1-1-2002

The asymptotic behavior of the integer solutions of the Rosenberger equations

Kensaku Umeda
University of Nevada, Las Vegas

THE ASYMPTOTIC BEHAVIOR OF THE INTEGER SOLUTIONS OF THE ROSENBERGER EQUATIONS

by

Kensaku Umeda

Bachelor of Science
Eastern Kentucky University
2000

A thesis submitted in partial fulfillment
of the requirements for the

Master of Science Degree
Department of the Mathematical Sciences
College of Sciences

Graduate College
University of Nevada, Las Vegas
May 2002

UMI Number: 1411224

UMI Microform 1411224

ProQuest Information and Learning Company
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Thesis Approval

The Graduate College
University of Nevada, Las Vegas

April 19, 2002

The Thesis prepared by

Kensaku Umeda

Entitled

The Asymptotic Behavior of the Integer Solutions of the Rosenberger Equations

is approved in partial fulfillment of the requirements for the degree of

Master of Science

Examination Committee Chair

Dean of the Graduate College

Examination Committee Member

Examination Committee Member

Graduate College Faculty Representative

1017-53

ABSTRACT

The Asymptotic Behavior of the Integer Solutions of the Rosenberger Equations

by

Kensaku Umeda

Dr. Arthur Baragar, Examination Committee Chair
Professor of Mathematics
University of Nevada, Las Vegas

The Rosenberger equations are equations of the form: $ax^2 + by^2 + cz^2 = dxyz$, where the sets of coefficients (a, b, c, d) are all integers such that each of a, b, and c divides d, and the equations themselves have infinitely many integer solutions. Rosenberger has shown that there are only six such sets of coefficients: one of which is the Markoff equation, $x^2 + y^2 + z^2 = 3xyz$. Zagier investigated the asymptotic behavior of the integer solutions of the Markoff equation. In this paper, we apply Zagier's techniques to the Rosenberger equations and show that the number $N(T)$ of positive integer solutions that are bounded by T is $N(T) = C(\log T)^2 + O\left(\log T(\log\log T)^2\right)$, where C is an explicitly computable constant that depends on the equation.

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ACKNOWLEDGMENT

Most of all. I owe my deepest gratitude to Dr. Arthur Baragar in the department of mathematical sciences for being my thesis advisor. Dr. Baragar arranged a meeting whenever I needed his help to develop the paper. With his brilliant insight for the topic. Dr. Baragar has shifted the highly intricate material down to the reasonably challenging topic for my master thesis. In addition. Dr. Baragar helped me implement the computer program we shall see in Chapter 4. Without his boundless help. this paper could not be completed.

I would also like to thank Dr. Michelle Schultz and Dr. Corran Webster in the department of mathematical sciences for being the thesis committee members. and Dr. Wolfgang Bein in the department of computer science for appreciation of being the graduate faculty representative.

CHAPTER 1

THE MARKOFF EQUATION

The Markoff equation is the equation: $x^2 + y^2 + z^2 = 3xyz$. This equation was first studied by Markoff[9] because of its enriched application to Diophantine approximation. We, hereby, begin discussion with Diophantine approximation.

One may consider how closely an arbitrary irrational number, say θ, may be approximated by rational fractions $\frac{p}{q}$ where $p, q \in \mathbb{Z}$. According to Hurwitz[8], if $\theta \in \mathbb{R} \setminus \mathbb{Q}$, there exist infinitely many fractions $\frac{p}{q}$ with $p, q \in \mathbb{Z}$ such that

$$\left| \theta - \frac{p}{q} \right| < \left(\frac{1}{\sqrt{5}} \right) \frac{1}{q^2}.$$

and that the constant $\frac{1}{\sqrt{5}}$ is best possible for an arbitrary θ. Markoff[9], furthermore, has indicated that we may improve the value of the constant if θ has some given conditions. Markoff investigated the behavior of $\varpi(\theta)$ where

$$\varpi(\theta) = \liminf_{p,q \in \mathbb{Z}} q \mid q\theta - p \mid .$$

By the result of Hurwitz[8], it is known that $0 < \varpi(\theta) \leq \frac{1}{\sqrt{5}}$ if $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Moreover, Markoff[9] has shown that we may improve the bounds of $\varpi(\theta)$ if the constant θ is not rationally equivalent to $\frac{\sqrt{5}-1}{2}$ or 1. Two numbers θ and α are rationally equivalent if there exist $p, q, r, s \in \mathbb{Z}$ such that $\alpha = \frac{p\theta+q}{r\theta+s}$ where $ps - qr \neq 0$. According to Markoff[9], if θ

is rationally equivalent to $\frac{\sqrt{5}-1}{2}$, which is a root of $\theta^2+\theta-1=0$, then $\varpi(\theta)=\frac{1}{\sqrt{5}}$. If, however, θ is not rationally equivalent to $\frac{\sqrt{5}-1}{2}$ or 1, then we may improve the value of the constant to $\varpi(\theta)\le\frac{1}{2\sqrt{2}}$. Similarly, if θ is rationally equivalent to $\sqrt{2}-1$, which is a root of $\theta^2+2\theta-1=0$, then $\varpi(\theta)=\frac{1}{2\sqrt{2}}$, and if θ is not rationally equivalent to $\sqrt{2}-1$, $\frac{\sqrt{5}-1}{2}$, or 1, we may further improve the constant. In fact, there exist sequences $\theta_1,\theta_2,\cdots$, and $c_1,c_2,\cdots$, such that if θ is rationally equivalent to θ_i, then $\varpi(\theta)=c_i$, and if θ is not rationally equivalent to $\theta_1,\theta_2,\cdots,\theta_{i-1}$, then $\varpi(\theta)\le c_i$. Markoff[9] has further shown that there is an injective correspondence between the ordered pairs (θ_i,c_i) and integer solutions to the Markoff equation, and that

$$\lim_{i\to\infty} c_i=\frac{1}{3}.$$

The sequence $\{c_i\}$ forms a portion of the so called Markoff spectrum, the portion larger than $\frac{1}{3}$. For information on the full Markoff spectrum, we refer the reader to Cusick and Flahive[6].

The Markoff equation is quadratic in x. Given $x^2+y^2+z^2=3xyz$, we may emphasize its quadratic nature as $X^2-3yzX+(y^2+z^2)=0$, and this, indubitably, has two solutions, say X and X'. Knowing $X+X'=3yz$, a solution (x,y,z) to the Markoff equation produces a new solution $(3yz-x,y,z)$. Similarly, we obtain two more maps:

$$\begin{aligned}
\varphi_1: &\quad (x,y,z) \longrightarrow (3yz-x,y,z),\\
\varphi_2: &\quad (x,y,z) \longrightarrow (x,3xz-y,z),\\
\varphi_3: &\quad (x,y,z) \longrightarrow (x,y,3xy-z).
\end{aligned}$$

Using these three maps, and the obvious solution $(1,1,1)$, we obtain the tree of solutions

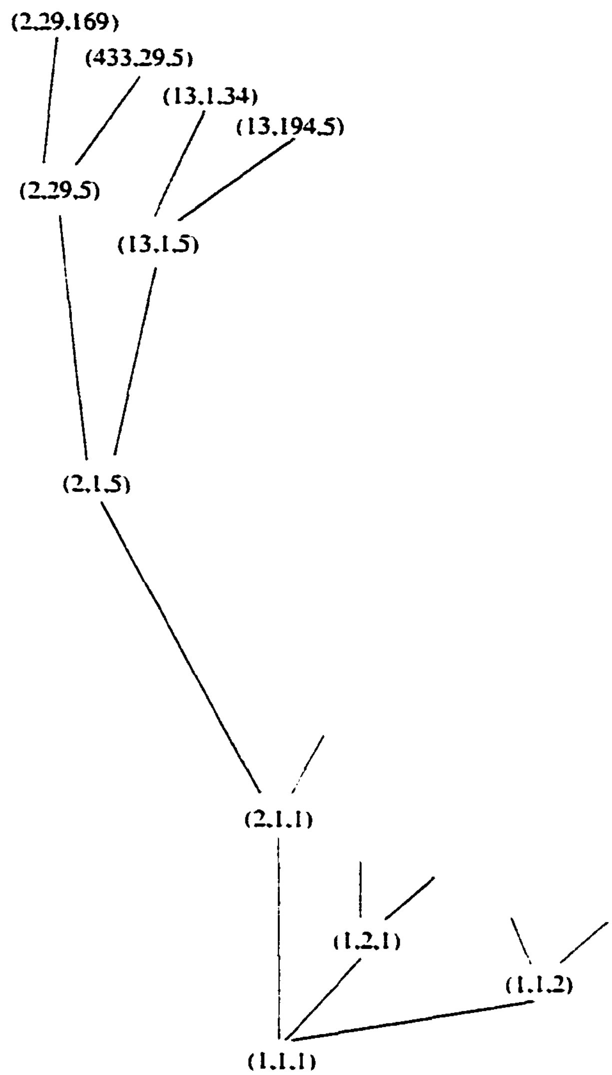

Figure 1.1: The Markoff Tree

shown in Figure 1.1. Now, we show that those solutions in the tree are all positive integer solutions to the Markoff equation. As the tree in Figure 1.1 suggests, by the map φ_1, φ_2, and φ_3, a triple $(x, y, z) \neq (1, 1, 1)$ generates two larger solutions and one smaller solution. We suppose $x \leq y \leq z$ and think of the Markoff equation as a quadratic in z. We let

$f(Z) = x^2 + y^2 + Z^2 - 3xyZ$. Then.

$$\begin{aligned} f(y) &= 2y^2 + x^2 - 3xy^2 \\ &\leq 3y^2 - 3xy^2 \\ &\leq 3y^2(1 - x) \\ &\leq 0. \end{aligned}$$

This inequality is strict unless $x = y = 1$. Thus if $(x, y) \neq (1, 1)$, then $f(y) < 0$. Let $z' = 3xy - z$ be the other root of $f(Z) = 0$. Then, since $f(Z)$ is an ends up parabola that opens up, we know $z' < y < z$. That is, $\varphi_3(x, y, z)$ is smaller solution than (x, y, z). Thus, given an arbitrary positive integer solution (x, y, z), we may always find a smaller solution except when $x = y = 1$. However, in this case, $z = 1$ or 2, and since both these solutions are in the tree generated by $(1, 1, 1)$, we know that all positive integer solutions are in this tree. For this reason, we call the triple $(1, 1, 1)$ the fundamental triple of the Markoff equation. This argument of decent was done by Markoff[9], and is summarized in Cassels[5], an excellent reference.

Regarding the importance of the Markoff equation, there have been two obvious questions. The first question is now known as the unicity conjecture for Markoff numbers and was proposed by Fröbenius[7] in 1907. The unicity conjecture states that, for each positive integer m, there exists at most one pair of integers (x, y) with $0 \leq x \leq y \leq m$ such that (x, y, m) is a solution to the Markoff equation. Note that solutions to the Markoff equation are symmetric in x, y, and z, so ordering the triple makes sense. Although computations indicate that this conjecture is almost certainly true, there is no known proof. Nonetheless, Baragar[2] has given a partial solution to this conjecture: there exists at most one pair

(x, y) such that $x \leq y \leq m$, and if either m, $3m-2$, or $3m+2$ is prime, twice a prime, or four times a prime. Button[4] has improved the result and has shown that m is unique if $m = kp^r$ where p is prime and $k^4 < m$.

The second question is to describe the growth of the number of Markoff numbers below an arbitrary given bound. For this question, Zagier[13] has found the asymptotic behavior. Let $\mathsf{M}(T)$ be the number of Markoff numbers below a given bound T. He has shown that

$$\mathsf{M}(T) = C(\log T)^2 + O(\log T(\log\log T)^2) \tag{1.1}$$

where C is an absolute constant and $C \approx 0.180717104711$. We remark that there is a typo in Zagier's paper regarding the value of C: the 7th decimal digit of C was omitted.

Mathematicians have studied a number of generalizations of the Markoff equation. For instance, Hurwitz[8] generalized the Markoff equation as follows:

$$x_0^2 + x_1^2 + \cdots + x_n^2 = ax_0x_1\cdots x_n \quad \text{where} \quad a \in \mathbb{Z}^+.$$

Hurwitz[8] has investigated what conditions are necessary for the equation to produce infinitely many solutions. Let $N(T)$ be the number of positive integer solutions to the Markoff-Hurwitz equation with the largest component less than T. Baragar[1] has shown that

$$\lim_{T\to\infty} \frac{\log N(T)}{\log\log T} = \alpha(n)$$

exists and $\alpha(n)$ depends only upon n. For $n = 3$, which is the Markoff equation, $\alpha(3) = 2$, but thereafter, $\alpha(n)$ is rarely an integer.

Baragar[3], has also investigated the asymptotic behavior of

$$x^2 + y^2 + z^2 = axyz + b \quad \text{where} \quad a, b \in \mathbb{Z},\ a \geq 1$$

which were first studied by Mordell[10, 11]. Baragar[3] has shown that the number of solutions $\gamma_{a,b}(T)$ below a given bound T is either

1) empty;

2) not empty but finite, and this case occurs only if b is a perfect square;

3) infinite and either

   a) $\gamma_{a,b}(T) = 12T + O\left(\sqrt{T}(\log T)^2\right)$ if $(a, b) = (1, 4)$ or $(2, 1)$;

   b) $\gamma_{a,b}(T) = 24T + O\left(\sqrt{T}(\log T)^2\right)$ if $(a, b) = (1, s^2 + 4)$ or $(2, s^2 + 1)$ with $s \in \mathbb{Z}^+$;

   c) or $\gamma_{a,b}(T) = C(\log T)^2 + O\left(\log T(\log\log T)\right)$ where C is computable.

In this paper, we investigate the asymptotic behavior regarding another sort of a generalized Markoff equation, which we call the Rosenberger equations.

Rosenberger Equations

Definition 1. The Rosenberger equations are equations of the form: $ax^2 + by^2 + cz^2 = dxyz$, where the sets of coefficients (a, b, c, d) are all positive integers such that each of a, b, and c divide d, and the equations themselves have infinitely many integer solutions (x, y, z).

According to Rosenberger[12], there are only finitely many such sets of coefficients; in

fact. there are only six. The following equations are the Rosenberger equations and their fundamental triples.

$$\begin{aligned}
&\mathsf{R}_1: && x^2 + y^2 + z^2 = xyz, && (3,3,3).\\
&\mathsf{R}_2: && x^2 + y^2 + z^2 = 3xyz, && (1,1,1).\\
&\mathsf{R}_3: && x^2 + y^2 + 2z^2 = 4xyz, && (1,1,1).\\
&\mathsf{R}_4: && x^2 + 2y^2 + 3z^2 = 6xyz, && (1,1,1).\\
&\mathsf{R}_5: && x^2 + y^2 + 2z^2 = 2xyz, && (2,2,2).\\
&\mathsf{R}_6: && x^2 + y^2 + 5z^2 = 5xyz, && (1,2,1) \quad \text{and} \quad (2,1,1).
\end{aligned}$$

Rosenberger[12] has indicated the maps to derive all solutions to each equation as the generalization of those maps found for solutions to the Markoff equation:

$$\begin{aligned}
\phi_1: &\quad (x,y,z) \to \left(\left(\frac{d}{a}\right) yz - x, y, z\right),\\
\phi_2: &\quad (x,y,z) \to \left(x, \left(\frac{d}{b}\right) xz - y, z\right),\\
\phi_3: &\quad (x,y,z) \to \left(x, y, \left(\frac{d}{c}\right) xy - z\right).
\end{aligned}$$

As a result. we have trees of solutions to Rosenberger equations. Note that the fundamental triples for equations R_1 and R_2 generate three triples as a result of the maps defined by Rosenberger[12] while the fundamental triples for the other equations generate only two triples. These trees are shown in Figure 1.2. 1.3. 1.4. 1.5. 1.6. 1.7. and 1.8. Because of the symmetry in some of these equations. portions of their trees of solutions are symmetric. and we have dropped these portions.

