline{Y} \in \omega^{\omega\cdot 2}$ that is according to s, define $g_{\overline{X},\overline{Y}} \in \{0,1\}^\omega$ by:

$g_{\overline{X},\overline{Y}} = \bigcup_{i=0}^{\infty} g_{\overline{X},\overline{y_i}}$ where $\overline{y_i} = \overline{Y} \restriction i$, $i \in \omega$.

We shall now define the strategy s for player I in the game A.

Definition 4.24 We inductively define s for Case 1 (the case where s^{*} is a w.s. for player I) so that the following properties hold:

(i) For any $\bar{x} \in \omega^{<\omega}$ that is according to s, $\widehat{f_{\bar{x}}} = \bar{x}$.

(ii) For any $\bar{X}^\frown \bar{y} \in \omega^{\omega} \times \omega^{<\omega}$ that is according to s, $\widehat{f_{\bar{X}}}^\frown \widehat{g_{\bar{X},\bar{y}}} = \bar{X}^\frown \bar{y}$.

From these properties, it will follow that:

(iii) For any $\bar{X} \in \omega^{\omega}$ that is according to s, $\widehat{f_{\bar{X}}} = \bar{X}$.

(iv) For any $\bar{X}^\frown \bar{Y} \in \omega^{\omega \cdot 2}$ that is according to s, $\widehat{f_{\bar{X}}}^\frown \widehat{g_{\bar{X},\bar{Y}}} = \bar{X}^\frown \bar{Y}$.

As the inductive hypothesis, assume that $\bar{x} = (x(0), x(1), \ldots, x(2k\text{-}1)) \in \omega^{<\omega}$ is according to s and that $\widehat{f_{\bar{x}}} = \bar{x}$. We shall define $s(\bar{x})$ so as to ensure that $\widehat{f_{\bar{x}^\frown s(\bar{x})}} = \bar{x}^\frown s(\bar{x})$. By the definition of $f_{\bar{x}}$, $f_{\bar{x}}$ is according to s^{*}, so that $s^{*}(f_{\bar{x}})$ makes sense. We now set up some notation to extend $f_{\bar{x}} \in \{0,1\}^{<\omega}$.

Definition 4.25 Let $f_{\bar{x}}^{j} \in \{0,1\}^{<\omega}$ be defined by:

(i) $f_{\bar{x}}^{0} = f_{\bar{x}}$

(ii) $f_{\bar{x}}^{j+1} \supset f_{\bar{x}}^{j}$

(iii) $f_{\bar{x}}^{2j+1}(\mathrm{lh}(f_{\bar{x}}^{2j})) = s^{*}(f_{\bar{x}}^{2j})$

(iv) $f_{\bar{x}}^{2j+2}(\mathrm{lh}(f_{\bar{x}}^{2j+1})) = 0$.

Remark 4.26 Notice that $f_{\bar{x}}^{j} = f_{\bar{x}}{}^\frown t_{\bar{x}}^{j}$ for some $t_{\bar{x}}^{j}$ where $\mathrm{lh}\left(t_{\bar{x}}^{j}\right) = j$ and $t_{\bar{x}}^{j} \subset t_{\bar{x}}^{j+1}$. Also note that player I must play a zero in $\bigcup_{j=0}^{\infty} t_{\bar{x}}^{j}$; for otherwise any play extending $\bigcup_{j=0}^{\infty} f_{\bar{x}}^{j}$ that is according to s^{**} is a loss for player I.

We define $s(\bar{x}) = x(2k)$ to be the number of ones played by player I in $\bigcup_{j=0}^{\infty} t_{\bar{x}}^{j}$ preceding the first zero played by player I in $\bigcup_{j=0}^{\infty} t_{\bar{x}}^{j}$. That is, $s(\bar{x}) = \mu j$ such that $s^{**}\left(f_{\bar{x}}^{2j}\right) = 0.$

Remark 4.27 Setting $t_{\bar{x}} = t_{\bar{x}}^{2s(\bar{x})+1}$, $\widehat{f_{\bar{x}}{}^\frown t_{\bar{x}}} = \bar{x}(2k+1)$ by Remark 4.26, by the definition of $s(\bar{x})$, and by the definition of $\widehat{f_{\bar{x}}{}^\frown t_{\bar{x}}}$.

Let $\bar{x}' = (x(0), x(1), \ldots, x(2k+1))$. We must show that $\widehat{f_{\bar{x}'}} = \bar{x}'$. By the inductive hypothesis, $\widehat{f_{\bar{x}}} = \bar{x} = \bar{x}(2k)$. Since $\bar{x} \subseteq \bar{x}'$, $f_{\bar{x}} \subseteq f_{\bar{x}'}$ by the definition of $f_{\bar{x}}$ and $f_{\bar{x}'}$, so that $\bar{x} = \widehat{f_{\bar{x}}} \subseteq \widehat{f_{\bar{x}'}}$. Therefore, it is enough to show that $\widehat{f_{\bar{x}'}}(2k) = x(2k)$ and $\widehat{f_{\bar{x}'}}(2k+1) = x(2k+1)$. By the definition of $x(2k)$ and the definition of $f_{\bar{x}'}$, $f_{\bar{x}'} \supseteq f_{\bar{x}}{}^\frown t_{\bar{x}}$, so that $\widehat{f_{\bar{x}'}}(2k) = \widehat{f_{\bar{x}}{}^\frown t_{\bar{x}}}(2k) = x(2k)$ by Remark 4.27. By property (ii) of the definition

of $f_{\bar{x}}$, $\widehat{f_{\bar{x}}}(2k+1)=x(2k+1)$. Therefore, $\widehat{f_{\bar{x}}}=\bar{x}'$. Note that once $\bar{X}\in\omega^{\omega}$ is completely defined, $\widehat{f_{\bar{X}}}=\bar{X}$.

We now define s on the second round analogous to our definition of s on the first round. Inductively assume that $\bar{X}^\frown\bar{y}$, where $\bar{X}=(X(0),X(1),X(2),\ldots)\in\omega^{\omega}$ and $\bar{y}=(y(0),y(1),\ldots,y(2k-1))\in\omega^{<\omega}$, is according to s, that $\widehat{f_{\bar{X}}}=\bar{X}$, and that $\widehat{g_{\bar{X},\bar{y}}}=\bar{y}$. Recall that we defined $g_{\bar{X},\bar{y}}$ in Definition 4.22. We shall define $s(\bar{X}^\frown\bar{y})$ so as to ensure that $\widehat{g_{\bar{X},\bar{y}^\frown s(\bar{X}^\frown\bar{y})}}=\bar{X}^\frown\bar{y}^\frown s(\bar{X}^\frown\bar{y})$. By the definition of $f_{\bar{X}}$ and $g_{\bar{X},\bar{y}}$, $f_{\bar{X}}{}^\frown g_{\bar{X},\bar{y}}$ is according to s^{**}, so that $s^{**}(f_{\bar{X}}{}^\frown g_{\bar{X},\bar{y}})$ makes sense. Analogous to the definition of $f_{\bar{x}}^{j}$, we define $g_{\bar{X},\bar{y}}^{j}$ and define $s(\bar{X}^\frown\bar{y})=y(2k)=\mu j$ such that $s^{**}(g_{\bar{X},\bar{y}}^{2j})=0$. Recall that player II provides the move $y(2k+1)$. Let $\bar{y}'=(y(0),y(1),\ldots,y(2k+1))$. Then $\widehat{g_{\bar{X},\bar{y}'}}=\bar{y}'$ is shown inductively just as $\widehat{f_{\bar{x}}}=\bar{x}'$ was shown above. Note that once $\bar{Y}\in\omega^{\omega}$ is completely defined, $\widehat{g_{\bar{X},\bar{Y}}}=\bar{Y}$. Therefore, $\widehat{f_{\bar{X}}}{}^\frown\widehat{g_{\bar{X},\bar{Y}}}=\bar{X}^\frown\bar{Y}$. Consequently, we have defined s so that properties (i), (ii), (iii), and (iv) hold. ■ (Definition 4.24)

Claim 4.28 The strategy s is a w.s. for player I in the game A.

Proof. Let $\bar{X}^\frown\bar{Y}$ be according to s. By the definition of s, $f_{\bar{X}}{}^\frown g_{\bar{X},\bar{Y}}$ is according to s^{**}. Since s^{**} is a w.s. for player I in the game A^{**}, $f_{\bar{X}}{}^\frown g_{\bar{X},\bar{Y}}\in A^{**}$. Since $f_{\bar{X}}{}^\frown g_{\bar{X},\bar{Y}}$ is according to s^{**} and s^{**} is a w.s. for player I, player I did not play a tail end of ones in the first or second round. By the definition of $f_{\bar{X}}$ and $g_{\bar{X},\bar{Y}}$, player II's moves were

simulated so that player II did not play a tail end of ones in the first or second round. Therefore since neither player played a tail end of ones and $f_{\bar{X}}{}^\frown g_{\bar{X},\bar{Y}} \in A^{**}$, $f_{\bar{X}}{}^\frown g_{\bar{X},\bar{Y}} \in A^{*}$, so that $\widehat{f_{\bar{X}}}{}^\frown \widehat{g_{\bar{X},\bar{Y}}} \in A$. By the definition of s, $\bar{X}{}^\frown \bar{Y} = \widehat{f_{\bar{X}}}{}^\frown \widehat{g_{\bar{X},\bar{Y}}}$, so that $\bar{X}{}^\frown \bar{Y} \in A$. Therefore, s is a w.s. for player I in the game A. ■ (Claim 4.28 and Case 1)

Case 2 Player II has a w.s., s^{**}, for A^{**}.

