Representation of special polynomials by the cycle indicator

Jack Lund Schofield
University of Nevada, Las Vegas

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REPRESENTATION OF SPECIAL POLYNOMIALS

BY THE CYCLE INDICATOR

by

Jack L. Schofield Jr.

Bachelor of Science
University of Nevada, Las Vegas
1997

A thesis submitted in partial fulfillment
of the requirements for the

Master of Science degree
Department of Mathematical Sciences
College of Sciences

Graduate College
University of Nevada, Las Vegas
August 2002
Thesis Approval
The Graduate College
University of Nevada, Las Vegas

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Jack L. Schofield Jr.

Entitled
Representation of Special Polynomials by the Cycle Indicator

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Master of Science in Mathematical Sciences

Examination Committee Chair

Dean of the Graduate College

Examination Committee Member

Examination Committee Member

Graduate College Faculty Representative
ABSTRACT

Representation of Special Polynomials
By the Cycle Indicator

by

Jack L. Schofield Jr.

Dr. Peter Shiue, Examination Committee Chair
Professor of Mathematics
University of Nevada, Las Vegas

Certain classes of special functions have been shown to have Cycle Indicator representations as well as recurrence relations. Most remarkably, it is shown how the Cycle Indicator can be used to unify or generalize special functions. Methods of unifying special functions are elaborated on. It follows that classical special functions with simple logarithms of their generating functions can be classified in this way. Also, there are counter-examples where the Cycle Indicator doesn’t represent the special functions given.

This thesis represents a study of a journal article called “Cycle Indicators and Special Functions” by Leetsch C. Hsu and Peter Jau-Shyong Shiue in the Annals of Combinatorics (cf. Hsu and Shiue, [10]).
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PREFACE

Dedicated to my new bride, Janette, and my valiant and loyal parents.

This thesis will introduce the Cycle Indicator, as well as theorems, corollaries, proofs, definitions, examples, applications, and research related to the Cycle Indicator relations, identities, formulas and special functions. It will demonstrate the computations and derivations involving the use of the cycle indicator to discover some generalizations of special functions for which, heretofore, there has been no unifying theory.

The introduction has three parts laying the foundation of the cycle indicator. Chapters 1, 2, 3 introduces cycle classes, permutation groups, the formula for calculating the number of permutations of a cycle group, proofs, and examples. It defines the cycle indicator and gives examples, and gives the $C_n$-type representation formulation of the formal power series $\Phi(t)$ with corollaries and proofs. In Chapters 4, 5 examples of cycle indicator representations of the generating functions of some special functions are given, along with a definition of $C_n$-representability. This is followed by perhaps the most interesting results of cycle indicator applications involving the generalization or unification of special functions, where, heretofore, there have been none. Also, a counter example is given where the cycle indicator fails to apply. Chapters 6, 7, 8 on applications, explore cycle indicator representations of a whole class of special functions called the Sheffer special function polynomials. Likewise the same is done with the Gegenbauer
special function polynomials. Also Recurrence Relations of Special Functions are explored.

ACKNOWLEDGMENTS

I would like to thank the committee members for their time and assistance. In particular, thank you to Dr. Shiue for his most dedicated direction, counsel, ideas, inspiration, etc. From the outset he has been the most enthusiastic supporter. He is so knowledgeable about so many thesis subjects. I had indicated that I wanted to research some area that spanned number theory, and analysis and he came up with, what for me was, a perfect match of pure and applied math. I was so excited and became more excited as my understanding of the subject grew. As it stands now I couldn’t be more excited. I am forever grateful to him for leading me to and into a subject of extreme interest. He has worked tirelessly in my behalf.

Also, because I may never get another chance to formally express how utterly thankful I am to the whole Math Dept. at UNLV, including the professors, the students, my graduate advisor, who allowed me so much freedom to try to get the most out of my graduate experience, the graduate coordinator Dr. Burke, and most especially to the previous and current chairs of the Math Department, for their support for and faith in me, to my boss Pat, and to, perhaps the ones most of us are most grateful to for their heartfelt service to us all, the math secretaries and support staff, I do so now express my utter appreciation, and the warmest feelings of highest regard, for all you’ve done for me, forever. Thank you all.
CHAPTER I

INTRODUCTION: CYCLE CLASSES

Key Words:

Cycle Indicator, $C_n$-representation, $C_n$-representability

Definition 1.1:

Let $k_1, k_2, \ldots, k_n$ be non-negative integers such that $k_1 + 2k_2 + \cdots + nk_n = n$, where $n$ is a natural number. Let $S_n$ be the symmetric group of $n$ letters. If $\sigma(n) \in S_n$ contains $k_1$ unit cycles, $k_2$ 2-cycles, $\ldots$, $k_n$ $n$-cycles, then $\sigma(n)$ is called the cycle class $1^{k_1} 2^{k_2} \cdots n^{k_n}$.

Example 1.1:

$(123) \in S_3$ is of cycle class $1^0 2^0 3^1$ because $k_1 + 2k_2 + 3k_3 = 3$, $k_i \geq 0$. $k_1 = 0, k_2 = 0, k_3 = 1$, so the cycle class is $1^0 2^0 3^1$.

Example 1.2:

In $S_3$, there are six permutations of 3 elements, and they can be divided into cycle classes as follows:

<table>
<thead>
<tr>
<th>Class</th>
<th>Permutations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1^0 2^0 3^1$</td>
<td>(123), (132)</td>
</tr>
<tr>
<td>$1^1 2^1 3^0$</td>
<td>(12)(3), (13)(2), (23)(1)</td>
</tr>
<tr>
<td>$1^1 2^0 3^0$</td>
<td>(1)(2)(3)</td>
</tr>
</tbody>
</table>
Example 1.3:

The permutations in $S_4$ provide another example: there are 24 permutations:

<table>
<thead>
<tr>
<th>Class</th>
<th>Permutations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1^6 2^0 3^0 4^4$</td>
<td>(1234), (1243), (1324), (1342), (1423), (1432)</td>
</tr>
<tr>
<td>$1^6 2^0 3^1 4^0$</td>
<td>(123)(4), (124)(3), (132)(4), (134)(2),</td>
</tr>
<tr>
<td></td>
<td>(142)(3), (143)(2), (234)(1), (243)(1)</td>
</tr>
<tr>
<td>$1^6 2^1 3^0 4^0$</td>
<td>(12)(3)(4), (13)(2)(4), (14)(2)(3),</td>
</tr>
<tr>
<td>$1^6 2^2 3^0 4^0$</td>
<td>(12)(34), (13)(24), (14)(23)</td>
</tr>
<tr>
<td>$1^6 2^0 3^2 4^0$</td>
<td>(1)(2)(3)(4). □</td>
</tr>
</tbody>
</table>

Theorem 1.1:

Let $\sigma(n) \in S_n$. If $C(k_1, k_2, \ldots, k_n)$ is the number of permutations of cycle class $1^{k_1} 2^{k_2} \cdots n^{k_n}$ then

$$C(k_1, k_2, \ldots, k_n) = \frac{n!}{1^{k_1} 2^{k_2} \cdots n^{k_n} k_1! k_2! \cdots k_n!} \quad (1.1)$$

where $k_1 + 2k_2 + \cdots + nk_n = n$, $k_i \geq 0$.

Proof: (Riordan [14])

To prove this formula, take an arbitrary permutation, $\sigma(n) \in S_n$ of cycle class $1^{k_1} 2^{k_2} \cdots n^{k_n}$, and permute in all $n!$ ways. The resulting permutations are not distinct for the following 2 reasons:
First, cycles containing the same elements in the same cyclic order are the same. In an
r cycle, there are r possible initial elements and therefore r possible duplications; the total
number of such duplications is $k_r 2^{k_r} \cdots n^{k_r}$, therefore, these are divided out.