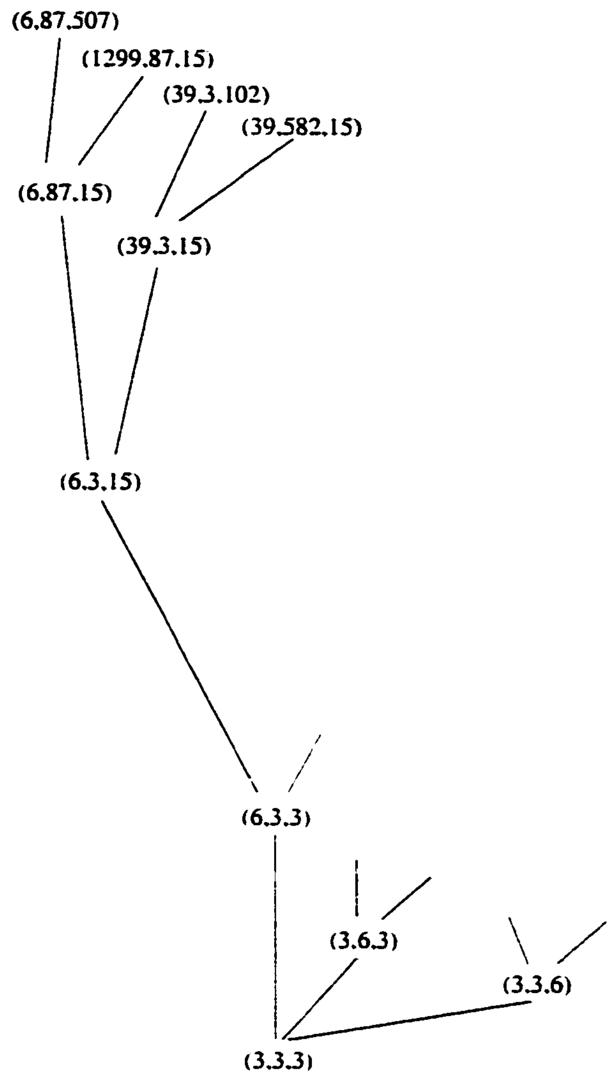

Figure 1.2: The Tree of Solutions for Eq. R_1 with the Fundamental Triple $(3,3,3)$

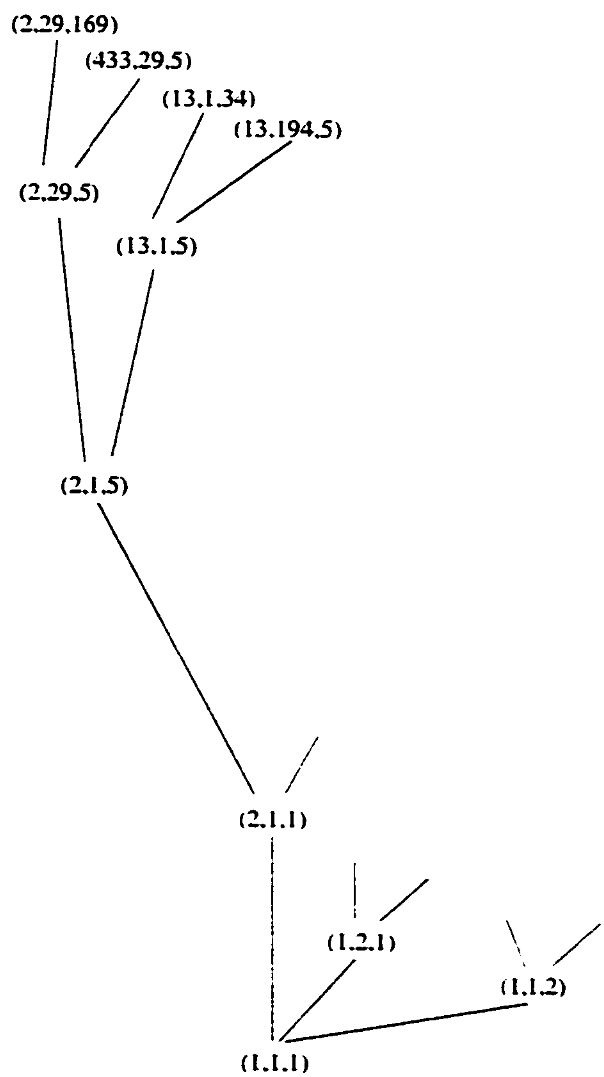

Figure 1.3: The Tree of Solutions for Eq. R_2 with the Fundamental Triple (1, 1, 1)

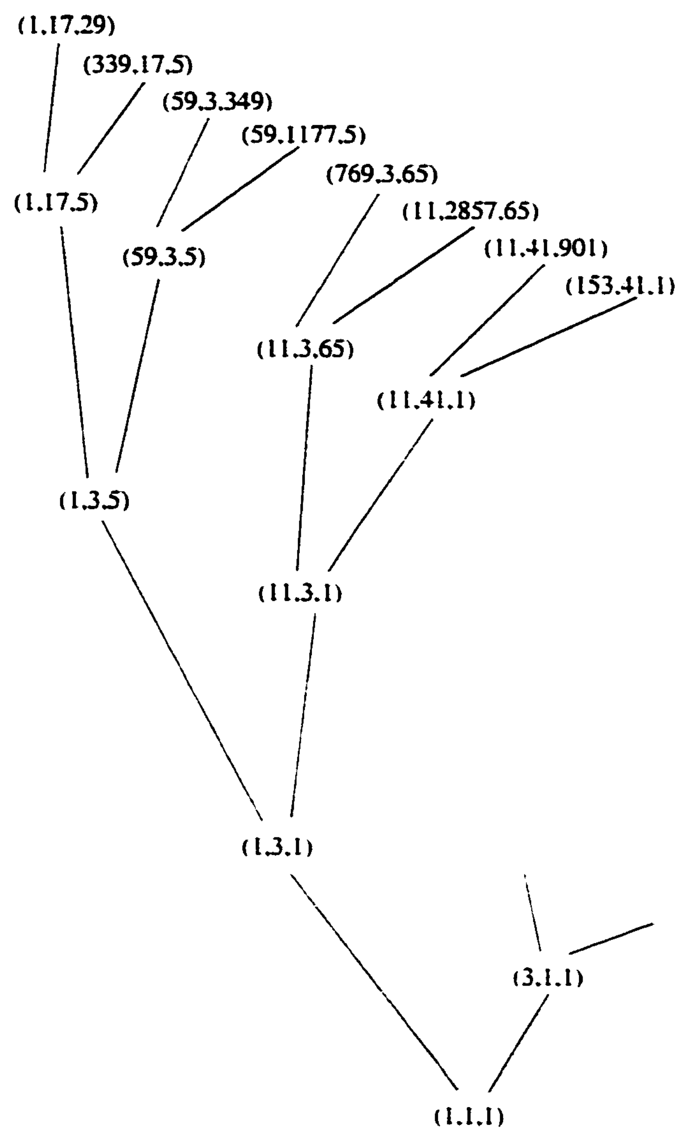

Figure 1.4: The Tree of Solutions for Eq. R_3 with the Fundamental Triple (1. 1. 1)

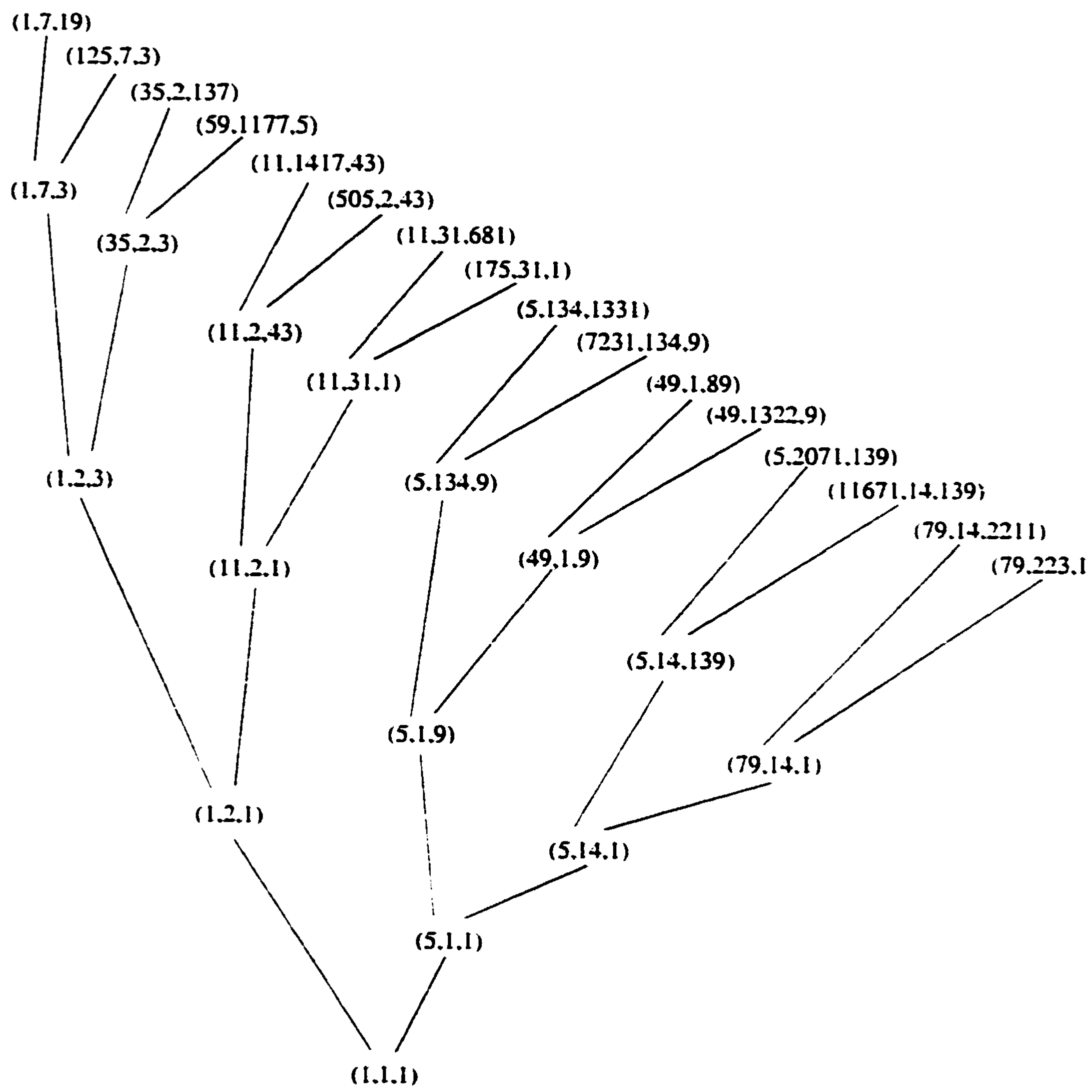

Figure 1.5: The Tree of Solutions for Eq. R_4 with the Fundamental Triple $(1.1.1)$

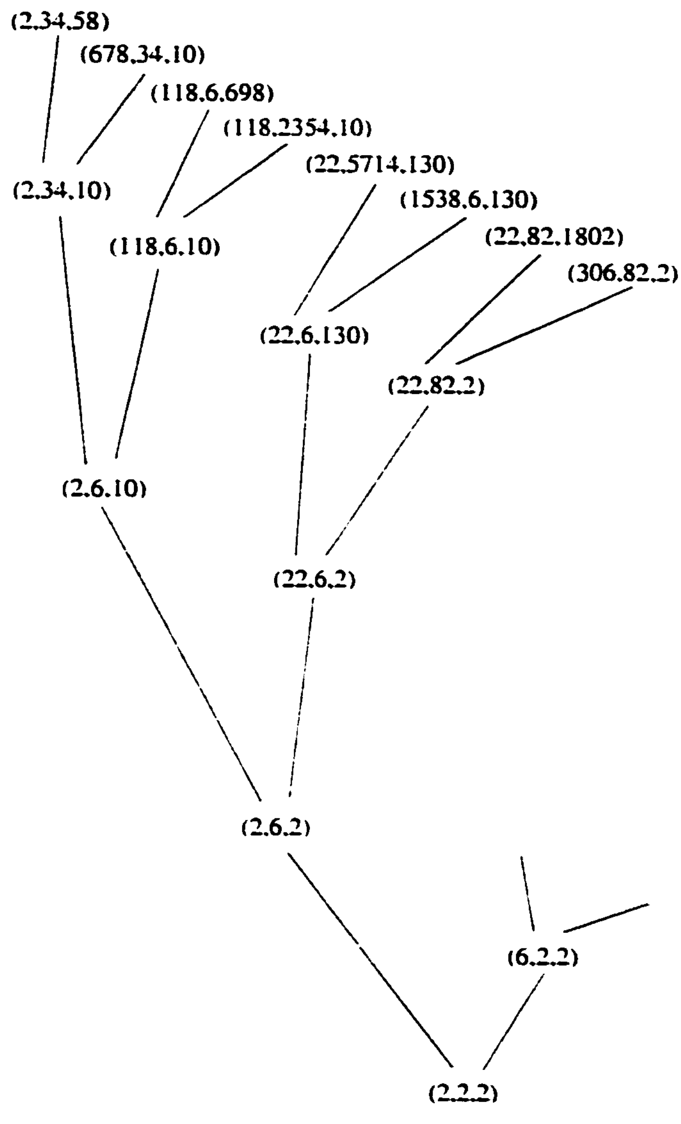

Figure 1.6: The Tree of Solutions for Eq. R_5 with the Fundamental Triple (2, 2, 2)

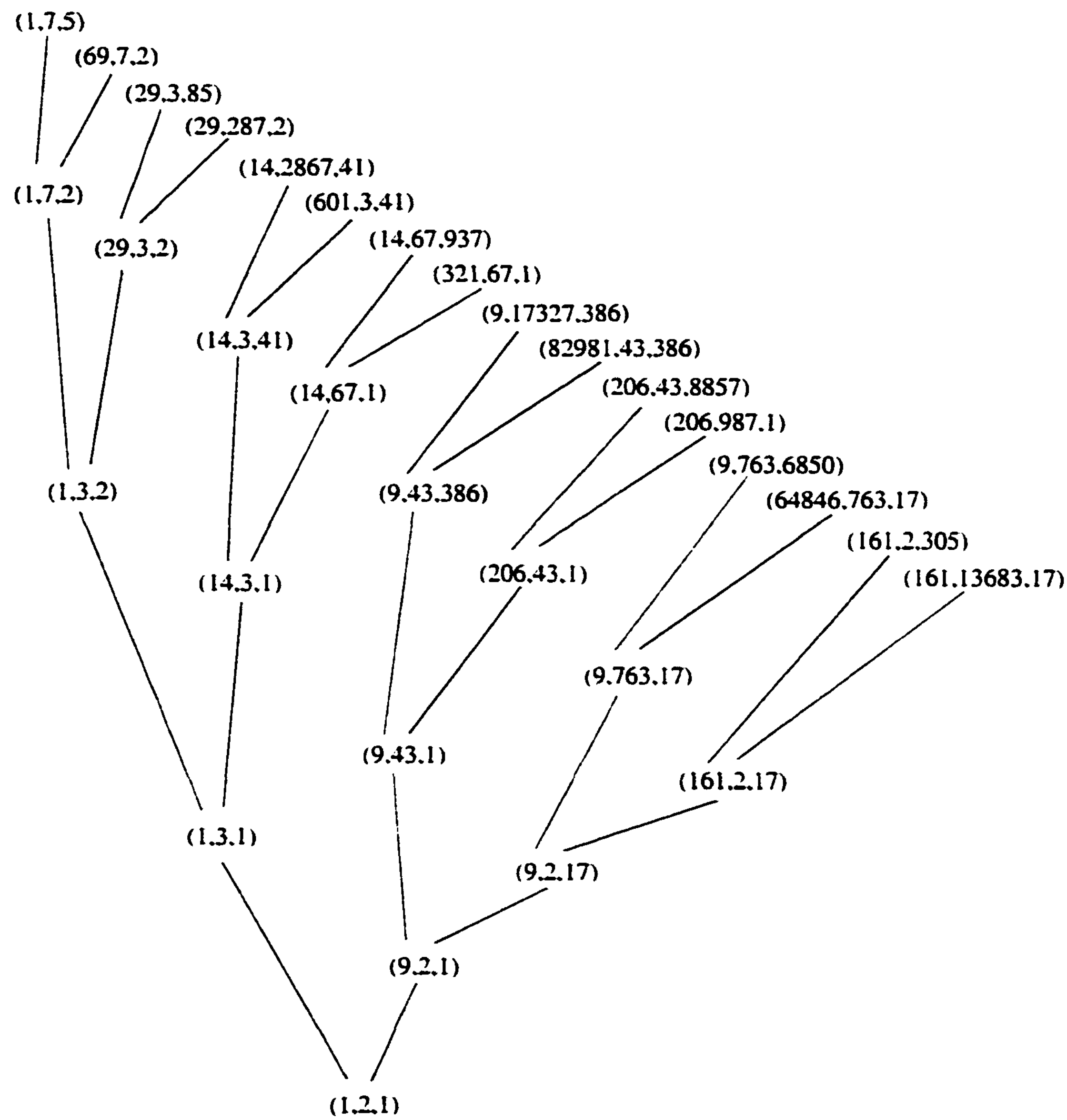

Figure 1.7: The Tree of Solutions for Eq. R_6 with the Fundamental Triple (1, 2, 1)

