From equi-graphical sets to graphical permutations: A problem of degrees in graphs

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FROM EQUI-GRAphICAL SETS TO GRAphICAL PERMUTATIONS:

A PROBLEM OF DEGREES IN GRAPHS

by

Michael Lee Watson

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ABSTRACT

From Equi-graphical Sets to Graphical Permutations: A Problem of Degrees in Graphs

by

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A set \( \{a_1, a_2, \ldots, a_n\} \) of positive integers with \( a_1 < a_2 < \cdots < a_n \) is said to be equi-graphical if there exists a graph with exactly \( a_i \) vertices of degree \( a_i \) for each \( i \) with \( 1 \leq i \leq n \). It is known that such a set is equi-graphical if and only if \( \sum_{i=1}^{n} a_i \) is even and \( a_n \leq \sum_{i=1}^{n-1} a_i^2 \). This concept is now generalized to the following problem: Given a set \( S \) of positive integers and a permutation \( \pi \) on \( S \), determine when there exists a graph containing exactly \( a_i \) vertices of degree \( \pi(a_i) \) for each \( i \) \((1 \leq i \leq n)\). If such a graph exists, then \( \pi \) is called a graphical permutation.

In this paper, the graphical permutations on sets of size four are characterized and using a criterion of Fulkerson, Hoffman, and McAndrew, we show that a permutation \( \pi \) of \( S = \{a_1, a_2, \ldots, a_n\} \), where \( 1 \leq a_1 < a_2 < \cdots < a_n \) and such that \( \pi(a_n) = a_n \), is graphical if and only if \( \sum_{i=1}^{n} a_i \pi(a_i) \) is even and \( a_n \leq \sum_{i=1}^{n-1} a_i \pi(a_i) \).
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A graph is a finite nonempty set of objects called vertices together with a set of unordered pairs of distinct vertices called edges. If \( u \) and \( v \) are vertices and \( \{u, v\} \) is an edge, then we say that \( u \) and \( v \) are adjacent or joined by an edge. For simplicity we denote the edge \( \{u, v\} \) by \( uv \). The degree of a vertex \( v \) in a graph is the number of vertices adjacent to \( v \). The vertex set of a graph \( G \) is denoted by \( V(G) \) and the edge set of \( G \) is denoted by \( E(G) \). The cardinality of the vertex set of a graph \( G \) is the order of \( G \) and the cardinality of its edge set is the size of \( G \). We let \( G \) be a graph with vertex set \( V(G) = \{v_1, v_2, \ldots, v_n\} \). For each \( i \ (1 \leq i \leq n) \) we denote the degree of \( v_i \) by \( d_i \) and say that \( d_1, d_2, \ldots, d_n \) is a degree sequence for \( G \).

Certainly every graph has a degree sequence but not every sequence of nonnegative integers has a corresponding graph. Let \( s: d_1, d_2, \ldots, d_n \) be a nonincreasing sequence of nonnegative integers. If there is a graph on \( n \) vertices with degree sequence \( s \), then we say that \( s \) is a graphical sequence or that \( s \) is graphical. For example, if \( d_i = n - 1 \) for every \( i \ (1 \leq i \leq n) \), then we can take the graph with \( n \) vertices in which every pair of vertices is adjacent. Thus in this instance, the sequence \( s: d_1, d_2, \ldots, d_n \) is graphical and this corresponding graph we have defined is called the complete graph of order \( n \); that is, in a complete graph every vertex is adjacent to every other vertex. We denote the
complete graph on \( n \) vertices with \( K_n \). Let \( G \) be a graph with \( V(G) = A \cup B \), where \( A \) and \( B \) are nonempty and disjoint, such that each vertex of \( A \) is adjacent to each vertex of \( B \). If \( |A| = a \) and \( |B| = b \) then such a graph will be written \( K_{a,b} \) and is called the complete bipartite graph with partite sets \( A \) and \( B \).

We notice that for any graphical sequence \( s : d_1, d_2, ... , d_n \) it is necessary that \( d_i \leq n - 1 \) for each \( i \) \( (1 \leq i \leq n) \) and that the sum \( \sum_{i=1}^{n} d_i \) must be even. We see this second condition by simply summing the degrees of each vertex in our graph and noticing that each edge is being counted twice. Hence the sum of the degrees in any graph is even. As it turns out these two conditions are necessary but not sufficient for a sequence \( s \) to be graphical. For example, the sequence \( 3, 3, 1, 1 \) is not graphical but satisfies both conditions.

Many necessary and sufficient conditions have been found for graphical sequences. Perhaps the most well know are those found by Havel [5] and Hakimi [3] and by Erdős and Gallai [1].

**THEOREM 1.1** (Havel and Hakimi) A sequence \( s : d_1, d_2, ... , d_n \) of nonnegative integers with \( d_1 \geq d_2 \geq ... \geq d_n \), where \( n \geq 2 \) and \( d_i \geq 1 \), is graphical if and only if the sequence \( s_1 \), is graphical, where

\[
s_1 : d_2 - 1, d_3 - 1, ... , d_{d_i+1} - 1, d_{d_i+2}, ... , d_n.
\]
THEOREM 1.2 (Erdős and Gallai) A sequence $s: d_1, d_2, \ldots, d_n$ of nonnegative integers with $n \geq 2$ and $d_1 \geq d_2 \geq \cdots \geq d_n$ is graphical if and only if $\sum_{i=1}^{n} d_i$ is even and for each integer $k$, $1 \leq k \leq n-1$, the following holds:

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}.$$ 

As an illustration we will show that the sequence

$s: 8, 8, 8, 8, 8, 8, 5, 5, 5, 3, 3, 3$

is graphical using the Havel-Hakimi criterion. Applying Theorem 1.1 to $s$ we get

$s_1': 7, 7, 7, 7, 7, 4, 5, 5, 5, 3, 3, 3$

Reordering the sequence we obtain

$s_1: 7, 7, 7, 7, 7, 7, 5, 5, 5, 4, 3, 3, 3$

Continuing, we have

$s_2': 6, 6, 6, 6, 6, 4, 5, 5, 5, 3, 3, 3$

$s_2: 6, 6, 6, 6, 6, 5, 5, 4, 4, 3, 3, 3$

$s_3': 5, 5, 5, 5, 4, 4, 4, 4, 3, 3, 3$

$s_3: 5, 5, 5, 5, 4, 4, 4, 4, 3, 3, 3$

$s_4': 4, 4, 4, 4, 4, 4, 3, 3, 3$

$s_4: 5, 4, 4, 4, 4, 4, 3, 3, 3$

$s_5': 3, 3, 3, 3, 4, 4, 3, 3, 3$

$s_5: 4, 4, 4, 3, 3, 3, 3, 3, 3$
We may proceed in this manner until we reach $s_{13}: 0, 0, 0, 0$ or until we have obtained a sequence we know to be the degree sequence of a graph. We recognize the degree sequence $s_8$ represents a (non-unique) graph on eight vertices, every vertex of which has degree two. Two possible graphs with degree sequence $s_8$ are provided in Figure 1.1. Therefore, we stop here and conclude that $s$ is graphical.

Figure 1.1 Two graphs with degree sequence $s_8$.

Now to show the same sequence $s$ is graphical using the Erdős-Gallai criterion we must show that the sum of the degrees is even and for each integer $k$, where $1 \leq k \leq 15$,

\[ \sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=1}^{16} \min\{k, d_i\} \]
holds. The sum of the degrees is ninety-eight which is even and we are now left with fifteen inequalities to check.

For \( k = 1 \), \( \sum_{i=1}^{1} d_i = 8 \) and \( k(k - 1) + \sum_{i=2}^{16} \min \{k, d_i\} = 15 \).

For \( k = 2 \), \( \sum_{i=1}^{2} d_i = 16 \) and \( k(k - 1) + \sum_{i=3}^{16} \min \{k, d_i\} = 30 \).

For \( k = 3 \), \( \sum_{i=1}^{3} d_i = 24 \) and \( k(k - 1) + \sum_{i=4}^{16} \min \{k, d_i\} = 45 \).

For \( k = 4 \), \( \sum_{i=1}^{4} d_i = 32 \) and \( k(k - 1) + \sum_{i=5}^{16} \min \{k, d_i\} = 57 \).

For \( k = 5 \), \( \sum_{i=1}^{5} d_i = 40 \) and \( k(k - 1) + \sum_{i=6}^{16} \min \{k, d_i\} = 69 \).

For \( k = 6 \), \( \sum_{i=1}^{6} d_i = 48 \) and \( k(k - 1) + \sum_{i=7}^{16} \min \{k, d_i\} = 76 \).

For \( k = 7 \), \( \sum_{i=1}^{7} d_i = 56 \) and \( k(k - 1) + \sum_{i=8}^{16} \min \{k, d_i\} = 83 \).

For \( k = 8 \), \( \sum_{i=1}^{8} d_i = 64 \) and \( k(k - 1) + \sum_{i=9}^{16} \min \{k, d_i\} = 90 \).

For \( k = 9 \), \( \sum_{i=1}^{9} d_i = 69 \) and \( k(k - 1) + \sum_{i=10}^{16} \min \{k, d_i\} = 101 \).

For \( k = 10 \), \( \sum_{i=1}^{10} d_i = 74 \) and \( k(k - 1) + \sum_{i=11}^{16} \min \{k, d_i\} = 114 \).

For \( k = 11 \), \( \sum_{i=1}^{11} d_i = 79 \) and \( k(k - 1) + \sum_{i=12}^{16} \min \{k, d_i\} = 129 \).

For \( k = 12 \), \( \sum_{i=1}^{12} d_i = 84 \) and \( k(k - 1) + \sum_{i=13}^{16} \min \{k, d_i\} = 146 \).

For \( k = 13 \), \( \sum_{i=1}^{13} d_i = 89 \) and \( k(k - 1) + \sum_{i=14}^{16} \min \{k, d_i\} = 165 \).

For \( k = 14 \), \( \sum_{i=1}^{14} d_i = 92 \) and \( k(k - 1) + \sum_{i=15}^{16} \min \{k, d_i\} = 188 \).

For \( k = 15 \), \( \sum_{i=1}^{15} d_i = 95 \) and \( k(k - 1) + \sum_{i=16}^{16} \min \{k, d_i\} = 213 \).
So we see that for every \( k \) the inequality holds and therefore the sequence \( s \) is graphical.

It has been observed that, given sequences of a particular form, there are easier ways to determine if a sequence is graphical. Having seen that the Hakimi-Havel and Erdös-Gallai criteria may become progressively more difficult as the sequences get longer, characterizing such sequences is of particular usefulness.

Let \( S = \{a_1, a_2, \ldots, a_n\} \) be a set of \( n \) distinct positive integers. We say that \( S \) is equi-graphical if there exists a graph of order \( \sum_{i=1}^{n} a_i \) that contains exactly \( a_i \) vertices of degree \( a_i \) for every \( i \) (\( 1 \leq i \leq n \)). For example, since the sequence \( s : 8, 8, 8, 8, 8, 8, 5, 5, 5, 5, 3, 3, 3 \) is graphical, we can now say that the set \( \{3, 5, 8\} \) is equi-graphical. Figure 1.2 shows a graph with 8 vertices of degree 8, 5 vertices of degree 5, and 3 vertices of degree 3. A characterization of equi-graphical sets is already known and is stated in the following theorem, which appeared in [4].

**THEOREM 1.3** (Hansen and Schultz) Let \( S = \{a_1, a_2, \ldots, a_n\} \) with \( n > 2 \), be a set of positive integers such that \( a_1 < a_2 < \cdots < a_n \) and \( \sum_{i=1}^{n} a_i \) is even. Then \( S \) is equi-graphical if and only if \( a_n \leq a_1^2 + a_2^2 + \cdots + a_{n-1}^2 \).

The set \( \{3, 5, 8\} \) is seen to be equi-graphical by Theorem 1.3 since \( 3 + 5 + 8 \) is even and \( 8 \leq 9 + 25 \). Of course, it is much easier to verify that \( \{3, 5, 8\} \) is equi-graphical using Theorem 1.3 than using the Havel-Hakimi or Erdös-Gallai criteria.
Later we will need the idea of an induced subgraph. So if \( U \) is a nonempty subset of the vertex set \( V(G) \) of a graph \( G \), then the subgraph \( \langle U \rangle \) of \( G \) induced by \( U \) is the graph having vertex set \( U \) and whose edge set consists of those edges \( uv \) of \( G \) such that \( u, v \in U \). For example, if we let \( U \) consist of the vertices of degree eight in the graph of Figure 1.2, then \( \langle U \rangle = K_8 \).