We plan to construct a w.s., s, for player II in the game A. Our definition of s and the notation provided to define s are the same as in Case 1, except for slight changes due to the fact that player II plays moves $f(2i+1)$ instead of moves $f(2i)$. Our definition of s is by induction. We shall define s on sequences of length $2k+1$ assuming s has been defined on sequences of smaller length. Due to this assumption, we can set up some notation $f_{\bar{x}}$ for finite sequences, $\bar{x} = (x(0), x(1), \ldots, x(2k)) \in \omega^{<\omega}$, that are according to s.

Definition 4.29 Given $\bar{x} = (x(0), x(1), \ldots, x(2k)) \in \omega^{<\omega}$ that is according to s, define $f_{\bar{x}} \in \{0,1\}^{<\omega}$ by:

(i) $f_{\bar{x}}$ is according to s^{**}

(ii) $\widehat{f_{\bar{x}}}(2i) = x(2i)$ for $0 \le i \le k$

(iii) set $f_{\bar{x}}(2i) = 0$ if $f_{\bar{x}}(2i)$ does not affect the value of $\widehat{f_{\bar{x}}}$ (that is, set

$$f_{\bar{x}}(2i) = 0 \text{ if } \overline{\left(\overline{f_{\bar{x}}}(2i)\right)^\frown(0)} = \overline{\left(\overline{f_{\bar{x}}}(2i)\right)^\frown(1)})$$

(iv) $f_{\bar{x}}$ is the shortest sequence satisfying (i)-(iii).

Remark 4.30 Notice that by property (iv), the last value of $f_{\bar{x}}$ is a move for player I that is a zero.

We now extend the definition of $f_{\bar{x}}$ and define $f_{\bar{X}}$ for infinite sequences $\bar{X}=(X(0),X(1),X(2),...)\in\omega^{\omega}$ that are according to s. Our definition of $f_{\bar{X}}$ for $\bar{X}\in\omega^{\omega}$ is exactly the same definition as in Case 1.

Definition 4.31 Given $\bar{X}=(X(0),X(1),X(2),...)\in\omega^{\omega}$ that is according to s, define $f_{\bar{X}}\in\{0,1\}^{\omega}$ by: $f_{\bar{X}}=\bigcup_{i=0}^{\infty}f_{\bar{x}_i}$ where $\bar{x}_i=\bar{X}\restriction i,\ i\in\omega$.

We now set up similar notation, $g_{\bar{X},\bar{y}}$, for infinite sequences $\bar{X}=(X(0),X(1),X(2),...)\in\omega^{\omega}$ and finite sequences $\bar{y}=(y(0),y(1),...,y(2k))\in\omega^{<\omega}$ such that $\bar{X}^\frown\bar{y}$ is according to s.

Definition 4.32 Given $\bar{X}=(X(0),X(1),X(2),...)\in\omega^{\omega}$ and $\bar{y}=(y(0),y(1),...,y(2k))\in\omega^{<\omega}$ such that $\bar{X}^\frown\bar{y}$ is according to s, define $g_{\bar{X},\bar{y}}\in\{0,1\}^{<\omega}$ by:

(i) $f_{\bar{X}}{}^\frown g_{\bar{X},\bar{y}}$ is according to s^{*}

(ii) $\widehat{g_{\overline{X},\bar{y}}}(2i) = y(2i)$ for $0 \le i \le k$

(iii) set $g_{\overline{X},\bar{y}}(2i) = 0$ if $g_{\overline{X},\bar{y}}(2i)$ does not affect the value of $\widehat{g_{\overline{X},\bar{y}}}$ (that is,

set $g_{\overline{X},\bar{y}}(2i) = 0$ if $\overline{\left(\overline{g_{\overline{X},\bar{y}}}(2i)\right)^\frown(0)} = \overline{\left(\overline{g_{\overline{X},\bar{y}}}(2i)\right)^\frown(1)}$)

(iv) $g_{\overline{X},\bar{y}}$ is the shortest sequence satisfying (i)-(iii).

We now extend the definition of $g_{\overline{X},\bar{y}}$ and define $g_{\overline{X},\overline{Y}}$ for infinite sequences $\overline{X} = (X(0), X(1), X(2), \ldots) \in \omega^\omega$ and $\overline{Y} = (Y(0), Y(1), Y(2), \ldots) \in \omega^\omega$ such that $\overline{X}^\frown\overline{Y}$ is according to s. Our definition of $g_{\overline{X},\overline{Y}}$ for $\overline{X}^\frown\overline{Y} \in \omega^{\omega \cdot 2}$ is exactly the same definition as in Case 1.

Definition 4.33 Given $\overline{X}^\frown\overline{Y} \in \omega^{\omega \cdot 2}$ that is according to s, define $g_{\overline{X},\overline{Y}} \in \{0,1\}^\omega$ by:

$g_{\overline{X},\overline{Y}} = \bigcup_{i=0}^{\infty} g_{\overline{X},\overline{y_i}}$ where $\overline{y_i} = \overline{Y} \restriction i$, $i \in \omega$.

We shall now define the strategy s for player II in the game A.

Definition 4.34 We inductively define s for Case 2 (the case where s^{**} is a w.s. for player II) so that the following same properties as in Case 1 hold:

(i) For any $\bar{x} \in \omega^{<\omega}$ that is according to s, $\widehat{f_{\bar{x}}} = \bar{x}$.

(ii) For any $\overline{X}^\frown\bar{y} \in \omega^\omega \times \omega^{<\omega}$ that is according to s, $\widehat{f_{\overline{X}}}^\frown\widehat{g_{\overline{X},\bar{y}}} = \overline{X}^\frown\bar{y}$.

From these properties, it will follow that:

(iii) For any $\overline{\mathrm{X}} \in \omega^{\omega}$ that is according to s, $\widehat{f_{\overline{\mathrm{X}}}} = \overline{\mathrm{X}}$.

(iv) For any $\overline{\mathrm{X}}^\frown \overline{\mathrm{Y}} \in \omega^{\omega \cdot 2}$ that is according to s, $\widehat{f_{\overline{\mathrm{X}}}}^\frown \widehat{g_{\overline{\mathrm{X}},\overline{\mathrm{Y}}}} = \overline{\mathrm{X}}^\frown \overline{\mathrm{Y}}$.

As the inductive hypothesis, assume that $\bar{\mathrm{x}} = (\mathrm{x}(0), \mathrm{x}(1), \ldots, \mathrm{x}(2k)) \in \omega^{<\omega}$ is according to s and that $\widehat{f_{\bar{\mathrm{x}}}} = \bar{\mathrm{x}}$. We shall define $s(\bar{\mathrm{x}})$ so as to ensure that $\widehat{f_{\bar{\mathrm{x}}^\frown s(\bar{\mathrm{x}})}} = \bar{\mathrm{x}}^\frown s(\bar{\mathrm{x}})$. By the definition of $f_{\bar{\mathrm{x}}}$, $f_{\bar{\mathrm{x}}}$ is according to s^{-}, so that $s^{-}(f_{\bar{\mathrm{x}}})$ makes sense. Recall that in Case 1, $f_{\bar{\mathrm{x}}}^{j}$ was defined to extend $f_{\bar{\mathrm{x}}} \in \{0,1\}^{<\omega}$. $f_{\bar{\mathrm{x}}}^{j}$ has exactly the same definition in Case 2.

Definition 4.35 Let $f_{\bar{\mathrm{x}}}^{j} \in \{0,1\}^{<\omega}$ be defined by:

(i) $f_{\bar{\mathrm{x}}}^{0} = f_{\bar{\mathrm{x}}}$

(ii) $f_{\bar{\mathrm{x}}}^{j+1} \supset f_{\bar{\mathrm{x}}}^{j}$

(iii) $f_{\bar{\mathrm{x}}}^{2j+1}\left(\mathrm{lh}\left(f_{\bar{\mathrm{x}}}^{2j}\right)\right) = s^{-}\left(f_{\bar{\mathrm{x}}}^{2j}\right)$

(iv) $f_{\bar{\mathrm{x}}}^{2j+2}\left(\mathrm{lh}\left(f_{\bar{\mathrm{x}}}^{2j+1}\right)\right) = 0.$

Remark 4.36 Notice that $f_{\bar{\mathrm{x}}}^{j} = f_{\bar{\mathrm{x}}}{}^\frown t_{\bar{\mathrm{x}}}^{j}$ for some $t_{\bar{\mathrm{x}}}^{j}$ where $\mathrm{lh}\left(t_{\bar{\mathrm{x}}}^{j}\right) = j$ and $t_{\bar{\mathrm{x}}}^{j} \subset t_{\bar{\mathrm{x}}}^{j+1}$. Also note that player II must play a zero in $\bigcup_{j=0}^{\infty} t_{\bar{\mathrm{x}}}^{j}$; for otherwise any play extending $\bigcup_{j=0}^{\infty} f_{\bar{\mathrm{x}}}^{j}$ that is according to s^{-} is a loss for player II.