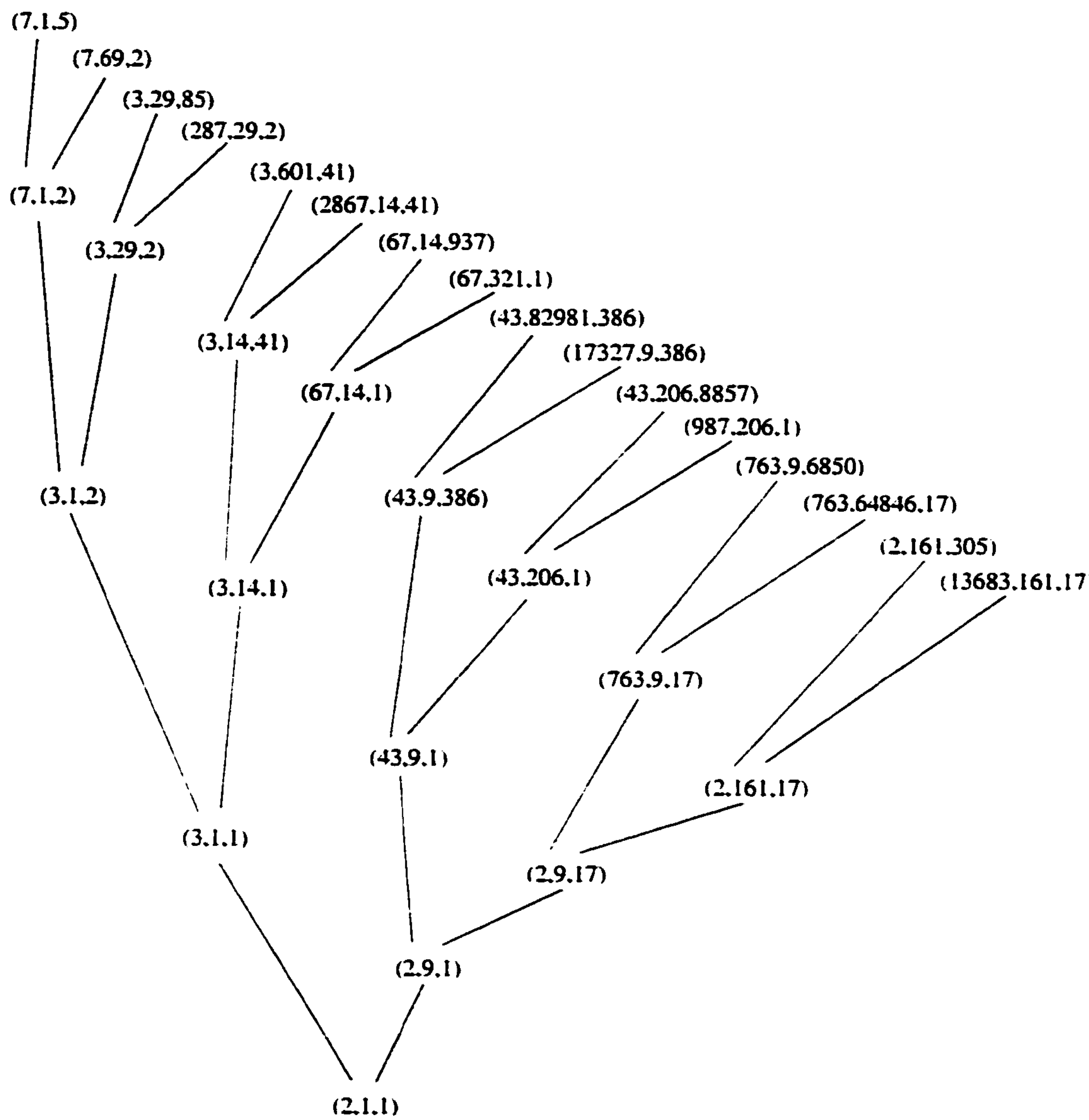

Figure 1.8: The Tree of Solutions for Eq. R_6 with the Fundamental Triple (2, 1, 1)

Our Main Result

Definition 2. Let R_n be the nth Rosenberger equation. and let $\mathsf{R}_n(\mathbb{Z}^+)$ be the set of positive integer solutions to equation R_n. For equation R_n. we define the number of solutions below a given bound to be:

$$\mathcal{N}_n(T) = \left|\{(x, y, z) \in \mathsf{R}_n(\mathbb{Z}^+) : \max\{x, y, z\} \leq T\}\right|.$$

The main result of this thesis is the following:

Theorem 1. *The number $\mathcal{N}_n(T)$ of solutions to a Rosenberger equation and below a given bound T has the asymptotic formula*

$$\mathcal{N}_n(T) = C_n(\log T)^2 + O\left(\log T(\log\log T)^2\right)$$

where C_n is the explicitly computable constant of the nth Rosenberger equation. The values of C_n are

$$C_1 = C_2 \approx 1.084302628266.$$

$$C_3 = C_5 \approx 0.5438094447296.$$

$$C_4 \approx 0.5542391311152.$$

$$C_6 \approx 1.1761039814344.$$

The observant reader might notice that C_2 is not equal to the constant C of equation 1.1 found by Zagier[13]. This is because we are counting all solutions. not just ordered solutions. Hence. $C_2 = 6C$.

Some Congruent Asymptotic Behaviors

Since our purpose in this paper is to prove Theorem 1, we *a priori* must investigate the asymptotic behavior for six different Rosenberger equations. However, Lemma 2 followed by Theorem 3 shows that there are some congruent asymptotic growths, so that we only need to consider four of them. Moreover, since Zagier[13] has investigated the asymptotic behavior for equation R_2, knowing the result, we only need to consider three cases.

Lemma 2. *A triple* (x, y, z) *is an integer solution to equation* R_2 *if and only if* $(3x, 3y, 3z)$ *is an integer solution to equation* R_1. *Likewise, a triple* (x, y, z) *is an integer solution to equation* R_3 *if and only if* $(2x, 2y, 2z)$ *is an integer solution to equation* R_5.

Proof. If the triple (x, y, z) is an integer solution to equation R_2, then $9x^2 + 9y^2 + 9z^2 = 27xyz$. Therefore,

$$(3x)^2 + (3y)^2 + (3z)^2 = (3x)(3y)(3z).$$

Thus, the triple $(3x, 3y, 3z)$ is an integer solution to equation R_1. Conversely, if a triple (x, y, z) is an integer solution to equation R_1, then $\frac{x^2}{9} + \frac{y^2}{9} + \frac{z^2}{9} = \frac{xyz}{9}$. Hence,

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{3}\right)^2 + \left(\frac{z}{3}\right)^2 = 3\left(\frac{x}{3}\right)\left(\frac{y}{3}\right)\left(\frac{z}{3}\right).$$

Therefore, the triple $\left(\frac{x}{3}, \frac{y}{3}, \frac{z}{3}\right)$ is a solution to equation R_2. Note that if (x, y, z) is an integer solution of equation R_1, then 3 divides each of x, y, and z, so $\left(\frac{x}{3}, \frac{y}{3}, \frac{z}{3}\right)$ is, in fact, an integer triple. To see this, note that the fundamental solution to equation R_1 is $(3, 3, 3)$; see Figure 1.2, and that the congruence to zero modulo 3 for all solutions to equation R_1, is preserved by the branch operations, so every integer solution to equation R_1 has components equivalent to 0 modulo 3.

Similarly, if a triple (x, y, x) is an integer solution to equation R_3, then $4x^2+4y^2+8z^2 = 16xyz$, thus

$$(2x)^2 + (2y)^2 + 2(2z)^2 = 2(2x)(2y)(2z).$$

As a result, a triple $(2x, 2y, 2z)$ is the solution to equation R_5. On the other hand, if a triple (x, y, z) is an integer solution to equation R_5, then $\frac{x^2}{4} + \frac{y^2}{4} + \frac{z^2}{2} = \frac{xyz}{2}$. Thus,

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 + 2\left(\frac{z}{2}\right)^2 = 4\left(\frac{x}{2}\right)\left(\frac{y}{2}\right)\left(\frac{z}{2}\right).$$

Therefore, $\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right)$ is a solution to R_3. Furthermore, $(2, 2, 2)$ is the fundamental triple to equation R_5. Since parity is preserved by the branch operations, all solutions to equation R_5 are even, so $\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right)$ is, in fact, an integer solution to R_3. $\square$

The growth rate of numbers of solutions to equations R_1 and R_2 below a given bound is logarithmic, thus Lemma 2 implies that numbers of solutions to equations R_1 and R_2 below a given bound are asymptotically equivalent, and the same may be said for equations R_3 and R_5. Theorem 3 demonstrates this.

Theorem 3. *Let $N_1(T)$ and $N_2(T)$ be numbers of solutions below a given bound for equations R_1 and R_2, respectively. Then, $N_1(T) = N_2(T) + O(\log T(\log\log T)^2)$ and $C_1 = C_2$.*

Proof. By Zagier[13], it is known that $N_2(T) = C_2(\log T)^2 + O(\log T(\log\log T)^2)$ where $C_2 \approx 1.084302628266$. Now, by Lemma 2, we know that

$$\begin{aligned} N_1(T) &= N_2\left(\frac{T}{3}\right) \\ &= C_2\left(\log\left(\frac{T}{3}\right)\right)^2 + O\left(\log\left(\frac{T}{3}\right)\left(\log\log\left(\frac{T}{3}\right)\right)^2\right) \\ &= C_2(\log T - \log 3)^2 + O\left((\log T - \log 3)\left(\log(\log T - \log 3)\right)^2\right) \end{aligned}$$

$$= C_2(\log T)^2 - 2C_2 \log T \log 3 + C_2 \log^2 3 + O\left(\log T\,(\log\log T)^2\right)$$

$$= C_2(\log T)^2 + O(\log T(\log\log T)^2)$$

$$= N_2(T) + O(\log T(\log\log T)^2).$$

Thus, $C_1 = C_2$. $\square$

Using a similar argument, we may conclude that $C_3 = C_5$, once we establish the first part of Theorem 1.

The Euclid Tree

We define the Euclid Tree to be the tree generated by the branching operations

$$\begin{aligned} \chi_1 &: \quad (s,t,u) \;\rightarrow\; (t+u,t,u), \\ \chi_2 &: \quad (s,t,u) \;\rightarrow\; (s,s+u,u), \\ \chi_3 &: \quad (s,t,u) \;\rightarrow\; (s,t,s+t), \end{aligned}$$

and rooted at $(1,1,2)$. Note that, if (s,t,u) is in the Euclid tree and $u = \max\{s,t,u\}$, then $u = s+t$ and $\chi_3(s,t,u) = (s,t,u)$, so we may ignore this branch. A similar remark may be made for $s = \max\{s,t,u\}$ and $t = \max\{s,t,u\}$. Thus, the Euclid tree is a 2-branch tree. We denote this tree with $\mathfrak{E}_{1,1}$ and depict it in Figure 1.9. If $(s,t,s+t)$ is in the Euclid tree, then $gcd(s,t) = 1$, and going down the tree is the Euclidean algorithm. This is why we call this tree the Euclid tree.

We also consider a similar tree rooted at $(\alpha,\beta,\alpha+\beta)$ where α and β are two arbitrary positive real numbers. We denote this tree with $\mathfrak{E}_{\alpha,\beta}$ and depict it in Figure 1.10. In our

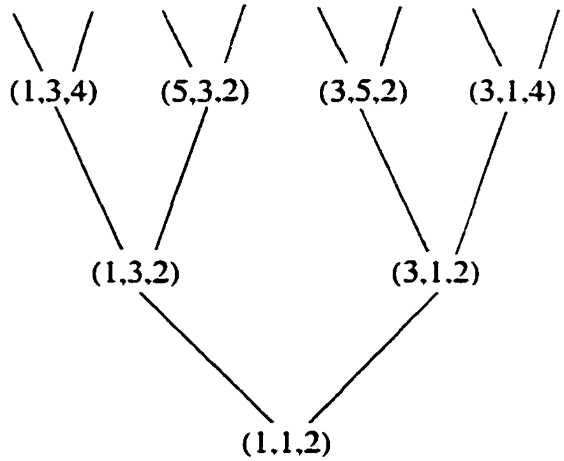

Figure 1.9: The Euclid Tree with the Root $(1, 1, 2)$

study of the Rosenberger equations, we will need to count the number of elements of the Euclid tree with the root $(\alpha, \beta, \alpha + \beta)$ and bounded by T. Thus, we first define

$$\mathsf{E}_{\alpha,\beta}(T) = \left|\{(s,t,u) \in \mathfrak{E}_{\alpha,\beta} : \max\{s,t,u\} \leq T\}\right|.$$

Fortunately, Zagier[13] has already estimated the value of this function.

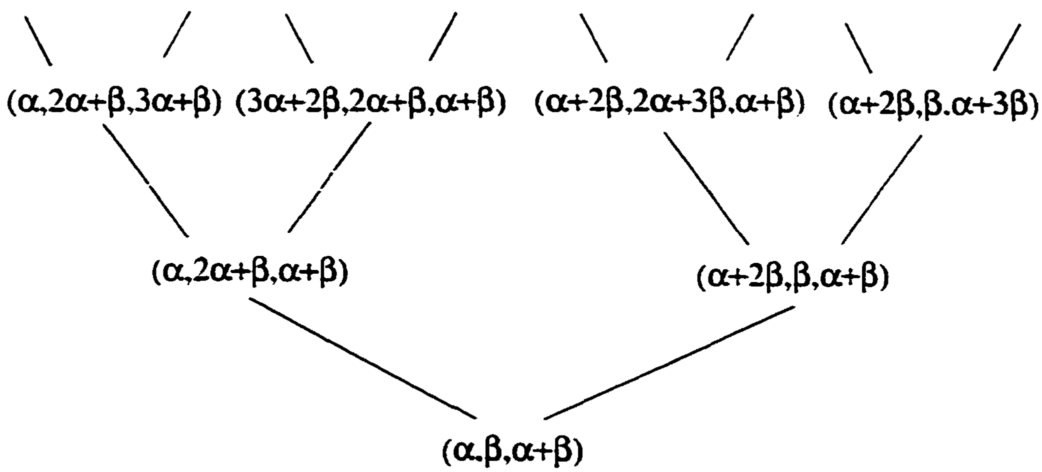

Figure 1.10: The Euclid Tree with the Root $(\alpha, \beta, \alpha + \beta)$

Theorem 4 (Zagier). *Let T, α, β be positive real numbers. The estimation for $\mathsf{E}_{\alpha,\beta}(T)$ with the root $(\alpha, \beta, \alpha + \beta)$ is given by:*

$$\mathsf{E}_{\alpha,\beta}(T) = \frac{3}{\pi^2}\frac{T^2}{\alpha\beta} + O\left(\frac{T}{\alpha}\right) + O\left(\frac{T}{\beta}\log\frac{T}{\beta}\right).$$

CHAPTER 2

PRELIMINARY INVESTIGATIONS

Before we derive the asymptotic formula in Chapter 3, we must find crude bounds on $\frac{N_n(T)}{\log^2(T)}$. That is because if we know $\frac{N_n(T)}{\log^2(T)}$ is bounded by some positive real numbers, then that helps us investigate the asymptotic behavior of $N_n(T)$. To find the bounds, we need fundamental knowledge regarding Rosenberger equations.