Second, the relative position of cycles is immaterial. Since $k_r$ r-cycles may be
permuted in $k_r!$ ways, duplications are enumerated by $k_1!k_2! \cdots k_n!$ and divided out.

Finally, the total number of such duplications for the first reason times the total
number of duplications for the second reason gives: $k_r 2^{k_r} \cdots n^{k_r}$ multiplied by
$k_1!k_2! \cdots k_n!$ Hence, this product in the denominator divides out the duplications and
hence gives the formula:

$$C(k_1, k_2, \ldots, k_n) = \frac{n!}{k_1!k_2! \cdots k_n! \cdot k_r 2^{k_r} \cdots n^{k_r}}.$$

**Example 1.4:**

In $S_3$, there are six permutations of 3 elements, and they can be divided into cycle
classes with the number of permutations as follows:

<table>
<thead>
<tr>
<th>Class</th>
<th>Permutations</th>
<th>Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1^2 2^0 3^1$</td>
<td>(123), (132)</td>
<td>$C(0, 0, 1) = \frac{3!}{1^2 2^0 3^1 0! 1!} = 2$</td>
</tr>
<tr>
<td>$1^1 2^1 3^0$</td>
<td>(12)(3), (13)(2), (23)(1)</td>
<td>$C(1, 1, 0) = \frac{3!}{1^1 2^1 3^0 1! 1! 0!} = 3$</td>
</tr>
<tr>
<td>$1^1 2^0 3^0$</td>
<td>(1)(2)(3)</td>
<td>$C(3, 0, 0) = \frac{3!}{1^1 2^0 3^0 3! 0! 0!} = 1.$</td>
</tr>
</tbody>
</table>

**Example 1.5:**

The permutations in $S_4$ provides another example; there are 24 permutations:
<table>
<thead>
<tr>
<th>Class</th>
<th>Permutations</th>
<th>Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l^23^04^0$</td>
<td>(1234), (1243), (1324), (1342), (1423), (1432)</td>
<td>$C(0,0,0,1) = \frac{4!}{1^02^03^04^00!0!0!0!} = 6$</td>
</tr>
<tr>
<td>$l^12^03^14^0$</td>
<td>(123)(4), (124)(3), (132)(4), (134)(2), (142)(3), (143)(2), (234)(1), (243)(1)</td>
<td>$C(1,0,1,0) = \frac{4!}{1^12^03^14^00!0!0!0!} = 8$</td>
</tr>
<tr>
<td>$l^22^13^04^0$</td>
<td>(12)(3)(4), (13)(2)(4), (14)(2)(3), (23)(1)(4), (24)(1)(3), (34)(1)(2)</td>
<td>$C(2,1,0,0) = \frac{4!}{1^22^13^04^02!0!0!0!} = 6$</td>
</tr>
<tr>
<td>$l^02^23^04^0$</td>
<td>(12)(34), (13)(24), (14)(23)</td>
<td>$C(0,2,0,0) = \frac{4!}{1^02^23^04^00!2!0!0!} = 3$</td>
</tr>
<tr>
<td>$l^12^03^04^0$</td>
<td>(1)(2)(3)(4)</td>
<td>$C(4,0,0,0) = \frac{4!}{1^12^03^04^04!0!0!0!} = 1$</td>
</tr>
</tbody>
</table>
CHAPTER 2

THE CYCLE INDICATOR

**Definition 2.1:** (Polya[13] and Riordan[14])

Let \(n\) be a natural number and let \(k_1, k_2, \ldots, k_s\) be non-negative integers such that

\[k_1 + 2k_2 + \cdots + nk_s = n.\]

The cycle indicator \(C_n(t_1, t_2, \ldots, t_s)\) of the symmetric group \(S_n\) is defined by

\[C_n(t_1, t_2, \ldots, t_s) = \sum_{\sigma \in S_n} C(k_1, k_2, \ldots, k_s) t_1^{k_1} t_2^{k_2} \cdots t_s^{k_s}.\]  \hspace{1cm} (2.1)

where the sum is taken over all possible \(k_1, k_2, \ldots, k_s\) satisfying \(k_1 + 2k_2 + \cdots + nk_s = n\), or equivalently, \(\sigma(n) = 1^{k_1} 2^{k_2} \cdots n^{k_s}\). By (1.1), (2.1) can be written as follows

\[C_n(t_1, t_2, \ldots, t_s) = \sum_{\sigma \in S_n} \frac{n!}{k_1! k_2! \cdots k_s!} \left(\frac{t_1}{1}\right)^{k_1} \left(\frac{t_2}{2}\right)^{k_2} \cdots \left(\frac{t_s}{n}\right)^{k_s}.\]  \hspace{1cm} (2.2)

**Example 2.1:**

Examples include computations for the first four instances, followed by a listing up to

\[C_9(t_1, t_2, \ldots, t_9)\]

(1) \(C_1(t_1) = t_1\), because for \(k_1 = 1 = n\).

Thus \(C_1(t_1) = \frac{1!}{1!} \left(\frac{t_1}{1}\right)^1 = t_1\).
(2) \( C_2(t_1, t_2) = t_1^2 + t_2 \), because for \( k_1 + 2k_2 = 2 = n \), then either:

\[ k_1 = 0, \ k_2 = 1, \ \text{as} \ 0 + 2 \cdot 1 = 2, \ \text{or} \]
\[ k_1 = 2, \ k_2 = 0 \ \text{as} \ 2 + 2 \cdot 0 = 2. \]

Thus \( C_2(t_1, t_2) = \left( \frac{2!}{0!0!1!} \right) \left( \frac{t_1}{1} \right)^0 \left( \frac{t_2}{2} \right)^1 + \left( \frac{2!}{2!0!1!} \right) \left( \frac{t_1}{1} \right)^2 \left( \frac{t_2}{2} \right)^0 = t_1 + t_2^2. \)

(3) \( C_3(t_1, t_2, t_3) = t_1^3 + 3t_1t_2 + 2t_3 \), because for \( k_1 + 2k_2 + 3k_3 = 3 = n \), then:

\[ k_1 = 0, k_2 = 0, k_3 = 1, \ \text{as} \ 0 + 2 \cdot 0 + 3 \cdot 1 = 3, \ \text{or} \]
\[ k_1 = 1, k_2 = 1, k_3 = 0, \ \text{as} \ 1 + 2 \cdot 1 + 3 \cdot 0 = 3, \ \text{or} \]
\[ k_1 = 3, k_2 = 0, k_3 = 0, \ \text{as} \ 3 + 2 \cdot 0 + 3 \cdot 0 = 3. \]

Thus

\[ C_3(t_1, t_2, t_3) = \left( \frac{3!}{0!0!1!} \right) \left( \frac{t_1}{1} \right)^0 \left( \frac{t_2}{2} \right)^1 \left( \frac{t_3}{3} \right)^1 + \left( \frac{3!}{1!1!1!} \right) \left( \frac{t_1}{1} \right)^1 \left( \frac{t_2}{2} \right)^1 \left( \frac{t_3}{3} \right)^0 + \left( \frac{3!}{3!0!0!} \right) \left( \frac{t_1}{1} \right)^3 \left( \frac{t_2}{2} \right)^0 \left( \frac{t_3}{3} \right)^0 = 2t_3 + 3t_1t_2 + t_1^3. \]