We define $s(\bar{x}) = x(2k+1)$ to be the number of ones played by player II in $\bigcup_{j=0}^{\infty} t_{\bar{x}}^{j}$ preceding the first zero played by player II in $\bigcup_{j=0}^{\infty} t_{\bar{x}}^{j}$. That is, $s(\bar{x}) = \mu j$ such that $s^{*}(f_{\bar{x}}^{2j}) = 0$.

Remark 4.37 Setting $t_{\bar{x}} = t_{\bar{x}}^{2s(\bar{x})+1}$, $\widehat{f_{\bar{x}}{}^\frown t_{\bar{x}}} = \bar{x}(2k+2)$ by Remark 4.36, by the definition of $s(\bar{x})$, and by the definition of $\widehat{f_{\bar{x}}{}^\frown t_{\bar{x}}}$.

Let $\bar{x}' = (x(0), x(1), \ldots, x(2k+2))$. We must show that $\widehat{f_{\bar{x}'}} = \bar{x}'$. By the inductive hypothesis, $\widehat{f_{\bar{x}}} = \bar{x} = \bar{x}(2k+1)$. Since $\bar{x} \subseteq \bar{x}'$, $f_{\bar{x}} \subseteq f_{\bar{x}'}$ by the definition of $f_{\bar{x}}$ and $f_{\bar{x}'}$, so that $\bar{x} = \widehat{f_{\bar{x}}} \subseteq \widehat{f_{\bar{x}'}}$. Therefore, it is enough to show that $\widehat{f_{\bar{x}'}}(2k+1) = x(2k+1)$ and $\widehat{f_{\bar{x}'}}(2k+2) = x(2k+2)$. By the definition of $x(2k+1)$ and the definition of $f_{\bar{x}'}$, $f_{\bar{x}'} \supseteq f_{\bar{x}}{}^\frown t_{\bar{x}}$, so that $\widehat{f_{\bar{x}'}}(2k+1) = \widehat{f_{\bar{x}}{}^\frown t_{\bar{x}}}(2k+1) = x(2k+1)$ by Remark 4.37. By property (ii) of the definition of $f_{\bar{x}'}$, $\widehat{f_{\bar{x}'}}(2k+2) = x(2k+2)$. Therefore, $\widehat{f_{\bar{x}'}} = \bar{x}'$. Note that once $\bar{X} \in \omega^{\omega}$ is completely defined, $\widehat{f_{\bar{X}}} = \bar{X}$.

We now define s on the second round analogous to our definition of s for the first round. Inductively assume that $\bar{X}{}^\frown \bar{y}$, where $\bar{X} = (X(0), X(1), X(2), \ldots) \in \omega^{\omega}$ and $\bar{y} = (y(0), y(1), \ldots, y(2k)) \in \omega^{<\omega}$, is according to s, that $\widehat{f_{\bar{X}}} = \bar{X}$, and that $\widehat{g_{\bar{X}, \bar{y}}} = \bar{y}$. Recall that we defined $g_{\bar{X}, \bar{y}}$ in Definition 4.32. We shall define $s(\bar{X}{}^\frown \bar{y})$ so as to ensure

that $\widehat{g_{\bar{X},\bar{y}^\frown s(\bar{X}^\frown\bar{y})}} = \bar{X}^\frown\bar{y}^\frown s(\bar{X}^\frown\bar{y})$. By the definition of $f_{\bar{X}}$ and $g_{\bar{X},\bar{y}}$, $f_{\bar{X}}{}^\frown g_{\bar{X},\bar{y}}$ is according to s^{**}, so that $s^{**}(f_{\bar{X}}{}^\frown g_{\bar{X},\bar{y}})$ makes sense. Analogous to the definition of $f_{\bar{x}}^{j}$, we define $g_{\bar{X},\bar{y}}^{j}$ and define $s(\bar{X}^\frown\bar{y}) = y(2k+1) = \mu j$ such that $s^{**}(g_{\bar{X},\bar{y}}^{2j}) = 0$. Recall that player I provides the move $y(2k+2)$. Let $\bar{y}' = (y(0), y(1), \ldots, y(2k+2))$. Then $\widehat{g_{\bar{X},\bar{y}}} = \bar{y}'$ is shown inductively just as $\widehat{f_{\bar{x}}} = \bar{x}'$ was shown above. Note that once $\bar{Y} \in \omega^{\omega}$ is completely defined, $\widehat{g_{\bar{X},\bar{Y}}} = \bar{Y}$. Therefore, $\widehat{f_{\bar{X}}}{}^\frown\widehat{g_{\bar{X},\bar{Y}}} = \bar{X}^\frown\bar{Y}$. Consequently, we have defined s so that properties (i), (ii), (iii), and (iv) hold. ■ (Definition 4.34)

Claim 4.38 The strategy s is a w.s. for player II in the game A.

Proof. Let $\bar{X}^\frown\bar{Y}$ be according to s. By the definition of s, $f_{\bar{X}}{}^\frown g_{\bar{X},\bar{Y}}$ is according to s^{**}. Since s^{**} is a w.s. for player II in the game A^{**}, $f_{\bar{X}}{}^\frown g_{\bar{X},\bar{Y}} \notin A^{**}$. By the definition of $f_{\bar{X}}$ and $g_{\bar{X},\bar{Y}}$, player I's moves were simulated so that player I did not play a tail end of ones in the first or second round. Since $f_{\bar{X}}{}^\frown g_{\bar{X},\bar{Y}}$ is according to s^{**} and s^{**} is a w.s. for player II, player II did not play a tail end of ones in the first or second round. Therefore since neither player played a tail end of ones and $f_{\bar{X}}{}^\frown g_{\bar{X},\bar{Y}} \notin A^{**}$, $f_{\bar{X}}{}^\frown g_{\bar{X},\bar{Y}} \notin A^{*}$, so that $\widehat{f_{\bar{X}}}{}^\frown\widehat{g_{\bar{X},\bar{Y}}} \notin A$. By the definition of s, $\bar{X}^\frown\bar{Y} = \widehat{f_{\bar{X}}}{}^\frown\widehat{g_{\bar{X},\bar{Y}}}$, so that $\bar{X}^\frown\bar{Y} \notin A$. Hence, s is a w.s. for player II in the game A. ■ (Claim 4.38 and Case 2)

Therefore, if player I has a w.s., s^{*}, for $A^{*} \subseteq \{0,1\}^{\omega \cdot 2}$, then player I has a w.s. s as defined in Definition 4.24 for the game $A \subseteq \omega^{\omega \cdot 2}$. If player II has a w.s., s^{*}, for $A^{*} \subseteq \{0,1\}^{\omega \cdot 2}$, then player II has a w.s. s as defined in Definition 4.34 for the game $A \subseteq \omega^{\omega \cdot 2}$. Thus, the determinacy of $\left(\Pi_3^0 \upharpoonright \{0,1\}^{\omega \cdot 2}\right)$ implies the determinacy of $\left(\Pi_3^0 \upharpoonright \omega^{\omega \cdot 2}\right)$. ■ (Lemma 4.14)

We now prove the converse of Lemma 4.14 as stated in Lemma 4.39.

Lemma 4.39 The determinacy of $\left(\Pi_3^0 \upharpoonright \omega^{\omega \cdot 2}\right)$ implies the determinacy of $\left(\Pi_3^0 \upharpoonright \{0,1\}^{\omega \cdot 2}\right)$.

Proof. Assume the determinacy of $\left(\Pi_3^0 \upharpoonright \omega^{\omega \cdot 2}\right)$. Let $A \in \left(\Pi_3^0 \upharpoonright \{0,1\}^{\omega \cdot 2}\right)$. We will define the auxiliary game $A^{*} \subseteq \omega^{\omega \cdot 2}$ by:

$$f^\frown g \in A^{*} \text{ if and only if } (f,g)_{\mathrm{I}} \in \{0,1\}^{\omega \cdot 2} \text{ and } \left((f,g)_{\mathrm{II}} \notin \{0,1\}^{\omega \cdot 2} \text{ or } f^\frown g \in A\right).$$

Claim 4.40 $A^{*} \in \left(\Pi_3^0 \upharpoonright \omega^{\omega \cdot 2}\right)$.

Proof. Note that $(f,g)_{\mathrm{I}} \in \{0,1\}^{\omega \cdot 2}$ is a closed condition in the space $\omega^{\omega \cdot 2}$ since the following are equivalent:

(i) $(f,g)_{\mathrm{I}} \in \{0,1\}^{\omega\cdot 2}$

(ii) $(f)_{\mathrm{I}} \in \{0,1\}^{\omega}$ and $(g)_{\mathrm{I}} \in \{0,1\}^{\omega}$

(iii) $\forall n\big(f(2n) \in \{0,1\}$ and $g(2n) \in \{0,1\}\big)$.

Note that $(f,g)_{\mathrm{II}} \notin \{0,1\}^{\omega\cdot 2}$ is an open condition in the space $\omega^{\omega\cdot 2}$ since the following are equivalent:

(i) $(f,g)_{\mathrm{II}} \notin \{0,1\}^{\omega\cdot 2}$

(ii) $(f)_{\mathrm{II}} \notin \{0,1\}^{\omega}$ or $(g)_{\mathrm{II}} \notin \{0,1\}^{\omega}$

(iii) $\exists n\big(f(2n+1) \notin \{0,1\}$ or $g(2n+1) \notin \{0,1\}\big)$.