Lemma 5. *Suppose x, y, and z are large. Then,*

$$dxyz \leq ax^2 + o(x^2) \quad \textit{if} \quad x = \max\{x, y, z\},$$

$$dxyz \leq by^2 + o(y^2) \quad \textit{if} \quad y = \max\{x, y, z\},$$

$$dxyz \leq cz^2 + o(z^2) \quad \textit{if} \quad z = \max\{x, y, z\}.$$

Proof. Suppose x, y, and z are large, and suppose $x = \max\{x, y, z\}$. From $dxyz = ax^2 + by^2 + cz^2$, we obtain that

$$dxyz < (a + b + c)x^2.$$

So, $yz < x\left(\frac{a+b+c}{d}\right)$. Therefore, we may assume that $yz < xL$ where $L = \frac{a+b+c}{d}$. Furthermore, this result implies that, for y and z large, xL is much larger than yz. Thus,

$$by^2 + cz^2 < c\left(\left(\frac{xL}{z}\right)^2 + \left(\frac{xL}{y}\right)^2\right) = x^2\left(\frac{cL^2}{z^2} + \frac{cL^2}{y^2}\right) = o(x^2).$$

Therefore, $ax^2 + by^2 + cz^2 < ax^2 + o(x^2)$. Finally, this yields $dxyz < ax^2 + o(x^2)$. Similarly, if $y = \max\{x, y, z\}$, we are able to deduce $dxyz < by^2 + o(y^2)$. Likewise, $dxyz < cz^2 + o(z^2)$ given $z = \max\{x, y, z\}$. $\square$

Lemma 6. *Let $k_1 = \frac{d}{\sqrt{bc}}$, $k_2 = \frac{d}{\sqrt{ac}}$, and $k_3 = \frac{d}{\sqrt{ab}}$. For x, y, z large, we obtain that*

$$\log k_2 y + \log k_3 z = \log k_1 x + o(1) \quad \text{if} \quad x = \max\{x, y, z\}. \tag{2.1}$$

$$\log k_1 x + \log k_2 z = \log k_2 y + o(1) \quad \text{if} \quad y = \max\{x, y, z\}. \tag{2.2}$$

$$\log k_1 x + \log k_2 y = \log k_3 z + o(1) \quad \text{if} \quad z = \max\{x, y, z\}. \tag{2.3}$$

Proof. We first assume that $x = \max\{x, y, z\}$. For any arbitrary solution (x, y, z) to a Rosenberger equation, we have the trivial inequality $ax^2 < dxyz$. Thus, by using this result and the result in Lemma 5, we have $ax^2 < dxyz \le ax^2 + o(x^2)$. Hence,

$$x < \left(\frac{d}{a}\right) yz \le x(1 + o(1)).$$

$$\log k_1 x < \log k_1 \left(\frac{d}{a}\right) yz \le \log k_1 x + \log(1 + o(1)).$$

Note that $\left(\frac{d}{a}\right) k_1 = k_2 k_3$; the constants k_1, k_2, and k_3 have been chosen to satisfy this, as well as equations $\left(\frac{d}{b}\right) k_2 = k_1 k_3$ and $\left(\frac{d}{c}\right) k_3 = k_1 k_2$. Thus,

$$\log k_1 x < \log k_2 y + \log k_3 z \le \log k_1 x + \log(1 + o(1)).$$

Hence, it is now sufficient to show that $\log(1 + o(1)) = o(1)$. To do so, note that $\frac{1}{1-x} =$

$1 + x + x^2 + \ldots$ for $|x| < 1$. Integrating both sides, we obtain that

$$-\log(1 - x) = c_0 + x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots \quad |x| < 1$$

where c_0 is the constant of the integration. By letting $x = 0$, we find $c_0 = 0$. Therefore, this implies that

$$\log(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots \quad |x| < 1.$$

If we let $x = o(1)$, then we obtain $\log(1 + o(1)) = o(1)$. This shows that equation (2.1) holds. The other two equations (2.2) and (2.3) may be shown by similar arguments. $\square$

The results in Lemma 6 suggests that there exists a relationship between the Rosenberger trees and the Euclid tree.

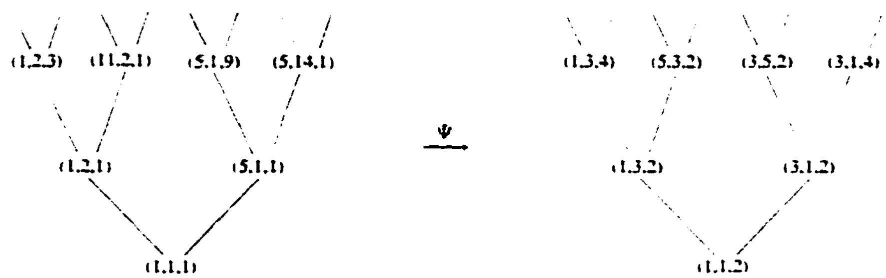

Figure 2.1: The Tree of Solutions for Equation R_4 and the Euclid Tree.

Definition 3. Let $\mathfrak{R}$ be a tree of solutions to a Rosenberger equation, and let $\mathfrak{E}_{1,1}$ be the Euclid tree. Define $\Psi : \mathfrak{R} \longrightarrow \mathfrak{E}_{1,1}$ to be the map that sends nodes in the Rosenberger tree to corresponding nodes in the Euclid tree.

For example, for $\mathrm{R}_n = \mathrm{R}_4$, as in Figure 2.1, we have $\Psi(5,1,1) = (3,1,2)$, $\Psi(11,2,1) = (5,3,2)$, and so on. Lemma 6 suggests that Ψ may be approximated by the map $(x, y, z) \longrightarrow$

$(\log k_1 x, \log k_2 y, \log k_3 z)$.

Finally, in order to investigate the lower bound and the upper bound of the ratio, it is crucial to distinguish the difference between the maximal value of a triple in a Rosenberger tree and the maximal value of the corresponding triple in the Euclid tree.

Definition 4. Let $\Psi(x, y, z) = (s, t, u)$. Define $\max^*\{x, y, z\}$ to be x if $\max\{s, t, u\} = s$, y if $\max\{s, t, u\} = t$, and z if $\max\{s, t, u\} = u$.

Note that $\max\{x, y, z\} = \max^*\{x, y, z\}$ in many cases, but there are exceptions. For example, for one of the fundamental triples of equation R_6, we have $\max\{1, 2, 1\} = 2$ but $\max^*\{1, 2, 1\} = 1$ since $\Psi(1, 2, 1) = (1, 1, 2)$.

Lemma 7. *If (x, y, z) is a positive integer solution to equation R_3 or R_4, then*

$$\max\{x, y, z\} = \max{}^*\{x, y, z\}.$$

If (x, y, z) is a positive integer solution to equation R_6 with $x \neq 1$ and $y \neq 1$, then

$$\max\{x, y, z\} = \max{}^*\{x, y, z\}.$$

Proof. We look at the case when (x, y, z) is a solution to equation R_6, and leave the other cases to the reader. Suppose $x \leq y \leq z$. Let

$$f(Z) = 5Z^2 - 5xyZ + (x^2 + y^2).$$

Then. $f(z) = 0$. Consider.

$$\begin{aligned} f(y) &= 5y^2 - 5xy^2 + x^2 + y^2 \\ &\leq (7 - 5x)y^2 \end{aligned}$$

with equality only if $x = y$. Thus. if $x \geq 2$. then $f(y) < 0$. so $z' = xy - z$ must be smaller than y. Hence. ϕ_3 gives decent. so $z = \max^*\{x. y. z\}$.

If $x \leq z \leq y$. then we let

$$f(Y) = Y^2 - 5xzY + \left(x^2 + 5z^2\right).$$

Then.

$$f(z) \leq (7 - 5x)z^2.$$

Therefore. again. if $x \geq 2$. then $f(z) < 0$. thus $y' = 5xz - y < z \leq y$. Hence. ϕ_2 gives descent. so $\max\{x. y. z\} = \max^*\{x. y. z\}$. The other four cases are similar to one of these two cases. $\square$

The Lower Bound

In this section. we find a lower bound for the ratio $\frac{N_n(T)}{\log^2 T}$.

Theorem 8.

$$\liminf_{T\to\infty} \frac{N_n(T)}{\log^2 T} \geq \frac{3}{\pi^2 \log^2 5} \approx 0.1173.$$

Proof. Let $(x_0. y_0. z_0)$ be a fundamental solution to a Rosenberger equation. We define λ so that $\lambda = \max\left\{\log(k_1 x_0). \log(k_2 y_0). \frac{\log(k_3 z_0)}{2}\right\}$. Note that $\lambda > 0$. Let $(x. y. z)$ be

an arbitrary solution in the tree generated by (x_0, y_0, z_0), and let $\Psi(x, y, z) = (s, t, u)$ so (s, t, u) is an element in the Euclid tree. We claim that

$$s \geq \frac{\log k_1 x}{\lambda}, \quad t \geq \frac{\log k_2 y}{\lambda}, \quad u \geq \frac{\log k_3 z}{\lambda}. \tag{2.4}$$

and prove this using mathematical induction. For the base case, we use the fundamental triple $(x, y, z) = (x_0, y_0, z_0)$. Since $\Psi(x_0, y_0, z_0) = (1, 1, 2)$, we have: $1 \geq \frac{\log(k_1 x_0)}{\lambda}$, $1 \geq \frac{\log(k_2 y_0)}{\lambda}$, and $2 \geq \frac{\log(k_3 z_0)}{\lambda}$. Thus, inequalities (2.4) hold for the base case. Now, we assume that inequalities (2.4) hold for an arbitrary triple (x', y', z').

Then, we must show that the two triples generated from the triple (x', y', z') also satisfy inequalities (2.4). We, therefore, have to consider the three cases: $x' = \max^*\{x', y', z'\}$, $y' = \max^*\{x', y', z'\}$, and $z' = \max^*\{x', y', z'\}$. We suppose that $x' = \max^*\{x', y', z'\}$. Let $\Psi(x', y', z') = (s', t', u')$ so that $s' = t' + u'$. We consider the two branches of (x', y', z'):

$$\Psi : \quad (x', y', z') < \begin{matrix} (x', y'', z') \\ (x', y', z'') \end{matrix} \quad \longrightarrow \quad (s', t', u') < \begin{matrix} (s', s' + u', u') \\ (s', t', s' + t') \end{matrix}.$$

Consider the triple (x', y'', z'). Since $b(y'')^2 < dx'y''z'$, it follows that $y'' < \left(\frac{d}{b}\right) x'z'$. Multiplying by $k_2 > 0$, we obtain:

$$k_2(y'') < k_2 \left(\frac{d}{b}\right) x'z' = \frac{d}{\sqrt{ac}} \left(\frac{d}{b}\right) x'z' = \left(\frac{d}{\sqrt{ab}}\right) \left(\frac{d}{\sqrt{bc}}\right) x'z' = k_1 k_3 x'z'.$$

Hence, it yields that

$$\frac{\log k_2 y''}{\lambda} < \frac{\log k_1 x'}{\lambda} + \frac{\log k_3 z'}{\lambda} = s' + u' = t''.$$

where, in the above, we have used our inductive hypothesis. The arguments for the other branch and the cases when y' or $z' = \max^*\{x', y', z'\}$ are similar. Consequently, if inequalities (2.4) are true for some triple (x', y', z') in the tree generated by (x_0, y_0, z_0), then it is true for its branches. Hence, by mathematical induction, it is true for all triples in the tree generated by (x_0, y_0, z_0).

We are now ready to estimate the lower bound of the ratio. First of all, for the case of the Markoff equation, Zagier[13] has proven the following:

$$\liminf_{T\to\infty} \frac{\mathcal{N}_2(T)}{\log^2 T} \geq \frac{3}{2\pi^2 \log^2 3} \approx 0.7555. \tag{2.5}$$

For the non-Markoff case, we first define

$$\mathcal{N}_n^*(T) = \left|\{(x, y, z) \in \mathsf{R}_n(\mathbb{Z}^+) : \max\{k_1 x, k_2 y, k_3 z\} \leq T\}\right|$$

where R_n is nth Rosenberger equation, and $\mathsf{R}_n(\mathbb{Z}^+)$ is the set of positive integer solutions to equation R_n. By using inequalities (2.4) and applying Theorem 4, we obtain

$$\mathcal{N}_n^*(T) \geq \mathsf{E}_{1,1}\left(\frac{\log T}{\lambda}\right). \tag{2.6}$$

To see this, let $\Psi(x, y, z) = (s, t, u)$, and $\max\{s, t, u\} < \frac{\log T}{\lambda}$. Then, it follows that

$$\frac{\log k_1 x}{\lambda} < s < \frac{\log T}{\lambda}.$$

Thus,

$$k_1 x < T.$$

Similarly, $k_2y < T$, $k_3z < T$, so for every element in the Euclid tree that is bounded by $\frac{\log T}{\lambda}$, there is an element of the Rosenberger tree that is bounded by T. Thus, the inequality (2.6) is true. Therefore,

$$
\begin{aligned}
\liminf_{T\to\infty} \frac{N_n^*(T)}{\log^2 T} &\geq \liminf_{T\to\infty} \frac{\mathsf{E}_{1,1}(\frac{\log T}{\lambda})}{\log^2 T} \\
&\geq \lim_{T\to\infty} \frac{\frac{3}{\pi^2}(\frac{\log T}{\lambda})^2 + O(\frac{\log T}{\lambda}) + O\left((\frac{\log T}{\lambda})\log(\frac{\log T}{\lambda})\right)}{\log^2 T} \\
&\geq \lim_{T\to\infty} \left(\frac{3}{\pi^2\lambda^2}\frac{\log^2 T}{\log^2 T} + O\left(\frac{1}{\lambda \log T}\right) + O\left(\frac{1}{\lambda \log T}\log\left(\frac{\log T}{\lambda}\right)\right)\right) \\
&\geq \frac{3}{\pi^2\lambda^2}. \qquad (2.7)
\end{aligned}
$$

Then, we choose (a, b, c, d) so that λ is maximal in inequality (2.7), and this occurs when $(a, b, c, d) = (1, 1, 5, 5)$. Thus, we have:

$$
\liminf_{T\to\infty} \frac{N_n^*(T)}{\log^2 T} \geq \frac{3}{\pi^2 \log^2 5} \approx 0.1173. \qquad (2.8)
$$

Since $a \leq b \leq c$, we have $k_1 \leq k_2 \leq k_3$. Thus, if $\max\{k_1x, k_2y, k_3z\} \leq T$, then we have $\max\{k_1x, k_1y, k_1z\} \leq T$, so $\max\{x, y, z\} \leq \frac{T}{k_1}$. Hence, $N_n\left(\frac{T}{k_1}\right) \geq N^*(T)$. Since $\log^2 T > 0$, this yields

$$
\frac{N_n(\frac{T}{k_1})}{\log^2 T} \geq \frac{N_n^*(T)}{\log^2 T}. \qquad (2.9)
$$

Let $u = \frac{T}{k_1}$. Then, consider:

$$
\begin{aligned}
\liminf_{T\to\infty} \frac{N_n(\frac{T}{k_1})}{\log^2 T} &= \liminf_{u\to\infty} \frac{N_n(u)}{\log^2 (k_1 u)} \\
&= \liminf_{u\to\infty} \frac{N_n(u)}{(\log u + \log k_1)^2}
\end{aligned}
$$

$$= \liminf_{u \to \infty} \frac{N_n(u)}{\log^2 u \left(1 + \frac{2 \log k_1}{\log u} + \frac{\log^2 k_1}{\log^2 u}\right)}$$

$$(2.10) \qquad = \liminf_{u \to \infty} \frac{N_n(u)}{\log^2 u}.$$

Hence, by comparing equations (2.5), (2.8), (2.9), and (2.10), we obtain the lower bound for the ratio. Thus,

$$\liminf_{T \to \infty} \frac{N_n(T)}{\log^2 T} = \liminf_{T \to \infty} \frac{N_n(\frac{T}{k_1})}{\log^2 T} \geq \liminf_{T \to \infty} \frac{N_n^*(T)}{\log^2 T} \geq 0.1173. \qquad \square$$

The Upper Bound

We, now, investigate the upper bound of the ratio. If we treat the investigation case by case, there are several cases. First of all, we have equations R_3, R_4, and R_6. Moreover, there would be three cases for each equation: when $x = \max^*\{x, y, z\}$, when $y = \max^*\{x, y, z\}$, and when $z = \max^*\{x, y, z\}$. Each case may be investigated by a particular method except this method, as we shall point out, is not quite applicable to equation R_6 when $y = \max^*\{x, y, z\}$ or $z = \max^*\{x, y, z\}$. Therefore, a modification of the method is required for these particular cases. Because of this reason, it is necessary to treat the investigation case by case.

Theorem 9.

$$\limsup_{T \to \infty} \frac{N_n(T)}{\log^2 T} \leq 14.7913.$$

Proof. For the first case, let us investigate the upper bound of the solutions to equation R_3. We first suppose that $x = \max^*\{x, y, z\} = \max\{x, y, z\}$. By the proof of Lemma 5, we know that $x - \left(\frac{1}{L}\right) yz > 0$ where $L = \frac{a+b+c}{d} = \frac{1+1+2}{4} = 1$. Thus, it yields that $x - yz > 0$.