(4) \( C_4(t_1, t_2, t_3, t_4) = t_1^4 + 6t_1^2t_2 + 3t_1^2t_3 + 8t_1t_3^2 + 6t_4 \), because for \( k_1 + 2k_2 + 3k_3 + 4k_4 = 4 = n \), then:

\[ k_1 = 4, k_2 = 0, k_3 = 0, k_4 = 0, \ \text{as} \ 4 + 2 \cdot 0 + 3 \cdot 0 + 4 \cdot 0 = 4, \ \text{or} \]
\[ k_1 = 2, k_2 = 1, k_3 = 0, k_4 = 0, \ \text{as} \ 2 + 2 \cdot 1 + 3 \cdot 0 + 4 \cdot 0 = 4, \ \text{or} \]
\[ k_1 = 1, k_2 = 0, k_3 = 1, k_4 = 0, \ \text{as} \ 1 + 2 \cdot 0 + 3 \cdot 1 + 4 \cdot 0 = 4, \ \text{or} \]
\[ k_1 = 0, k_2 = 2, k_3 = 0, k_4 = 0, \ \text{as} \ 0 + 2 \cdot 2 + 3 \cdot 0 + 4 \cdot 0 = 4, \ \text{or} \]
\[ k_1 = 0, k_2 = 0, k_3 = 0, k_4 = 1 \ \text{as} \ 0 + 2 \cdot 0 + 3 \cdot 0 + 4 \cdot 1 = 4. \]
Thus

\[ C_2 \left( t_1, t_2, t_3, t_4 \right) = \frac{4!}{4!} \left( \frac{t_1}{1} \right)^4 + \frac{4!}{2!1!} \left( \frac{t_1}{1} \right)^2 \left( \frac{t_2}{2} \right)^2 + \frac{4!}{1!1!1!} \left( \frac{t_1}{3} \right) + \frac{4!}{2!2!} \left( \frac{t_2}{2} \right)^2 - \frac{4!}{1!} \left( \frac{t_4}{4} \right) \]

\[ = t_1^4 + 6t_1^2t_2 + 8t_1t_3 + 3t_2^2 + 6t_4 \]

Similarly, we can obtain:

(5) \[ C_5 \left( t_1, t_2, t_3, t_4, t_5 \right) = t_1^5 + 10t_1^3t_2 + 15t_1^2t_3^2 + 20t_1t_2t_3 + 20t_1t_2t_4 + 30t_1t_4 + 24t_5 \]

(6) \[ C_6 \left( t_1, t_2, t_3, t_4, t_5, t_6 \right) \]

\[ = t_1^6 + 15t_1^4t_2 + 45t_1^3t_3^2 + 70t_1^2t_3t_4 + 120t_1t_2t_3 + 90t_1t_4 + 40t_2^2 + 90t_2t_4 + 144t_3t_5 + 120t_6 \]

(7) \[ C_7 \left( t_1, t_2, t_3, t_4, t_5, t_6, t_7 \right) \]

\[ = t_1^7 + 21t_1^5t_2 + 105t_1^4t_3^2 + 70t_1^3t_3t_4 + 105t_1^2t_4 + 420t_1t_2t_3 + 210t_1^2t_4 \]

\[ 210t_2t_3^2 + 280t_1t_4^2 + 630t_1t_2t_4 + 504t_2t_4^2 \]

\[ 420t_3t_4 + 504t_2t_5 + 840t_1t_6 + 720t_7 \]

(8) \[ C_8 \left( t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8 \right) \]

\[ = t_1^8 + 28t_1^6t_2 + 210t_1^5t_3^2 + 112t_1^4t_3t_4 + 420t_1^3t_2t_3 + 1120t_1^2t_2t_3 + 420t_1t_2^2t_3 + 105t_1^4 + 1680t_1^3t_2 + 1120t_1^2t_2t_3 + 2520t_1t_2t_3 + 1344t_1t_2 \]

\[ + 1120t_2t_3^2 + 1260t_2^2t_4 + 3360t_1t_3t_4 + 4032t_1t_2t_4 + 3360t_2t_4 \]

\[ + 1260t_3t_4 + 2688t_1t_5 + 3360t_2t_5 + 5760t_1t_6 + 5040t_8 \]

(9) \[ C_9 \left( t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9 \right) \]

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\[ t_0^4 = 36t_1^2t_2 + 378t_1^2t_3 + 168t_1^2t_4 + 1260t_1^2t_5 + 2520t_1^2t_6 + 756t_1^2t_7 \\
945t_1^4t_2 + 7560t_1^4t_3 + 3360t_1^4t_4 + 7560t_1^4t_5 + 3024t_1^4t_6 \\
+2520t_1^4t_7 + 10080t_1^4t_8 + 11340t_1^2t_2^2t_3 + 15120t_1^2t_2^2t_4 + 18144t_1^2t_2^2t_5 \\
+10080t_1^4t_6 + 2240t_1^4t_8 + 9072t_1^2t_2^2t_4 + 11340t_1^2t_2^2t_5 \\
+24192t_1^2t_2t_3 + 30240t_1^2t_2t_4 + 25920t_1^2t_2t_5 \\
+18144t_1^2t_3^2t_4 + 20160t_1^2t_3^2t_5 + 25920t_1^2t_3^2t_6 + 45360t_1^2t_3^2t_7 + 40320t_1^2t_3^2t_8. \]
CHAPTER 3

THE CYCLE INDICATOR AND $C_n$-TYPE REPRESENTATION

**Lemma 3.1: (Riordan[14])**

Let $A(t) = f(u)$, $u = g(t)$, $A_n = A_n(f, g_1, \ldots, g_n) = \frac{d^n A}{dt^n}$, and

$$f_n(t) = \frac{d^n f}{dt^n} \bigg|_{u=g(t)} , \quad \frac{d^n g}{dt^n}.$$

Then we have Di Bruno's Formula:

$$A_n(t) = \sum_{\sigma(n)} \frac{n! f_n}{k_1! k_2! \cdots k_n!} \left( \frac{g_1}{1!} \right)^{k_1} \left( \frac{g_2}{2!} \right)^{k_2} \cdots \left( \frac{g_n}{n!} \right)^{k_n}.$$

**Proof:** (Riordan[14])

By successive differentiation of $A(t)$:

- $A_1 = f_1 g_1$
- $A_2 = f_1 g_2 + f_2 g_1$
- $A_3 = f_1 g_3 + 3f_2 g_2 g_1 + f_3 g_1$

The general form may be written

$$A_n = f_1 \Gamma_{n_1} + f_2 \Gamma_{n_2} + \cdots + f_n \Gamma_{n_n} = \sum_{k=1}^{n} f_k \Gamma_{\alpha_k}(g_1, g_2, \ldots, g_n). \tag{3.1}$$

Note that the coefficients $\Gamma_{\alpha_k}$ depend only on the derivatives $g_1, g_2, \ldots, g_n$,

as indicated by the expanded notation of the second line, and not on the
\( f_k \). Hence they may be determined by a special choice of \( f \).

Let \( f(u) = e^{au} \) so \( f(t) = e^{aq(t)} \) and \( f_n(t) = a^n e^{aq(t)} \).

Then \( A_n(f, g_1, \cdots, g_n) = \sum_{k=1}^{n} a^k e^{aq(t)} \Gamma_{nk}(g_1, \cdots, g_n) \), which means that

\[
e^{-aq(t)} A_n(f, g_1, \cdots, g_n) = \sum_{k=1}^{n} a^k \Gamma_{nk}(g_1, \cdots, g_n). \tag{3.2}\]

Let

\[
B_n(a, g_1, \cdots, g_n) = e^{-aq(t)} A_n(f, g_1, \cdots, g_n). \tag{3.3}\]

We wish to determine the coefficients of \( a^k \) in \( B_n(a, g_1, \cdots, g_n) \). This will tell us the coefficients of \( \Gamma_{nk}(g_1, \cdots, g_n) \). Then by the Principle of Mathematical Induction and using Liebniz formula for differentiation of a product

\[
B_{n-1}(a, g_1, \cdots, g_n) = e^{-aq(t)} \frac{d^n}{dt^n} \left( \frac{d}{dt} e^{aq(t)} \right) = e^{-aq(t)} \frac{d^n}{dt^n} (ag_1(t) e^{aq(t)})
\]

\[
= a \sum_{k=0}^{n} \binom{n}{k} e^{-aq(t)} \frac{d^{n-k}}{dt^{n-k}} e^{aq(t)} \frac{d^k}{dt^k} g_1(t)
\]

\[
= a \sum_{k=0}^{n} \binom{n}{k} B_{n-k}(a, g_1, \cdots, g_n) g_{k-1}(t) = a g \{ B(a) + g \}^n \tag{3.4}\]

we get a Binomial Theorem formulaic expression where

\[
(A(a))^k = A_k(a), \quad g^k = g_k.
\]

Instances of (3.4) with \( B_0(a) = 1 \) are

\[
B_1(a) = ag_1,
B_2(a) = ag_2 + ag_1 B_1(a) = ag_2 + a^2 g_1^2,
B_3(a) = ag_3 + 2ag_2 B_1(a) + ag_1 B_2(a) = ag_3 + 3a^2 g_1 g_2 + a^3 g_1^3
\]

which agree with the results preceding (3.1).
Next, (3.4) implies the exponential generating function relation
\[
\exp uB(a) = \sum_{n=0}^{\infty} B_n(a)u^n / n! = \exp a\left[ u g_1 + u^2 g_2 / 2! + \cdots \right] = \exp a G(u) \quad (3.5)
\]
with \( G(u) = \exp(u g) - g_0 \), \( g^r = g_r \).