We now concentrate on the condition $f\,\hat{}\,g \in A$. Since $A \in \left(\Pi^0_3 \restriction \{0,1\}^{\omega\cdot 2}\right)$, there exist open sets $O_{i,j} \subseteq \{0,1\}^{\omega\cdot 2}$ such that $A = \bigcap_{i\in\omega}\bigcup_{j\in\omega}\left(O_{i,j}\right)^C$. For $i,j \in \omega$, let $\left\{\left(\overline{\mathrm{u}^k_{i,j}}, \overline{\mathrm{v}^k_{i,j}}\right) \middle| k \in K\right\}$ generate $O_{i,j}$. Let $\widehat{O_{i,j}}$ be the open subsets of $\omega^{\omega\cdot 2}$ generated by $\left\{\left(\overline{\mathrm{u}^k_{i,j}}, \overline{\mathrm{v}^k_{i,j}}\right) \middle| k \in K\right\}$. Let $\hat{A} = \bigcap_{i\in\omega}\bigcup_{j\in\omega}\left(\widehat{O_{i,j}}\right)^C \in \left(\Pi^0_3 \restriction \omega^{\omega\cdot 2}\right)$. Then for $f\,\hat{}\,g \in \{0,1\}^{\omega\cdot 2}$, $f\,\hat{}\,g \in \hat{A}$ if and only if $f\,\hat{}\,g \in A$. Hence for $f\,\hat{}\,g \in \omega^{\omega\cdot 2}$ such that $(f,g)_{\mathrm{I}} \in \{0,1\}^{\omega\cdot 2}$, we have:

$(f,g)_{\mathrm{II}} \notin \{0,1\}^{\omega\cdot 2}$ or $f\,\hat{}\,g \in \hat{A}$ if and only if $(f,g)_{\mathrm{II}} \notin \{0,1\}^{\omega\cdot 2}$ or $f\,\hat{}\,g \in A$.

Therefore for $f\,\hat{}\,g \in \omega^{\omega\cdot 2}$, $f\,\hat{}\,g \in A^*$ if and only if: $(f,g)_{\mathrm{I}} \in \{0,1\}^{\omega\cdot 2}$ and $\left(\left(f\,\hat{}\,g\right)_{\mathrm{II}} \notin \{0,1\}^{\omega\cdot 2}\right.$ or $\left.f\,\hat{}\,g \in \hat{A}\right)$. Therefore since $(f,g)_{\mathrm{I}} \in \{0,1\}^{\omega\cdot 2}$ is a closed condition,

$(f^\frown g)_{\text{II}} \notin \{0,1\}^{\omega\cdot 2}$ is an open condition, and $f^\frown g \in \hat{A}$ is a Π^0_3 condition, $A^* \in (\Pi^0_3 \restriction \omega^{\omega\cdot 2})$. ■ (Claim 4.40)

Since $A^* \in (\Pi^0_3 \restriction \omega^{\omega\cdot 2})$ by Claim 4.40, A^* is determined by the hypothesis to Lemma 4.39. Therefore, either player I or player II has a w.s., s^*, for A^*.

Case 1 Player I has a w.s., s^*, for A^*.

We shall now define a strategy s for player I in the game $A \subseteq \{0,1\}^{\omega\cdot 2}$. Our definition of s is by induction.

Definition 4.41 We shall define s so that the following properties hold for $\bar{x} = (f(0), f(1), \ldots f(n-1)) \in \{0,1\}^{<\omega}$, $\bar{y} = (g(0), g(1), \ldots, g(n-1)) \in \{0,1\}^{<\omega}$, $f = (f(0), f(1), f(2), \ldots) \in \{0,1\}^{\omega}$, and $g = (g(0), g(1), g(2), \ldots) \in \{0,1\}^{\omega}$:

(i) If $\bar{x}$ is according to s, then $\bar{x}$ is according to s^*.

(ii) If $f^\frown \bar{y}$ is according to s, then $f^\frown \bar{y}$ is according to s^*.

From these properties, it will follow that:

(iii) If f is according to s, then f is according to s^*.

(iv) If $f^\frown g$ is according to s, then $f^\frown g$ is according to s^*.

Assume that $\bar{x} \in \{0,1\}^{<\omega}$ has been defined according to s. Assume inductively that $\bar{x}$ is according to s^*. Without loss of generality, we can assume that player II played

the last move in $\bar{x}$; for otherwise, we can include player II's next move and the new $\bar{x}$ remains according to both s and s^*. Since $\bar{x}$ is according to s^* and s^* is a w.s. for player I in the game A^*, $s^*(\bar{x}) \in \{0,1\}$. (For otherwise, it is easy to extend $\bar{x}$ to a play that is according to s^*, but is a loss for player I.) Since $s^*(\bar{x}) \in \{0,1\}$, define $s(\bar{x}) = s^*(\bar{x})$. Then, s has been defined so that property (i) holds. Note that once f is completely defined, f is according to s^*, so that property (iii) holds.

Assume that $f^\frown \bar{y}$ has been defined according to s. Assume inductively that $f^\frown \bar{y}$ is according to s^* and without loss of generality assume that the last move of $f^\frown \bar{y}$ belongs to player II. Since $f^\frown \bar{y}$ is according to s^* and s^* is a w.s. for player I in the game A^*, $s^*(f^\frown \bar{y}) \in \{0,1\}$. (For otherwise, it is easy to extend $f^\frown \bar{y}$ to a play that is according to s^*, but is a loss for player I.) Since $s^*(f^\frown \bar{y}) \in \{0,1\}$, define $s(f^\frown \bar{y}) = s^*(f^\frown \bar{y})$. Then, s has been defined so that property (ii) holds. Note that once $f^\frown g$ is completely defined, $f^\frown g$ is according to s^*, so that property (iv) holds.

■ (Definition 4.41)

Claim 4.42 The strategy s is a w.s. for player I in the game A.

Proof. Let $f^\frown g \in \{0,1\}^{\omega \cdot 2}$ be a play of the game $A \subseteq \{0,1\}^{\omega \cdot 2}$ that is according to s. By the definition of s, $f^\frown g$ is according to s^*. Since s^* is a w.s. for player I in the game

A^{*}, $f^{\frown}g \in A^{*}$. Therefore by the definition of A^{*}, $f^{\frown}g \in A$ since $f^{\frown}g \in \{0,1\}^{\omega \cdot 2}$.

Hence, s is a w.s. for player I in the game A. ■ (Claim 4.42 and Case 1)

Case 2 Player II has a w.s., s^{*}, for A^{*}.

We shall now define a strategy s for player II in the game $A \subseteq \{0,1\}^{\omega \cdot 2}$. Our definition of s is by induction.

Definition 4.43 We shall define s so that the following same properties as in Case 1 hold for $\bar{x} = (f(0), f(1), \ldots f(n-1)) \in \{0,1\}^{<\omega}$, $\bar{y} = (g(0), g(1), \ldots, g(n-1)) \in \{0,1\}^{<\omega}$, $f = (f(0), f(1), f(2), \ldots) \in \{0,1\}^{\omega}$, and $g = (g(0), g(1), g(2), \ldots) \in \{0,1\}^{\omega}$:

(i) If $\bar{x}$ is according to s, then $\bar{x}$ is according to s^{*}.

(ii) If $f^{\frown}\bar{y}$ is according to s, then $f^{\frown}\bar{y}$ is according to s^{*}.

From these properties, it will follow that:

(iii) If f is according to s, then f is according to s^{*}.

(iv) If $f^{\frown}g$ is according to s, then $f^{\frown}g$ is according to s^{*}.

Assume that $\bar{x} \in \{0,1\}^{<\omega}$ has been defined according to s. Assume inductively that $\bar{x}$ is according to s^{*}. Without loss of generality, we can assume that player I played the last move in $\bar{x}$; for otherwise, we can include player I's next move and the new $\bar{x}$ remains according to both s and s^{*}. Since $\bar{x}$ is according to s^{*} and s^{*} is a w.s. for player II in the game A^{*}, $s^{*}(\bar{x}) \in \{0,1\}$. (For otherwise, it is easy to extend $\bar{x}$ to a play

that is according to s^{*}, but is a loss for player II.) Since $s^{*}(\bar{x}) \in \{0,1\}$, define $s(\bar{x}) = s^{*}(\bar{x})$. Then, s has been defined so that property (i) holds. Note that once f is completely defined, f is according to s^{*}, so that property (iii) holds.

Assume that $f^\frown \bar{y}$ has been defined according to s. Assume inductively that $f^\frown \bar{y}$ is according to s^{*} and without loss of generality assume that the last move of $f^\frown \bar{y}$ belongs to player I. Since $f^\frown \bar{y}$ is according to s^{*} and s^{*} is a w.s. for player II in the game A^{*}, $s^{*}(f^\frown \bar{y}) \in \{0,1\}$. (For otherwise, it is easy to extend $f^\frown \bar{y}$ to a play that is according to s^{*}, but is a loss for player II.) Since $s^{*}(f^\frown \bar{y}) \in \{0,1\}$, define $s(f^\frown \bar{y}) = s^{*}(f^\frown \bar{y})$. Then, s has been defined so that property (ii) holds. Note that once $f^\frown g$ is completely defined, $f^\frown g$ is according to s^{*}, so that property (iv) holds.