We also notice that the idea of equi-graphical sets can be extended to a more general form. Given a set \( S = \{a_1, a_2, \ldots, a_n\} \) of \( n \) distinct positive integers and a permutation \( \pi \) on \( S \), we are interested in knowing when there exists a graph \( G \) of order \( \sum_{i=1}^{n} a_i \) containing exactly \( a_i \) vertices of degree \( \pi(a_i) \) for \( i = 1, 2, \ldots, n \). When such a graph does exist we say that \( \pi \) is a graphical permutation. It should be noticed that equi-graphical sets are graphical permutations \( \pi \), where \( \pi \) is just the identity mapping, and that for a given set \( S \) of \( n \) distinct positive integers we have \( n! \).
permutations. General results for graphical permutations on sets with cardinality two and	hree are witnessed in [4] by Hansen and Schultz and are as follows:

**THEOREM 1.4** Let \( S = \{a, b\} \) with \( 1 \leq a < b \). Then the permutation

\[
(1) \quad \pi_1 = \begin{pmatrix} a & b \\ a & b \end{pmatrix} \text{ is graphical if and only if } a \text{ and } b \text{ are of the same parity and } b \leq a^2.
\]

and

\[
(2) \quad \pi_2 = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \text{ is graphical.}
\]

**THEOREM 1.5** Let \( S = \{a, b, c\} \) with \( 1 \leq a < b < c \). Then the permutation

\[
(1) \quad \pi_3 = \begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix} \text{ is graphical if and only if } a + b + c \text{ is even and } c \leq a^2 + b^2.
\]

\[
(2) \quad \pi_2 = \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix} \text{ is graphical if and only if } c \text{ is even and } c \leq 2ab.
\]

\[
(3) \quad \pi_3 = \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix} \text{ is graphical if and only if } b \text{ is even,}
\]

\[
(4) \quad \pi_4 = \begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix} \text{ is graphical if and only if } a \text{ is even,}
\]

\[
(5) \quad \pi_5 = \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix} \text{ is graphical if and only if at most one of } a, b, \text{ and } c \text{ is odd, and}
\]

\[
(6) \quad \pi_6 = \begin{pmatrix} a & b & c \\ c & a & b \end{pmatrix} \text{ is graphical if and only if at most one of } a, b, \text{ and } c \text{ is odd.}
\]
In this thesis we shall first add a necessary condition to, and provide a new proof for, the \( \pi_2 \) case in Theorem 1.5 using both a construction method and the Erdős-Gallai criterion. Next, following the methods for sets of cardinality two and three, we characterize the graphical permutations of sets with cardinality four and then provide two general proofs for permutations of \( n \) integers, making use of a criterion of Fulkerson, Hoffman, and McAndrew, which will be introduced later. In particular, we prove part (1) of the following conjecture, which was made in [4].

**CONJECTURE 1.6**  Given \( S = \{a_1, a_2, \ldots, a_n\} \), a set of positive integers with \( 1 \leq a_1 < a_2 < \cdots < a_n \) and a permutation \( \pi \) on \( S \) such that \( \sum_{i=1}^n a_i \pi(a_i) \) is even, then

1. if \( \pi(a_n) = a_n \), then \( \pi \) is graphical if and only if \( a_n \leq \sum_{i=1}^{n-1} a_i \pi(a_i) \) and

2. if \( \pi(a_n) \neq a_n \), then \( \pi \) is graphical.

In conclusion, we provide a modified conjecture for part (2) of Conjecture 1.6.
CHAPTER 2

A CORRECTION AND TWO PROOFS FOR WHEN $\pi_5$ IS GRAPHICAL

We begin this chapter by looking at one of the existing results from [4], namely Theorem 1.5 (5). Consider the $S = \{1, 2, 6\}$. Then $\pi_5 = \begin{pmatrix} 1 & 2 & 6 \\ 2 & 6 & 1 \end{pmatrix}$ is not graphical yet satisfies the conditions of Theorem 1.5 (5). We verify that $\pi_5$ is not graphical by applying Theorem 1.1 to the sequence $s: 6, 6, 2, 1, 1, 1, 1, 1$. Applying the Hakimi-Havel criterion we obtain the sequence $s': 5, 1, 0, 0, 0, 0, 1, 0$ and reordering, we have $s: 5, 1, 1, 0, 0, 0, 0$. Notice that constructing a graph with one vertex of degree five and three vertices of degree one is impossible. Thus, in general, we find it necessary that such a permutation meet a further restriction in order to conclude that it is graphical. We will show that the additional condition we need for $\pi_5 = \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}$ to be graphical is the inequality $bc \leq ac + ab + b(b - 1)$. In order to prove our condition we will need the following lemma, the proof of which can be found in [4].

**Lemma 2.1** Let $x, y,$ and $r$ be nonnegative integers such that $x + y > 0$ and $r < x + y - 1$. Then there exists a graph $G$ of order $x + y$ containing $x$ vertices of degree $r$ and $y$ vertices of degree $r + 1$ if and only if $rx + (r + 1)y$ is even.
In the proof of the following theorem, Case I was verified in [4], but we also include its proof here for completeness.

**THEOREM 2.2** The permutation $\pi = \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}$ is graphical if and only if at most one of $a, b, c$ is odd and $bc \leq ab + ac + b(b - 1)$.

**PROOF** Assume that $\pi$ is graphical. So there exists a graph $G$ with $a$ vertices of degree $b$, $b$ vertices of degree $c$, and $c$ vertices of degree $a$. Since the sum of the degrees of the vertices is even, we have $ab + bc + ac$ is even. This is true when either $a, b,$ and $c$ are all even or exactly one of $a, b, c$ is odd. Thus at most one of $a, b, c$ is odd. We define the sets $A, B,$ and $C$ so that the set $A$ contains the $a$ vertices of degree $b$, the set $B$ contains the $b$ vertices of degree $c$ and the set $C$ contains the $c$ vertices of degree $a$. Let $x$ denote the number of edges that join vertices of $B$ to vertices of $A \cup C$. Since the degree of each vertex of $A$ is $b$ and the degree of each vertex of $C$ is $a$, we know at most $ab + ac$ edges leave $A \cup C$. Thus $x \leq ab + ac$.

Further, a vertex of $B$ has degree at most $b - 1$ in $(B)$. So at least $c - b + 1$ edges for each vertex of $B$ must leave the set $B$. Hence $x \geq b(c - b + 1)$. Therefore $b(c - b + 1) \leq ab + ac$ or equivalently $bc \leq ab + ac + b(b - 1)$.

For the converse, suppose that at most one of $a, b, c$ is odd and $bc \leq ab + ac + b(b - 1)$. Define $V(G) = A \cup B \cup C$ where $|A| = a$, $B = \{v_1, v_2, \ldots, v_b\}$, and $C = \{w_1, w_2, \ldots, w_c\}$. Construct the graph $K_{a,b}$ on the sets...
A and B, using A and B as the partite sets, so that the vertices in A have degree b.

Now we consider two cases.

CASE 1 Suppose that \( b(c-a) \leq ac \). At this time, the vertices of A have degree b and the vertices of B have degree a. We proceed by joining the vertex \( v_1 \) of B to the vertices \( w_1, w_2, \ldots, w_{c-a} \) of C. Then joining vertex \( v_2 \) of B to the next \( c-a \) vertices of C, namely \( w_{c-a+1}, w_{c-a+2}, \ldots, w_{2(c-a)} \), where the subscripts are expressed modulo c. Continue in this manner until each vertex of B is joined to \( c-a \) vertices of C. Notice that since \( b(c-a) \leq ac \), the degrees of the vertices of C will not exceed a as a result of this distribution. We now denote this graph by \( G' \) and observe that the vertices of B have degree c in \( G' \). Define the integers \( q \) and \( r \) by

\[
b(c-a) = qc + r,
\]

where \( 0 \leq r \leq c-1 \). Then \( r \) vertices of C have degree \( q+1 \) in \( G' \) and \( c-r \) vertices of C have degree \( q \) in \( G' \). Thus, we need only show that there exists a graph \( H \) of order C such that \( r \) vertices have degree \( a-q-1 \) in \( H \) and \( c-r \) vertices have degree \( a-q \) in \( H \). Using Lemma 2.1, we find it possible to construct such a graph \( H \) if and only if \( (a-q-1)r + (a-q)(c-r) \) is even and \( a-q-1 < c-1 \). Clearly \( a-q-1 < c-1 \). Simplifying, we have

\[
(a-q-1)r + (a-q)(c-r) = ac - b(c-a).
\]

Since at most one of \( a, b \) and \( c \) is odd, it follows that \( ac - b(c-a) \) is even. Thus we can construct such a graph \( H \) on the set C and, consequently, the desired graph \( G \).
CASE 2 Suppose that \( b(c - a) > ac \). From the hypothesis we also know that \( bc \leq ab + ac + b(b - 1) \). The construction now depends on whether \( c - a \geq b - 1 \) or \( c - a < b - 1 \).

SUB-CASE 2.1 Suppose that \( c - a \geq b - 1 \). Construct the graph \( K_b \) on the set \( B \) so the vertices in \( B \) now each have degree \( a + b - 1 \). Since \( b(c - a) \leq ac + b(b - 1) \), we have \( b(c - a - b + 1) \leq ac \). We now add \( b(c - a - b + 1) \) edges in the following way.

Join the vertex \( v_i \) of \( B \) to the vertices \( w_1, w_2, \ldots, w_{c-a-b+1} \) of \( C \). Join the vertex \( v_2 \) to \( w_{c-a+b+2}, w_{c-a+b+3}, \ldots, w_{c(c-a-b+1)} \), where the subscripts are expressed modulo \( c \). Continue in this way until each vertex of \( B \) is joined to \( c - a - b + 1 \) vertices of \( C \). Since \( b(c - a - b + 1) \leq ac \), the degree of each vertex of \( C \) will not exceed \( a \). Denote this graph \( G' \). In \( G' \) we see that the vertices of \( A \) have degree \( b \) and the vertices of \( B \) have degree \( c \). Define the integers \( q \) and \( r \) by

\[
b(c - a - b + 1) = cq + r,
\]

where \( 0 \leq r \leq c - 1 \). Distributing \( b(c - a - b + 1) \) edges among the vertices of \( C \) we have \( r \) vertices that have degree \( q + 1 \) in \( G' \) and \( c - r \) have degree \( q \) in \( G' \). Thus it remains to show that there exists a graph \( H \) of order \( c \) such that \( r \) vertices have degree \( a - q - 1 \) and \( c - r \) have degree \( a - q \). By Lemma 2.1, this is possible if and only if

\[
(a - q - 1)r + (a - q)(c - r) \text{ is even and } a - q - 1 < c - 1.
\]

Clearly \( a - q - 1 < c - 1 \). Also

\[
(a - q - 1)r + (a - q)(c - r) = -r + ca - cq = ca - b(c - a - b + 1).
\]

Since at most one of \( a, b, c \) is odd, \( ca - bc + ab + b(b - 1) \) is even. Thus we may construct the graph \( H \) on \( C \) and thus the desired graph \( G \).
**SUB-CASE 2.2** Assume that $c-a < b-1$. Since we are in Case 2, we know $b(c-a) > ac$, and so we may add $ac$ edges in the following way. Join the vertex $w_1$ of $C$ to the vertices $v_1, v_2, \ldots, v_a$ of $B$. Then join the vertex $w_2$ to the vertices $v_{a+1}, v_{a+2}, \ldots, v_{2a}$, where subscripts are expressed modulo $b$. Proceed in this manner until each vertex of $C$ is joined to $a$ vertices of $B$. Since $b(c-a) > ac$, the degree of each vertex of $B$ will not exceed $c-a$. Denote this graph by $G'$. Then each vertex of $A$ has degree $b$ and each vertex of $C$ has degree $a$. Define integers $q$ and $r$ by

$$ac = bq + r,$$

where $0 \leq r \leq b-1$. So $r$ vertices of $B$ have degree $q+1$ in $G'$ and $b-r$ have degree $q$ in $G'$. Hence it remains to show that there exists a graph $H$ of order $b$ that has $r$ vertices of degree $c-a-q-1$ and $b-r$ vertices of degree $c-a-q$. By Lemma 2.1, this is possible if $(c-a-q-1)r + (c-a-q)(b-r)$ is even and $c-a-q-1 < b-1$. Since $c-a < b-1$, we have $c-a-q-1 < b-1$. Also,

$$(c-a-q-1)r + (c-a-q)(b-r) = -r + bc - ab - bq = -ac + bc - ab$$

and since at most one of $a, b, c$ is odd, it follows that $-ac + bc - ab$ is even. Thus such a graph $H$ exists and so does the desired graph $G$. \(\square\)

In order to illustrate an entirely different technique, we provide an alternate proof that the same permutation $\pi_5 = \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}$ is graphical when, at most one of $a, b, c$ is odd and $bc \leq ab + ac + b(b-1)$. In particular, we show $\pi_5$ is graphical using the Erdős-Gallai criterion.
PROOF Suppose that at most one of $a, b, c$ is odd and $bc \leq ab + ac + b(b - 1)$. For each integer $i$ with $1 \leq i \leq a + b + c$, we define

$$d_i = \begin{cases} 
c & \text{if } 1 \leq i \leq b \\
b & \text{if } b + 1 \leq i \leq a + b \\
a & \text{if } a + b + 1 \leq i \leq a + b + c
\end{cases}$$

We show that the sequence of integers $d_1, d_2, \ldots, d_n$ satisfies the inequality

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k}^{a+b+c} \min\{k, d_i\}$$

for each positive integer $k$, where $1 \leq k \leq a + b + c - 1$.

For convenience, we consider four cases depending on how large the integer $k$ is.

CASE 1: Suppose that $1 \leq k \leq a$. We see that $\sum_{i=1}^{k} d_i = kc$ while

$$k(k-1) + \sum_{i=k}^{a+b+c} \min\{k, d_i\} = k(k-1) + k(a + b + c - k) = k(a + b + c - 1)$$

and clearly $kc \leq k(a + b + c - 1)$.