Differentiation of (3.5) and equating coefficients of \( u^n \) gives (3.4).

Finally, expanding (3.4), using the multinomial theorem, and equating coefficients of \( u^n \) gives the explicit formula
\[
B_n = B_n(n) = \sum_{\sigma(n)} \frac{a^n n!}{\sigma(1)^1 \sigma(2)^2 \cdots \sigma(n)^n} \left( \frac{g_1}{1!} \right)^{\sigma(1)} \left( \frac{g_2}{2!} \right)^{\sigma(2)} \cdots \left( \frac{g_n}{n!} \right)^{\sigma(n)} \quad (3.6)
\]
with \( k = k_1 + k_2 + \cdots + k_n \), the sum over all solutions in non-negative integers of \( k_1 + 2k_2 + \cdots + nk_n = n \), or over all partitions of \( n \).

Then by (3.1) \( A_n = A_n(f) = \sum_{\sigma(n)} \frac{n! f_{\sigma(n)}}{\sigma(1)! \sigma(2)! \cdots \sigma(n)!} \left( \frac{g_1}{1!} \right)^{\sigma(1)} \left( \frac{g_2}{2!} \right)^{\sigma(2)} \cdots \left( \frac{g_n}{n!} \right)^{\sigma(n)} \)
which proves di Bruno’s formula. □

Table 1 in the Appendix shows these polynomials, in Bell’s notation (slightly modified), for \( n = 1 \) to 8.

Now let \( \Phi(t) = \sum_{n=0}^{\infty} \left[ \frac{\Phi}{n} \right] t^n \) be a formal power series with \( \Phi(0) > 0 \), where \( \left[ \frac{\Phi}{n} \right] \) denotes the nth coefficient of the power series. Let \( \Phi(t) = \ln \Phi(t) \) so that
\[
\Phi(t) = \sum_{n=0}^{\infty} \left[ \frac{\Phi}{n} \right] t^n \quad \text{where} \quad \left[ \frac{\Phi}{n} \right] = \left[ \frac{\ln \Phi}{n} \right].
\]
Then we have:
**Theorem 3.1:** (Hsu and Shiue[9], Gessel [5])

There holds the \( C_n \)-type representation

\[
\begin{bmatrix}
\Phi \\
n
\end{bmatrix} = \frac{\Phi(0)}{n!} C_n \left[ \begin{bmatrix}
\Phi \\
1
\end{bmatrix}, 2 \begin{bmatrix}
\Phi \\
2
\end{bmatrix}, \ldots, n \begin{bmatrix}
\Phi \\
n
\end{bmatrix} \right] (3.4)
\]

where \( \begin{bmatrix}
\Phi \\
j
\end{bmatrix} = \begin{bmatrix}
\ln \Phi(t) \\
j
\end{bmatrix}, \ (1 \leq j \leq n) \). (3.4) is called the \( C_n \)-type representation of the formal power series of \( \Phi(t) \).

**Proof:**

Since \( \Phi(0) > 0 \), we have \( \Phi(x) = \ln \Phi(0) + \ln \frac{\Phi(t)}{\Phi(0)} = \ln \Phi(0) + \ln \Phi(t) - \ln \Phi(0) \).

Now we use Di Bruno’s Formula (Lemma 3.1) which is given as:

Let

\[
A(t) = f \left[ g(t) \right], \ A_n = D^n A(t), \ \left[ D^n f(u) \right]_{u=g(t)} = f_n, \ D^n g(t) = g_n
\]

then, taking the chain rule and applying it to the cycle indicator.

\[
A_n = \sum_{\sigma(n)} \frac{n! f_s}{k_1! k_2! \cdots k_n!} \left( \frac{g_{k_1}}{1!} \right)^{i_1} \left( \frac{g_{k_2}}{2!} \right)^{i_2} \cdots \left( \frac{g_{k_n}}{n!} \right)^{i_n}
\]

Now let \( h(x) = \ln x \), so \( (h \circ \Phi)(x) = \ln \Phi(x) = \Phi(x) \). Then we have

\[
\Phi(t) = h^{-1} \circ (h \circ \Phi)(t) = h^{-1} \circ \Phi(t).
\]
Applying Di Bruno’s formula to the composite function \( \Phi(t) = h^{-1} \circ \Phi(t) \) (so, in Di Bruno’s formula we have \( A = \Phi, f = h^{-1} = e^t, g = \Phi \)), we get

\[
\begin{bmatrix} \Phi \\ n \end{bmatrix} = \sum_{\sigma \in \Sigma_n} \left[ D^\sigma e^t \left( x \right) \right]_{x=\Phi(0)} \prod_{i=1}^{n} \frac{1}{k_i!} \begin{bmatrix} \Phi^{\gamma^k} \\ i \end{bmatrix}
\]

\[
= \sum_{\sigma \in \Sigma_n} \left[ D^\sigma e^t \right]_{x=\Phi(0)} \prod_{i=1}^{n} \frac{1}{k_i!} \begin{bmatrix} \Phi^{\gamma^k} \\ i \end{bmatrix}
\]

\[
= \sum_{\sigma \in \Sigma_n} \begin{bmatrix} e^t \end{bmatrix}_{x=\Phi(0)} \begin{bmatrix} \Phi^{\gamma^k} \\ 1 \\ 2 \\ \vdots \\ n \end{bmatrix}_{k_1!k_2!\cdots k_n!}
\]

\[
= \frac{\Phi(0)}{n!} C_n \begin{bmatrix} \Phi^1 \\ 1 \\ \Phi^2 \\ 2 \\ \vdots \\ \Phi^n \\ n \end{bmatrix}.
\]

**Corollary 3.2:** (Hsu and Shiue[9],[10])

Let \( f_i(t) \) \( (1 \leq i \leq m) \) be formal power series with \( f_i(0) > 0 \). Then there holds the \( C_n \)-type representation for the product function \( \Phi(t) = \prod_{i=1}^{n} f_i(t) \):

\[
\begin{bmatrix} \Phi \\ n \end{bmatrix} = \frac{\Phi(0)}{n!} C_n \begin{bmatrix} \sum_{i=1}^{n} f_i^\gamma \\ 1 \\ 2 \\ \vdots \\ n \end{bmatrix}_{\sum_{i=1}^{n} f_i^\gamma} \quad (3.5)
\]

where \( f_i = \ln f_i \). Moreover, for all real numbers \( \alpha, \alpha \neq 0 \), there exists a
C -type representation for $\Phi(t) = (f(t))^a$:

$$\begin{bmatrix} \Phi \\ n \end{bmatrix} = \frac{f(0)^a}{n!} C_n \left( \begin{bmatrix} f_1 \\ 1 \\ 1 \end{bmatrix}, 2\alpha \begin{bmatrix} f_2 \\ 2 \\ 2 \end{bmatrix}, \ldots, n\alpha \begin{bmatrix} f_n \\ n \\ n \end{bmatrix} \right). \quad (3.6)$$

**Proof:**

(3.2) follows from (3.1), since $\dot{\Phi}(t) = \ln \Phi = \sum \ln f_i = \sum \dot{f}_i$, and

$$\begin{bmatrix} \dot{\Phi} \\ k \end{bmatrix} = [t^k] \sum \dot{f} = \sum \begin{bmatrix} f \\ k \end{bmatrix}.$$ 

(3.3) follows similarly with $\Phi(t) = (f(t))^a$. $\square$
CYCLE INDICATOR REPRESENTATIONS AND
GENERALIZATIONS OF SPECIAL FUNCTIONS

Cycle indicators can be used to express the following identities and
special polynomials by using (3.1), (3.2) and (3.3).