■ (Definition 4.43)

Claim 4.44 The strategy s is a w.s. for player II in the game A.

Proof. Let $f^\frown g \in \{0,1\}^{\omega \cdot 2}$ be according to s. By the definition of s, $f^\frown g$ is according to s^{*}. Since s^{*} is a w.s. for player II in the game A^{*}, $f^\frown g \notin A^{*}$. Therefore by the definition of A^{*}, $f^\frown g \notin A$ since $f^\frown g \in \{0,1\}^{\omega \cdot 2}$. Hence, s is a w.s. for player II in the game A. ■ (Claim 4.44 and Case 2)

Therefore, if player I has a w.s., s^*, for $A^* \subseteq \omega^{\omega \cdot 2}$, then player I has a w.s. s as defined in Definition 4.41 for the game $A \subseteq \{0,1\}^{\omega \cdot 2}$. If player II has a w.s., s^*, for $A^* \subseteq \omega^{\omega \cdot 2}$, then player II has a w.s. s as defined in Definition 4.43 for the game $A \subseteq \{0,1\}^{\omega \cdot 2}$. Hence, the determinacy of $\left(\Pi_3^0 \restriction \omega^{\omega \cdot 2}\right)$ implies the determinacy of $\left(\Pi_3^0 \restriction \{0,1\}^{\omega \cdot 2}\right)$. ■ (Lemma 4.39)

Thus by Lemma 4.14 and Lemma 4.39, the determinacy of $\left(\Pi_3^0 \restriction \{0,1\}^{\omega \cdot 2}\right)$ is equivalent to the determinacy of $\left(\Pi_3^0 \restriction \omega^{\omega \cdot 2}\right)$. ■ (Theorem 4.13)

The reader may have noticed that the proof to Theorem 4.13 easily generalizes. We formally note such generalizations in Chapter 5.

CHAPTER 5

EXTENSION OF PREVIOUS RESULTS

The goal of this chapter is to extend the results of Chapter 4 and to examine the following questions:

(Q1) For what collection of "small" complexity (least complexity if it exists) can we replace Π_3^0 with in Theorem 4.13 that will keep the theorem true?

(Q2) The determinacy of what collection of games of "small" complexity (least complexity if it exists) with length $\omega \cdot n$ and with moves from $\{0,1\}$ requires the existence of large cardinals for $n \geq 2$?

Notice that by the same proof to Theorem 4.13, we have:

Theorem 5.1 For $n \in \omega$ and $n \neq 0$, the determinacy of $\left(\Pi_3^0 \restriction \{0,1\}^{\omega \cdot n}\right)$ is equivalent to the determinacy of $\left(\Pi_3^0 \restriction \omega^{\omega \cdot n}\right)$.

The equivalency of the determinacy of $\left(\Pi_3^0 \restriction \{0,1\}^{\omega \cdot n}\right)$ and the determinacy of $\left(\Pi_3^0 \restriction \omega^{\omega \cdot n}\right)$ stated in Theorem 5.1 is interesting because for $n \geq 2$, the theorem is false if Π_3^0 is replaced with Σ_1^0. We slightly divert our attention from the main topics of this

thesis to note (through Corollary 5.3 and Theorem 5.4) that the existence of large cardinals is required to obtain the determinacy of $(\Pi_3^0 \upharpoonright \omega^{\omega \cdot n})$ and therefore (by Theorem 5.1) the determinacy of $(\Pi_3^0 \upharpoonright \{0,1\}^{\omega \cdot n})$ for $n \geq 2$. We return to the main development of this thesis by Definition 5.5, but we shall also use Corollary 5.3 and Theorem 5.4 to note that large cardinals are required for the determinacy of other classes considered in this chapter.

Theorem 5.2 The determinacy of $(\Sigma_1^0 \upharpoonright \omega^{\omega \cdot (n+1)})$ is equivalent to the determinacy of $(\Pi_n^1 \upharpoonright \omega^{\omega})$ for $n \geq 1$.

We refer the reader to *Descriptive Set Theory* by Yiannis N. Moschovakis [Mo] for the definition of Π_n^1 sets. We shall only use Theorem 5.2 for the case where $n = 1$. We provide an outline of the proof for this case below. The reader not familiar with Π_1^1 sets can continue through this thesis treating the determinacy of Π_1^1 sets as a sort of black box.

Corollary 5.3 The determinacy of $(\Sigma_1^0 \upharpoonright \omega^{\omega \cdot 2})$ is equivalent to the determinacy of $(\Pi_1^1 \upharpoonright \omega^{\omega})$.

Outline of proof. Assume the determinacy of $\left(\Sigma_1^0 \restriction \omega^{\omega \cdot 2}\right)$ and let $A \in \left(\Pi_1^1 \restriction \omega^\omega\right)$. Then $\exists O \in \left(\Sigma_1^0 \restriction \omega^{\omega \cdot 2}\right)$ such that: $f \in A$ if and only if $\forall h \in \omega^\omega$ $(f,h) \in O$. Let $\left\{\left(\overline{u_i}, \overline{v_i}\right) \middle| i \in I\right\}$ generate O. We define the auxiliary game A^* by:

$$(f,g) \in A^* \text{ if and only if } \left(f,(g)_{\mathrm{II}}\right) \in O.$$

One can show that $A^* \in \left(\Sigma_1^0 \restriction \omega^{\omega \cdot 2}\right)$, so that by the hypothesis, A^* is determined. Therefore, either player I or player II has a w.s., s^*, for A^*.

Case 1: Player I has a w.s., s^*, for A^*.

We will define a strategy s for player I in the game A. Let s follow s^* to obtain f according to both s and s^*. Pick any h. Let g be according to s^*, but with $(g)_{\mathrm{II}} = h$. Since s^* is a w.s. for player I, $(f,g) \in A^*$, so that $(f,h) \in O$. Consequently, $\forall h \in \omega^\omega$ $(f,h) \in O$, so that $f \in A$ and therefore s is a w.s. for player I in the game A.

Case 2: Player II has a w.s., s^*, for A^*.

We will define a strategy s for player II in the game A. Let s follow s^* to obtain f according to both s and s^*. Pick any g such that g is according to s^*. Then since s^* is a w.s. for player II, $(f,g) \notin A^*$, so that $\left(f,(g)_{\mathrm{II}}\right) \notin O$. Consequently, $\exists h \in \omega^\omega$ such that $(f,h) \notin O$, so that $f \notin A$ and therefore s is a w.s. for player II in the game A. Hence, the determinacy of $\left(\Sigma_1^0 \restriction \omega^{\omega \cdot 2}\right)$ implies the determinacy of $\left(\Pi_1^1 \restriction \omega^\omega\right)$.

For the other direction, assume the determinacy of $\left(\Pi_1^1 \restriction \omega^\omega\right)$ and let $A \in \left(\Sigma_1^0 \restriction \omega^{\omega \cdot 2}\right)$. Let $\left\{\left(\overline{\mathrm{u}_i}, \overline{\mathrm{v}_i}\right) \middle| i \in \omega\right\}$ generate A. We define the auxiliary game A^* by:

> $f \in A^*$ if and only if for all strategies τ for player II $\exists n$ $\exists\left(g(0), g(2), \ldots, g(2n)\right)$ $\exists i$ such that $\left(\overline{\mathrm{u}_i}, \overline{\mathrm{v}_i}\right) \subseteq \left(\overline{f}(2n+1), \overline{g}(2n+1)\right)$ where $g(2i+1) = \tau\left(\overline{g}(2i+1)\right)$ for $i < n$.

In the remainder of this proof, whenever τ, $g(0)$, $g(2), \ldots,$ $g(2n)$ are clear from the context, we shall write $\overline{g}(2n+1)$ assuming that $g(2i+1) = \tau\left(\overline{g}(2i+1)\right)$ for $i < n$. One can show that $A^* \in \left(\Pi_1^1 \restriction \omega^\omega\right)$, so that by the hypothesis, A^* is determined. Therefore, either player I or player II has a w.s., s^*, for A^*.

Case 1: Player I has a w.s., s^*, for A^*.

We will define a strategy s for player I in the game A. Let s follow s^* to obtain f according to both s and s^*. Then since f is according to s^*, $f \in A^*$. Let $A_f = \left\{g \middle| f^\frown g \in A\right\} \subseteq \omega^\omega$. Then A_f is open and therefore determined (by Gale-Stewart), so that either player I or player II has a w.s. for A_f. Arguing towards a contradiction, suppose player II has a w.s., τ, for A_f. Then let g be according to τ. Since τ is a w.s. for player II, $g \notin A_f$, so that $f^\frown g \notin A$. Since $f \in A^*$, $\exists n\, \exists\left(g(0), g(2), \ldots, g(2n)\right)$ $\exists i$ such that $\left(\overline{\mathrm{u}_i}, \overline{\mathrm{v}_i}\right) \subseteq \left(\overline{f}(2n+1), \overline{g}(2n+1)\right)$, so that $f^\frown g \in A$, contradicting $f^\frown g \notin A$. Consequently, player I has a w.s., σ, for A_f. Let s follow σ to obtain g. Since σ is a

w.s. for player I, $g \in A_f$, so that $f^\frown g \in A$ and therefore s is a w.s. for player I in the game A.

Case 2: Player II has a w.s., s^*, for A^*.