CASE 2: Assume that $a \leq k \leq b$. Then $\sum_{i=1}^{k} d_i = kc$ and

$$k(k-1) + \sum_{i=k}^{a+b+c} \min\{k, d_i\} = k(k-1) + k(a + b - k) + ac = k(a + b - 1) + ac.$$ 

By our original assumption $bc \leq ab + ac + b(b - 1)$ so we know that $c \leq \frac{b(a + b - 1)}{b - a}$. In this case it remains to show that $kc \leq k(a + b - 1) + ac$. Since $b \geq k$, it is easily seen that

$$\frac{b}{b-a} \leq \frac{k}{k-a}.$$ 

Thus $c \leq \frac{b}{b-a}(a + b - 1) \leq \frac{k}{k-a}(a + b - 1)$ so that

$$c(k-a) \leq k(a + b - 1) \text{ or } kc \leq k(a+b-1)+ac.$$ 

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CASE 3: Suppose that \( b \leq k \leq a + b \). Then \( \sum_{i=1}^{k} d_i = bc + (k-b)b \) while

\[
\sum_{i=k+1}^{a+b+c} \min \{k, d_i\} = k(k-1) + k(a+b-k) + ac.
\]

Since

\[
c(b-a) = b(a + b - 1) = b^2 + b(a - 1) \leq b^2 + k(a - 1),
\]

it follows that \( bc - b^2 \leq k(a-1) + ac \). Adding \( kb \) to both sides, we obtain

\[
bc + (k-b)b \leq k(a+b-1) + ac.
\]

CASE 4: Suppose that \( a+b \leq k \leq a+b+c \). Then \( \sum_{i=1}^{k} d_i = bc + ab + a(k-a-b) \) while

\[
k(k-1) + \sum_{i=k+1}^{a+b+c} \min \{k, d_i\} = k(k-1) + a(a+b+c-k).
\]

We proceed by noticing that

\[
k^2 - (2a+1)k + 2a^2 - b(b-1)\]

is a quadratic in \( k \), which we will call \( f(k) \) and that the function \( f(k) \) has two real roots we will call \( r_1 \) and \( r_2 \), where

\[
r_1 = \frac{2a+1 + \sqrt{(2a+1)^2 - 4(2a^2 - b^2 + b)}}{2}\quad\text{and}\quad r_2 = \frac{2a+1 - \sqrt{(2a+1)^2 - 4(2a^2 - b^2 + b)}}{2}.
\]

Observe that \( f(k) \geq 0 \) when either \( k \leq r_2 \) or \( k \geq r_1 \). Since \( a(a-1) \geq 0 \) we have

\[
b(b-1) - a(a-1) + \frac{1}{4} \leq b(b-1) + \frac{1}{4}
\]

or

\[
b(b-1) - a(a-1) + \frac{1}{4} \leq \left( \frac{b-1}{2} \right)^2
\]

so that

\[
b - \frac{1}{2} \geq \sqrt{b(b-1) - a(a-1) + \frac{1}{4}}.
\]

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Thus,

\[2b + 2a \geq 2a + 1 + 2\sqrt{b(b-1) - a(a-1) + \frac{1}{4}}\]

or

\[
a + b \geq \frac{2a + 1 + 2\sqrt{b(b-1) - a(a-1) + \frac{1}{4}}}{2} = r_i
\]

By assumption, \( k \geq a + b \), thus \( k \geq r_i \) and \( f(k) \geq 0 \). Finally, we know that \( bc \leq b(b-1) + ab + ac \) and \( f(k) \geq 0 \) for \( a + b \leq k \leq a + b + c \) so we see

\[bc \leq b(b-1) + ab + ac + (k^2 - (2a + 1)k + 2a^2 - b(b-1)),\]

which is equivalent to \( bc + ab + a(k - a - b) \leq k(k-1) + a(a + b + c - k)\). \( \square \)

Thus, we see that the inequality is satisfied for every positive integer \( k \leq a + b + c - 1 \) and therefore, by the Erdős-Gallai criterion, the sequence \( \pi_s \) is graphical.

It is interesting to notice that by utilizing the construction method, aside from proving \( \pi_s \) is graphical, we also provide an algorithm for constructing a graph for \( \pi_s \). On the other hand, the Erdős-Gallai method only guarantees that such a graph is possible but provides no method for its construction.

Of particular importance, by providing the necessary condition to \( \pi_s \), we have disproved (2) of Conjecture 1.6.
CHAPTER 3

GRAPHICAL PERMUTATIONS FOR SETS OF FOUR INTEGERS

In this chapter, we look at the necessary and sufficient conditions for graphical permutations on sets with four integers. Of course the identity permutation follows directly from Theorem 1.3.

**THEOREM 3.1** Let $S = \{a, b, c, d\}$ with $1 < a < b < c < d$. Then

$$\pi_1 = \begin{pmatrix} a & b & c & d \\ a & b & c & d \end{pmatrix}$$

is graphical if and only if $d \leq a^2 + b^2 + c^2$.

Let $S = \{a, b, c, d\}$, where $1 < a < b < c < d$, and consider the permutations that fix exactly two of $a, b, c, \text{ and } d$. We call these permutations $\pi_2, \ldots, \pi_7$ as given below. The constructions of $\pi_2, \ldots, \pi_7$ are all similar in nature and so we only provide the proof for $\pi_1$.

**THEOREM 3.2** Let $S = \{a, b, c, d\}$ with $1 < a < b < c < d$. Then

1. $\pi_2 = \begin{pmatrix} a & b & c & d \\ a & b & d & c \end{pmatrix}$ is graphical if and only if $a$ and $b$ are of the same parity,

2. $\pi_3 = \begin{pmatrix} a & b & c & d \\ a & d & c & b \end{pmatrix}$ is graphical if and only if $a$ and $c$ are of the same parity,
(3) \( \pi_4 = \begin{pmatrix} a & b & c & d \\ d & b & c & a \end{pmatrix} \) is graphical if and only if \( b \) and \( c \) are of the same parity,

(4) \( \pi_5 = \begin{pmatrix} a & b & c & d \\ a & c & b & d \end{pmatrix} \) is graphical if and only if \( a \) and \( d \) are of the same parity and 
\[
d \leq a^2 + 2bc,
\]

(5) \( \pi_6 = \begin{pmatrix} a & b & c & d \\ c & b & a & d \end{pmatrix} \) is graphical if and only if \( b \) and \( d \) are of the same parity and 
\[
d \leq b^2 + 2ac,
\]

(6) \( \pi_7 = \begin{pmatrix} a & b & c & d \\ b & a & c & d \end{pmatrix} \) is graphical if and only if \( c \) and \( d \) are of the same parity and
\[
d \leq c^2 + 2ab.
\]

**PROOF** Assume that \( \pi_3 \) is graphical. This means that there exists a graph \( G \) of order \( a + b + c + d \) containing \( a \) vertices of degree \( a \), \( b \) vertices of degree \( c \), \( c \) vertices of degree \( b \), and \( d \) vertices of degree \( d \). Since the sum of the degrees of the vertices in \( G \) is even, \( a^2 + 2bc + d^2 \) must be even. Since \( 2bc \) is always even, \( a^2 + d^2 \) must also be even, thus it is necessary that \( a \) and \( d \) are of the same parity. Now define \( V(G) = A \cup B \cup C \cup D \), where \( A \) contains \( a \) vertices of degree \( a \), \( B \) contains \( b \) vertices of degree \( c \), \( C \) contains \( c \) vertices of degree \( b \), and \( D \) contains \( d \) vertices of degree \( d \). Let \( x \) denote the number of edges between the vertices of \( D \) and the vertices of \( A \cup B \cup C \). Since the degree of each vertex of \( A \) is \( a \), the degree of each vertex of \( B \) is \( c \), and the degree of each vertex of \( C \) is \( b \), we know at most \( a^2 + 2bc \) edges leave \( A \cup B \cup C \). Thus \( x \leq a^2 + 2bc \). Further, a vertex of \( D \) has degree at most \( d - 1 \)
in \( \langle D \rangle \). So at least \( d \) edges must leave the set \( D \). Hence \( x \geq d \). Therefore \( d \leq a^2 + 2bc \).

For the converse, assume that \( d \leq a^2 + 2bc \) and that \( a \) and \( d \) are of the same parity. Let \( V(G) = A \cup B \cup C \cup D \), where \( A = \{w_1, w_2, \ldots, w_a\} \), \( D = \{z_1, z_2, \ldots, z_d\} \), \( |B| = b \), and \( |C| = c \). Begin by placing \( K_{b,c} \) on the sets \( B \) and \( C \) so that the vertices of \( B \) all have degree \( c \) and the vertices of \( C \) all have degree \( b \). Place \( K_a \) on the set \( A \) and \( K_d \) on the set \( D \) so that the vertices of \( A \) and \( D \) need their degree increased by one. Place an edge between the vertices \( w_i \) and \( z_i \) for \( 1 \leq i \leq a \). Now the vertices of \( A \) have degree \( a \) and \( d - a \) vertices of \( D \) still need their degree increased by one. Since \( a \) and \( d \) are of the same parity, the difference \( d - a \) is even. Pair up the \( d - a \) vertices of \( D \) into \( (d - a)/2 \) pairs \( \{z_{a-1}, z_{a-2}\}, \ldots, \{z_{d-1}, z_d\} \). Notice that for each such pair \( \{z_i, z_{i+1}\} \) \( (a + 1 \leq i \leq d - 1) \) in \( D \), deleting an edge joining vertices \( v \) and \( v' \) in \( \langle A \cup B \cup C \rangle \) and adding two new edges \( z_i v \) and \( z_{i+1} v' \) does not affect the degree of any vertex in \( \langle A \cup B \cup C \rangle \) and increases the degree of each of the vertices \( z_i \) and \( z_{i+1} \) by one. Continue in this manner until all \( d - a \) vertices have their degree increased by one. Note that we have \( a(a - 1) + bc \) edges available from \( \langle A \cup B \cup C \rangle \) and that we need \( (d - a)/2 \) edges from \( \langle A \cup B \cup C \rangle \). Since \( d \leq a^2 + 2bc \) was part of our original assumption, we have enough edges to raise our \( d - a \) vertices of \( D \) one degree each and the construction is complete. □
We next consider the permutations \( \pi_8, \ldots, \pi_{15} \) that fix exactly one of \( a, b, c, d \). Two of these, which we denote by \( \pi_8 \) and \( \pi_9 \), have similar proofs and are included in the next result. In fact, since the constructions for \( \pi_8 \) and \( \pi_9 \) are similar in nature we only provide the proof for \( \pi_8 \).

**THEOREM 3.3** Let \( S = \{a, b, c, d\} \) with \( 1 \leq a < b < c < d \). Then

1. \( \pi_8 = \begin{pmatrix} a & b & c & d \\ a & c & d & b \end{pmatrix} \) is graphical if and only if \( a^2 + bc + cd + bd \) is even and
   \[ cd \leq a^2 + bc + bd + c(c-1), \]
   and

2. \( \pi_9 = \begin{pmatrix} a & b & c & d \\ c & b & d & a \end{pmatrix} \) is graphical if and only if \( ac + b^2 + cd + ad \) is even and
   \[ cd \leq ac + b^2 + ad + c(c-1). \]

**PROOF** Assume that \( \pi_8 \) is graphical. Then there exists a graph \( G \) of order \( a+b+c+d \) containing \( a \) vertices of degree \( a \), \( b \) vertices of degree \( c \), \( c \) vertices of degree \( d \), and \( d \) vertices of degree \( b \). Since the sum of the degrees of the vertices of \( G \) is even we have \( a^2 + bc + cd + db \) is even. Define \( V(G) = A \cup B \cup C \cup D \), where \( A \) contains \( a \) vertices of degree \( a \), \( B \) contains \( b \) vertices of degree \( c \), \( C \) contains \( c \) vertices of degree \( d \), and \( D \) contains \( d \) vertices of degree \( b \). Let \( x \) denote the number of edges that join vertices of \( C \) to vertices of \( A \cup B \cup D \). Since the degree of each vertex of \( A \) is \( a \), the degree of each vertex of \( B \) is \( c \), and the degree of each vertex of \( D \) is \( b \), we know at most \( a^2 + bc + bd \) edges leave \( A \cup B \cup D \). Thus \( x \leq a^2 + bc + bd \).
Further, a vertex of $C$ has degree at most $c - 1$ in $\langle C \rangle$. So at least $d - c + 1$ edges for each vertex of $C$ must leave the set $C$. Hence $x \geq c(d - c + 1)$. Therefore $c(d - c + 1) \leq a^2 + bc + bd$, or equivalently

$$cd \leq a^2 + bc + bd + c(c - 1).$$

For the converse, assume that $a^2 + bc + cd + db$ is even and $cd \leq a^2 + bc + bd + c(c - 1)$. Let $V(G) = A \cup B \cup C \cup D$, where $A = \{w_1, w_2, \ldots, w_a\}$, $C = \{y_1, y_2, \ldots, y_c\}$, and $D = \{z_1, z_2, \ldots, z_d\}$. Begin by placing the complete bipartite graph $K_{b,c}$ on the sets $B$ and $C$, using $B$ and $C$ as the partite sets, so that the vertices of $B$ have degree $c$ and the vertices of $C$ need degree their degree increased by $d - b$, and for all but Case 3, place $K_z$ on the set $A$ so that each vertex of $A$ needs its degree increased by one. The remainder of the construction is now divided into three cases depending whether $cd < bc + bd$, $bc + bd \leq cd \leq bc + bd + a^2$, or $bc + bd + a^2 < cd \leq bc + bd + a^2 + c(c - 1)$.