Example 4.1: (Hsu and Shiue[10])

Using the generating function $\Phi(t) = \frac{1}{1-t}$ we can show $C_n(1.1.1\ldots.1) = n!$.

This is equivalent to the Cauchy Identity

$$\sum_{a_{1n}/k_1!k_2!\ldots k_n!} 2^{k_1} \ldots n^{k_n} = 1.$$ (4.1)

Proof:

Let $\Phi(t) = \frac{1}{1-t} = 1 + t + t^2 + \ldots + t^n + \ldots$, so that $\left[\Phi\right] = 1$, for all $n$.

and $\Phi(0) = \frac{1}{1-0} = 1$.

Now take $\Phi = \ln \Phi(t) = \ln \left(\frac{1}{1-t}\right) = -\ln(1-t) = t + \frac{t^2}{2} + \frac{t^3}{3} + \ldots + \frac{t^n}{n} + \ldots$.

by the Taylor's series.
Therefore \[
\begin{bmatrix}
\Phi \\
1
\end{bmatrix} = 1, \begin{bmatrix}
\Phi \\
2
\end{bmatrix} = \frac{1}{2}, \begin{bmatrix}
\Phi \\
3
\end{bmatrix} = \frac{1}{3}, \ldots, \begin{bmatrix}
\Phi \\
n
\end{bmatrix} = \frac{1}{n}.
\]

Now by Theorem 3.1, \[
\begin{bmatrix}
\Phi \\
n
\end{bmatrix} = \frac{\Phi(0)}{n!} C_n \left( 1, \begin{bmatrix}
\Phi \\
1
\end{bmatrix}, 2, \begin{bmatrix}
\Phi \\
2
\end{bmatrix}, \ldots, n, \begin{bmatrix}
\Phi \\
n
\end{bmatrix} \right).
\]

This implies \[
1 = \frac{1}{n!} C_n \left( 1, \frac{1}{2}, \frac{1}{3}, \ldots, n, \frac{1}{n} \right) = \frac{1}{n!} C_n (1,1,\ldots,1) \text{ if and only if }
\]

\[n! = C_n (1,1,\ldots,1).\]

Now by definition of \[C_n (t_1, t_2, \ldots, t_n) = \sum_{\sigma(n)} C(k_1, k_2, \ldots, k_n) t_1^{k_1} t_2^{k_2} \ldots t_n^{k_n}\]

we have

\[
\sum_{\sigma(n)} \frac{n!}{k_1! k_2! \cdots k_n!} \left( \frac{1}{1} \right)^{k_1} \left( \frac{1}{2} \right)^{k_2} \cdots \left( \frac{1}{n} \right)^{k_n} = 1. \quad \Box
\]

**Example 4.2**: (Konvallina[11] and MacMahon[12]).

A generalization of Cauchy’s identity by Sylvester shows that the rising factorial \[\langle \rho \rangle_n\], given by \[\langle \rho \rangle_n = (\rho)(\rho+1)\cdots(\rho+n-1)\] generated by the function

\[(1-t)^{-\rho} = \sum \langle \rho \rangle_n \frac{t^n}{n!}\]

has a \(C_n\)-type representation

\[\langle \rho \rangle_n = C_n (\rho, \rho, \ldots, \rho) (4.2)\]

where \(\rho\) is any Real or Complex number.
Proof:

By Taylor's Formula, $\Phi(t) = (1-t)^{-\rho} = 1 + \rho t + \frac{\rho(\rho+1)}{2!} t^2 + \frac{\rho(\rho+1)(\rho+2)}{3!} t^3 + \ldots$.

Therefore, we get $\left[ \frac{\Phi}{n!} \right] = \frac{(\rho)_n}{n!}$, and $\Phi(0) = 1$.

Now take $\dot{\Phi}(t) = \ln \Phi(t) = \ln(1-t)^{-\rho} = -\rho \ln(1-t) = \rho t + \frac{\rho t^2}{2} + \frac{\rho t^3}{3} + \ldots$.

So $\left[ \frac{\Phi}{1} \right] = \rho, \left[ \frac{\Phi}{2} \right] = \frac{\rho}{2}, \left[ \frac{\Phi}{3} \right] = \frac{\rho}{3}, \ldots, \left[ \frac{\Phi}{n} \right] = \frac{\rho}{n}$.

Now, by Theorem (3.1), $\left[ \frac{\Phi}{n!} \right] = \frac{\Phi(0)}{n!} C_n \left[ \begin{array}{c} \Phi \\ 1 \\ 2 \\ 3 \\ \vdots \\ n \end{array} \right] = \left[ \begin{array}{c} \dot{\Phi} \\ \ddot{\Phi} \\ \dddot{\Phi} \\ \vdots \\ \frac{\Phi(n)}{n!} \end{array} \right]$.

So, $\left[ \frac{\Phi}{n!} \right] = \frac{1}{n!} C_n \left( 1 \cdot \rho, 2 \cdot \rho, 3 \cdot \rho, \ldots, n \cdot \rho \right) = \frac{1}{n!} C_n(\rho, \rho, \ldots, \rho)$.

and, $\frac{(\rho)_n}{n!} = \left[ \frac{\Phi}{n!} \right] = \frac{1}{n!} C_n(\rho, \rho, \ldots, \rho)$ or $\langle \rho_n \rangle = C_n(\rho, \rho, \ldots, \rho)$.

To illustrate:

$\langle \rho \rangle_1 = C_1(\rho) = \rho$

because $C_1(t_1) = t_1$, 

$\langle \rho \rangle_2 = C_2(\rho, \rho) = \rho^2 + \rho = \rho(\rho+1)$

because $C_2(t_1, t_2) = t_1^2 + t_2$

$\langle \rho \rangle_3 = C_3(\rho, \rho, \rho) = \rho^3 + 2\rho + 3\rho^2 = \rho(\rho+1)(\rho+2)$

because $C_3(t_1, t_2, t_3) = t_1^3 + 3t_1 t_2 + 2t_3$. \(\square\)
Example 4.3: (Hsu and Shiue[10])

The Bell numbers, $B_n$, generated by the function $\exp(e^t - 1) = \sum B_n \frac{t^n}{n!}$, can be expressed using the cycle indicator as

$$B_n = C_n \left( \frac{1}{0!} \cdot \frac{1}{1!} \cdot \ldots \cdot \frac{1}{(n-1)!} \right). \quad (4.3)$$

Proof:

Given that $\Phi(t) = \exp(e^t - 1) = \sum B_n \frac{t^n}{n!}$ yields $\left[ \frac{\Phi}{n} \right] = \frac{B_n}{n!}$, and

$$\Phi(0) = \exp(e^0 - 1) = \exp(1 - 1) = 1.$$

Let

$$\Phi(t) = \ln \Phi(t) = \ln(\exp(e^t - 1)) = e^t - 1 = -1 + t + \frac{t^2}{2!} + \ldots + \frac{t^n}{n!} + \ldots$$

since $f'(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$.