We will define a strategy s for player II in the game A. Let s follow s^* to obtain f according to both s and s^*. Then since f is according to s^*, $f \notin A^*$. Then there exists a strategy $\tau = \tau_f$ for player II such that:

(i) $\forall n\ \forall(g(0), g(2), \ldots, g(2n))\ \forall i\ \left(\overline{\mathrm{u}_i}, \overline{\mathrm{v}_i}\right) \not\subset \left(\overline{f}(2n+1), \overline{g}(2n+1)\right)$.

Recalling that f is given according to both s and s^*, let s follow τ_f to obtain g that satisfies (i). Then $(f, g) \notin A$, so that s is a w.s. for player II in the game A. Hence, the determinacy of $\left(\Pi_1^1 \restriction \omega^\omega\right)$ implies the determinacy of $\left(\Sigma_1^0 \restriction \omega^{\omega \cdot 2}\right)$. Thus, the determinacy of $\left(\Sigma_1^0 \restriction \omega^{\omega \cdot 2}\right)$ is equivalent to the determinacy of $\left(\Pi_1^1 \restriction \omega^\omega\right)$. ■ (Corollary 5.3)

It is well-known that Martin has shown that Π_1^1 determinacy implies the large cardinal property:

(i) $\forall x \in \omega^\omega \left(x^\# \text{ exists}\right)$.

We do not want to divert our attention by defining $x^\#$ (x sharp), but instead note that it is well-known that large cardinal properties cannot be proved in ZFC (the usual axioms for set theory). This, together with Corollary 5.3, will be cited to observe that the determinacy of certain classes considered in this thesis cannot be proved in ZFC. We now state the theorem of Martin.

Theorem 5.4 [Ma3] The determinacy of $\left(\Pi_1^1 \upharpoonright \omega^\omega\right)$ implies $\forall x \in \omega^\omega \left(x^\# \text{ exists}\right)$.

Recall we showed the determinacy of $\left(\Sigma_1^0 \upharpoonright \{0,1\}^{\omega \cdot n}\right)$ for $n \geq 2$ in Chapter 3 without assuming any large cardinal hypotheses. A natural question is: For what collection of "small" complexity (least complexity if it exists) can we replace Π_3^0 with in Theorem 5.1 that will keep the theorem true? To address this question, we next recall the definition of the algebra, $\mathcal{Q}\left(\Pi_2^0\right)$, of Π_2^0 sets and eventually note that Theorem 5.1 remains true if Π_3^0 is replaced by $\mathcal{Q}\left(\Pi_2^0\right)$.

Definition 5.5 $\mathcal{Q}$ is an algebra of sets if and only if the following hold:

(i) $\mathcal{Q}$ is closed under finite intersections. That is, if $A, B \in \mathcal{Q}$, then $A \cap B \in \mathcal{Q}$.

(ii) $\mathcal{Q}$ is closed under complements. That is, if $A \in \mathcal{Q}$, then $A^C \in \mathcal{Q}$.

Definition 5.6 Define $\mathcal{Q}(\Gamma)$ to be the smallest algebra containing Γ. That is, if $\mathcal{Q}'$ is an algebra and $\Gamma \subseteq \mathcal{Q}'$, then $\mathcal{Q} \subseteq \mathcal{Q}'$.

Remark 5.7 It is well known that $\mathcal{Q}(\Gamma)$ can be "internally" defined by: $\mathcal{Q}(\Gamma) = \bigcup_{n \in \omega} \Gamma_n$ if we set $\Gamma_0 = \Gamma$ and $\Gamma_{n+1} = \left\{A \cap B,\ A^C \,\middle|\, A, B \in \Gamma_n\right\}$.

Recall from the proof of Lemma 4.14, that given A, one can define A^* and A^{**}. Note that one can naturally extend the definition of the sets, $E_{f\,\mathrm{NTE}}$, $E_{f\,\mathrm{TE}}^{\mathrm{II\ before\ I}}$, $E_{f\,\mathrm{NTE},g\,\mathrm{TE}}^{\mathrm{II\ before\ I}}$,

and $E_{f^{\text{NTE}}, g^{\text{NTE}}}$, defined in Definition 4.11, for games of length $\omega \cdot n$ for $n \in \omega$ (not just for $n = 2$). We use this extended definition to define A^{*} and A^{**}.

Definition 5.8 Given $A \subseteq \omega^{\omega \cdot n}$, define:

$$A^{*} = \left\{ f^{1\frown} f^{2\frown} \ldots {}^{\frown} f^{n} \in \{0,1\}^{\omega \cdot n} \,\middle|\, f^{1\frown} f^{2\frown} \ldots {}^{\frown} f^{n} \in \bigcap_{n=0}^{\infty} E_{f^{n}\,\text{NTE}} \text{ and } \widehat{f^{1}}^{\frown} \widehat{f^{2}}^{\frown} \ldots {}^{\frown} \widehat{f^{n}} \in A \right\}$$

and

$$A^{**} = E^{\text{II before I}}_{f^{1}\,\text{TE}} \cup \left(\bigcup_{k=2}^{n} E^{\text{II before I}}_{f^{1}\,\text{NTE}, f^{2}\,\text{NTE}, \ldots, f^{k-1}\,\text{NTE}, f^{k}\,\text{TE}} \right) \cup A^{*}.$$

Proposition 5.9 If $A \in \mathcal{G}\left(\Pi^0_2\right)$, then $A^{**} \in \mathcal{G}\left(\Pi^0_2\right)$.

Outline of proof. Assume that $A \in \mathcal{G}\left(\Pi^0_2\right)$. Similar to the proof of Claim 4.17, one can show that $A^{*} \in \mathcal{G}\left(\Pi^0_2\right)$. By the definition of Σ^0_2 and Π^0_2 sets of the Borel hierarchy, $E^{\text{II before I}}_{f^{1}\,\text{TE}} \in \Sigma^0_2$ and $E^{\text{II before I}}_{f^{1}\,\text{NTE}, f^{2}\,\text{NTE}, \ldots, f^{k-1}\,\text{NTE}, f^{k}\,\text{TE}} \in \Pi^0_2 \wedge \Sigma^0_2$ for $2 \le k \le n$. Therefore, $A^{**} \in \mathcal{G}\left(\Pi^0_2\right)$. ■ (Proposition 5.9)

Therefore by Proposition 5.9, Theorem 5.1, and the proof of Theorem 4.13, we have the following theorem:

Theorem 5.10 For $n \in \omega$ and $n \neq 0$, the determinacy of ($\mathcal{G}\left(\Pi^0_2\right) \upharpoonright \{0,1\}^{\omega \cdot n}$) is equivalent to the determinacy of ($\mathcal{G}\left(\Pi^0_2\right) \upharpoonright \omega^{\omega \cdot n}$).

We shall next note in Theorem 5.12 that one can also extend Theorem 5.1 by showing that Theorem 5.1 holds for more complex sets.

Proposition 5.11 For $k \geq 3$, if $A \in \Pi_k^0$, then $A^{**} \in \Pi_k^0$.

Outline of proof. Assume $A \in \Pi_k^0$. Similar to the proof of Claim 4.17, one can show that $A^* \in \Pi_k^0$. Therefore since $E_{f^1 \text{ TE}}^{\text{II before I}} \in \Sigma_2^0$ and $E_{f^1 \text{ NTE}, f^2 \text{ NTE}, \ldots, f^{k-1} \text{ NTE}, f^k \text{ TE}}^{\text{II before I}} \in \Pi_2^0 \wedge \Sigma_2^0$ for $2 \leq k \leq n$, $A^{**} \in \Pi_k^0$. ■ (Proposition 5.11)

Therefore by Proposition 5.11, Theorem 5.1, and the proof of Theorem 4.13, we have the following theorem:

Theorem 5.12 For $n \in \omega$, $n \neq 0$, and $k \geq 3$, the determinacy of $\left(\Pi_k^0 \restriction \{0,1\}^{\omega \cdot n}\right)$ is equivalent to the determinacy of $\left(\Pi_k^0 \restriction \omega^{\omega \cdot n}\right)$.

Remark 5.13 Note that for $n \in \omega$ and $n \neq 0$, the determinacy of $\left(\Pi_k^0 \restriction \omega^{\omega \cdot n}\right)$ implies the determinacy of $\left(\Pi_k^0 \restriction \{0,1\}^{\omega \cdot n}\right)$ for all $k \geq 1$. This implication is true for $k = 1$ since the consequent of the implication is true; for the determinacy of $\left(\Pi_1^0 \restriction \{0,1\}^{\omega \cdot n}\right)$ is equivalent to the determinacy of $\left(\Sigma_1^0 \restriction \{0,1\}^{\omega \cdot n}\right)$ which was shown to be true in Chapter 3. For $k \geq 2$, the proof is the same as the proof of Lemma 4.39.

In light of Theorems 5.10 and Theorem 5.12, it is natural to ask the question:

(Q3) For $n \geq 2$, is the determinacy of $\left(\Pi_2^0 \restriction \{0,1\}^{\omega \cdot n}\right)$ equivalent to the determinacy of $\left(\Pi_2^0 \restriction \omega^{\omega \cdot n}\right)$?