CASE 1: Suppose that $cd < bc + bd$. Since $c(d - b) \leq bd$, we add $c(d - b)$ edges in the following way. Join the vertex $y_1$ of $C$ to the vertices $z_1, z_2, \ldots, z_{d - b}$ of $D$. Join the vertex $y_2$ to $z_{d - b + 1}, z_{d - b + 2}, \ldots, z_{2(d - b)}$, where the subscripts are expressed modulo $d$. Continue in this way until each vertex of $C$ is joined to $d - b$ vertices of $D$. Denote this graph by $G'$. In $G'$ we see that the vertices of $B$ have degree $c$ and the vertices of $C$ have degree $d$. Define the integers $q$ and $r$ by

$$c(d - b) = dq + r,$$

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where \(0 \leq r \leq d - 1\). Distributing \(c(d - b)\) edges among the vertices of \(D\) we have \(r\) vertices that have degree \(q + 1\) in \(G'\) and \(d - r\) have degree \(q\) in \(G'\). Thus it remains to show that there exists a graph \(H\) of order \(d\) such that \(r\) vertices have degree \(b - q - 1\) and \(d - r\) have degree \(b - q\). By Lemma 2.1, this is possible if and only if
\[
(b - q - 1)r + (b - q)(d - r) = \text{even} \quad \text{and} \quad b - q - 1 < d - 1.
\]
Clearly \(b - q - 1 < d - 1\). Also
\[
(b - q - 1)r + (d - r)(b - q) = -r + db - dq = -cd + bc + bd,
\]
which is even if and only if \(a\) is even. We now consider two possibilities.

**SUBCASE 1.1** Assume that \(a\) is even. Then \(-cd + bc + bd\) is even and such a graph \(H\) exists and the vertices of \(A\) each need their degree increased by one. Pair up the \(a\) vertices of \(A\) into \(a/2\) pairs, \(\{w_1, w_2\}, \ldots, \{w_{a-1}, w_a\}\). Notice that for each such pair \(\{w_i, w_{i+1}\}\) \((1 \leq i \leq a - 1)\) in \(A\), deleting an edge that joins vertices \(v\) and \(v'\) in \(<B \cup C \cup D>\) and adding two new edges \(w_iv\) and \(w_{i+1}v'\) does not affect the degree of any vertex in \(<B \cup C \cup D>\) and increases the degree of each of the vertices \(w_i\) and \(w_{i+1}\) by one. Continue in this manner until all \(a\) vertices of \(A\) have their degree increased by one. Note that we have \(1/2(bc + cd + bd)\) edges available from \(<B \cup C \cup D>\) for this procedure and that we need \(a/2\) edges from \(<B \cup C \cup D>\). Since \(\frac{a}{2} \leq \frac{(bc + cd + bd)}{2}\), we have enough edges to raise the degree of each of our \(a\) vertices of \(A\) by one and the construction is complete.
SUBCASE 1.2 Assume that \( a \) is odd. Then we add an edge from the vertex \( w_a \) of \( A \) to a vertex of \( D \) that has degree \( q \) in \( G' \) (such a vertex exists since \( d - r \geq 1 \)). In this way, we have \( r + 1 \) vertices of degree \( q + 1 \) in \( G' \) and \( d - r - 1 \) of degree \( q \) in \( G' \). Thus it remains to show that there exists a graph \( H' \) of order \( d \) such that \( r + 1 \) vertices have degree \( b - q - 1 \) and \( d - r - 1 \) have degree \( b - q \). By Lemma 2.1, this is possible if and only if \( (b - q - 1)(r + 1) + (b - q)(d - r - 1) \) is even and \( b - q - 1 < d - 1 \).

Clearly \( b - q - 1 < d - 1 \). Also

\[
(b - q - 1)(r + 1) + (b - q)(d - r - 1) = -r + bd - dq - 1 = -cd + bc + bd - 1.
\]

Since \( a \) is odd we notice that \( -cd + bc + bd \) is odd and that \( -cd + bc + bd + 1 \) is even so that such a graph \( H' \) exists. Proceed as in Subcase 1.1 by pairing the \( a - 1 \) vertices \( w_1, w_2, \ldots, w_{a-1} \) of \( A \) into \( \frac{a - 1}{2} \) pairs and increase their degree each by one in the same manner.

CASE 2: Suppose that \( bc + bd \leq cd \leq bc + bd + a^2 \). Since \( bd \leq c(d - b) \), we now add \( bd \) edges in the following way. Join the vertex \( z_1 \) of \( D \) to the vertices \( y_1, y_2, \ldots, y_b \), of \( C \). Join the vertex \( z_2 \) to \( y_{b+1}, y_{b+2}, \ldots, y_{2b} \), where the subscripts are expressed modulo \( c \). Continue in this way until each vertex of \( D \) is joined to \( b \) vertices of \( C \). Denote this graph \( G' \). In \( G' \) we see that the vertices of \( B \) have degree \( c \) and the vertices of \( D \) have degree \( b \). Define the integers \( q \) and \( r \) by

\[
bd = cq + r,
\]

where \( 0 \leq r \leq c - 1 \).
Distributing \( bd \) edges among the vertices of \( C \) we have \( r \) vertices of degree \( q + 1 \) in \( G' \) and \( c - r \) of degree \( q \) in \( G' \). Thus it remains to show that there exists a graph \( H \) of order \( c \) such that \( r \) vertices have degree \( d - b - q - 1 \) and \( c - r \) have degree \( d - b - q \). By Lemma 2.1, this is possible if and only if \((d - b - q - 1)r + (d - b - q)(c - r)\) is even and \( d - b - q - 1 < c - 1 \). By definition, \( r \leq c - 1 \) and it is easily seen that \( c - 1 < (c + a)(c - a) \) so that \( r < c^2 - a^2 \), or equivalently \( a^2 < c^2 - r \). By assumption \( cd \leq bc + bd + a^2 \) and \( bc + bd + a^2 < bc + bd + c^2 - r \) so that \( cd < bc + bd + c^2 - r \).

Reordering terms we see that

\[
|d - b - (bd - r)| < c^2
\]

and dividing by \( c \) and subtracting 1 we have

\[
d - b - \frac{(bd - r)}{c} - 1 < c - 1
\]

or equivalently, \( d - b - q - 1 < c - 1 \). Also,

\[
(d - b - q - 1)r + (d - b - q)(c - r) = -r + cd - bc - cq = -bd + cd - bc.
\]

If \( a \) is even, we see that \( -bd + cd - bc \) is even and such a graph \( H \) exists and the vertices of \( A \) each need their degree increased by one. Proceed to pair up the vertices of \( A \) as in Subcase 1.1 to complete the construction. Now if \( a \) is odd, we add an edge from the vertex \( w_a \) of \( A \) to a vertex of \( C \) that has degree \( q \) so that now \( r + 1 \) vertices have degree \( q + 1 \) in \( G' \) and \( c - r - 1 \) have degree \( q \) in \( G' \). Thus it remains to show that there exists a graph \( H' \) of order \( c \) such that \( r + 1 \) vertices have degree \( d - b - q - 1 \) and \( c - r - 1 \) have degree \( d - b - q \). By Lemma 2.1, this is
possible if and only if \((d-b-q-1)(r+1)+(d-b-q)(c-r-1)\) is even and \(d-b-q-1 < c-1\). We have already seen \(d-b-q-1 < c-1\). Also

\[(d-b-q-1)(r+1)+(d-b-q)(c-r-1) = -r + cd - bc - cq - 1 = -bc + cd - bd - 1.\]

Since \(a\) is odd we notice that \(-bc + cd - bd\) is odd so that \(-bc + cd - bd - 1\) is even and such a graph \(H'\) exists. Proceed as before by pairing the vertices \(w_1, w_2, ..., w_{d-1}\) of \(A\) into \(\frac{a-1}{2}\) pairs and increase the degree of each vertex by one in the same manner as Subcase 1.2.

**CASE 3** Suppose that \(bc + bd + a^2 \leq cd \leq bc + bd + a^2 + c(c-1)\). Recall that in this case, the construction begins with \(K_{b,c}\) on the partite sets \(B\) and \(C\), but not with \(K_a\) on \(A\). Each vertex of \(C\) now has degree \(b\) and needs \(d-b\). Since \(bd + a^2 < c(d-b)\), we add \(a^2 + bd\) edges in the following way. Join the vertex \(z_1\) of \(D\) to the vertices \(y_1, y_2, ..., y_b\) of \(C\). Join the vertex \(z_2\) to \(y_{b+1}, y_{b+2}, ..., y_{2b}\), where the subscripts are expressed modulo \(c\). Continue in this way until each vertex of \(D\) is joined to \(b\) vertices of \(C\). Let \(y_j\) be the last vertex of \(C\) that was connected to \(z_d\). Next, we joint the vertex \(w_1\) of \(A\) to the vertices \(y_{j+1}, y_{j+2}, ..., y_{j+a}\) of \(C\). Join the vertex \(w_2\) to \(y_{j+a+1}, y_{j+a+2}, ..., y_{2(j+a)}\). Continue in this way until each vertex of \(A\) is joined to \(a\) vertices of \(C\). Denote this graph \(G'\). In \(G'\) we see that the vertices of \(B\) have degree \(c\), the vertices of \(D\) have degree \(b\), and the vertices of \(A\) have degree \(a\). Define the integers \(q\) and \(r\) by

\[a^2 + bd = cq + r,\]
where \( 0 \leq r \leq c-1 \).

Distributing \( a^2 + bd \) edges among the vertices of \( C \) we have \( r \) vertices of degree \( q + 1 \) in \( G' \) and \( c-r \) of degree \( q \) in \( G' \). Thus it remains to show that there exists a graph \( H \) of order \( c \) such that \( r \) vertices have degree \( d-b-q-1 \) and \( c-r \) have degree \( d-b-q \). By Lemma 2.1, this is possible if and only if \((d-b-q-1)r + (d-b-q)(c-r)\) is even and \( d-b-q-1 < c-1 \). By definition, \( 0 \leq r \leq c-1 \) so that \( 0 < c-r \). By assumption \( cd \leq bc + bd + a^2 + c(c-1) \) and

\[
bc + bd + a^2 + c(c-1) < bc + bd + a^2 + c(c-1) + c-r
\]

and thus \( cd < bc + bd + a^2 + c(c-1) + c-r \). Reordering terms we notice

\[
cd - bc - (a^2 + bd - r) - c < c(c-1).
\]

Dividing by \( c \) we have

\[
d - b - \frac{a^2 + bd - r}{c} - 1 < c-1
\]

or \( d-b-q-1 < c-1 \).

Also

\[
(d-b-q-1)r + (d-b-q)(c-r) = -r + cd - bc - cq = -a^2 - bd + cd - bc,
\]

where we see that \( -a^2 - bd + cd - bc \) is even. Thus such a graph \( H \) exists. \( \square \)

Recall that the permutations \( \pi_{10}, \ldots, \pi_{13} \) fix exactly one of \( a, b, c, \) and \( d \). The proofs for these four permutations are similar so we will just outline the construction for the permutation \( \pi_{10} = \begin{pmatrix} a & b & c & d \\ a & d & b & c \end{pmatrix} \).
Define the sets $A$, $B$, $C$, and $D$ in the usual way. Place the graph $K_a$ on the set $A$, and the graph $K_{b,d}$ on the sets $B$ and $D$. Now the vertices of $B$ have degree $d$, while the vertices of $C$ need degree $b$ and the vertices of $D$ need degree $c-b$. Divide the construction into two cases.

The first case is when $d(c-b)<bc$. Proceed to distribute $d(c-b)$ edges of $D$ among the vertices of $C$. Use Lemma 2.1 to complete the graph on $\langle C \rangle$. To finish the construction we pair up the vertices of $A$ (all of them or all but one, depending on the parity of $a$) that need their degree increased by one and raise each degree using edges from $\langle B \cup C \cup D \rangle$ as described earlier.

The second case is when $bc \leq d(c-b)$. Now distribute $bc$ edges of $C$ among the vertices of $D$. Use Lemma 2.1 to complete the graph on $\langle D \rangle$. Again, pair up the vertices of $A$ (or all but one) that need their degree increased by one and raise each degree by one using edges from $\langle B \cup C \cup D \rangle$.

**Theorem 3.4** Let $S = \{a, b, c, d\}$ with $1 \leq a < b < c < d$. Then

1. \(\pi_{10} = \begin{pmatrix} a & b & c & d \\ a & d & b & c \end{pmatrix}\) is graphical if and only if $a^2 + bd + bc + cd$ is even,

2. \(\pi_{11} = \begin{pmatrix} a & b & c & d \\ d & b & a & c \end{pmatrix}\) is graphical if and only if $ad + b^2 + ac + cd$ is even,

3. \(\pi_{12} = \begin{pmatrix} a & b & c & d \\ d & a & c & b \end{pmatrix}\) is graphical if and only if $ad + ab + c^2 + bd$ is even, and
(4) \( \pi_{13} = \begin{pmatrix} a & b & c & d \\ c & a & b & d \end{pmatrix} \) is graphical if and only if \( ac + ab + bc + d^2 \) is even and \( d \leq ab + bc + ac \).

Of the remaining two permutations \( \pi_{14} \) and \( \pi_{15} \) that fix exactly one of \( a, b, c, \) and \( d \), we will provide the characterization for \( \pi_{14} \) next and will handle \( \pi_{15} \) in Chapter 4.

**THEOREM 3.5** Let \( S = \{a, b, c, d\} \) with \( 1 \leq a < b < c < d \). Then

\[
\pi_{14} = \begin{pmatrix} a & b & c & d \\ b & d & c & a \end{pmatrix}
\]

is graphical if and only if \( ab + bd + c^2 + ad \) is even and \( bd \leq ab + bc + ad + b(b-1) \).