Consider the product

$$(x - 3yz)(x - yz) = x^2 - 4xyz + 3y^2z^2$$
$$= x^2 - (x^2 + y^2 + 2z^2) + 3y^2z^2$$
$$= 3y^2z^2 - y^2 - 2z^2$$
$$= 2z^2(y^2 - 1) + y^2(z^2 - 1). \tag{2.11}$$

Now, by observing Figure 1.4, we may assume that $y \geq 1$ and $z \geq 1$. Hence, inequality (2.11) is nonnegative. Since $x - yz > 0$, it yields $x - 3yz \geq 0$. Thus,

$$x \geq 3yz.$$

By symmetry, then

$$y \geq 3xz.$$

If $z = \max^*\{x, y, z\} = \max\{x, y, z\}$, we again know that $z - xy > 0$. By symmetry, we may assume that $z > y > x$. Then, $x \geq 1$ and $y \geq 3$; see Figure 1.5. Thus,

$$\left(z - \left(\frac{5}{3}\right)xy\right)\left(z - \left(\frac{1}{3}\right)xy\right) = z^2 - 2xyz + \frac{5x^2y^2}{9}$$
$$= z^2 - \left(\frac{x^2}{2} + \frac{y^2}{2} + z^2\right) + \frac{5x^2y^2}{9}$$
$$= \frac{5x^2y^2}{9} - \frac{x^2}{2} - \frac{y^2}{2}$$
$$= \frac{y^2}{2}(x^2 - 1) + \frac{x^2}{18}(y^2 - 9)$$
$$\geq 0.$$

Since $z - xy > 0$ implies that $z - \left(\frac{1}{3}\right) xy > z - xy \geq 0$, we conclude that

$$z \geq \left(\frac{5}{3}\right) xy.$$

We use these to show

$$s \leq \frac{\log(\sqrt{5}x)}{\log\sqrt{3}}, \quad t \leq \frac{\log(\sqrt{5}y)}{\log\sqrt{3}}, \quad u \leq \frac{\log(3z)}{\log\sqrt{3}}, \tag{2.12}$$

where $\Psi(x, y, z) = (s, t, u)$. We prove this using mathematical induction. So, let us observe the injective relationship between the tree of solutions for equation R_3 and the tree of solutions for the Euclidean algorithm. For the base case, we use the fundamental

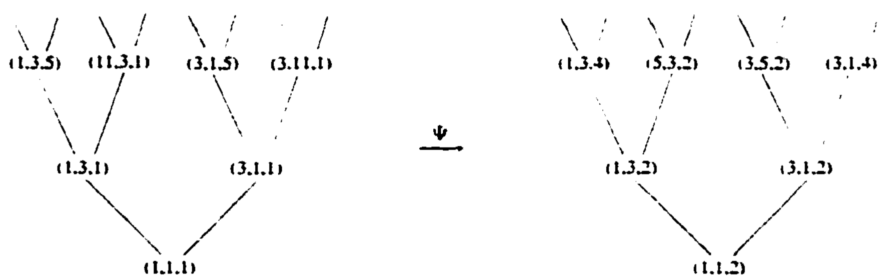

Figure 2.2: The Tree of Solutions for Equation R_3 and the Euclid Tree.

triple $(1, 1, 1)$. Since $\Psi(1, 1, 1) = (1, 1, 2)$, inequality (2.12) is clearly satisfied. Then, we assume that inequalities (2.12) are satisfied for an arbitrary solution (x', y', z'). Let $\Psi(x', y', z') = (s', t', u')$. We first assume that $x' = \max^*\{x', y', z'\}$, and we consider the

two branches of (x', y', z'):

$$\Psi: \quad (x', y', z') \begin{array}{l} \nearrow (x', y'', z') \\ \searrow (x', y', z'') \end{array} \quad — \quad (s', t', u') \begin{array}{l} \nearrow (s', s' + u', u') \\ \searrow (s', t', s' + t') \end{array}.$$

Then, consider the triple (x', y'', z'). By knowing $y'' \geq 3x'z'$, it yields that

$$t'' = s' + u' \leq \frac{\log(\sqrt{5}x')}{\log\sqrt{3}} + \frac{\log(3z')}{\log\sqrt{3}} = \frac{\log(3\sqrt{5}x'z')}{\log\sqrt{3}} \leq \frac{\log(\sqrt{5}y'')}{\log\sqrt{3}}.$$

where, in the above, we have used our inductive hypothesis. The arguments for the other branch and the cases when y' or $z' = \max^*\{x', y', z'\}$ are similar. Consequently, if inequalities (2.12) are true for some triple (x', y', z') for equation R_3, then it is true for its branches. Hence, by mathematical induction, it is true for all triples in the tree of Figure 1.4. Suppose that $\max\{x, y, z\} < T$. Then,

$$\max\{s, t, u\} \leq \frac{\log(3T)}{\log\sqrt{3}} = \frac{\log T}{\log\sqrt{3}} + 2 = \frac{2\log T}{\log 3} + 2. \tag{2.13}$$

Therefore,

$$\frac{N_3(T)}{\log^2 T} \leq \frac{E_{1,1}\left(\frac{2\log T}{\log 3} + 2\right)}{\log^2 T}.$$

It yields that

$$\begin{aligned} \limsup_{T\to\infty} \frac{N_3(T)}{\log^2 T} &\leq \limsup_{T\to\infty} \frac{E_{1,1}\left(\frac{2\log T}{\log 3} + 2\right)}{\log^2 T} \\ &\leq \lim_{T\to\infty} \frac{\frac{3\left(\frac{2\log T}{\log 3}+2\right)^2}{\pi^2}}{\log^2 T} + \lim_{T\to\infty} \frac{O\left(\frac{2\log T}{\log 3} + 2\right)}{\log^2 T} \\ &\quad + \lim_{T\to\infty} \frac{O\left(\left(\frac{2\log T}{\log 3} + 2\right)\log\left(\frac{2\log T}{\log 3} + 2\right)\right)}{\log^2 T} \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{12}{\pi^2 \log^2 3} + \lim_{T \to \infty} \frac{O\left(\left(\frac{2\log T}{\log 3}\right) \log\left(\frac{2\log T}{\log 3}\right)\right)}{\log^2 T} \\
&\leq \frac{12}{\pi^2 \log^2 3} \\
&\approx 1.0074.
\end{aligned}
$$

Similarly, we shall investigate the upper bound of the ratio for equation R_4. We first suppose that $x = \max^*\{x, y, z\} = \max\{x, y, z\}$. Then, we may assume that $y \geq 1$, $z \geq 1$, and $x - yz > 0$. Consider,

$$
\begin{aligned}
(x - 5yz)(x - yz) &= x^2 - 6xyz + 5y^2z^2 \\
&= x^2 - (x^2 + 2y^2 + 3z^2) + 5y^2z^2 \\
&= 5y^2z^2 - 2y^2 - 3z^2 \\
&= 3z^2(y^2 - 1) + 2y^2(z^2 - 1) \\
&\geq 0.
\end{aligned}
$$

Thus,

$$x \geq 5yz.$$

If $y = \max^*\{x, y, z\} = \max\{x, y, z\}$, we may assume that $x \geq 1$, $z \geq 1$, and $y - xz > 0$. Similarly, it implies that

$$
\begin{aligned}
(y - 2xz)(y - xz) &= y^2 - 3xyz + 2x^2z^2 \\
&= y^2 - \left(\frac{x^2}{2} + y^2 + \frac{3z^2}{2}\right) + 2x^2z^2 \\
&= 2x^2z^2 - \frac{x^2}{2} - \frac{3z^2}{2}
\end{aligned}
$$

$$= \frac{3z^2}{2}(x^2-1) + \frac{y^2}{2}(y^2-1)$$

$$\geq 0.$$

Thus, $y - xz > 0$ yields

$$y \geq 2xz.$$

If $z = \max^*\{x, y, z\} = \max\{x, y, z\}$, we also assume that $z - xy > 0$. In this case, there are two possibilities which are $z > y > x$ and $z > x > y$. First, we consider the case of $z > y > x$. From Figure 1.5, we know that $x \geq 1$ and $y \geq 2$, so

$$\begin{aligned}
\left(z - \left(\frac{3}{2}\right)xy\right)\left(z - \left(\frac{1}{2}\right)xy\right) &= z^2 - 2xyz + \frac{3x^2y^2}{4} \\
&= z^2 - \left(\frac{x^2}{3} + \frac{2y^2}{3} + z^2\right) + \frac{3x^2y^2}{4} \\
&= \frac{3x^2y^2}{4} - \frac{x^2}{3} - \frac{2y^2}{3} \\
&= \frac{2y^2}{3}(x^2-1) + \frac{x^2}{12}(y^2-4) \\
&\geq 0.
\end{aligned}$$

By the assumption, it implies that $z - \left(\frac{1}{2}\right)xy > z - xy > 0$, thus $z \geq \left(\frac{3}{2}\right)xy$. On the other hand, if $z > x > y$, then $x \geq 5$ and $y \geq 1$; see Figure 1.5. Thus,

$$\begin{aligned}
\left(z - \left(\frac{9}{5}\right)xy\right)\left(z - \left(\frac{1}{5}\right)xy\right) &= z^2 - 2xyz + \frac{9x^2y^2}{25} \\
&= z^2 - \left(\frac{x^2}{3} + \frac{2y^2}{3} + z^2\right) + \frac{9x^2y^2}{25} \\
&= \frac{9x^2y^2}{25} - \frac{x^2}{3} - \frac{2y^2}{3} \\
&= \frac{2y^2}{75}(x^2-25) + \frac{x^2}{3}(y^2-1)
\end{aligned}$$

$$\geq 0.$$

Whence. the assumption $z - \left(\frac{1}{5}\right) xy > z - xy > 0$ implies that $z \geq \left(\frac{9}{5}\right) xy$. Combining these two inequalities. we have

$$z \geq \left(\frac{3}{2}\right) xy.$$

Now. suppose that $\Psi(x, y, z) = (s, t, u)$. We claim that

$$s \leq \frac{\log\left(\sqrt{3}x\right)}{\log\sqrt{3}}, \quad t \leq \frac{\log\left(\sqrt{\frac{15}{2}y}\right)}{\log\sqrt{3}}, \quad u \leq \frac{\log\left(\sqrt{10}z\right)}{\log\sqrt{3}}. \tag{2.14}$$

To see this. we again observe the injective relationship between the tree of solutions for equation R_4 and the tree of elements in the Euclidean tree as in Figure 2.3. For the base

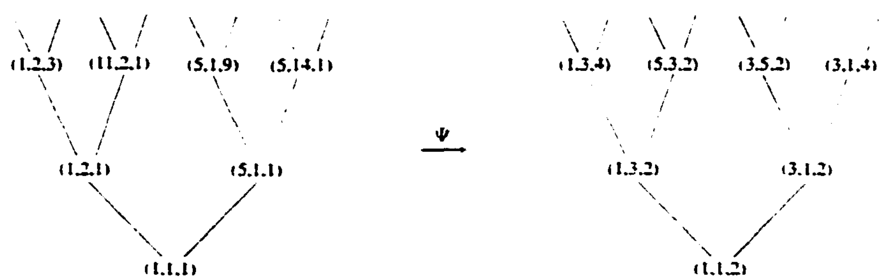

Figure 2.3: The Tree of Solutions for Equation R_4 and the Euclid Tree.

case. we use the fundamental triple $(1, 1, 1)$. Since $\Psi(1, 1, 1) = (1, 1, 2)$. inequality (2.14) is satisfied. Then. we assume that inequalities (2.14) is true for an arbitrary solution (x', y', z'). We assume that $x' = \max^{*}\{x', y', z'\}$. and we consider the two branches of

(x', y', z'):

$$\Psi: \quad (x', y', z') \begin{array}{l} \nearrow (x', y'', z') \\ \searrow (x', y', z'') \end{array} \quad \longrightarrow \quad (s', t', u') \begin{array}{l} \nearrow (s', s' + u', u') \\ \searrow (s', t', s' + t') \end{array}.$$

Then, consider the triple (x', y'', z'). By knowing $y'' \geq 2x'z'$, we conclude that

$$t'' = s' + u' \leq \frac{\log(\sqrt{3}x')}{\log\sqrt{3}} + \frac{\log(\sqrt{10}z')}{\log\sqrt{3}} = \frac{\log(\sqrt{30}x'z')}{\log\sqrt{3}} \leq \frac{\log(\sqrt{\frac{15}{2}}y'')}{\log\sqrt{3}}.$$

where, in the above, we have used our inductive hypothesis. The arguments for the other branch and the cases when y' or $z' = \max^*\{x', y', z'\}$ are similar. Consequently, if inequalities (2.14) are true for some triple (x', y', z') for equation R_4, then it is true for its branches. Hence, by mathematical induction, it is true for all triples in the tree of Figure 1.5. Therefore, we obtain that the upper bound of almost identical to the upper bound in (2.13) except the $+2$ is replaced as the $+1$. Nonetheless, the terms involved these constants converges to 0 as the denominator $\log^2 T$ grow sufficiently large. Hence, to obtain the upper bound for the ratio, the rest of the argument is identical as we have shown for equation R_3. Thus, we obtain that

$$\frac{N_4(T)}{\log^2 T} \leq \frac{12}{\pi^2 \log^2 3} \approx 1.0074.$$

We, at last, investigate the upper bound of the ratio for equation R_6 with $L = \frac{7}{5}$. Although there are two fundamental triples for equation R_6, we note that these two trees are symmetric to each other. Thus, without loss of generality, we consider the tree with the fundamental triple $(1, 2, 1)$. We first suppose that $x = \max^*\{x, y, z\} = \max\{x, y, z\}$.

so $x - \left(\frac{5}{7}\right) yz > 0$. By observing the tree in Figure 1.7. we may assume that $y \geq 2$ and $z \geq 1$. So.

$$\begin{aligned}
\left(x - \left(\frac{9}{2}\right) yz\right)\left(x - \left(\frac{1}{2}\right) yz\right) &= x^2 - 5xyz + \frac{9y^2z^2}{4} \\
&= x^2 - (x^2 + y^2 + 5z^2) + \frac{9y^2z^2}{4} \\
&= \frac{9y^2z^2}{4} - y^2 - 5z^2 \\
&= \frac{5z^2}{4}(y^2 - 4) + y^2(z^2 - 1) \\
&\geq 0.
\end{aligned}$$

Since $x - \left(\frac{5}{7}\right) yz > 0$. it yields $x - \left(\frac{1}{2}\right) yz > x - \left(\frac{5}{7}\right) yz > 0$. thus.

$$x \geq \left(\frac{9}{2}\right) yz.$$

We. now. suppose that $y = \max^*\{x, y, z\}$. This case is different from those we have already studied. and we deal with it in two cases: the case when $x = 1$ and when $x \neq 1$. If $x \neq 1$. then $y = \max\{x, y, z\}$ by Lemma 7. If $x = 1$. by observing Figure 1.7. we claim that

$$y \geq 3z. \tag{2.15}$$

To see this. consider the branch operations:

$$\cdots — (1, y, z) — (1, y, y - z) — (1, 4y - 5z, y - z) — \cdots.$$

This gives a sequence $(1, y_i, z_i)$ with $y_i = \max^*\{x, y, z\}$. $y_1 = 3$. $z_1 = 1$. $y_{i+1} = 5z_i - y_i$.

and $z_{i+1} = 4z_i - y_i$. Then, note that

$$5yz > y^2 + 5z^2$$

$$4yz - 5z^2 > y^2 - yz$$

Hence, it implies that

$$\frac{4y - 5z}{y - z} > \frac{y}{z}.$$

Therefore, when $x = 1$, we have $\frac{y_t}{z_t} > \frac{y_1}{z_1}$. Thus, inequality (2.15) is true. On the other hand, if $x \neq 1$ with $y = \max^*\{x, y, z\}$, by observing Figure 1.7, we note that $x \geq 9$ and $z \geq 1$. Therefore,

$$\begin{aligned}\left(y - \left(\frac{43}{9}\right) xz\right)\left(y - \left(\frac{2}{9}\right) xz\right) &= y^2 - 5xyz + \frac{86x^2z^2}{81} \\ &= y^2 - (x^2 - y^2 + 5z^2) + \frac{86x^2z^2}{81} \\ &= \frac{86x^2z^2}{81} - x^2 - 5z^2 \\ &= \frac{5z^2}{81}(x^2 - 81) + x^2(z^2 - 1) \\ &\geq 0.\end{aligned}$$

Since $y - \left(\frac{2}{9}\right) xz > y - \left(\frac{5}{7}\right) xz > 0$, we may conclude that $y \geq \left(\frac{43}{9}\right) xz$. Now, $y \geq \left(\frac{43}{9}\right) xz \geq 3xz$, so for $y = \max^*\{x, y, z\}$, we always have

$$y \geq 3xz.$$

Lastly, we consider the case: $z = \max^*\{x, y, z\}$. If $x \neq 1$, then $z = \max\{x, y, z\}$ by Lemma 7. So, $z > \left(\frac{5}{7}\right) xy$. If $x = 1$, then this inequality does not work, for instance, for

the fundamental triple $(1, 2, 1)$. In this case, we claim that

$$z \geq \left(\frac{1}{2}\right) y. \tag{2.16}$$

To see this, consider the branch operations:

$$\cdots \text{ — } (1, y, z) \text{ — } (1, 5z - y, z) \text{ — } (1, 5z - y, 4z - y) \text{ — } \cdots.$$