It is well-known that the determinacy of $\left(\Pi_2^0 \restriction X^{\omega \cdot n}\right)$ is equivalent to the determinacy of $\left(\Sigma_2^0 \restriction X^{\omega \cdot n}\right)$, so our question can be reformulated in terms of Σ_2^0 sets as follows:

(Q4) For $n \geq 2$, is the determinacy of $\left(\Sigma_2^0 \restriction \{0,1\}^{\omega \cdot n}\right)$ equivalent to the determinacy of $\left(\Sigma_2^0 \restriction \omega^{\omega \cdot n}\right)$?

Recall that in Remark 5.13, we noted that for $n \in \omega$ and $n \neq 0$, the determinacy of $\left(\Pi_k^0 \restriction \omega^{\omega \cdot n}\right)$ implies the determinacy of $\left(\Pi_k^0 \restriction \{0,1\}^{\omega \cdot n}\right)$ for all $k \geq 1$. Therefore, the determinacy of $\left(\Pi_2^0 \restriction \omega^{\omega \cdot n}\right)$ implies the determinacy of $\left(\Pi_2^0 \restriction \{0,1\}^{\omega \cdot n}\right)$ and the determinacy of $\left(\Sigma_2^0 \restriction \omega^{\omega \cdot n}\right)$ implies the determinacy of $\left(\Sigma_2^0 \restriction \{0,1\}^{\omega \cdot n}\right)$. We note the following theorem to see that the determinacy of only a slightly more complex collection than Π_2^0 is needed to prove the first implication in the other direction for $n \geq 2$ and that the determinacy of only a slightly more complex collection than Σ_2^0 is needed to prove the second implication in the other direction for $n \geq 2$.

Theorem 5.14

(i) The determinacy of $\left(\Sigma_2^0 \vee \Pi_2^0 \restriction \{0,1\}^{\omega\cdot n}\right)$ implies the determinacy of $\left(\Pi_2^0 \restriction \omega^{\omega\cdot n}\right)$ for $n \geq 2$.

(ii) The determinacy of $\left(\Pi_2^0 \wedge \Sigma_2^0 \restriction \{0,1\}^{\omega\cdot n}\right)$ implies the determinacy of $\left(\Sigma_2^0 \restriction \omega^{\omega\cdot n}\right)$ for $n \geq 2$.

Outline of proof of (i) for $n = 2$. Let $A \in \left(\Pi_2^0 \restriction \omega^{\omega\cdot n}\right)$. Then there exists open sets O_i such that $A = \bigcap_{i\in\omega} O_i$. For $i \in \omega$, let $\left\{\left(\overline{\mathrm{u}_i^j}, \overline{\mathrm{v}_i^j} \mid j \in J\right)\right\}$ generate O_i. Then one can show the determinacy of A using the following auxiliary game $A_{\Sigma_2^0 \vee \Pi_2^0} \subseteq \{0,1\}^{\omega\cdot 2}$:

$$f^\frown g \in A_{\Sigma_2^0 \vee \Pi_2^0} \text{ if and only if } \left((f,g)_{\mathrm{II}} \text{ has a tail end of ones}\right) \text{ or}$$

$$\left(\forall i\, \exists j \left(\overline{\mathrm{u}_i^j} \subseteq \hat{f} \text{ and } \overline{\mathrm{v}_i^j} \subseteq \hat{g}\right)\right).$$

One can show that the complexity of the first disjunct is Σ_2^0 and that the complexity of the second disjunct is Π_2^0, so that $A_{\Sigma_2^0 \vee \Pi_2^0}$ is $\Sigma_2^0 \vee \Pi_2^0$. Hence there is a w.s. $s_{\Sigma_2^0 \vee \Pi_2^0}$ for $A_{\Sigma_2^0 \vee \Pi_2^0}$ by the hypothesis to the theorem. We define a w.s. for A from $s_{\Sigma_2^0 \vee \Pi_2^0}$ using the same techniques as in Chapter 4. ■ (Theorem 5.14(i))

Outline of proof of (ii) for $n = 2$. Let $A \in \left(\Sigma_2^0 \restriction \omega^{\omega\cdot n}\right)$. Then there exists open sets O_i such that $A = \bigcup_{i\in\omega} (O_i)^C$. For $i \in \omega$, let $\left\{\left(\overline{\mathrm{u}_i^j}, \overline{\mathrm{v}_i^j} \mid j \in J\right)\right\}$ generate O_i. Then one can show the determinacy of A using the following auxiliary game $A_{\Pi_2^0 \wedge \Sigma_2^0} \subseteq \{0,1\}^{\omega\cdot 2}$:

$$f^\frown g \in A_{\Pi_2^0 \wedge \Sigma_2^0} \text{ if and only if } \left((f,g)_1 \text{ has no tail end of ones} \right) \text{ and}$$

$$\left(\exists i \; \forall j \; \left(\overline{\mathrm{u}_i^j} \not\subset \hat{f} \text{ or } \overline{\mathrm{v}_i^j} \not\subset \hat{g} \right) \right).$$

One can show that the complexity of the first conjunct is Π_2^0 and that the complexity of the second conjunct is Σ_2^0, so that $A_{\Pi_2^0 \wedge \Sigma_2^0}$ is $\Pi_2^0 \wedge \Sigma_2^0$. Hence there is a w.s. $s_{\Pi_2^0 \wedge \Sigma_2^0}$ for $A_{\Pi_2^0 \wedge \Sigma_2^0}$ by the hypothesis to the theorem. We define a w.s. for A from $s_{\Pi_2^0 \wedge \Sigma_2^0}$ using the same techniques as in Chapter 4. ■ (Theorem 5.14(ii))

It follows from Corollary 5.3 and Theorem 5.4 that large cardinals are needed to obtain the determinacy of $\left(\Sigma_1^0 \restriction \omega^{\omega \cdot n}\right)$ and are therefore needed to obtain the determinacy of $\left(\Sigma_2^0 \restriction \omega^{\omega \cdot n}\right)$ for $n \geq 2$. If the equivalency from question (Q4) is true, then large cardinals are needed to obtain the determinacy of $\left(\Sigma_2^0 \restriction \{0,1\}^{\omega \cdot n}\right)$ for $n \geq 2$. By Theorem 3.18, the determinacy of $\left(\Sigma_1^0 \restriction \{0,1\}^{\omega \cdot n}\right)$ is true regardless of the existence of large cardinals. Thus, a weaker question than (Q4) (than whether the determinacy of $\left(\Sigma_2^0 \restriction \{0,1\}^{\omega \cdot n}\right)$ is equivalent to the determinacy of $\left(\Sigma_2^0 \restriction \omega^{\omega \cdot n}\right)$) is whether large cardinals are needed to obtain the determinacy of $\left(\Sigma_2^0 \restriction \{0,1\}^{\omega \cdot n}\right)$ for $n \geq 2$. Dr. DuBose has some observations that relate to these questions.

Proposition 5.15 (DuBose) For $n \in \omega$ and $n \neq 0$, the determinacy of $\left(\Sigma_2^0 \restriction \{0,1\}^{\omega \cdot n}\right)$ implies the determinacy of $\left(\Sigma_1^0 \restriction \omega^{\omega \cdot n}\right)$.

Outline of proof. Let $A \in \left(\Sigma_1^0 \restriction \omega^{\omega \cdot n}\right)$. Then some $\left\{\left(\overline{\mathrm{u}_i^1}, \overline{\mathrm{u}_i^2}, \ldots, \overline{\mathrm{u}_i^n}\right) \middle| i \in I\right\}$ generates A. One can show the determinacy of A using the following auxiliary game $A_{\Sigma_2^0} \subseteq \{0,1\}^{\omega \cdot n}$:

$$f^1 {}^\frown f^2 {}^\frown \ldots {}^\frown f^n \in A_{\Sigma_2^0} \text{ if and only if } \exists j \leq n \text{ such that } \left(f^j\right)_{\mathrm{II}} \text{ has a tail end of ones or}$$

$$\exists i \in I \text{ such that } \forall j \leq n,\ \overline{\mathrm{u}_i^j} \subseteq \widehat{f^j}.$$

(In consideration of $\widehat{f^j}$ above, recall the decoding function, $\hat{}$, from Definition 4.3.) One can show that the complexity of the first disjunct is Σ_2^0 and the complexity of the second disjucnt is Σ_1^0, so that the complexity of $A_{\Sigma_2^0}$ is Σ_2^0. Hence there is a w.s. $s_{\Sigma_2^0}$ for $A_{\Sigma_2^0}$ by the hypothesis to the proposition. We define a w.s. for A from $s_{\Sigma_2^0}$ using the same techniques as in Chapter 4. ■ (Proposition 5.15)

Therefore by Corollary 5.3, Theorem 5.4, and Proposition 5.15, the determinacy of $\left(\Sigma_2^0 \restriction \{0,1\}^{\omega \cdot n}\right)$ requires the existence of large cardinals for $n \geq 2$.

Although we do not know the answer to the original question of whether the determinacy of $\left(\Sigma_2^0 \restriction \{0,1\}^{\omega \cdot n}\right)$ is equivalent to the determinacy of $\left(\Sigma_2^0 \restriction \omega^{\omega \cdot n}\right)$, we do know that the existence of large cardinals is needed to prove the determinacy of each for $n \geq 2$. Considering that the existence of large cardinals is not needed to prove the

determinacy of $\left(\Sigma_1^0 \restriction \{0,1\}^{\omega \cdot n}\right)$, a natural question is whether the existence of large cardinals is needed to prove the determinacy of slightly more complex games. To address this question, we note the following observation:

Proposition 5.16 (DuBose) The determinacy of $\left(\Sigma_1^0 \wedge \Pi_1^0 \restriction \{0,1\}^{\omega+\omega+\omega}\right)$ implies the determinacy of $\left(\Sigma_2^0 \restriction \{0,1\}^{\omega+\omega}\right)$.