**PROOF** Assume that \( \pi_{14} \) is graphical. Then there exists a graph \( G \) of order \( a+b+c+d \) containing \( a \) vertices of degree \( b \), \( b \) vertices of degree \( d \), \( c \) vertices of degree \( c \), and \( d \) vertices of degree \( a \). Since the sum of the degrees of the vertices of \( G \) is even we have \( ab + bd + c^2 + ad \) is even. Define \( V(G) = A \cup B \cup C \cup D \), where \( A \) contains \( a \) vertices of degree \( b \), \( B \) contains \( b \) vertices of degree \( d \), \( C \) contains \( c \) vertices of degree \( c \), and \( D \) contains \( d \) vertices of degree \( a \). Let \( x \) denote the number of edges that join vertices of \( B \) to vertices of \( A \cup C \cup D \). Since the degree of each vertex of \( A \) is \( b \), the degree of each vertex of \( C \) is \( c \) but each vertex of \( C \) can be adjacent only to \( b \) vertices of \( B \), and the degree of each vertex of \( D \) is \( a \), we know at most \( ab + bc + ad \) edges leave \( A \cup C \cup D \). Thus \( x \leq ab + bc + ad \). Further, a vertex
of \( B \) has degree at most \( b-1 \) in \( \langle B \rangle \). So at least \( d-b+1 \) edges for each vertex of \( B \) must leave the set \( B \). Hence \( x \geq b(d-b+1) \). Therefore \( b(d-b+1) \leq ab + bc + ad \), or equivalently

\[
bd \leq ab + bc + ad + b(b-1).
\]

For the converse, assume that \( ab + bd + c^2 + ad \) is even and \( bd \leq ab + bc + ad + b(b-1) \). Let \( V(G) = A \cup B \cup C \cup D \), where \( |A| = a, \ |B| = b, \ |C| = c, \) and \( |D| = d \). Begin by placing the complete bipartite graph \( K_{a,b} \) on the sets \( A \) and \( B \), using \( A \) and \( B \) as the partite sets, so that the vertices of \( A \) have degree \( b \) and the vertices of \( B \) need degree their degree increased by \( d-a \). The remainder of the construction is now divided into three cases depending whether \( bd \leq ab + ad \), \( ab + ad < bd \leq ab + ad + bc \), or \( ab + ad + bc < bd \leq ab + ad + bc + b(b-1) \).

**CASE 1** Suppose that \( bd \leq ab + ad \). Begin by placing the complete graph \( K_c \) on the set \( C \). Since \( b(d-a) \leq ad \), we will distribute \( b(d-a) \) edges from the vertices of \( B \) among the vertices of \( D \). Define the integers \( q \) and \( r \) by

\[
b(d-a) = dq + r,
\]

where \( 0 \leq r \leq d-1 \). After distributing \( b(d-a) \) edges among the vertices of \( D \) we have \( r \) vertices that need degree \( a-q-1 \) and \( d-r \) vertices that need degree \( a-q \).

By Lemma 2.1, this is possible if and only if \( (a-q-1)r + (a-q)(d-r) \) is even and \( a-q-1 < d-1 \). Clearly \( a-q-1 < d-1 \). Also

\[
(a-q-1)r + (d-r)(a-q) = aq + ad - dq = ad - bd + ab,
\]
which is even if and only if \( c \) is even. If \( c \) is odd, we add an edge from a vertex of \( D \) that needed \( a - q \) and a vertex of \( C \) so that \((a-q-1)(r+1)+(d-r-1)(a-q)\) is even and we can get the degrees of the vertices in \( D \) to be \( a \) as desired. Finally depending on whether \( c \) is even or odd, we pair up the \( c \) or \( c - 1 \) vertices of \( C \) needing their degree increased by one and use edges from \( \langle B \cup D \rangle \) to get their degrees increased by one.

**CASE 2** Suppose that \( ab + ad < bd \leq ab + ad + bc \). We now distribute \( ad \) edges from the set \( D \) among the vertices of \( B \). Define the integers \( q' \) and \( r' \) by

\[
ad = q' b + r',
\]

where \( 0 \leq r' \leq b - 1 \). Now the vertices of \( D \) each have degree \( a \) while \( r' \) vertices of \( B \) need degree \( d - a - q' - 1 \) and \( b - r' \) vertices need degree \( d - a - q' \).

**SUBCASE 2.1** Suppose that \( d - a - q' - 1 < b - 1 \). Now the total degree needed in \( B \) is \( bd - ab - ad \). Since \( bd - ab - ad \leq bd \), we distribute \( bd - ab - ad \) edges from the set \( B \) among the vertices of \( C \). Define the integers \( q \) and \( r \) by

\[
bd - ab - ad = qc + r,
\]

where \( 0 \leq r \leq c - 1 \). The construction further depends on \( q \). If \( q \geq 1 \), we use Lemma 2.1 to raise the vertices of \( C \) to degree \( c \) each; whereas if \( q = 0 \), we place the complete graph \( K_c \) on \( C \) so that \( r \) vertices have degree \( c \) and \( c - r \) vertices need their degree increased by one. By noticing that \( c - r = c - bd + ab + ad \) we find that...
$c - r$ is even and so we pair these vertices up and increase the degree of each pair by one using edges from $(B \cup D)$.

**SUBCASE 2.2** Suppose that $d - a - q' - 1 \geq b - 1$. Place the complete graph $K_b$ on the set $B$ so that the degree sum still needed in $B$ is $bd - ab - ad - b(b - 1)$. This is accomplished by distributing $bd - ab - ad - b(b - 1)$ edges from the set $B$ among the vertices of $C$. Define the integers $q$ and $r$ by

$$bd - ab - ad - b(b - 1) = qc + r,$$

where $0 \leq r \leq c - 1$. The construction further depends on $q$. If $q \geq 1$, the construction is completed by using Lemma 2.1 to raise each vertex of $C$ to degree $c$; whereas if $q = 0$, we place the complete graph $K_c$ on $C$ so that $r$ vertices have degree $c$ and $c - r$ vertices need their degree increased by one. Again, $c - r$ is even so we pair up these vertices and increase each degree by one using edges from $(B \cup D)$.

**CASE 3** Suppose that $ab + ad + bc \leq bd \leq ab + ad + bc + b(b - 1)$. Place the complete bipartite graph $K_{b,c}$ on the sets $B$ and $C$ and distribute $ad$ edges from the vertices of $D$ among the vertices of $B$. Define the integers $q$ and $r$ by

$$ad = bq + r,$$

where $0 \leq r \leq b - 1$. After the distribution, we find that in $B$, $r$ vertices need their degree increased by $d - a - c - q - 1$ and $b - r$ need their degree increased by $d - a - c - q$. In addition, the $c$ vertices of $C$ each need their degree increased by
The construction is further divided into two cases depending on the parity of \( c(c-b) \).

If \( c(c-b) \) is even then we use Lemma 2.1 on the vertices of \( C \) to raise their degree to \( c \) and also use Lemma 2.1 on \( B \) to raise the degree of each vertex to degree \( d \). On the other hand, if \( c(c-b) \) is odd, we remove an edge from the complete bipartite graph \( K_{b,c} \), where the edge removed is incident to a vertex of \( B \) that needs degree \( d-a-c-q \). Now the parity conditions in \( B \) and \( C \) hold and we are able to apply Lemma 2.1 to each set to complete the construction. □

The next three permutations \( \pi_{16}, \pi_{17}, \pi_{18} \) consist of the product of two transpositions. Their proofs are all similar so we only verify that \( \pi_{16} \) is graphical.

**Theorem 3.6** Let \( S = \{a, b, c, d\} \) with \( 1 \leq a < b < c < d \). Then

1. \( \pi_{16} = \begin{pmatrix} a & b & c & d \\
                         b & a & d & c \end{pmatrix} \) is always graphical,
2. \( \pi_{17} = \begin{pmatrix} a & b & c & d \\
                         c & d & a & b \end{pmatrix} \) is always graphical, and
3. \( \pi_{18} = \begin{pmatrix} a & b & c & d \\
                         d & c & b & a \end{pmatrix} \) is always graphical.

**Proof** Let \( V(G) = A \cup B \cup C \cup D \), where \( |A| = a \), \( |B| = b \), \( |C| = c \), and \( |D| = d \).

Place the complete bipartite graph \( K_{a,b} \) on the sets \( A \) and \( B \) so that the vertices of \( A \) have degree \( b \) and the vertices of \( B \) have degree \( a \). Place the complete bipartite graph
K_{c,d} on the sets C and D so that the vertices of C have degree d and the vertices of D have degree c. Thus, regardless of our choice of a, b, c or d, such a graph will always exist. □

The permutations π_{19}, ..., π_{24} are all similar in that each is a cycle of length four. The constructions of π_{19}, π_{30}, and π_{21} are similar in nature and therefore we will provide the proof for π_{19} but omit the remaining two.

**THEOREM 3.7** Let S = \{a, b, c, d\} with 1 ≤ a < b < c < d. Then

1. \(\pi_{19} = \begin{pmatrix} a & b & c & d \\ b & c & d & a \\ d & a & b & c \end{pmatrix}\) is graphical if and only if a and d or b and c have the same parity,

2. \(\pi_{20} = \begin{pmatrix} a & b & c & d \\ d & c & a & b \end{pmatrix}\) is graphical if and only if a and b or c and d have the same parity, and

3. \(\pi_{21} = \begin{pmatrix} a & b & c & d \\ d & a & b & c \end{pmatrix}\) is graphical if and only if a and c or b and d have the same parity.

**PROOF** Assume that \(\pi_{19}\) is graphical. Then there exists a graph G of order \(a+b+c+d\) containing a vertices of degree d, b vertices of degree c, c vertices of degree a, and d vertices of degree b. Since the sum of the degrees of the vertices of G
is even we have \( ad + bc + ac + bd \) is even. Notice that \( ad + bc + ac + bd = (a + b)(c + d) \), which is even when \( a \) and \( b \) or \( c \) and \( d \) have the same parity.

For the converse, assume that \( a \) and \( b \) or \( c \) and \( d \) have the same parity. Let \( V(G) = A \cup B \cup C \cup D \), where \( B = \{x_1, x_2, \ldots, x_a\} \) and \( D = \{z_1, z_2, \ldots, z_d\} \). Begin placing the complete bipartite graph \( K_{a,d} \) on the sets \( A \) and \( D \), using \( A \) and \( D \) as the partite sets, so that the vertices of \( A \) have degree \( d \) and the vertices of \( D \) need degree their degree increased \( b - a \). Also place the complete bipartite graph \( K_{a,c} \) on the vertices \( x_1, x_2, \ldots, x_a \) of \( B \) and the set \( C \), using the \( a \) vertices of \( B \) and the set \( C \) as the partite sets. Then the vertices of \( C \) have degree \( a \) and the vertices \( x_1, x_2, \ldots, x_a \) of \( B \) have degree \( c \) while the vertices \( x_{a+1}, x_{a+2}, \ldots, x_b \) of \( B \) still need degree \( c \).

We now add \( c(b - a) \) edges in the following way. Join the vertex \( x_{a+1} \) of \( B \) to the vertices \( z_1, z_2, \ldots, z_c \) of \( D \). Join the vertex \( x_{a+2} \) to \( z_{c+1}, z_{c+2}, \ldots, z_{2c} \), where the subscripts are expressed modulo \( d \). Continue in this way until each of the \( b - a \) vertices of \( B \) are joined to \( c \) vertices of \( D \). Denote this graph by \( G' \). In \( G' \) we see that the vertices of \( B \) have degree \( c \). Define the integers \( q \) and \( r \) by

\[
c(b - a) = dq + r,
\]

where \( 0 \leq r \leq d - 1 \). Distributing \( c(b - a) \) edges among the vertices of \( D \) we have \( r \) vertices that have degree \( q + 1 \) in \( G' \) and \( d - r \) have degree \( q \) in \( G' \). Thus it remains to show that there exists a graph \( H \) of order \( d \) such that \( r \) vertices have degree \( b - a - q - 1 \) and \( d - r \) have degree \( b - a - q \). By Lemma 2.1, this is possible if and only if \( (b - a - q - 1)r + (b - a - q)(d - r) \) is even and \( b - a - q - 1 < d - 1 \). Clearly \( b - a - q - 1 < d - 1 \). Also

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\[(b-a-q-1)r+(d-r)(b-a-q) = -r + bd - ad - dq = -bc + ac + bd - ad,\]

which is even if and only if \(a\) and \(b\) or \(c\) and \(d\) are of the same parity. Therefore the graph \(H\) exists and the construction is complete. \(\square\)

The constructions for \(\pi_{22}\) and \(\pi_{23}\), both cycles of length four, are similar. Therefore we will just outline the construction for \(\pi_{22} = (a b c d, b c d a)\).

Define the sets \(A, B, C,\) and \(D\) in the usual way. We begin the construction by placing the complete bipartite graph \(K_{b,c}\) on the sets \(B\) and \(C\). Now the vertices of \(B\) have degree \(c\) and the vertices of \(C\) need degree \(d-b\). The construction is now divided into three cases.

The first case is when \(c(d-b) \leq ab\). Place the complete bipartite graph \(K_{a,b}\) on the set \(A\) and \(b\) vertices of the set \(D\). Notice that \(d-b\) vertices of the set \(D\) need degree \(a\). Distribute \(a(d-b)\) edges from the set \(D\) among the vertices of \(C\) and use Lemma 2.1 on \(\langle C\rangle\) to complete the construction.