Hence,

$$\left[ \frac{\Phi}{1} \right] = 1, \quad \left[ \frac{\Phi}{2} \right] = \frac{1}{2!}, \quad \left[ \frac{\Phi}{3} \right] = \frac{1}{3!}, \ldots, \quad \left[ \frac{\Phi}{n} \right] = \frac{1}{n!}.$$

By Theorem (3.1)

$$\left[ \frac{\Phi}{n} \right] = \frac{\Phi(0)}{n!} C_n \left( \frac{1}{0!} \cdot \frac{1}{1!} \cdot \ldots \cdot \frac{1}{(n-1)!} \right) = \frac{1}{n!} C_n \left( \frac{1}{0!} \cdot \frac{1}{1!} \cdot \frac{1}{2!} \cdot \frac{1}{3!} \cdot \ldots \cdot \frac{1}{n!} \right) = \frac{1}{n!} C_n \left( \frac{1}{0!} \cdot \frac{1}{1!} \cdot \frac{1}{(n-1)!} \right) = \frac{1}{n!} C_n \left( \frac{1}{0!} \cdot \frac{1}{1!} \cdot \ldots \cdot \frac{1}{(n-1)!} \right),$$

so

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After cancelling, \( B_n = C_n \left( \frac{1}{0!} \frac{1}{1!} \cdots \frac{1}{(n-1)!} \right) \), the identity for the Bell numbers.

To illustrate:

\[
C_1(t_1) = 1. \\
C_2(t_1, t_2) = t_1^2 + 1 = 2. \\
C_3(t_1, t_2, t_3) = t_1^3 + 2 \cdot \frac{1}{2} + 3 \cdot 1 = 5.
\]

**Example 4.4:** (Gessel[5])

The special polynomial \( H_n \), the Hermite polynomials, generated by the function

\[
\exp(2zt - t^2) = \sum_{n=0}^{\infty} H_n(z) \frac{t^n}{n!},
\]

can be expressed using the cycle indicator as

\[
H_n(z) = C_n(2z, -2, 0, \ldots, 0). (4.4)
\]

**Proof:**

Take \( \Phi(t, z) = \exp(2zt - t^2) = \sum_{n=0}^{\infty} H_n(z) \frac{t^n}{n!} \), so that \( \begin{bmatrix} \Phi \\ n \end{bmatrix} = \frac{H_n(z)}{n!} \) and \( \Phi(0, z) = \exp(0) = 1. \)

Now let \( \Phi(t, z) = \ln \Phi(t, z) = \ln(\exp(2zt - t^2)) = 2zt - t^2. \)
So \[
\begin{bmatrix}
\Phi \\
1 \\
2 \\
3 \\
n
\end{bmatrix} = 2z, \begin{bmatrix}
\Phi \\
1 \\
2 \\
3 \\
n
\end{bmatrix} = -1, \begin{bmatrix}
\Phi \\
1 \\
2 \\
3 \\
n
\end{bmatrix} = 0, \ldots, \begin{bmatrix}
\Phi \\
1 \\
2 \\
3 \\
n
\end{bmatrix} = 0.
\]

Now by Theorem (3.1),
\[
\begin{bmatrix}
\Phi \\
1 \\
2 \\
3 \\
n
\end{bmatrix} = \frac{\Phi(0)}{n!} C_n \left( \begin{bmatrix}
\Phi \\
1 \\
2 \\
3 \\
n
\end{bmatrix} \frac{1}{1}, \begin{bmatrix}
\Phi \\
1 \\
2 \\
3 \\
n
\end{bmatrix} \frac{2}{2}, \ldots, n \begin{bmatrix}
\Phi \\
1 \\
2 \\
3 \\
n
\end{bmatrix} \frac{n}{n} \right)
\]
\[
= \frac{1}{n!} C_n (1 \cdot 2z \cdot 2 \cdot (-1) \cdot 3 \cdot 0 \cdot \ldots \cdot n \cdot 0) = \frac{1}{n!} C_n (2z \cdot 2 \cdot 0 \cdot \ldots \cdot 0).
\]

So, \[
\frac{H_n(z)}{n!} = \begin{bmatrix}
\Phi \\
1 \\
2 \\
3 \\
n
\end{bmatrix} = \frac{1}{n!} C_n (2z, -2, 0, \ldots) \text{ or } H_n(z) = C_n (2z, -2, 0, \ldots).
\]

We can expand to get
\[
H_n(z) = \sum_{\sigma(e)} \frac{n!}{k_1!k_2! \ldots k_n!} \left( \left( \frac{2z}{1} \right)^{k_1} \left( \frac{-2}{2} \right)^{k_2} \right) = \sum_{\sigma(e)} \frac{n! (2z)^{k_1} (-1)^{k_1}}{k_1!k_2! \ldots k_n!}.
\]

To illustrate:
\[
H_1(z) = C_1 (2z) = 2z \text{ because } C_1 (1) = 1.
\]
\[
H_2(z) = C_2 (2z, -2) = 4z^2 - 2 \text{ because } C_2 (1, 1) = 1^2 + 1^2.
\]
\[
H_3(z) = C_3 (2z, -2, 0) = 8z^3 - 12z \text{ because } C_3 (1, 1, 1) = 1^3 + 2 \cdot 1 + 3 \cdot 1. \quad \Box
\]

Example 4.5: (Gessel[5])

The special polynomials, Laguerre polynomials, \( L_n^{\rho} (z) \), \( \rho > 0 \) are generated by the function \((1-t)^{-\rho} \exp \left( \frac{tz}{t-1} \right) = \sum L_n^{(\rho)} (z) t^n \). and are expressed using the cycle indicator as

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\[ n! L_{n}^{\rho-1}(z) = C_n(\rho - z, \rho - 2z, \ldots, \rho - nz). \quad (4.5) \]

**Proof:**

Let \( \Phi(t, z) = (1-t)^{-\rho} \exp \left( \frac{tz}{t-1} \right) = \sum L_{n}^{\rho-1}(z) t^n; \) so, \( [\Phi] = L_{n}^{\rho-1}(z), \) and \( \Phi(0, z) = 1. \)

Next

\[
\Phi(t, z) = \ln \Phi(t, z) = \ln \left[ (1-t)^{-\rho} \exp \left( \frac{tz}{t-1} \right) \right] = -\rho \ln (1-t) + \ln \left( \exp \left( \frac{tz}{t-1} \right) \right)
\]

\[
= \left( \rho t + \rho \frac{t^2}{2} + \rho \frac{t^3}{3} + \cdots \right) + \frac{tz}{t-1} = \left( \rho t + \rho \frac{t^2}{2} + \rho \frac{t^3}{3} + \cdots \right) + z \left( -t - \frac{t^2}{2} - \frac{t^3}{3} - \cdots \right)
\]

\[
= (\rho t - zt) + \left( \frac{\rho}{2} t^2 - zt^2 \right) + \left( \frac{\rho}{3} t^3 - zt^3 \right) + \cdots
\]

\[
= (\rho - z) t + \left( \frac{\rho - 2z}{2} \right) t^2 + \left( \frac{\rho - 3z}{3} \right) t^3 + \cdots.
\]

(since \( \frac{t}{t-1} = 1 + \frac{1}{t-1} = 1 - \frac{1}{1-t} = 1 - (1 + t + t^2 + \cdots) \).

Therefore, \( \Phi = \rho - z, \quad \Phi = \rho - 2z, \quad \Phi = \rho - 3z, \quad \cdots, \quad \Phi = \rho - nz. \)

Now,

\[
[\Phi] = \frac{1}{n!} C_n \left( \frac{\rho - z}{1}, \frac{\rho - 2z}{2}, \ldots, \frac{\rho - nz}{n} \right) = \frac{1}{n!} C_n(\rho - z, \rho - 2z, \ldots, \rho - nz) = L_{n}^{\rho-1}(z).
\]

So, \( C_n(\rho - z, \rho - 2z, \ldots) = n! L_{n}^{\rho-1}(z) \).
To illustrate

\[ L_{1}^{(z+1)} = C_1(\rho - z) = \rho - z, \]

because \( C_1(t) = t \), and

\[ 2! L_{2}^{(z+1)} = C_2(\rho - z, \rho - 2z) = (\rho - z)^2 + (\rho - 2z) = \rho^2 - 2\rho z + z^2 + \rho - 2z \]

\[ = \rho^2 - 4\rho z + \rho + z^2 \]

because \( C_2(t_1, t_2) = t_1^2 + t_2 \). \( \Box \)

Notice that (4.8) implies (4.2) with \( z = 0 \).