Outline of proof. Let $A \in \Sigma_2^0 \restriction \{0,1\}^{\omega+\omega}$. Then there exists open sets $O_i \subseteq \{0,1\}^{\omega+\omega}$ such that $A = \bigcup_{i\in\omega}(O_i)^C$. For $i \in \omega$, let $\left\{\left(\overline{\mathrm{u}_i^j}, \overline{\mathrm{v}_i^j}\right) \middle| j \in J\right\}$ generate O_i. Then $f^\frown g \in A$ if and only if $\exists i$ such that $\forall j$, $\overline{\mathrm{u}_i^j} \not\subset f$ or $\overline{\mathrm{v}_i^j} \not\subset g$. One can show the determinacy of A using the following auxiliary game $A_{\Sigma_1^0 \wedge \Pi_1^0} \subseteq \{0,1\}^{\omega+\omega+\omega}$:

$f^\frown g^\frown h \in A_{\Sigma_1^0 \wedge \Pi_1^0}$ if and only if $\exists i$ such that $h_1(i) = 0$ and

$$\forall i\ \forall j\left((i) \not\subset \left(\hat{h}\right)_1 \text{ or } \overline{\mathrm{u}_i^j} \not\subset f \text{ or } \overline{\mathrm{v}_i^j} \not\subset g\right).$$

One can show that the complexity of the first conjunct is Σ_1^0. The condition "$(i) \not\subset \left(\hat{h}\right)_1$" (of the second conjunct) is equivalent to the condition "$\overline{\mathrm{e}_i} \not\subset (h)_1$" where $\overline{\mathrm{e}_i} = (1,1,\ldots,1,0)$ is the sequence of i ones followed by one zero. Therefore, the second conjunct is equivalent to the Π_1^0 condition: $\forall i\ \forall j\left(\overline{\mathrm{e}_i} \not\subset (h)_1 \text{ or } \overline{\mathrm{u}_i^j} \not\subset f \text{ or } \overline{\mathrm{v}_i^j} \not\subset g\right)$. Hence, the complexity of $A_{\Sigma_1^0 \wedge \Pi_1^0}$ is $\Sigma_1^0 \wedge \Pi_1^0$.

Since $A_{\Sigma_1^0 \wedge \Pi_1^0} \in \left(\Sigma_1^0 \wedge \Pi_1^0 \upharpoonright \{0,1\}^{\omega+\omega+\omega}\right)$, $A_{\Sigma_1^0 \wedge \Pi_1^0}$ is determined by the hypothesis to the proposition. Therefore, either player I or player II has a w.s., $s_{\Sigma_1^0 \wedge \Pi_1^0}$, for $A_{\Sigma_1^0 \wedge \Pi_1^0}$.

Case 1: Player I has a w.s., $s_{\Sigma_1^0 \wedge \Pi_1^0}$, for $A_{\Sigma_1^0 \wedge \Pi_1^0}$.

We define a strategy s for player I in the game A. Let s follow $s_{\Sigma_1^0 \wedge \Pi_1^0}$ to obtain $f^\frown g$ according to both s and $s_{\Sigma_1^0 \wedge \Pi_1^0}$. Use $s_{\Sigma_1^0 \wedge \Pi_1^0}$ to also obtain h, simulating player II's moves $h(2n+1)$ your favorite way. Since $s_{\Sigma_1^0 \wedge \Pi_1^0}$ is a w.s. for player I in the game $A_{\Sigma_1^0 \wedge \Pi_1^0}$, there is a least i such that $h_1(i)=0$ and $\forall j\left(\overline{\mathrm{u}_i^j} \not\subset f \text{ or } \overline{\mathrm{v}_i^j} \not\subset g\right)$. Hence $f^\frown g \in A$ and therefore s is a w.s. for player I in the game A.

Case 2: Player II has a w.s., $s_{\Sigma_1^0 \wedge \Pi_1^0}$, for $A_{\Sigma_1^0 \wedge \Pi_1^0}$.

We define a strategy s for player II in the game A. Let s follow $s_{\Sigma_1^0 \wedge \Pi_1^0}$ to obtain $f^\frown g$ according to both s and $s_{\Sigma_1^0 \wedge \Pi_1^0}$. Pick an arbitrary $i \in \omega$. Let h be such that $f^\frown g^\frown h$ is according to $s_{\Sigma_1^0 \wedge \Pi_1^0}$ and $\overline{\mathrm{e}_i} \subseteq (h)_1$. Since $s_{\Sigma_1^0 \wedge \Pi_1^0}$ is a w.s. for player II in the game $A_{\Sigma_1^0 \wedge \Pi_1^0}$, $f^\frown g^\frown h \notin A_{\Sigma_1^0 \wedge \Pi_1^0}$, so that $\exists j\left(\overline{\mathrm{u}_i^j} \subseteq f \text{ and } \overline{\mathrm{v}_i^j} \subseteq g\right)$. Hence $f^\frown g \notin A$ and therefore s is a w.s. for player II in the game A. ■ (Proposition 5.16)

Note that by the same proof, one can obtain the following more general proposition.

Proposition 5.17 (DuBose) For $n \geq 1$, the determinacy of $\left(\Sigma_1^0 \wedge \Pi_1^0 \restriction \{0,1\}^{\omega\cdot(n+1)}\right)$ implies the determinacy of $\left(\Sigma_2^0 \restriction \{0,1\}^{\omega\cdot n}\right)$.

Note that with a similar proof to Proposition 5.16, one can also show:

Proposition 5.18 (DuBose) Let Γ be a collection of sets. Let A be in the collection $\left(\exists^\omega \Gamma \restriction X^{\omega+\omega}\right)$ if and only if $\exists B \in \left(\Gamma \restriction X^{\omega+\omega} \times \omega\right)$ such that $A = \left\{ f^\frown g \in X^{\omega+\omega} \middle| \exists n < \omega \text{ such that } f^\frown g^\frown (n) \in B \right\}$. Then the determinacy of $\left(\Gamma \restriction X^{\omega+\omega} \times \omega\right)$ is equivalent to the determinacy of $\left(\exists^\omega \Gamma \restriction X^{\omega+\omega}\right)$.

In particular:

(i) The determinacy of $\left(\Gamma \restriction \omega^{\omega+\omega} \times \omega\right)$ is equivalent to the determinacy of $\left(\exists^\omega \Gamma \restriction \omega^{\omega+\omega}\right)$.

(ii) The determinacy of $\left(\Gamma \restriction \{0,1\}^{\omega+\omega} \times \omega\right)$ is equivalent to the determinacy of $\left(\exists^\omega \Gamma \restriction \{0,1\}^{\omega+\omega}\right)$.

Recall that for $n \geq 2$, the determinacy of $\left(\Sigma_2^0 \restriction \{0,1\}^{\omega\cdot n}\right)$ requires the existence of large cardinals. Therefore by Proposition 5.17, the determinacy of $\left(\Sigma_1^0 \wedge \Pi_1^0 \restriction \{0,1\}^{\omega\cdot(n+1)}\right)$ requires the existence of large cardinals for $n \geq 2$. Hence, the remaining interesting question is:

(Q5) For what classes Γ such that Γ properly contains Σ_1^0, but is properly contained in Σ_2^0, does the determinacy of $\left(\Gamma \restriction \{0,1\}^{\omega \cdot 2}\right)$ require the existence of large cardinals?

Obvious classes to consider are $\Gamma = \Sigma_1^0 \wedge \Pi_1^0$, $\Gamma = \mathscr{Q}\left(\Sigma_1^0\right) = \mathscr{Q}\left(\Pi_1^0\right)$, and $\Gamma = \Delta_2^0$.

In conclusion, we have shown that the determinacy of $\left(\Sigma_1^0 \restriction \{0,1\}^{\omega \cdot n}\right)$ is true regardless of the existence of large cardinals for $n \geq 2$. For $k \geq 3$ and $n \geq 2$, we have shown that the determinacy of $\left(\Pi_k^0 \restriction \{0,1\}^{\omega \cdot n}\right)$ is equivalent to the determinacy of $\left(\Pi_k^0 \restriction \omega^{\omega \cdot n}\right)$ and therefore requires the existence of large cardinals. We do not know if this equivalency holds for Π_2^0, but we do know that this equivalency holds for $\mathscr{Q}\left(\Pi_2^0\right)$ and we also know that the determinacy of $\left(\Pi_2^0 \restriction \{0,1\}^{\omega \cdot n}\right)$ requires the existence of large cardinals for $n \geq 2$. Finally, we noted that for $n \geq 2$, the determinacy of the slightly more complex than open game $\left(\Sigma_1^0 \wedge \Pi_1^0 \restriction \{0,1\}^{\omega \cdot (n+1)}\right)$ also requires the existence of large cardinals. These results are interesting and further enhance the original result from Chapter 3 that for $n \geq 2$, the determinacy of $\left(\Sigma_1^0 \restriction \{0,1\}^{\omega \cdot n}\right)$ is true regardless of the existence of large cardinals.

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