The next case is when \(ab \leq c(d-b) \leq ab + ad\). The construction is further divided into two subcases depending on whether or not we are able to place a complete graph on the set \(C\). If \(d-b < c-1\), distribute \(ab\) edges from the set \(A\) to the vertices of \(C\) and use Lemma 2.1 to complete the graph on \(\langle C\rangle\). Depending on the parity of \(ad\), either \(d\) vertices will need \(a\) or \(d-1\) vertices will need \(a\) while one vertex will need degree \(a-1\). Again, we use Lemma 2.1 to complete the graph on \(D\). If \(d-b \geq c-1\), place the complete graph \(K_c\) on \(C\), and distribute \(ad\) edges from \(A\) into \(C\) using Lemma 2.1.
to complete the graph on \( (C) \). Once again, depending on the parity of \( ad \), either \( d \) vertices will need \( a \) or \( d - 1 \) vertices will need \( a \) while one vertex will need degree \( a - 1 \). Finally, as in the first subcase, use Lemma 2.1 on the set \( D \) thus completing the construction.

Finally, we consider the case when \( ab + ad \leq c(d - b) \leq ab + ad + c(c - 1) \). Here we distribute \( ab \) edges from the set \( A \) and \( ad \) edges from the set \( D \). The vertices of \( A \) now have degree \( b \) and the vertices of \( D \) have degree \( a \), whereas \( r \) vertices of \( C \) need degree \( d - b - q - 1 \) and \( c - r \) need degree \( d - b - q \). By using Lemma 2.1 on the set \( C \), we find that such a graph is possible and the construction is complete.

**Theorem 3.8** Let \( S = \{a, b, c, d\} \) with \( 1 \leq a < b < c < d \). Then

1. \( \pi_{22} = \begin{pmatrix} a & b & c & d \\ b & c & d & a \end{pmatrix} \) is graphical if and only if \( a \) and \( c \) or \( b \) and \( d \) have the same parity and \( cd \leq ab + bc + ad + c(c - 1) \), and

2. \( \pi_{23} = \begin{pmatrix} a & b & c & d \\ c & a & d & b \end{pmatrix} \) is graphical if and only if \( a \) and \( d \) or \( b \) and \( c \) have the same parity and \( cd \leq ac + ab + bd + c(c - 1) \).

Finally, we will provide the result for \( \pi_{24} \), the remaining cycle of length four.

**Theorem 3.9** Let \( S = \{a, b, c, d\} \) with \( 1 \leq a < b < c < d \). Then
\[ \pi_{24} = \begin{pmatrix} a & b & c & d \\ c & d & b & a \end{pmatrix} \] is graphical if and only if \( a \) and \( b \) or \( c \) and \( d \) have the same parity and either (1) if \( c \leq a + b - 1 \) then \( bd \leq ab + bc + ad + b(b-1) \) or (2) if \( c \geq a + b - 1 \) then \( ac + bd \leq bc + ad + (a+b)(a+b-1) \).

**Proof** Assume that \( \pi_{24} \) is graphical. Then there exists a graph \( G \) of order \( a+b+c+d \) containing \( a \) vertices of degree \( c \), \( b \) vertices of degree \( d \), \( c \) vertices of degree \( b \), and \( d \) vertices of degree \( a \). Since the sum of the degrees of the vertices of \( G \) is even we have \( ac + bd + bc + ad = (a+b)(c+d) \) is even. So either \( a \) and \( b \) have the same parity or \( c \) and \( d \) have the same parity. Define \( V(G) = A \cup B \cup C \cup D \), where \( A \) contains \( a \) vertices of degree \( c \), \( B \) contains \( b \) vertices of degree \( d \), \( C \) contains \( c \) vertices of degree \( b \), and \( D \) contains \( d \) vertices of degree \( a \). Suppose that \( c \leq a + b - 1 \) and let \( x \) denote the number of edges that join vertices of \( B \) to vertices of \( A \cup C \cup D \). Since the degree of each vertex of \( A \) is \( c \) but each vertex of \( A \) can be adjacent to only \( b \) vertices of \( B \), the degree of each vertex of \( C \) is \( b \), and the degree of each vertex of \( D \) is \( a \), we know at most \( ab + bc + ad \) edges leave \( A \cup B \cup D \). Thus \( x \leq ab + bc + ad \). Further, a vertex of \( B \) has degree at most \( b - 1 \) in \( \langle B \rangle \). So at least \( d - b + 1 \) edges for each vertex of \( B \) must leave the set \( B \). Hence \( x \geq b(d - b + 1) \). Therefore \( b(d - b + 1) \leq ab + bc + ad \), or equivalently

\[ bd \leq ab + bc + ad + b(b-1) . \]

Now suppose that \( c > a + b - 1 \) and let \( x \) denote the number of edges that join vertices of \( A \cup B \) to vertices of \( C \cup D \). Since the degree of each vertex of \( C \) is \( b \) and the degree of each vertex of \( D \) is \( a \), we know at most \( bc + ad \) edges leave \( C \cup D \). Thus
Further, a vertex of $A \cup B$ has degree at most $a + b - 1$ in $(A \cup B)$. So at least $d - a - b + 1$ edges for each vertex of $B$ and at least $c - a - b + 1$ edges for each vertex of $A$ must leave the set $A \cup B$. Hence $x \geq b(d - a - b + 1) + a(c - a - b + 1)$. Therefore $b(d - a - b + 1) + a(c - a - b + 1) \leq bc + ad$, or equivalently

$$ac + bd \leq bc + ad + (a + b)(a + b - 1).$$

For the converse, assume that $a$ and $b$ or $c$ and $d$ have the same parity and let $V(G) = A \cup B \cup C \cup D$, where $|A| = a$, $|B| = b$, $|C| = c$, and $|D| = d$. Begin by placing the complete bipartite graph $K_{2x2}$ on the sets $B$ and $C$, using $B$ and $C$ as the partite sets, so that the vertices of $C$ have degree $b$ and the vertices of $B$ need degree their degree increased by $d - c$. The remainder of the construction is now divided into three cases depending whether $bd \leq ab + bc$, $ab + bc < bd \leq ab + ad + bc$, or $ab + ad + bc < bd \leq ab + ad + bc + b(b - 1)$.

**CASE 1:** Suppose that $bd \leq ab + bc$. Since $b(d - c) \leq ab$, we distribute $b(d - c)$ edges from the set $B$ to $A$. Define the integers $q$ and $r$ by

$$b(d - c) = aq + r,$$

where $0 \leq r \leq a - 1$. Denote this graph by $G'$. In $G'$ we see that the vertices of $B$ have degree $d$, the vertices of $C$ have degree $b$, and in $A$ we have $r$ vertices that have degree $q + 1$ in $G'$ and $a - r$ have degree $q$ in $G'$. We now consider two possibilities.
SUBCASE 1.1 Assume that $c-q-1 < a-1$. It remains to construct a graph with $r$ vertices of degree $c-q-1$, $a-r$ vertices of degree $c-q$, and $d$ vertices of degree $a$. Place the complete bipartite graph $K_{a,c-q-1}$ on the vertices of $A$ and $c-q-1$ vertices of $D$. In $A$, $r$ vertices now have degree $c$ and $a-r$ vertices have degree $c-1$ while in $D$, $d-c+q+1$ vertices need degree $a$. Now if $a$ and $r$ have the same parity use Lemma 2.1 to raise the remaining vertices of $D$ to degree $a$, while if $a$ and $r$ are of opposite parity, add an edge between a vertex of $D$ needing degree $a$ and a vertex of $A$ needing its degree increased by one and then use Lemma 2.1. The remaining $a-r$ or $a-r-1$ vertices can be paired and their degree increased by one using edges from $(B\cup C)$.

SUBCASE 1.2 Assume that $c-q-1 \geq a-1$. Proceed by placing the complete graph $K_a$ on the set $A$, so that $r$ vertices need their degree increased by $c-q-a$ and $a-r$ need their degree increased by $c-q-a+1$. Place $K_{a,c-q-a}$ on the vertices of $A$ and $c-q-a$ vertices of $D$ so that $a-r$ vertices of $A$ need their degree increased by one and $d-c+q+a$ vertices of $D$ need their degree increased by $a$. Notice that $d-c+q+a > a-r$, so we add $a-r$ edges between the vertices in $A$ who need their degree increased by one and $a-r$ vertices of the $d-c+q+a$ in $D$ who need their degree increased by $a$. Thus it remains to show that there exists a graph $H$ of order $d-c+q+a$ such that $a-r$ vertices have degree $a-1$ and $d-c+q+r$ have degree $a$. By Lemma 2.1, this is possible if and only if $(a-r)(a-1)+a(d-c+q+r)$ is even and $a < d-c+q+a-1$. Clearly $a < d-c+q+a-1$. Also
\[(a-r)(a-1)+a(d-c+q+r)=a(a-1)+ad-ac+bd-bc\] is even and such a graph \(H\) exists.

**CASE 2:** Suppose that \(ab+bc<bd\leq ab+bc+ad\). Place \(K_{a,b}\) on the sets \(A\) and \(B\) and notice that the vertices of \(A\) need their degree increased by \(c-b\) while the vertices of \(B\) need their degree increased by \(d-c-a\). The construction is further divided into two cases.

**SUBCASE 2.1** Suppose that \(c\leq a+b-1\). Proceed by distributing \(b(d-c-a)\) edges among the vertices of \(D\). Define the integers \(q\) and \(r\) by

\[b(d-c-a)=dq+r,\]

where \(0\leq r\leq d-1\). Distributing \(b(d-c-a)\) edges among the vertices of \(D\) we have \(r\) vertices of \(D\) that have degree \(q+1\) and \(d-r\) have degree \(q\). Thus it remains to show that there exists a graph \(H\) of order \(d\) such that \(r\) vertices have degree \(a-q-1\) and \(d-r\) have degree \(a-q\). By Lemma 2.1, this is possible if and only if \((a-q-1)r+(a-q)(d-r)\) is even and \(a-q-1<d-1\). Clearly \(a-q-1<d-1\).

Also \((a-q-1)r+(a-q)(d-r)=-r+ad-dq=bc-bd+ad+ab\), and

\[bc-bd+ad+ab=bc-bd+ad+ac-a(c+b)\]

which is even if and only if \(a(c+b)\) is even. When \(a(c+b)\) is even the graph on \(D\) may be finished and since \(c-b\leq a-1\) we use Lemma 2.1 to obtain the desired degrees for the vertices in \(A\). If \(a(c+b)\) is odd, we add an edge from a vertex in \(A\) to a vertex.
in $D$ needing degree $a-q$ so that the parity condition now holds and we may finish both $A$ and $D$ using Lemma 2.1.

**SUBCASE 2.2** Suppose that $c > a + b - 1$. Proceed by placing the graph $K_{a-b}$ on the sets $A$ and $B$. Now each vertex of $A$ needs its degree increased by $c - a - b + 1$ while each vertex of $B$ needs its degree increased by $d - c - b - a + 1$. We accomplish this by distributing $a(c - a - b + 1) + b(d - c - b - a + 1)$ edges from $A \cup B$ to the set $D$. Notice that $a(c - a - b + 1) + b(d - c - b - a + 1) \leq ad$ since $c > a + b - 1$. Define the integers $q$ and $r$ by

$$a(c - a - b + 1) + b(d - c - b - a + 1) = qd + r,$$

where $0 \leq r \leq d - 1$. Distributing these edges among the vertices of $D$ we have $r$ vertices that need their degree increased by $a - q - 1$ and $d - r$ that need degree $a - q$. By Lemma 2.1, this is possible if and only if $(a - q - 1)r + (a - q)(d - r)$ is even and $a - q - 1 < d - 1$. Clearly $a - q - 1 < d - 1$. Also

$$(a - q - 1)r + (a - q)(d - r) = -r + ad - dq = ad - ac - bd + bc + 2ab + (a + b)(a + b - 1),$$

which is even.

**CASE 3** Suppose that $ab + ad + bc < bd \leq ab + ad + bc + b(b - 1)$. The construction is once again divided into two subcases.
SUBCASE 3.1 Suppose that $c \leq a + b - 1$. Proceed by placing the graph $K_{a,b}$ on the sets $A$ and $B$ and distribute $ad$ edges from the set $D$ to the vertices of $B$. Define the integers $q$ and $r$ by

$$ad = bq + r,$$

where $0 \leq r \leq b - 1$. Thus it remains to show that there exists a graph $H$ of order $b$ such that $r$ vertices have degree $d - c - a - q - 1$ and $b - r$ have degree $d - c - a - q$. By Lemma 2.1, this is possible if and only if $d - c - a - q - 1 < b - 1$. By definition $q = \frac{ad - r}{b}$ and the inequality reduces to $bd < ab + bc + ad + b(b - 1) + b - r$. By assumption $bd \leq ab + bc + ad + b(b - 1)$ and since $b - r > 0$, the inequality holds. Also,

$$(d - c - a - q - 1)r + (d - c - a - q)(b - r) = -r + bd - bc - ab - bq$$

and

$$-r + bd - bc - ab - bq = -ad + bd - bc + ac - a(b + c).$$

Thus, if $a(b + c)$ is even then such a graph $H$ is possible on $B$ and since $c - b \leq a - 1$ we can use Lemma 2.1 to complete the set $A$. If $a(b + c)$ is odd, we remove an edge from the complete bipartite graph $K_{a,b}$ so the necessary parity conditions will hold and again the construction can be finished by using Lemma 2.1 on both $A$ and $B$.