**Example 4.6**: (Hsu and Shiue[10])

The special polynomials, the Touchard polynomials \( \tau_n(z) \), are generated by

\[ \exp(z(e^t - 1)) = \sum_{n=0}^{\infty} \tau_n(z) t^n \]  

(4.6)

and are expressed using the cycle indicator as

\[ n! \tau_n(z) = C_n \left( \frac{z}{0!}, \frac{z}{1!} \frac{z}{2!}, \ldots, \frac{z}{(n-1)!} \right). \]

**Proof:**

Let \( \Phi(t, z) = \exp(z(e^t - 1)) = \sum_{n=0}^{\infty} \tau_n(z) t^n \). so that \( \left[ \frac{\Phi}{n} \right] = \tau_n(z) \). for all \( n \) and

\[ \Phi(0, z) = \exp(z(e^0 - 1)) = \exp(z \cdot 0) = \exp(0) = 1. \]

Therefore, we can say

\[ \Phi(0, z) = \ln \Phi(0, z) = \ln \left( \exp(z(e^t - 1)) \right) = z(e^t - 1) = z \left( 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots \right) - 1 \]

\[ = zt + \frac{zt^2}{2!} + \frac{zt^3}{3!} + \ldots. \]
Thus, \[
\begin{bmatrix}
\hat{\Phi} \\
1
\end{bmatrix} = z, \quad \begin{bmatrix}
\hat{\Phi} \\
2
\end{bmatrix} = \frac{z}{2!}, \quad \begin{bmatrix}
\hat{\Phi} \\
3
\end{bmatrix} = \frac{z}{3!}, \ldots .
\]

Now, by Theorem 3.1,

\[
\begin{bmatrix}
\Phi \\
n
\end{bmatrix} = \frac{\Phi(0,z)}{n!} C_n \left( \begin{bmatrix}
1 \\
1
\end{bmatrix} \cdot \frac{\Phi}{2!} \cdot \frac{\Phi}{3!} \cdot \ldots \cdot \begin{bmatrix}
1 \\
n
\end{bmatrix} \right) = \frac{1}{n!} C_n \left( \frac{1 \cdot z \cdot \frac{z}{2!} \cdot \frac{z}{3!} \cdot \ldots \cdot \frac{z}{n!}}{n!} \right)
\]

\[
= \frac{1}{n!} C_n \left( \frac{z \cdot \frac{z}{0!} \cdot \frac{z}{1!} \cdot \frac{z}{2!} \cdot \ldots \cdot \frac{z}{(n-1)!}}{(n-1)!} \right) = r_n(z).
\]

Hence \( n!r_n(z) = C_n \left( \frac{z \cdot \frac{z}{0!} \cdot \frac{z}{1!} \cdot \frac{z}{2!} \cdot \ldots \cdot \frac{z}{(n-1)!}}{(n-1)!} \right) \).

To illustrate:

\[
1!r_1(z) = C_1 \left( \frac{z}{1!} \right) = z
\]

because \( C_1(t) = t \), and

\[
2!r_2(z) = C_2 \left( \frac{z \cdot \frac{z}{2!}}{1!} \right) = z^2 + \frac{z}{2}
\]

because \( C_2(t_1, t_2) = t_1^2 + t_2 \cdot \square \)

**Definition 4.1:** (Hsu and Shiue[10])

A sequence of special functions or polynomials \( \{ p_n(z) \} \), defined by

\[
p_n(z) := [t^n] f(t, z),
\]

is said to be \( C_n \)-representable if:

1. \( f(0, z) \) is a positive constant and

2. \[
\begin{bmatrix}
f \\
n
\end{bmatrix} = [t^n] \ln f(t, z)
\]

has finite expression.
Similarly, a number sequence $\alpha_n := [t_n] f(t)$ is called $C_n$-representable whenever

1. $f(0) > 0$ and

2. $\left[ \begin{array}{c} f \\ n \end{array} \right] = \left[ t^n \right] \ln f(t)$ has finite expression.

**Example 4.7:** (Hsu and Shiue[10])

Both $\{\rho_n\}$ and $\{B_n\}$ are $C_n$-representable sequences. Also, the Hermite polynomials $H_n(z)$, the Laguerre polynomials $L_n^{(\alpha)}(z)$, Cauchy’s identity, and the Touchard polynomials are $C_n$-representable sequences of functions.

**Proof:**

To see this, recall $\frac{\rho_n}{n!} = \left[ t^n \right] f(t)$ where $f(t) = (1 - t)^{-\alpha}$, so $f(0) = 1 > 0$, and we computed $\left[ \begin{array}{c} f \\ n \end{array} \right] = P_n$, which has finite expression.

Also, $\frac{B_n}{n!} = \left[ t^n \right] f(t)$, where $f(t) = e^{-t}$. Then $f(0) = 1 > 0$, and we computed

$\left[ \begin{array}{c} \Phi \\ n \end{array} \right] = \frac{1}{n!}$, which has finite expression.

Finally, $\frac{H_n(z)}{n!} = \left[ t^n \right] f(t, z)$, where $f(t, z) = e^{2z - t}$. $f(0, z) = e^0 = 1$, a positive
constant, and \[ \begin{bmatrix} f \\ n \\ 1 \\ 2 \end{bmatrix} = 0, \ n > 2, \ \begin{bmatrix} f \\ 1 \\ 2 \end{bmatrix} = 2z, \ \begin{bmatrix} f \\ 2 \end{bmatrix} = -1, \ \text{so} \ \begin{bmatrix} f \\ n \end{bmatrix} \] has finite expression. □

Clearly, Corollary 3.2 implies that \( \Phi(t) = \prod_{i=1}^{n} f_{i}(t) \) and \( (f(t))^\alpha \). \( \alpha \neq 0 \), will be \( C_{n} \)-representable whenever each of \( f_{i}(t) \) and \( f(t) \) are \( C_{n} \)-representable.

Now, one can find a representation that includes (4.1) (4.2), (4.3), and (4.6) which are

\[ \sum_{\pi(n)k_{1}!k_{2}! \cdots k_{n}!} \frac{1}{2^{k_{1}} \cdots n^{k_{n}}} = 1 \] (4.1),

\[ \langle \rho \rangle_{n} = C_{n}(\rho, \rho, \ldots, \rho) \] (4.2).

\[ B_{n} = C_{n} \left( \frac{1}{0!}, \frac{1}{1!}, \ldots, \frac{1}{(n-1)!} \right) \] (4.3), and

\[ \exp(z(e^{\alpha} - 1)) = \sum_{n=0}^{\infty} \tau_{n}(z)\alpha^{n} \] (4.6)

as special cases by considering a kind of weighted Touchard polynomial \( T_{n}^{[\mu]}(z) \) defined by the following:

**Example 4.8:** The Cycle Indicator and Generalization of Special Functions:

Let \( \Phi(t, z) = (1-t)^{-\alpha} \exp(z(e^{\alpha} - 1)) = \sum_{n=0}^{\infty} T_{n}^{[\mu]}(z) t^{n} \).