This gives a sequence $(1, y_j, z_j)$ with $z_j = \max^*\{1, y_j, z_j\}$, $y_1 = 2, z_1 = 1$, $y_{j+1} = 5z_j - y_j$, and $z_{j+1} = 4z_j - y_j$. Note that

$$\begin{aligned} 5yz &> 5z^2 - y^2 \\ 4yz - y^2 &> 5z^2 - yz \\ \frac{y}{z} &> \frac{5z - y}{4z - y}. \end{aligned}$$

Hence,

$$\frac{y_j}{z_j} > \frac{y_{j+1}}{z_{j+1}}$$

Thus, when $x = 1$, we have $\frac{y_1}{z_1} > \frac{y_j}{z_j}$ where $\frac{y_1}{z_1} = 2$. Therefore, we have inequality (2.16). Combining the inequality (2.16) with $z > \left(\frac{5}{7}\right) xy$ when $x \neq 1$, we always have, for $z = \max^*\{x, y, z\}$,

$$z \geq \left(\frac{1}{2}\right) xy.$$

Now, suppose $\Psi(x, y, z) = (s, t, u)$. We claim that

$$s \leq \frac{\log\left(\sqrt{\frac{3}{2}x}\right)}{\log\sqrt{\frac{3}{2}}}, \quad t \leq \frac{\log\left(\left(\frac{3}{2}\right)y\right)}{\log\sqrt{\frac{3}{2}}}, \quad u \leq \frac{\log\left(\sqrt{\frac{27}{2}z}\right)}{\log\sqrt{\frac{3}{2}}}. \tag{2.17}$$

We observe the injective relationship between the tree of solutions for equation R_6 with its fundamental solution $(1, 2, 1)$ and the tree of solutions in the Euclid tree as in Figure 2.4. For the base case, we use the fundamental triple $(1, 2, 1)$. Since $\Psi(1, 2, 1) = (1, 1, 2)$,

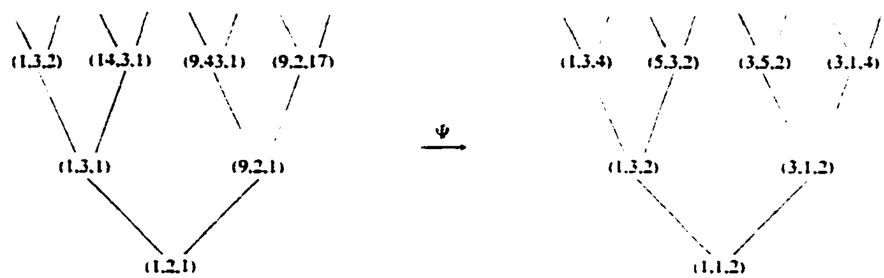

Figure 2.4: The Tree of Solutions for Equation R_6 and the Euclid Tree.

inequalities (2.17) is true. Then, we assume that inequalities (2.17) is true for an arbitrary solution (x', y', z'). Suppose first that $x' = \max^*\{x', y', z'\}$, and we consider the two branches of (x', y', z'):

$$\Psi: \quad (x', y', z') < \begin{matrix} (x', y'', z') \\ \\ (x', y', z'') \end{matrix} \quad \longrightarrow \quad (s', t', u') < \begin{matrix} (s', s' + u', u') \\ \\ (s', t', s' + t') \end{matrix}.$$

Consider the triple (x', y'', z'). Since $y'' > 3x'z'$, it implies that

$$t'' = s' + u'$$

$$
\begin{aligned}
&\le \frac{\log\left(\sqrt{\frac{3}{2}}x'\right)}{\log\sqrt{\frac{3}{2}}} + \frac{\log\left(\sqrt{\frac{27}{2}}z'\right)}{\log\sqrt{\frac{3}{2}}} \\
&\le \frac{\log\left(\sqrt{\frac{3}{2}}\sqrt{\frac{27}{2}}x'z'\right)}{\log\sqrt{\frac{3}{2}}} \\
&\le \frac{\log\left(\frac{1}{3}\left(\frac{9}{2}\right)y''\right)}{\log\sqrt{\frac{3}{2}}} \\
&\le \frac{\log\left(\left(\frac{3}{2}\right)y''\right)}{\log\sqrt{\frac{3}{2}}}.
\end{aligned}
$$

where. in the above. we have used our inductive hypothesis. The arguments for the other branch and the cases when y' or $z' = \max^*\{x'. y'. z'\}$ are similar. Consequently. if inequalities (2.17) are true for some triple $(x'. y'. z')$ for equation R_6. then it is true for its branches. Hence. by mathematical induction. it is true for all triples in the tree of Figure 1.7. Thus. let $\max\{x. y. z\} < T$. and $\Psi(x. y. z) = (s. t. u)$. then

$$
\max\{s. t. u\} \le \frac{\log\left(\sqrt{\frac{27}{2}}T\right)}{\log\sqrt{\frac{3}{2}}}.
$$

By applying Theorem 4. we obtain the following inequality:

$$
\frac{\mathcal{N}_6(T)}{\log^2 T} \le \frac{2\mathsf{E}_{1.1}\left(\frac{\log\left(\sqrt{\frac{27}{2}}T\right)}{\log\sqrt{\frac{3}{2}}}\right)}{\log^2 T}.
$$

Note that we multiply the numerator by 2 because for equation R_6 since there are two symmetric trees of solutions: see Figure 1.7 and 1.8. Then.

$$
\limsup_{T\to\infty}\frac{\mathcal{N}_6(T)}{\log^2 T} \le \limsup_{T\to\infty}\frac{2\left(\mathsf{E}_{1.1}\left(\frac{\log\left(\sqrt{\frac{27}{2}}T\right)}{\log\sqrt{\frac{3}{2}}}\right)\right)}{\log^2 T}
$$

$$
\begin{aligned}
&\le \lim_{T\to\infty} \frac{6\left(\frac{2\log T+\log\left(\frac{27}{2}\right)}{\log\left(\frac{3}{2}\right)}\right)^2}{\pi^2\log^2 T} \\
&\quad + \lim_{T\to\infty} O\left(\frac{2\left(\frac{2\log T}{\log\left(\frac{3}{2}\right)}\right)\log\left(\frac{\log T}{\log\left(\frac{3}{2}\right)}\right)}{\log^2 T}\right) \\
&\le \frac{24}{\pi^2\log^2\left(\frac{3}{2}\right)} \\
&\approx 14.7913.
\end{aligned}
$$

Hence. we have the conclusion:

$$\frac{N_3(T)}{\log^2 T} \le \frac{N_4(T)}{\log^2 T} < \frac{N_6(T)}{\log^2 T} < 14.7913.$$

Furthermore. according to Zagier[13]. we know that $\frac{N_2(T)}{\log^2 T} < 1.4082$. Thus. in general.

$$\frac{N_n(T)}{\log^2 T} < 14.7913. \qquad \square$$

The Crude Bounds

Corollary 10. *For any Rosenberger equation. the ratio* $\frac{N_n(T)}{\log^2 T}$ *is bounded as*

$$0.1173 < \frac{N_n(T)}{\log^2 T} < 14.7913.$$

Proof. The proof is trivial according to Theorem 8 and Theorem 9. $\square$

Therefore. the ratio $\frac{N_n(x)}{\log^2 T}$ is bounded. and it is bounded away from 0. In the next chapter. we will use these crude bounds to find the precise asymptotic formulae given in Theorem 1.

CHAPTER 3

SEARCHING FOR AN ASYMPTOTIC FORMULA

In the previous chapter, we approximated the function Ψ with the map $(x, y, z) \rightarrow (\log k_1 x, \log k_2 y, \log k_3 z)$. In this chapter, we use a refined approximation, the map

$$(x, y, z) \rightarrow (f_1(x), f_2(y), f_3(z))$$

such that

$$f_i(v(i)) = \log\left(\frac{k_i v(i) + \sqrt{k_i^2 (v(i))^2 - 4}}{2}\right) \tag{3.1}$$

where $i \in \{1, 2, 3\}$ and $v(i) = (x, y, z)$ with $v(i)$ indicating ith element of v. The function $f_i(v(i))$ arises for the following reason. Let (x_0, y, z) be any arbitrary solution to a Rosenberger equation where the value of x_0 is fixed. Then, the branch of the tree with x_0 fixed looks like

$$\cdots - \left(x_0, y, z\right) - \left(x_0, y, \frac{dx_0 y}{c} - z\right) \\ - \left(x_0, \frac{dx_0}{b}\left(\frac{dx_0 y}{c} - z\right) - y, \frac{dx_0 y}{c} - z\right) - \cdots.$$

Note that the action of the composition on y and z may be represented by the matrix operation:

$$\begin{bmatrix} \left(\frac{d^2}{bc}\right) x_0^2 - 1 & -\left(\frac{d}{b}\right) x_0 \\ \left(\frac{d}{c}\right) x_0 & -1 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} \left(\left(\frac{d^2}{bc}\right) x_0^2 - 1\right) y - \left(\frac{d}{b}\right) x_0 z \\ \left(\frac{d}{c}\right) x_0 y - z \end{bmatrix}.$$

Thus, in the long run, multiplication by this matrix begins to look like multiplication by its largest eigenvalue. We solve for the eigenvalues:

$$\det \begin{bmatrix} \left(\frac{d^2}{bc}\right) x_0{}^2 - 1 - \lambda_{x_0} & -\left(\frac{d}{b}\right) x_0 \\ \left(\frac{d}{c}\right) x_0 & -1 - \lambda_{x_0} \end{bmatrix} = \left(\frac{d^2 x_0^2}{bc} - 1 - \lambda_{x_0}\right)(-1 - \lambda_{x_0}) - \left(\frac{d^2 x_0^2}{bc}\right)$$

$$= \lambda_{x_0}^2 + \left(2 - \frac{d^2 x_0^2}{bc}\right)\lambda_{x_0} + 1.$$

We let this equal zero, and obtain λ_{x_0}:

$$\begin{aligned} \lambda_{x_0} &= \frac{\frac{d^2 x_0^2}{bc} - 2 \pm \sqrt{\left(2 - \frac{d^2 x_0^2}{bc}\right)^2 - 4}}{2} \\ &= \frac{\frac{d^2 x_0^2}{bc} - 2 \pm \sqrt{\left(\frac{d^2 x_0^2}{bc}\right)\left(\frac{d^2 x_0^2}{bc} - 4\right)}}{2} \\ &= \frac{k_1^2 x_0^2 - 2 \pm \sqrt{k_1^2 x_0^2 (k_1^2 x_0^2 - 4)}}{2} \\ &= \left(\frac{k_1 x_0 \pm \sqrt{k_1^2 x_0^2 - 4}}{2}\right)^2. \end{aligned}$$

We take the larger value for λ_{x_0}, and this explains our choice of $f_1(x) = \frac{1}{2}\log \lambda_x$. Similarly,

$$\lambda_{y_0} = \left(\frac{k_2 y + \sqrt{k_2^2 y^2 - 4}}{2}\right)^2 \quad \text{and} \quad \lambda_{z_0} = \left(\frac{k_3 z + \sqrt{k_3^2 z^2 - 4}}{2}\right)^2.$$

The map $(x, y, z) \to (f_1(x), f_2(y), f_3(z))$ gives a better approximation to Ψ than the map $(x, y, z) \to (\log k_1 x, \log k_2 y, \log k_3 z)$. To see this, let us suppose $\log k_3 z = \log k_1 x + \log k_2 y$. Then, $k_3 z = k_1 k_2 xy$, so $\frac{dz}{\sqrt{ab}} = \frac{d^2 xy}{\sqrt{ab}\sqrt{bc}}$. Therefore,

$$cz^2 = dxyz.$$

This is not a very good approximation to the equation $ax^2 + by^2 + cz^2 = dxyz$. On the other hand, suppose $f_3(z) = f_1(x) + f_3(z)$. Then, $\lambda_z = \lambda_x \lambda_y$. Note that $\lambda_x^{\frac{1}{2}} + \lambda_x^{-\frac{1}{2}} = k_1 x$, $\lambda_y^{\frac{1}{2}} + \lambda_y^{-\frac{1}{2}} = k_2 y$, and $\lambda_z^{\frac{1}{2}} + \lambda_z^{-\frac{1}{2}} = k_3 z$. Thus,

$$\begin{aligned}
ax^2 + by^2 + cz^2 - dxyz &= \frac{a}{k_1^2}\left(\lambda_x^{\frac{1}{2}} + \lambda_x^{-\frac{1}{2}}\right)^2 + \frac{b}{k_2^2}\left(\lambda_y^{\frac{1}{2}} + \lambda_y^{-\frac{1}{2}}\right)^2 + \frac{c}{k_3^2}\left(\lambda_z^{\frac{1}{2}} + \lambda_z^{-\frac{1}{2}}\right)^2 \\
&\quad - \frac{d}{k_1 k_2 k_3}\left(\lambda_x^{\frac{1}{2}} + \lambda_x^{-\frac{1}{2}}\right)\left(\lambda_y^{\frac{1}{2}} + \lambda_y^{-\frac{1}{2}}\right)\left(\lambda_z^{\frac{1}{2}} + \lambda_z^{-\frac{1}{2}}\right) \\
&= \frac{abc}{d^2}\left(\lambda_x + \lambda_x^{-1} + 2 + \lambda_y + \lambda_y^{-1} + 2 + \lambda_z + \lambda_z^{-1} + 2\right) \\
&\quad - \frac{abc}{d^2}\left(\lambda_x + \lambda_x^{-1} + \lambda_y + \lambda_y^{-1} + \lambda_z + \lambda_z^{-1} + 2\right) \\
&= \frac{4abc}{d^2}.
\end{aligned}$$

Furthermore, by symmetry, if $f_1(x) + f_3(z) = f_2(y)$ or $f_1(x) = f_2(y) + f_3(z)$, then we obtain the same equation. Thus, the map from a tree of solutions to the equation

$$ax^2 + by^2 + cz^2 - dxyz = \frac{4abc}{d^2} \tag{3.2}$$

to a Euclid tree is exactly $\Psi(x, y, z) = (f_1(x), f_2(y), f_3(z))$.

For further discussion, ordering a triple is necessary. Hence, we shall define the order of a triple. By observing Figure 2.2, 2.3, and 2.4, a function Ψ maps any given arbitrary

solution (x, y, z) of a Rosenberger equation to an arbitrary solution (s, t, u) of the Euclid algorithm. Then, we define $\varrho = \max^*\{x, y, z\}$, and

$$\begin{array}{llll} \text{if} \quad \varrho = x, & \text{then} \quad p = \min^*\{y, z\} & \text{and} & q = \max^*\{y, z\}, \\ \text{if} \quad \varrho = y, & \text{then} \quad p = \min^*\{x, z\} & \text{and} & q = \max^*\{x, z\}, \\ \text{if} \quad \varrho = z, & \text{then} \quad p = \min^*\{x, y\} & \text{and} & q = \max^*\{x, y\}. \end{array}$$

¿From now on, we modify the definition of $f_i(v(i))$ when it is necessary. We define a function f as we have defined in (3.1) but replace i as $i \in \{p, q, \varrho\}$ and $v(i)$ as $v(i) = i$, so that each of $f_i(v(i))$ and k_i picks each appropriate element of each set $\{f_1(x), f_2(y), f_3(z)\}$ and $\{k_1, k_2, k_3\}$ respectively. Note that $f_p(p) \approx \log k_p p$, $f_q(q) \approx \log k_q q$, and $f_\varrho(\varrho) \approx \log k_\varrho \varrho$.