SUBCASE 3.2 Suppose that $c > a + b - 1$. To complete the construction we place the graph $K_{a,b}$ on the vertices of $A$ and $B$. Now each vertex of $A$ needs its degree increased by $c - a - b + 1$ while each vertex of $B$ needs its degree increased by $d - c - b - a + 1$. We accomplish this by distributing $a(c - a - b + 1) + b(d - c - b - a + 1)$ edges from the set $D$ to the vertices of $B$. Define the integers $q$ and $r$ by

$$ad = bq + r,$$

where $0 \leq r \leq b - 1$. Thus it remains to show that there exists a graph $H$ of order $b$ such that $r$ vertices have degree $d - c - a - q - 1$ and $b - r$ have degree $d - c - a - q$. By Lemma 2.1, this is possible if and only if $d - c - a - q - 1 < b - 1$. By definition $q = \frac{ad - r}{b}$ and the inequality reduces to $bd < ab + bc + ad + b(b - 1) + b - r$. By assumption $bd \leq ab + bc + ad + b(b - 1)$ and since $b - r > 0$, the inequality holds. Also,

$$(d - c - a - q - 1)r + (d - c - a - q)(b - r) = -r + bd - bc - ab - bq$$

and

$$-r + bd - bc - ab - bq = -ad + bd - bc + ac - a(b + c).$$

Thus, if $a(b + c)$ is even then such a graph $H$ is possible on $B$ and since $c - b \leq a - 1$ we can use Lemma 2.1 to complete the set $A$. If $a(b + c)$ is odd, we remove an edge from the complete bipartite graph $K_{a,b}$ so the necessary parity conditions will hold and again the construction can be finished by using Lemma 2.1 on both $A$ and $B$. 

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edges from \( A \cup B \) to the set \( D \). Notice that 
\[ a(c-a-b+1)+b(d-c-b-a+1) \leq ad \]
since \( c > a+b-1 \). Define the integers \( q \) and \( r \) by 
\[ a(c-a-b+1)+b(d-c-b-a+1) = qd + r, \]
Thus it remains to show that there exists a graph \( H \) of order \( d \) such that \( r \) vertices have 
degree \( a-q-1 \) and \( d-r \) have degree \( a-q \). By Lemma 2.1, this is possible if and
only if \( (a-q-1)r+(a-q)(d-r) \) is even and \( a-q-1 < d-1 \). Clearly
\( a-q-1 < d-1 \). Also
\[ (a-q-1)r+(a-q)(d-r) = -r + ad - dq = ad - ac - bd + bc + 2ab + (a+b)(a+b-1), \]
which is even. □
CHAPTER 4

GRAPHICAL PERMUTATIONS FOR SETS OF N INTEGERS

As mentioned in Chapter 1, in [4], it was conjectured that for a set $S = \{a_1, a_2, \ldots, a_n\}$ with $1 \leq a_1 < a_2 < \cdots < a_n$, (1) a permutation $\pi$ of $S$ such that

$$\sum_{i=1}^{n} a_i \pi(a_i)$$

is even and $\pi(a_n) = a_n$ is graphical if and only if $a_n \leq \sum_{i=1}^{n-1} a_i \pi(a_i)$ and (2) a permutation $\pi$ such that $\sum_{i=1}^{n} a_i \pi(a_i)$ is even and $\pi(a_n) \neq a_n$ is graphical. In this chapter we study a particular class of permutations and characterize when they are graphical, thus providing evidence to support the truth of the conjecture in certain cases. Of course, we know from Theorem 2.2 that (2) is false in general. First we give necessary and sufficient conditions for a permutation that interchanges exactly two integers to be graphical and then we prove part (1) of the conjecture.

**THEOREM 4.1** Let $S = \{a_1, a_2, \ldots, a_n\}$ be a set of positive integers with $a_1 < a_2 < \cdots < a_n$, and for $1 \leq i < j \leq n$, let $\pi_{i,j}$ be the permutation of $S$ that interchanges $a_i$ and $a_j$ but fixes $a_k$ for every $k \neq i, j$. Then (1) for $j \neq n$, the permutation $\pi_{i,j}$ is graphical if and only if $\sum_{k \neq i, j} a_k$ is even and the inequality

$$\sum_{i=1}^{n} a_i \pi(a_i)$$

is even and $\pi(a_n) = a_n$ is graphical. In this chapter we study a particular class of permutations and characterize when they are graphical, thus providing evidence to support the truth of the conjecture in certain cases. Of course, we know from Theorem 2.2 that (2) is false in general. First we give necessary and sufficient conditions for a permutation that interchanges exactly two integers to be graphical and then we prove part (1) of the conjecture.
\[ a_n \leq 2a_j + \sum_{k=n}^j a_k^2 \] holds, and (2) for \( j = n \), the permutation \( \pi_{i,j} \) is graphical if and only if \( \sum_{k=n}^j a_k \) is even.

**PROOF** To simplify the notation we reindex the sequence \( a_1, a_2, \ldots, a_n \) in the following way calling the new sequence \( b_1, b_2, \ldots, b_n \). When \( j \neq n \), define

\[
b_k = \begin{cases} 
  a_k & 1 \leq k \leq i-1 \\
  a_{k+1} & i \leq k \leq j-2 \\
  a_{k+2} & j-1 \leq k \leq n-2 \\
  a_i & k = n-1 \\
  a_j & k = n 
\end{cases}
\]

while if \( j = n \), define

\[
b_k = \begin{cases} 
  a_k & 1 \leq k \leq i-1 \\
  a_{k+1} & i \leq k \leq n-2 \\
  a_i & k = n-1 \\
  a_n & k = n 
\end{cases}
\]

and define the permutation \( \sigma \) so that \( \sigma = \begin{pmatrix} b_1 & b_2 & \cdots & b_{n-2} & b_{n-1} & b_n \\ b_1 & b_2 & \cdots & b_{n-2} & b_n & b_{n-1} \end{pmatrix} \). Thus we see that \( \sigma \) simply shifts the two integers \( a_i \) and \( a_j \) to the end of the sequence and reindexes the terms. We may also notice that \( b_1 < b_2 < \cdots < b_{n-2} \). We now restate the theorem using our new notation. The permutation \( \sigma \) of \( \{b_1, b_2, \ldots, b_n\} \) is graphical if and only if \( \sum_{k=1}^{n-2} b_k \) is even and, in case \( j \neq n \), \( b_{n-2} \leq 2b_{n-1} + b_n + \sum_{k=1}^{n-1} b_k^2 \).
Assume that $\sigma$ is graphical. Thus there exists a graph $G$ of order $\sum_{k=1}^{n} b_k$ containing $b_k$ vertices of degree $b_k$ for $1 \leq k \leq n-2$, $b_{n-1}$ vertices of degree $b_n$, and $b_n$ vertices of degree $b_{n-1}$. Since the sum of degrees of the vertices is even,

$$\sum_{k=1}^{n-2} b_k^2 + 2b_{n-1}b_n$$

is even. Notice that $2b_{n-1}b_n$ is always even, which implies the sum $\sum_{k=1}^{n-2} b_k$ is even. Now consider when $j \neq n$. For each $k$ ($1 \leq k \leq n$), let $B_k$ denote the set of $b_k$ vertices of degree $\sigma(b_k)$. Let $x$ denote the number of edges that join the vertices of $B_{n-2}$ to the vertices of $V(G) - B_{n-2}$. Since the degree of each vertex of $B_k$ is $b_k$ for $1 \leq k \leq n-3$ and the degree of each vertex in $B_{n-1}$ is $b_n$, while the degree of each vertex of $B_n$ is $b_{n-1}$, we see that at most $\sum_{k=1}^{n-3} b_k^2 + 2b_{n-1}b_n$ edges leave $V(G) - B_{n-2}$. Thus $x \leq \sum_{k=1}^{n-3} b_k^2 + 2b_{n-1}b_n$. Further, a vertex in $B_{n-2}$ has at most degree $b_{n-2} - 1$ in $\langle B_{n-2} \rangle$. So at least one edge per vertex of $B_{n-2}$ must leave the set $B_{n-2}$. Hence $x \geq b_{n-2}$. Therefore $b_{n-2} \leq \sum_{k=1}^{n-1} b_k^2 + 2b_{n-1}b_n$.

Now assume that $\sum_{k=1}^{n-2} b_k$ is even and when $j \neq n$, suppose further that $b_{n-2} \leq 2b_{n-1}b_n + \sum_{k=1}^{n-1} b_k^2$. We now construct a graph $G$ with the desired properties. Begin by defining the sets $B_k$ for every $k$ ($1 \leq k \leq n$), so that the set $B_k$ has $b_k$ vertices. On the sets $B_{n-1}$ and $B_n$ place the complete bipartite graph $K_{b_{n-1}, b_n}$ so that each vertex of $B_{n-1}$ has degree $b_n$ and each vertex of $B_n$ has degree $b_{n-1}$. On each of the
remaining $B_k$ sets $(1 \leq k \leq n-2)$ we place the complete graph $K_{b_k}$. Now in the graph $\bigcup_{k=1}^{n-2} B_k$ each vertex needs one more edge to satisfy its degree requirements. Proceed by joining each vertex of $B_1$ to one vertex of $B_2$ (so that no vertex of $B_2$ gets its degree increased by more than one). Now the vertices of $B_1$ all have degree $b_1$ and $b_2 - b_1$ vertices of $B_2$ still need their degree increased by one. Again, connect each of the $b_2 - b_1$ vertices of $B_2$ to one vertex of $B_3$ (making sure no two vertices of $B_2$ get joined to the same vertex of $B_3$). Now all of the vertices of $B_2$ have degree $b_2$ and $b_3 - b_2 + b_1$ vertices of $B_3$ need their degree increased by one. Continue in this manner until all the vertices of $B_k$ have degree $b_k$ for each $k$ $(1 \leq k \leq n-3)$. See Figure 3.1.

Figure 4.1 First step in the process of constructing a graph associated with $\pi_{n,i}$.

The remainder of the construction depends on the parity of $n$. 
CASE 1 Assume that \( n \) is even. Now we have \( \sum_{i=1}^{n-1} (b_{2i} - b_{2i-1}) \) vertices of \( B_{n-2} \) who need their degree increased by one. Let \( B'_{n-2} \) denote this subset of \( B_{n-2} \). By assumption, \( \sum_{k=1}^{n-2} b_k \) is even so that \( \sum_{i=1}^{n-1} (b_{2i} - b_{2i-1}) \) is even. Since the number of vertices in \( B'_{n-2} \) is even, we may form \( \frac{1}{2} \sum_{i=1}^{n-1} (b_{2i} - b_{2i-1}) \) pairs. Then for each pair \( u, v \) in \( B'_{n-2} \) we remove one edge \( xy \) from \( \bigcup_{k=1}^{n-3} B_k \cup B_{n-1} \cup B_n \) and add the two edges \( ux \) and \( vy \). This process does not change the degrees of \( x \) and \( y \) but increases the degree of each \( u \) and \( v \) by one. We must now ensure there are enough edges available to do this.

We have \( \frac{1}{2} \sum_{k=1}^{n-2} b_k (b_k - 1) \) edges from the complete graphs placed on sets \( B_1, B_2, \ldots, B_{n-3} \). Counting the edges between pairs of (consecutive) sets \( B_1, B_2, \ldots, B_{n-3}, B_{n-4} \), we see that there are \( \sum_{i=1}^{n-2} b_{2i-1} \) edges joining the sets \( B_1, \ldots, B_{n-4} \).

Now, joining the sets \( B_{n-3} \) and \( B_{n-4} \) we have \( \sum_{i=1}^{n-2} (b_{2i} - b_{2i-1}) \) edges for a total of \( \sum_{i=1}^{n-2} b_{2i} \) edges joining the sets \( B_1, B_2, \ldots, B_{n-3} \). Finally, we have \( b_{n-1} b_n \) edges from our complete bipartite graph on the sets \( B_{n-1} \) and \( B_n \). To have enough edges to increase each pair of vertices of \( B_{n-2} \) we must have the following inequality hold:
\[ \frac{1}{2} \sum_{i=1}^{\frac{n-1}{2}} (b_{2i} - b_{2i-1}) \leq \frac{1}{2} \sum_{k=1}^{\frac{n-1}{2}} b_k (b_k - 1) + \sum_{i=1}^{\frac{n-2}{2}} b_{2i} + b_{n-1} b_n. \]

By the hypothesis, we have
\[ b_{n-2} \leq \sum_{k=1}^{\frac{n-1}{2}} b_k^2 + 2b_{n-1} b_n. \]

Thus
\[ b_{n-2} \leq \sum_{k=1}^{\frac{n-1}{2}} b_k^2 + \sum_{k=1}^{\frac{n-1}{2}} b_k - \sum_{k=1}^{\frac{n-1}{2}} b_{2i} + 2b_{n-1} b_n \]

or
\[ b_{n-2} \leq \sum_{k=1}^{\frac{n-1}{2}} b_k^2 + \left( \sum_{i=1}^{\frac{n-1}{2}} b_{2i} + \sum_{i=1}^{\frac{n-1}{2}} b_{2i-1} \right) - \sum_{k=1}^{\frac{n-1}{2}} b_k + 2b_{n-1} b_n \]

so that by adding \( \sum_{i=1}^{\frac{n-1}{2}} b_{2i} - \sum_{i=1}^{\frac{n-1}{2}} b_{2i-1} \) to both sides, we obtain
\[ b_{n-2} + \sum_{i=1}^{\frac{n-1}{2}} b_{2i} - \sum_{i=1}^{\frac{n-1}{2}} b_{2i-1} \leq \sum_{k=1}^{\frac{n-1}{2}} b_k^2 - \sum_{k=1}^{\frac{n-1}{2}} b_k + 2 \sum_{i=1}^{\frac{n-1}{2}} b_{2i} + 2b_{n-1} b_n. \]

Therefore
\[ \sum_{i=1}^{\frac{n-1}{2}} (b_{2i} - b_{2i-1}) \leq \sum_{k=1}^{\frac{n-1}{2}} b_k (b_k - 1) + 2 \sum_{i=1}^{\frac{n-2}{2}} b_{2i} + 2b_{n-1} b_n \]

and by multiplying through by \( \frac{1}{2} \), we obtain the desired inequality.

**CASE 2** Assume that \( n \) is odd. In this case there are \( b_{n-2} + \sum_{i=1}^{\frac{n-1}{2}} (b_{2i-1} - b_{2i}) \) vertices of \( B_{n-2} \) that need their degree increased by one. Let \( B'_{n-2} \) denote these vertices. As in
Case 1, it is clear that $B'_{n-2}$ contains an even number of vertices. Thus we proceed as before and delete one edge $xy$ of $\left( \bigcup_{k=1}^{n-3} B_k \bigcup B_{n-1} \bigcup B_n \right)$ for each pair $u, v$ of vertices in $B'_{n-2}$ and add the two edges $ux$ and $vy$, thereby increasing the degrees of $u$ and $v$ each by one and leaving the degrees of all other vertices unchanged. Again, we must count the available edges and make sure there are enough to increase the degree of each vertex in $B'_{n-2}$. We have $\frac{1}{2} \sum_{k=1}^{n-3} b_k (b_k - 1)$ edges from the complete graphs placed on the sets $B_1, B_2, \ldots, B_{n-3}$. There are also $\sum_{i=1}^{n-3} b_{2i-1}$ edges between the complete graphs on $B_1, B_2, \ldots, B_{n-3}$ and $b_{n-1} b_n$ edges from the complete bipartite graph placed on the sets $B_{n-1}$ and $B_n$. Thus we must have the following inequality hold:

$$\frac{1}{2} \left( b_{n-2} + \sum_{i=1}^{n-3} (b_{2i-1} - b_{2i}) \right) \leq \frac{1}{2} \sum_{k=1}^{n-3} b_k (b_k - 1) + \sum_{i=1}^{n-3} b_{2i-1} + b_{n-1} b_n$$

Using similar algebraic steps as in Case 1, we see that this inequality is equivalent to $b_{n-2} \leq 2b_{n-1} b_n + \sum_{k=1}^{n-3} b_k^2$, which is provided by the initial hypothesis and thus concludes the construction. □

In Chapter 1, we have seen how the Erdős-Gallai and Havel-Hakimi criterion can be used to verify that a given sequence is graphical and in Chapters 2 and 3 we have utilized a construction method to provide the proofs that some specific permutations are
graphical. We now introduce a criterion of Fulkerson-Hoffman-McAndrew [2] (see also [6]) that determines if a given sequence is graphical.

**THEOREM 4.2** (Fulkerson-Hoffman-McAndrew) Let \( s : d_1, d_2, \ldots, d_n \) be a sequence with \( d_1 \geq d_2 \geq \cdots \geq d_n \geq 1 \), where \( n \geq 2 \). Then \( s \) is graphical if and only if for each \( k = 1, 2, \ldots, n, \) and \( m \) with \( k + m \leq n \),

\[
\sum_{i=1}^{k} d_i \leq k(n-m-1) + \sum_{i=n-m+1}^{n} d_i .
\]

Using this theorem, we now prove part (1) of Conjecture 1.6.

**THEOREM 4.3** Let \( S = \{a_1, a_2, \ldots, a_n\} \) be a set of integers such that \( 1 \leq a_1 < a_2 < \cdots < a_n \) and let \( \pi \) be a permutation of \( S \) such that \( \pi(a_n) = a_n \). Then \( \pi \) is graphical if and only if \( \sum_{i=1}^{n} a_i \pi(a_i) \) is even and \( a_n \leq \sum_{i=1}^{n-1} a_i \pi(a_i) \).

**PROOF** Let \( s = a_1 + a_2 + \cdots + a_n \) and let \( \sigma = \pi^{-1} \). For a given \( r \) \((1 \leq r \leq n)\), we define \( d_r = a_r \) if and only if \( \sum_{j=r}^{n} \sigma(a_j) + 1 \leq i \leq \sum_{j=r}^{n} \sigma(a_j) \). We use Theorem 4.2 to show that the sequence \( d_1, d_2, \ldots, d_n \) is graphical. Notice that this sequence is the sequence

\[ a_n, \ldots, a_n, a_{n-1}, \ldots, a_{n-1}, \ldots, a_2, \ldots, a_2, a_1, \ldots, a_1, \]

where the first \( a_n \) terms are \( a_n \), the next \( \sigma(a_{n-1}) \) terms are \( a_{n-1} \), the next \( \sigma(a_{n-2}) \) terms are \( a_{n-2} \), \ldots, and the last \( \sigma(a_1) \) terms are \( a_1 \). This sequence corresponds to \( \pi \).
Thus we must show that for each \( k = 1, 2, \ldots, s \) and \( m \) with \( 0 \leq m \leq s-k \), the following inequality holds

\[
\sum_{i=1}^{s} d_i \leq k(s-m-1) + \sum_{i=m}^{s} d_i \tag{4.1}
\]

We now divide the proof into three cases that relate \( k \) to \( a_n \).

CASE 1 Suppose that \( k = a_n \). By the hypothesis, we have \( a_n \leq \sum_{i=1}^{n-1} a_i \sigma(a_i) \) so that

\[
\sum_{i=1}^{s} d_i \leq a_n^2 - a_n + \sum_{i=1}^{n-1} a_i \sigma(a_i) = a_n^2 - a_n + \sum_{i=1}^{n-1} a_i \sigma(a_i)
\]

Since \( k = a_n \), we have \( \sum_{i=1}^{a_n} d_i = a_n^2 \). If \( m = s-a_n \), then the right hand side of (4.1) is

\[
a_n(s-(s-a_n)-1) + \sum_{i=s-(s-a_n)-1}^{n-1} d_i = a_n^2 - a_n + \sum_{i=1}^{n-1} a_i \sigma(a_i) = a_n^2 - a_n + \sum_{i=1}^{n-1} a_i \sigma(a_i)
\]

and we see that inequality (4.1) holds for \( k = a_n \) and \( m = s-a_n \). Now let

\[
0 \leq m < s-a_n. \text{ Then } s-a_n - m - 1 \geq 0 \text{ and } \sum_{i=m}^{s} d_i \geq 0 \text{ so that }
\]

\[
\sum_{i=1}^{s} d_i = a_n^2 \leq a_n(s-a_n + (s-a_n - m - 1)) + \sum_{i=m}^{s} d_i = a_n(s-m-1) + \sum_{i=m}^{s} d_i
\]

and thus inequality (4.1) holds for \( k = a_n \) and for every \( m \) \((0 \leq m \leq s-a_n-1)\).

CASE 2 Suppose that \( k > a_n \). Since \( k > a_n \), we know that \( d_k \) must fall somewhere in the sequence \( d_{a_n-1}, \ldots, d_s \). Thus there exists an integer \( r \) \((1 \leq r \leq n-1)\) such that \( d_k = a_r \) and

\[
1 + \sum_{i=r+1}^{n} \sigma(a_i) \leq k \leq \sum_{i=r}^{n} \sigma(a_i).
\]
We define the integer \( j \) \((1 \leq j \leq \sigma(a_r))\) so that \( k = \sum_{i=r}^{n} \sigma(a_i) + j \). Now we see that

\[
\sum_{i=1}^{k} d_i = a^2 + a_{n-1} \sigma(a_{n-1}) + \cdots + a_{r+1} \sigma(a_{r+1}) + ja_r = \sum_{i=r}^{n} a_i \sigma(a_i) + ja_r.
\]

Let \( m \) be an integer with \( 0 \leq m \leq s - k \), or equivalently, we can say \( k \leq s - m \leq s \).

Since \( k > a_n \) we see that \( k \geq a_n + 1 \) so that \( a_n + 1 \leq s - m \), or \( a_n \leq s - m - 1 \). Observe that the right hand side of (4.1) is

\[
k(s - m - 1) + \sum_{i=s-m+1}^{s} d_i \geq \left( \sum_{i=r}^{n} \sigma(a_i) + j \right) a_n = \sum_{i=r}^{n} a_i \sigma(a_i) + ja_n \geq \sum_{i=r}^{n} a_i \sigma(a_i) + ja_r,
\]

and thus inequality (4.1) holds for \( k > a_n \) and for every \( m \) \((0 \leq m \leq s - k)\).

CASE 3 Suppose that \( k < a_n \). Observe that \( \sum_{i=1}^{k} d_i = ka_n \). First, let \( m \) be a nonnegative integer such that \( m \leq s - a_n - 1 = a_1 + a_2 + \cdots + a_{n-1} - 1 \). Notice that \( \sum_{i=s-m+1}^{s} d_i \geq 0 \) and by assumption \( a_n \leq s - m - 1 \) so that \( ka_n \leq k(s - m - 1) + \sum_{i=s-m+1}^{s} d_i \). Thus inequality (4.1) holds for every \( m \) with \( 0 \leq m \leq a_1 + a_2 + \cdots + a_{n-1} - 1 \). Next let \( a_1 + a_2 + \cdots + a_{n-1} \leq m \leq s - k \). We define the integer \( j \) \((0 \leq j \leq a_n - k)\) such that \( m = a_1 + a_2 + \cdots + a_{n-1} + j \). Notice that since \( k < a_n \) we have \( kj \leq a_n j \) and by assumption \( a_n \leq \sum_{i=1}^{n} a_i \pi(a_i) \) so that \( k \leq \sum_{i=1}^{n} a_i \pi(a_i) \). Using these facts, we observe that

\[
kj + k \leq a_n j + \sum_{i=1}^{n} a_i \pi(a_i) \quad \text{and by adding} \quad ka_n - (kj + k) \quad \text{to both sides of this inequality,}
\]

we have
$$ka_n \leq k(a_n - j - 1) + a_n j + \sum_{i=1}^{s-1} a_i \pi(a_i).$$

Noticing that $a_n - j - 1 = s - m - 1$ and $a_n j + \sum_{i=1}^{s-1} a_i \pi(a_i) = \sum_{i=s-m}^{s-m-1} d_i$, we see that inequality (4.1) holds for $k < a_n$ and for every $m$ ($a_1 + a_2 + \cdots + a_{n-1} \leq m \leq s - k$) completing the proof. □

Using Theorem 4.3, we are now able to state the conditions for $\pi_{15}$ without having to use a construction based proof.

**THEOREM 4.4** Let $S = \{a, b, c, d\}$ with $1 \leq a < b < c < d$. Then

$$\pi_{15} = \begin{pmatrix} a & b & c & d \\ b & c & a & d \end{pmatrix}$$

is graphical if and only if $ab + bc + ac + d^2$ is even and $d \leq ab + bc + ac$. 

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CHAPTER 5

CONJECTURES AND CONCLUSION

Recall that in Chapter 1, we stated a standing conjecture that had two parts. The first, Conjecture 1.6 (1) stated that, for a permutation \( \pi \) on the set \( S = \{a_1, a_2, \ldots, a_n\} \) with \( 1 \leq a_1 < a_2 < \cdots < a_n \) such that \( \sum_{i=1}^{n} a_i \pi(a_i) \) is even and \( \pi(a_n) = a_n \), the permutation \( \pi \) is graphical if and only if \( a_n \leq \sum_{i=1}^{n-1} a_i \pi(a_i) \). Using the criterion of Fulkerson, Hoffman, and McAndrew we proved that this conjecture is true. The second, Conjecture 1.6 (2), stated that a permutation \( \pi \) meeting the same conditions as (1) but with \( \pi(a_n) \neq a_n \), is always graphical. As we have seen in Chapter 2, this conjecture is false and \( \pi \) provides a counterexample to the conjecture.

We conclude with a revision of Conjecture 1.6 (2).

CONJECTURE 5.1 Let \( S = \{a_1, a_2, \ldots, a_n\} \) be a set of integers with \( 1 \leq a_1 < a_2 < \cdots < a_n \) \((n \geq 2)\) and let \( \pi \) be a permutation on \( S \) with \( \pi(a_n) \neq a_n \) and \( \sum_{i=1}^{n} a_i \pi(a_i) \) even. Let \( m \) be the smallest integer from \( \{1, 2, \ldots, n\} \) such that

\[
a_n \pi(a_n) = \max\{a_i \pi(a_i) \mid 1 \leq i \leq n\}
\]

and
\[ L = \{ a_i \mid \pi(a_i) > a_m, \ 1 \leq i \leq n \}. \]

Then,

(1) if \( a_m < \pi(a_m) \), then \( \pi \) is graphical if and only if

\[ a_m \pi(a_m) \leq \sum_{i \in L} a_i \pi(a_i) + \sum_{i \in \bar{L}} a_i a_m + a_m (a_m - 1), \]

and

(2) if \( a_m \geq \pi(a_m) \), then \( \pi \) is graphical.
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