Lemma 11. *Let (x, y, z) be a solution to a Rosenberger equation. Then, there exists a constant κ such that*

$$f_\varrho(\varrho) < f_p(p) + f_q(q) < f_\varrho(\varrho) + \frac{\kappa}{\varrho^2}.$$

Proof. We first suppose that $\varrho = x$. Let x_m be the larger real number such that $f_1(x_m) = f_2(y) + f_3(z)$. Then

$$ax_m^2 + by^2 + cz^2 - dx_m yz - \frac{4abc}{d^2} = 0. \tag{3.3}$$

Also, from equation (3.2), we obtain that

$$ax^2 + by^2 + cz^2 - dxyz = 0. \tag{3.4}$$

Subtracting equation (3.4) from equation (3.3), we obtain:

$$\frac{4abc}{d^2} = (x_m - x)(ax + ax_m - dyz). \tag{3.5}$$

Now, by solving equations (3.3) and (3.4) for x_m and x, we have:

$$x_m = \frac{dyz \pm \sqrt{d^2y^2z^2 - 4a\left(by^2 + cz^2 - \frac{4ac}{d^2}\right)}}{2a}$$

and

$$x = \frac{dyz \pm \sqrt{d^2y^2z^2 - 4a(by^2 + cz^2)}}{2a}.$$

Now, $x_m > x$, so $x_m - x > 0$. By the proof of Theorem 9, we know that $x \geq 3yz$ for the case of equation R_3 if $x = \max^*\{x, y, z\}$. Thus, $(a, b, c, d) = (1, 1, 2, 4)$ implies that

$$\frac{1}{2} = (x_m - x)(x + x_m - 4yz) > x(x_m - x)\left(2 - \frac{4}{3}\right).$$

Note that $\left(2 - \frac{4}{3}\right) = \frac{2}{3} > 0$. The other cases of equation R_3 and any cases of the other equation also give us the factors of the right hand side to be positive. Without loss of generality, we assume

$$\frac{1}{2} = (x_m - x)(x + x_m - 4yz) > \left(\frac{2}{3}\right) x(x_m - x).$$

By simplifying that, we obtain:

$$x < x_m < x + \frac{3}{x}. \tag{3.6}$$

We may write $x_m = f_1^{-1}(f_2(y) + f_3(z))$: this makes sense because the function $f_1(v(1))$ is a monotone increasing function, and $f_1(x_m) = f_2(y) + f_3(z)$. Then, we apply $f_i(v(i))$ to inequalities (3.6). Knowing $x_m > x$, this yields

$$f_1(x) < f_2(y) + f_3(z) < f_1\left(x + \frac{3}{x}\right).$$

Thus, it is now sufficient to show that $f_1\left(x + \frac{3}{x}\right) \leq f_1(x) + \frac{\kappa}{x^2}$. Let $L(x)$ be the tangent line of $f_1(x)$ at point x_0. Thus,

$$\begin{aligned} L(x) &= f_1(x_0) + f'(x_0)(x - x_0) \\ &= f_1(x_0) + \left(\frac{k_1^2 x_0 + k_1\sqrt{k_1 x_0^2 - 4}}{k_1^2 x_0^2 - 4 + k_1 x_0\sqrt{k_1^2 x_0^2 - 4}}\right)(x - x_0). \end{aligned}$$

Then, let $\epsilon > 0$, so that

$$L(x + \epsilon) = f_1(x_0) + \left(\frac{k_1^2 x_0 + k_1\sqrt{k_1 x_0^2 - 4}}{k_1^2 x_0^2 - 4 + k_1 x_0\sqrt{k_1^2 x_0^2 - 4}}\right)(x + \epsilon - x_0).$$

Therefore, $f(x + \epsilon) < L(x + \epsilon)$ since $f_1''(x + \epsilon) < 0$, and this completes the proof for $\varrho = x$. Similarly, this argument works for the other two cases. $\square$

Definition 5. Let $\mathfrak{R}'$ be a finite connected subset of a tree $\mathfrak{R}$ which contains the fundamental triple. Let $\partial\mathfrak{R}'$ be the set of solutions $(x', y', z') \in \mathfrak{R} \setminus \mathfrak{R}'$ such that $\mathfrak{R}' \cup \{(x', y', z')\}$ is connected.

For $\mathfrak{R}$, $\mathfrak{R}'$, and $\partial\mathfrak{R}'$, we rather suppress the index variable n of that indicates nth Rosenberger equation since we are now studying an arbitrary Rosenberger tree.

Lemma 12. *We let $|\partial\mathfrak{R}'|$ and $|\mathfrak{R}'|$ be the numbers of elements in $\partial\mathfrak{R}'$ and $\mathfrak{R}'$ respectively.*

Then, for equations R_3, R_4, *and* R_6,

$$|\partial\mathfrak{R}'| = |\mathfrak{R}'| + 1. \tag{3.7}$$

Proof. We prove this using mathematical induction on the number of elements in $\mathfrak{R}'$. For the base case, if $|\mathfrak{R}'| = 1$, then only the fundamental triple is contained in $\mathfrak{R}'$. Then, it is obvious, by looking at the trees, that we have that $|\partial\mathfrak{R}'| = 2$. We now assume that equation (3.7) holds for $\mathfrak{R}'$ such that $|\mathfrak{R}'| = k - 1$. Let $\mathfrak{R}''$ be a subtree with k elements. Let (x', y', z') be an element in $\mathfrak{R}''$ such that $\mathfrak{R}' = \mathfrak{R}'' \setminus \{(x', y', z')\}$ is still a subtree. Then, $\mathfrak{R}'$ has $k - 1$ elements, so

$$|\partial\mathfrak{R}'| = |\mathfrak{R}'| + 1.$$

by the inductive hypothesis. However, $\partial\mathfrak{R}''$ contains all elements of $\partial\mathfrak{R}'$ except for (x', y', z'), and also contains the two branches from (x', y', z'). Thus,

$$\begin{aligned} |\partial\mathfrak{R}''| &= |\partial\mathfrak{R}'| - 1 + 2 \\ &= |\partial\mathfrak{R}'| + 2 \\ &= |\partial\mathfrak{R}''| - 1. \end{aligned}$$

∎

Definition 6. Let $\vec{r} = (x', y', z')$, and let $\mathfrak{R}_{\vec{r}}$ be the infinite tree of arbitrary Rosenberger equation rooted at $\vec{r}$: that is, let $\mathfrak{R}_{\vec{r}}$ be the set of all triples in $\mathfrak{R}$ lying above the triple $\vec{r}$. Then, $\mathfrak{R}$ may be written as a disjoint union as

$$\mathfrak{R} = \mathfrak{R}' \cup \bigcup_{\vec{r} \in \partial\mathfrak{R}'} \mathfrak{R}_{\vec{r}}.$$

Definition 7. Let

$$\mathcal{N}_{\vec{r}}(T) = \left|\{(x,y,z) \in \mathfrak{R}_{\vec{r}} : \max\{x,y,z\} \le T\}\right|.$$

Lemma 13. *For T larger than the ϱ of any $(x,y,z) \in \mathfrak{R}'$, we have:*

$$\mathcal{N}_n(T) = \sum_{\vec{r}\in\partial\mathfrak{R}'} (\mathcal{N}_{\vec{r}}(T) + 1) - 1. \tag{3.8}$$

Proof. By using equation (3.7), for sufficiently large T, it yields

$$\begin{aligned}\mathcal{N}_n(T) &= |\mathfrak{R}'| + \sum_{\vec{r}\in\partial\mathfrak{R}'} \mathcal{N}_{\vec{r}}(T)\\ &= |\partial\mathfrak{R}'| - 1 + \sum_{\vec{r}\in\partial\mathfrak{R}'} (\mathcal{N}_{\vec{r}}(T) + 1) - |\partial\mathfrak{R}'|\\ &= \sum_{\vec{r}\in\partial\mathfrak{R}'} (\mathcal{N}_{\vec{r}}(T) + 1) - 1.\end{aligned}$$

□

Definition 8. For an arbitrary solution (x,y,z), let

$$\mathfrak{R}(\tau) = \{(x,y,z) \in \mathfrak{R} : \varrho \le \tau\}.$$

Lemma 14. *Let $\vec{r} = (x,y,z)$ be in $\mathfrak{R}$, and let κ be as in Lemma 11. We let $\alpha = f_1(p)$, $\beta = f_2(q)$, $\alpha' = f_1(p) - \frac{\kappa}{\varrho^2}$, and $\beta' = f_2(q) - \frac{c}{\varrho^2}$. Then,*

$$\mathsf{E}_{\alpha,\beta}(f_1(T)) \le \mathcal{N}_{\vec{r}}(T) \le \mathsf{E}_{\alpha',\beta'}(f_3(T)).$$

Proof. First, we assume that $\varrho = z$. We define a map $\Psi_{\vec{r}} : \mathfrak{R}_{\vec{r}} \to \mathfrak{E}_{\alpha,\beta}$ where $\mathfrak{E}_{\alpha,\beta}$ is the Euclid tree as in Figure 1.10. Let $(\hat{x},\hat{y},\hat{z}) \in \mathfrak{R}_{\vec{r}}$, and let $\Psi_{\vec{r}}(\hat{x},\hat{y},\hat{z}) = (\hat{s},\hat{t},\hat{u})$. Using an

induction argument similar to that used in the proof of Lemma 11, we show that

$$f_1(\hat{x}) < \hat{s}, \quad f_2(\hat{y}) < \hat{t}, \quad f_3(\hat{z}) < \hat{u}.$$

Thus, if $\max\{\hat{s}, \hat{t}, \hat{u}\} < f_1(T)$, then

$$f_1(\hat{x}) < \hat{s} < f_1(T),$$

$$f_2(\hat{y}) < \hat{t} < f_1(T) < f_2(T),$$

$$f_3(\hat{z}) < \hat{u} < f_1(T) < f_3(T),$$

thus,

$$\max\{\hat{x}, \hat{y}, \hat{z}\} < T.$$

It implies that

$$\mathsf{E}_{\alpha,\beta}(f_1(T)) \leq N_{\tilde{\gamma}}(T). \tag{3.9}$$

Furthermore, for the second inequality, we replace α, β by α', β'. By Lemma 11, it yields that

$$\max\{s, t, u\} = f_2(\hat{y}) - f_3(\hat{z}) - \frac{2\kappa}{\varrho^2} \leq f_1(\hat{x}) - \frac{\kappa}{\varrho^2} - \frac{2\kappa}{\varrho^2} \leq f_1(\hat{x}) - \frac{\kappa}{\varrho^2}.$$

Hence, by the induction, it follows that

$$N_{\tilde{\gamma}}(T) \leq \mathsf{E}_{\alpha',\beta'}\left(f_3(T) - \frac{\kappa}{\varrho^2}\right). \tag{3.10}$$

By inequalities (3.9) and (3.10), the proof is completed for $\varrho = z$. By the identical

argument, the other two cases may be proven, and this completes the proof. □

Lemma 15.

$$f_i(T) = \log T + O(1).$$

Proof. The proof is obvious. □

Lemma 16.

$$\mathcal{N}_n(T) = C_\tau(\log T + O(1))^2 + O\left(D_\tau(\log T + O(1))^2\right) \tag{3.11}$$
$$+ O\left(E_\tau(\log T + O(1))\right) + O\left(F_\tau(\log T + O(1)) \log\log T + O(1)\right).$$

where

$$C_\tau = \frac{3}{\pi^2} \sum_{\partial\mathfrak{R}(\tau)} \frac{1}{f_p(p) f_q(q)}, \quad D_\tau = \sum_{\partial\mathfrak{R}(\tau)} \frac{1}{\varrho^2 f_p(p)^2 f_q(q)},$$
$$E_\tau = \sum_{\partial\mathfrak{R}(\tau)} \frac{1}{f_p(p)}, \quad \textit{and} \quad F_\tau = \sum_{\partial\mathfrak{R}(\tau)} \frac{1}{f_q(q)}.$$

We, again, suppress the index n rather than setting $C_{n,\tau}$, $D_{n,\tau}$, $E_{n,\tau}$, or $F_{n,\tau}$.

Proof. Let us estimate the various terms in equation (3.8). By applying Lemma 11 and Lemma 14 to Theorem 4 where we set $\alpha = f_p(p) + O\left(\frac{1}{\varrho^2}\right)$ and $\beta = f_q(q) + O\left(\frac{1}{\varrho^2}\right)$, we obtain:

$$\mathcal{N}_{(x,y,z)}(T) = \frac{3}{\pi^2} \frac{(\log T + O(1))^2}{\left(f_p(p) + O\left(\frac{1}{\varrho^2}\right)\right)\left(f_q(q) + O\left(\frac{1}{\varrho^2}\right)\right)}$$
$$+ O\left(\frac{\log T + O(1)}{f_p(p)}\right) + O\left(\left(\frac{\log T + O(1)}{f_q(q)}\right) \log\left(\frac{\log T + O(1)}{f_q(q)}\right)\right)$$
$$= \frac{3}{\pi^2}\left(\frac{f_p(p)}{f_p(p) + O(\frac{1}{\varrho})}\right)\left(\frac{f_q(q)}{f_q(q) + O(\frac{1}{\varrho})}\right)\left(\frac{1}{f_p(p) f_q(q)}\right)(\log T + O(1))^2 \tag{3.12}$$

$$+ O\left(\frac{\log T + O(1)}{f_p(p)}\right) + O\left(\left(\frac{\log T + O(1)}{f_q(q)}\right)\log\left(\log T + O(1)\right)\right).$$

Then, we take look at a part of the first term of the equation:

$$\begin{aligned}
&\left(\frac{f_p(p)}{f_p(p) + O(\frac{1}{\varrho})}\right)\left(\frac{f_q(q)}{f_q(q) + O(\frac{1}{\varrho^2})}\right)\left(\frac{1}{f_p(p)f_q(q)}\right) \\
&= \left(\frac{f_p(p) + O(\frac{1}{\varrho^2}) - O(\frac{1}{\varrho^2})}{f_p(p) + O(\frac{1}{\varrho^2})}\right)\left(\frac{f_q(q) + O(\frac{1}{\varrho^2}) - O(\frac{1}{\varrho^2})}{f_q(q) + O(\frac{1}{\varrho^2})}\right)\left(\frac{1}{f_p(p)f_q(q)}\right) \\
&= \left(1 - O\left(\frac{1}{\varrho}\right)\right)\left(1 - O\left(\frac{1}{\varrho f_q(q)}\right)\right)\left(\frac{1}{f_p(p)f_q(q)}\right) \\
&= \frac{1}{f_p(p)f_q(q)} + O\left(\frac{1}{\varrho f_p(p)^2 f_q(q)}\right) + O\left(\frac{1}{\varrho f_p(p) f_q(q)^2}\right) + O\left(\frac{1}{x^4 f_p(p)^2 f_q(q)^2}\right) \\
&= \frac{1}{f_p(p)f_q(q)} + O\left(\frac{1}{\varrho f_p(p)^2 f_q(q)}\right).
\end{aligned}$$

As a result, equation (3.12) becomes:

$$\begin{aligned}
N_{(x,y,z)}(T) = \frac{3}{\pi^2}\left(\frac{1}{f_p(p)f_q(q)}\right)(\log T + O(1))^2 + O\left(\frac{1}{\varrho f_p(p)^2 f_q(q)}\right)(\log T + O(1))^2 \\
+ O\left(\frac{\log T + O(1)}{f_p(p)}\right) + O\left(\left(\frac{\log T + O(1)}{f_q(q)}\right)\log\left(\log T + O(1)\right)\right).
\end{aligned}$$

Furthermore, the various $+1$ in equation (3.8) may be absorbed in to the error term $O\left(\frac{\log T + O(1)}{f_p(p)}\right)$ since $p < T$. Therefore, we obtain equation (3.11). □

Theorem 17. *Let g be an arbitrary function of two variables.*

$$\sum_{(x,y,z)\in\partial\mathfrak{R}'} g(p,q) = g(x_0, y_0) + \sum_{(x,y,z)\in\mathfrak{R}'} \left(g(p,\varrho) + g(q,\varrho) - g(p,q)\right). \tag{3.13}$$

Proof. We prove this using mathematical induction. For the base case, we let $\mathfrak{R}'$ be only the fundamental triple, then $\partial\mathfrak{R}'$ is the two triples generated from the fundamental triple.

Since $\Psi(x_0, y_0, z_0) = (1, 1, 2)$, we know that $p = x_0$, $q = y_0$, $\varrho = z_0$. The left hand side of equation (3.13) is

$$\sum_{(x,y,z)\in\partial\mathfrak{R}} g(p,q) = g(x_0, z_0) + g(y_0, z_0).$$

On the other hand, the right hand side of the equality (3.13) is:

$$\begin{aligned}
g(x_0, y_0) + \sum_{(x_0,y_0,z_0)\in\mathfrak{R}'} &(g(p,\varrho) + g(q,\varrho) - g(p,q)) \\
&= g(x_0, y_0) + (g(x_0, z_0) + g(y_0, z_0) - g(x_0, y_0)) \\
&= g(x_0, z_0) + g(y_0, z_0).
\end{aligned}$$

Hence, equality (3.13) holds for the base case. Now, suppose equality (3.13) holds for any connected subtree $\mathfrak{R}'$ with k elements. Suppose $\mathfrak{R}''$ is a subtree that has $k+1$ elements. There exists a triple $(x', y' z') \in \mathfrak{R}''$ such that $\mathfrak{R}'' \setminus \{(x', y', z')\}$ is still connected. Then, $\mathfrak{R}' = \mathfrak{R}'' \setminus \{(x', y', z')\}$ has k elements. Thus, using the inductive hypothesis,

$$\begin{aligned}
\sum_{(x,y,z)\in\partial\mathfrak{R}''} g(p,q) &= \left(\sum_{(x,y,z)\in\partial\mathfrak{R}'} g(p,q)\right) + (g(p',\varrho') + g(q',\varrho') - g(p',q')) \\
&= g(x_0, y_0) + \left(\sum_{(x,y,z)\in\mathfrak{R}'} (g(p,\varrho) + g(q,\varrho) - g(p,q))\right) \\
&\quad + (g(p',\varrho') + g(q',\varrho') - g(p',q')) \\
&= g(x_0, y_0) + \left(\sum_{(x,y,z)\in\mathfrak{R}''} (g(p,\varrho) + g(q,\varrho) - g(p,q))\right).
\end{aligned}$$

We have obtained what we have desired. Thus, by the principle of mathematical induction, equation (3.13) is true for any subtree of a Rosenberger tree. $\square$

Lemma 18. *Let C_τ. D_τ. E_τ. and F_τ be as in Lemma* 16. *Then.*

$$C_\tau = C_n + O\left(\frac{1}{\tau^2}\right) \quad \textit{where} \quad C_n = \frac{3}{\pi^2} \sum_{(x.y.z)\in\mathfrak{R}} \frac{f_p(p) - f_q(q) - f_\varrho(\varrho)}{f_p(p)f_q(q)f_\varrho(\varrho)}. \tag{3.14}$$

$$D_\tau = O\left(\frac{1}{\tau^2}\right). \tag{3.15}$$

$$E_\tau = O(\log\tau). \tag{3.16}$$

$$F_\tau = O(\log\tau). \tag{3.17}$$

Proof. We first let $g(p.q) = \frac{1}{f_p(p)f_q(q)}$ in equation (3.13). Then. the equation becomes:

$$\sum_{(x.y.z)\in\partial\mathfrak{R}'} \frac{1}{f_p(p)f_q(q)} = \frac{1}{f_p(x_0)f_q(y_0)} + \sum_{(x.y.z)\in\mathfrak{R}'} \left(\frac{f_p(p) - f_q(q) - f_\varrho(\varrho)}{f_p(p)f_q(q)f_\varrho(\varrho)}\right). \tag{3.18}$$

By Lemma 16 and equation (3.18). this yields the following equalities.

$$\begin{aligned} C_\tau &= \frac{3}{\pi^2 f_1(x_0)f_2(y_0)} + \frac{3}{\pi^2} \sum_{q\le\tau<\varrho} \frac{1}{f_p(p)f_q(q)} \\ &= \frac{3}{\pi^2 f_1(x_0)f_2(y_0)} + \frac{3}{\pi^2} \sum_{\substack{(x.y.z)\in\mathfrak{R} \\ \varrho\le\tau}} \left(\frac{f_p(p) - f_q(q) - f_\varrho(\varrho)}{f_p(p)f_q(q)f_\varrho(\varrho)}\right) \\ &= C_n - \frac{3}{\pi^2} \sum_{\substack{(x.y.z)\in\mathfrak{R} \\ \varrho>\tau}} \left(\frac{f_p(p) + f_q(q) - f_\varrho(\varrho)}{f_p(p)f_q(q)f_\varrho(\varrho)}\right). \end{aligned} \tag{3.19}$$

Furthermore. by using the result in Lemma 11. we obtain:

$$C_\tau = C_n + \frac{3}{\pi^2} \sum_{\substack{(x.y.z)\in\mathfrak{R} \\ \varrho>\tau}} \left(\frac{O\left(\frac{1}{\varrho^2}\right)}{f_p(p)f_q(q)f_\varrho(\varrho)}\right).$$

Thus, it follows that

$$C_\tau = C_n + \frac{3}{\pi^2} \sum_{\substack{(x,y,z)\in\Re \\ \varrho>\tau}} O\left(\frac{1}{\varrho^2 \log(k_p p)\log(k_q q)\log(k_\varrho \varrho)}\right). \tag{3.20}$$

We, then, consider: $w_p p^2 + w_q q^2 + w_\varrho \varrho^2 = dpq\varrho$ where, depending on the subindexes, (w_p, w_q, w_ϱ) takes the coefficients of the Rosenberger equations a, b or c. It, therefore, yields that $w_\varrho \varrho < dpq$, and $w_\varrho \varrho < dq^2$. Thus, $\log(dw_\varrho \varrho) < \log(dq^2) = 2\log(dq)$. Consequently,

$$\log(dq) < \log(dw_\varrho \varrho) < 2\log(dq). \tag{3.21}$$

Thus, $O\left(\frac{1}{\log k_q q}\right) = O\left(\frac{1}{\log \varrho}\right)$. Also, $O\left(\frac{1}{\log k_p p}\right) = O(1)$. Combining these with equation (3.20) gives:

$$\begin{aligned} C_\tau &= C_n + \frac{3}{\pi^2} \sum_{\substack{(x,y,z)\in\Re \\ \varrho>\tau}} O\left(\frac{1}{\varrho^2 \log\varrho \log(k_\varrho \varrho)}\right) \\ &= C_n + \frac{3}{\pi^2} \sum_{\substack{(x,y,z)\in\Re \\ \varrho>\tau}} O\left(\frac{1}{\varrho^2 \log^2\varrho}\right) \\ &= C_n + \frac{3}{\pi^2} O\left(\sum_{\substack{(x,y,z)\in\Re \\ \varrho>\tau}} \left(\frac{1}{\varrho^2 \log^2\varrho}\right)\right). \end{aligned} \tag{3.22}$$

Then, we estimate the sum in equation (3.22) using Abel's summation:

$$\sum_{\substack{(x,y,z)\in\Re \\ \varrho>\tau}} \left(\frac{1}{\varrho^2 \log^2\varrho}\right) = \sum_{\varrho=\tau+1}^{\infty} \frac{N(\varrho) - N(\varrho-1)}{\varrho^2 \log^2\varrho}.$$

By listing the first few terms, one may realize that we may restate the sum as:

$$\begin{aligned}\sum_{\substack{(x,y,z)\in\Re\\ \varrho>\tau}}\left(\frac{1}{\varrho^2\log^2\varrho}\right) &= \frac{-N(\tau)}{(\tau+1)^2\log^2(\tau+1)}\\ &\quad+\sum_{\varrho=\tau+1}^{\infty}N(\varrho)\left(\frac{1}{\log^2\varrho}-\frac{1}{(\varrho+1)^2\log^2(\varrho+1)}\right)\\ &= O\left(\frac{N(\tau)}{\tau^2\log^2\tau}\right)+\sum_{\varrho=\tau+1}^{\infty}N(\varrho)\,O\left(\frac{\varrho\log^2\varrho}{\varrho^4\log^4\varrho}\right)\\ &= O\left(\frac{N(\tau)}{\tau^2\log^2\tau}\right)+\sum_{\varrho=\tau+1}^{\infty}N(\varrho)\,O\left(\frac{1}{\varrho^3\log^2\varrho}\right)\\ &= O\left(\left(\frac{N(\tau)}{\log^2\tau}\right)\frac{1}{\tau^2}\right).\end{aligned}$$

In the above, the bounds implied by the O notation in the infinite sum is uniformly bounded, so we may switch it to the infinite sum. According to the result in Corollary 10, it is the fact that $\frac{N(\tau)}{\log^2\tau}$ is bounded. Hence,

$$\sum_{\substack{(x,y,z)\in\Re\\ \varrho>\tau}}\left(\frac{1}{\varrho^2\log^2\varrho}\right)=O\left(\frac{1}{\tau^2}\right). \tag{3.23}$$

By equation (3.22) and equation (3.23), we obtain equation (3.14). Further, since $\varrho>\tau$, we observe that $D_\tau<\tau^{-2}C_\tau=O\left(\frac{1}{\tau^2}\right)$, which is equation (3.15). For equation (3.16), we apply $g(p,q)=\frac{1}{f_p(p)}$ to equation (3.13), and we, first of all, obtain that

$$\sum_{(x,y,z)\in\partial\Re'}\frac{1}{f_p(p)}=\frac{1}{f_1(x_0)}+\sum_{\substack{(x,y,z)\in\Re\\ \varrho\le\tau}}\frac{1}{f_q(q)}. \tag{3.24}$$

Since $f_q(q) \approx \log k_q q$, equation (3.24) with equation (3.21) becomes:

(3.25)
$$\sum_{(x,y,z)\in\partial\mathfrak{R}'} \frac{1}{f_p(p)} = \sum_{\substack{(x,y,z)\in\mathfrak{R}\\ \varrho\le\tau}} O\left(\frac{1}{\log k_q q}\right) = \sum_{\substack{(x,y,z)\in\mathfrak{R}\\ \varrho\le\tau}} O\left(\frac{1}{\log(dw_\varrho \varrho)}\right) \ll \sum_{\substack{(x,y,z)\in\mathfrak{R}\\ \varrho\le\tau}} \frac{1}{\log\varrho}$$

where $\ll$ indicates that the equation is bounded by $\sum \frac{M}{\log\varrho}$ where M is a positive real number. Thus, it is sufficient to show that equation (3.25) implies equation (3.16):

$$\begin{aligned}
\sum_{\substack{(x,y,z)\in\mathfrak{R}\\ \varrho\le\tau}} \frac{1}{\log\varrho} &= \sum_{\varrho=1}^{\tau} \frac{N(\varrho) - N(\varrho-1)}{\log\varrho} \\
&= \frac{N(\tau)}{\log\tau} + \sum_{\varrho=1}^{\tau-1} N(\varrho)\left(\frac{1}{\log\varrho} - \frac{1}{\log(\varrho+1)}\right) \\
&= O\left(\frac{N(\tau)}{\log\tau}\right) + \sum_{\varrho=1}^{\tau-1} O\left(\frac{N(\varrho)}{\varrho\log^2\varrho}\right).
\end{aligned}$$

By applying the result in Corollary 10, equation (3.25) indeed yields equation (3.16).

$$\sum_{\substack{(x,y,z)\in\mathfrak{R}\\ \varrho\le\tau}} \frac{1}{\log\varrho} = O(\log\tau) + \sum_{\varrho=1}^{\tau-1} O\left(\frac{1}{\varrho}\right) = O(\log\tau).$$

Finally, since $F_\tau \ll E_\tau$, knowing $E_\tau = O(\log\tau)$, we conclude that $F_\tau = O(\log\tau)$. Therefore, the proof is completed. □

Theorem 19. *Let C_n be as defined in equation (3.14). The number of solution (x, y, z) to a Rosenberger equation such that $\max\{x, y, z\} < T$ is given by*

(3.26)
$$N_n(T) = C_n(\log T)^2 + O\left(\log T(\log\log T)^2\right) \quad \text{as} \quad T \to \infty.$$

Proof. By plugging in each (3.14), (3.15), (3.16), and (3.17) into equation (3.11), we obtain

that

$$\begin{aligned}
N_n(T) = {} & \left(C_n + O\left(\frac{1}{\tau^2}\right)\right)(\log T + O(1))^2 \\
& + O\left(\frac{(\log T + O(1))^2}{\tau^2}\right) + ((O(\log \tau))(\log T + O(1))) \\
& + ((O(\log \tau))(\log T + O(1)) \log(\log T + O(1))) \\
= {} & C_n(\log T + O(1))^2 + O\left(\frac{(\log T + O(1))^2}{\tau^2}\right) + O\left((\log \tau) f(\tau)\right) \\
& + O\left((\log \tau)\log T\right) + O\left(O(1)\log(\log T + O(1))\right) \\
= {} & C_n(\log T + O(1))^2 + O\left(\frac{(\log T + O(1))^2}{\tau^2}\right) \\
& + O\left((\log \tau)\log T\right) + O\left(O(1)\log(\log T + O(1))\right).
\end{aligned}$$

We now choose the value of τ. Setting $\frac{(\log T + O(1))^2}{\tau^2} = (\log \tau)\log T + O(1)\log(\log T + O(1))$, we may approximate an appropriate value for τ and find $\tau = \frac{\sqrt{\log T}}{\log\log T}$. With this choice, we obtain:

$$\begin{aligned}
N_n(T) = {} & C_n(\log T + O(1))^2 + O\left(\frac{(\log T + O(1))^2}{\left(\frac{\sqrt{\log T}}{\log\log T}\right)^2}\right) \\
& + O\left(\left(\log\left(\frac{\sqrt{\log T}}{\log\log T}\right)\right)\log T\right) + O\left(O(1)\log(\log T + O(1))\right) \\
= {} & C_n(\log T + O(1))^2 + O\left(\frac{(\log T + O(1))^2(\log\log T)^2}{\log T}\right) \\
& + O\left((\log T + O(1))\log\log T \log(\log T + O(1))\right) \\
= {} & C_n(\log T)^2 + O\left(\log T\,(\log\log T)^2\right). \qquad \square
\end{aligned}$$

This completes the proof of Theorem 1.

CHAPTER 4

COMPUTATIONS

In this chapter, we describe how we compute approximate values of the constants C_n. This *Mathematica* program has been developed by Dr. Arthur Baragar in the Department of Mathematical Sciences and has been modified by the author. Note that the sentences inside of (* *) indicate comments. This program calculates the left hand side of equation (3.18).

```
(* A user stores the values of coefficient here. *)
a = Input["Type the value of the coefficient a."];
b = Input["Type the value of the coefficient b."];
c = Input["Type the value of the coefficient c."];
d = Input["Type the value of the coefficient d."];
(* A user stores the values of the fundamental triple here. *)
x = Input["Type the first element of the fundamental triple."];
y = Input["Type the second element of the fundamental triple."];
z = Input["Type the third element of the fundamental triple."];
(* A user defines the boundary value T here. *)
t = Input["Type the boundary value tau."];
(* It defines value of the constants k. *)
k = {d/Sqrt[b*c], d/Sqrt[a*c], d/Sqrt[a*b]};
(* The branch operations are executed here. *)
```

```
b1[P_]:= {(d/a)*P[[2]]*P[[3]]-P[[1]],P[[2]],P[[3]]};

b2[P_]:= {P[[1]],(d/b)*P[[1]]*P[[3]]-P[[2]],P[[3]]};

b3[P_]:= {P[[1]],P[[2]],(d/c)*P[[1]]*P[[2]]-P[[3]]};

(* The functions (3.1) is defined here. *)

f[T_,w_]:= N[Log[(k[[w]]*T+Sqrt[(k[[w]]^2)*(T^2)-4])/2],30];

(* The equation (3.18) is defined here. *)

g[P_,w_]:= N[f[P[[w]],w]/(f[P[[1]],1]*f[P[[2]],2]*f[P[[3]],3]),30];

(* Evaluate (3.14) by using (3.19). *)

cnt[P_,t_]:=
    If[t==1, If[P[[1]]<T, cnt[b2[P],2]+cnt[b3[P],3], g[P,1]],
      If[t==2, If[P[[2]]<T, cnt[b1[P],1]+cnt[b3[P],3], g[P,2]],
        If[P[[3]]<T, cnt[b1[P],1]+cnt[b2[P],2], g[P,3]]]];

(* The initialization of the program is executed here. *)

N[(3/(Pi^2))*(cnt[{x,y,z},3]),30]
```

As Table 4.1 indicates. these constants converge in the first 12 decimal digits around $T = 10^6$. We must note that the value of C_6 tends to converge to 0.588051990717 according to our program. but that equation R_6 has two fundamental triples. so we must double the value of the constant.

T	C_3	C_4	C_6
10^2	0.543808773432	0.554238122395	1.176097024748
10^4	0.543809447261	0.554239131118	1.176103981248
10^6	0.543809447296	0.554239131152	1.176103981434
10^8	0.543809447296	0.554239131152	1.176103981434

Table 4.1: A Table for the Constant C_n

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VITA

Graduate College
University of Nevada, Las Vegas

Kensaku Umeda

Local Address:
1600 E University Ave Apt 108
Las Vegas, NV 89119-5835

Home Address:
2568-1 Hatakeda, Ogi-City,
Saga-Prefecture 845-0002,
Japan

Degree:
Bachelor of Science, Mathematics, 2000
Eastern Kentucky University

Thesis title:
The Asymptotic Behavior of Integer Solutions of the Rosenberger Equations

Thesis Examination Committee:
Chairperson, Dr. Arthur Baragar, Ph. D.
Committee Member, Dr. Michelle Schultz, Ph. D.
Committee Member, Dr. Corran Webster, Ph. D.
Graduate Faculty Representative, Dr. Wolfgang Bein, Ph. D.