It is now necessary only to utilize (3.2) with

\[ \Phi = f_{1} f_{2}, \ \text{with} \ f_{1} = (1-t)^{-\alpha} \ \text{and} \ f_{2} = \exp(z(e^{\alpha} - 1)). \] (4.8)

Apparently (4.8) includes the following particular cases:
\[ T_n^{(0)}(z) = \tau(z) \text{ (Touchard)}; \quad n!T_n^{(0)}(1) = B_n \text{ (Bell)}; \]
\[ T_n^{(1)}(0) = \left( \frac{\rho + n - 1}{n} \right) \text{ (Sylvester)}; \quad T_n^{(1)}(0) = 1 \text{ (Cauchy)}. \]

**Remark:** (Hsu and Shiue[10])

Example 4.7 suggests that one can always unify various given \( C_n \)-type identities into a general representation by merely making use of (3.2) and (3.3).

Suppose that we are given \( m \) known \( C_n \)-type identities as follows:

\[ f_i \left( \frac{0}{n} \right) = \frac{f_i(0)}{n!} C_n \left( \begin{array}{cc} 1 & \frac{f_i}{1} \\ 2 & \frac{f_i}{2} \\ \vdots & \vdots \\ n & \frac{f_i}{n} \end{array} \right) \quad (4.9) \]

where \( f_i(0) > 0 \) and \( f_i = \ln f_i \) (I = 1,2,...,m).

Introducing a generating function of the form

\[ \Phi(t) = (f_1(t))^\alpha_1 (f_2(t))^\alpha_2 \cdots (f_m(t))^\alpha_m \quad (4.10) \]

with \( \alpha_1, \alpha_2, \ldots, \alpha_m \) being \( m \) real parameters, we obtain

\[ \Phi(t) = \frac{\Phi(0)}{n!} C_n \left( \sum_{i=1}^n \frac{f_i}{1} \alpha_1, 2 \sum_{i=1}^n \frac{f_i}{2} \alpha_2, \ldots, n \sum_{i=1}^n \frac{f_i}{n} \alpha_n \right) \quad (4.11) \]

Evidently, all the \( m \) identities given by (4.9) can be deduced from (4.11) just by taking \( (\alpha_1, \alpha_2, \ldots, \alpha_n) \) to be \((1,0,0,\ldots,0)\), \((0,1,0,\ldots,0)\), \((0,0,1,\ldots,0)\), respectively.

Hence (4.11) can be viewed as a unified generalization of the \( m \) \( C_n \)-type identities.
This also suggests that one can take \[ \left[ \begin{array}{c} f_i \\ j \end{array} \right] (1 \leq i \leq m, 1 \leq j \leq n) \] to be the basic coefficients which are generated from some simple typical functions \( f_i (i = 1, \ldots, m) \).

In this way, representations and recurrence relations for those special function polynomials having generating functions of the form \( (4.11) \) can be immediately written down in terms of linear combinations of \( \left[ \begin{array}{c} f_i \\ j \end{array} \right] \)'s.
CHAPTER 5

A COUNTEREXAMPLE

Example 5.1: (Hsu and Shiue[10])

A counterexample is as follows:

The Lerch polynomials \( p_n(z) \) are generated by the following:

\[
\Phi(t,z) = \left[ 1 - z \ln(1 + t) \right]^{-1} = \sum_{n=0}^{\infty} p_n(z)t^n. \tag{5.1}
\]

This polynomial cannot be represented by the Cycle Indicator.

Proof:

Using the Taylor Series, \( \ln(1-t) = -\sum_{k=1}^{\infty} \frac{t^k}{k} \), \( \Phi(0,z) = 1 \) and

\[
\Phi(t,z) = \ln \Phi(t,z) = -z \ln \left( 1 - z \ln(1+t) \right) = -z^2 \sum_{k=1}^{\infty} \frac{\ln(1+t)}{k} = -z^2 \sum_{k=1}^{\infty} \frac{(-t)^k}{k}.
\]

So

\[
\begin{bmatrix} \Phi \\ n \end{bmatrix} = \lambda \sum_{k=1}^{\infty} \frac{1}{k} \left( -z \sum_{j=1}^{\infty} \frac{(-t)^j}{j} \right)^{t^k}.
\]

After pulling off their coefficients, \( \left( zt - z^2 t + \cdots \right)^t \), and using Theorem (3.2),

\[
\begin{bmatrix} \Phi \\ n \end{bmatrix} = \frac{\Phi(0)}{n!} C_n \begin{bmatrix} 1 & \Phi \\ 2 & \Phi \\ \vdots & \vdots \\ n & \Phi \end{bmatrix},
\]

we conclude that evidently \( \begin{bmatrix} \Phi \\ n \end{bmatrix} \) is a
polynomial in $z$ of degree $n$. However, the computations of $\binom{\Phi}{n}$ show that the number of non-zero terms involved (depending on $n$) will increase indefinitely with $n$ approaching infinity. Hence, $p_n(z)$ cannot be represented by the cycle indicator. \qed
CHAPTER 6

CYCLE INDICATOR REPRESENTATIONS OF
SHEFFER SPECIAL POLYNOMIALS

Applications to Special Functions: two classes of special functions, the Sheffer-type polynomials and Gegenbauer-Humbert-type polynomials, constitute the major interest here, along with recurrence relations of Special Functions. (See Table 2 for a list of Sheffer-type polynomials.)

Example 6.1: (Boas and Buck[3])

Sheffer-type Polynomials \( p_n(z) \) are generated by functions of the form

\[
\Phi(t, z) = A(t) e^{zt(t)} = \sum_{n=0}^{\infty} P_n(z) t^n. \quad (6.1)
\]

where \( A(0) = 1, g(0) = 0, \) and \( [t]g(t) \neq 0. \)

Note that sequences of such polynomials \( p_n(z) \) as defined above, also called sequences of Sheffer A-type zero, have been treated thoroughly and systematically by Roman and Rota[17], using the method of modern classical umbral calculus (also Roman[16] and Rota[18]).

In accordance with (6.1), \( \Phi(0, z) = 1. \) and

\[
\begin{bmatrix}
\Phi \\
k
\end{bmatrix} = \begin{bmatrix}
t^k
\end{bmatrix} \ln \Phi(t, z) = \begin{bmatrix}
t^k
\end{bmatrix} \ln A(t) + \begin{bmatrix}
t^k
\end{bmatrix} z g(t), \quad k = 1, 2, 3, \ldots.
\]

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Thus, the Sheffer-type polynomials $p_n(z)$ are $C_n$-representable functions whenever

$\left[t^k\right]\ln A(t)$ and $\left[t^k\right]g(t)$ have finite expression. The following special polynomials are $C_n$-representable functions, and their explicit $C_n$-type expressions and recurrence relations are easily obtained via (3.1).

**Example 6.2:**

The Poisson-Charlier special function polynomial $(PC)_n(z)$ generated by the function

$$e^t(1+t)^z = \sum_{n=0}^{\infty} (PC)_n(z)t^n$$

(based on the definition of Poisson-Charlier using $\Phi(t, z) = A(t)e^{z(t)} = \sum_{n=0}^{\infty} p_n(z)t^n$ where $A(t) = e^t \cdot e^{z(t)} = e^{z(1+t)}$) has the $C_n$ type expression

$$(PC)_n(z) = \frac{1}{n!}C_n(1+z,-z, z, \ldots, (-1)^n z).$$

**Proof:**

Let $A(t) = e^t$ and $g(t) = \ln(1+t)$. Then it can be seen that the Poisson-Charlier polynomial is a Sheffer-type polynomial such that

$$\Phi(t, z) = A(t)e^{z(t)} = e^t \cdot e^{z(1+t)} = e^t \cdot e^{z(1+t)} = e^t (1+t)^z$$

where

$$\Phi(t, z) = A(t) \cdot e^{z(t)} = e^t \cdot e^{z(1+t)} = \sum_{n=0}^{\infty} p_n(z)t^n$$

will be the generating function for the Poisson-Charlier.

Then,$$\left[\Phi\right]_n = p_n(z) = (PC)_n(z),$$ since
\[ \dot{\Phi}(t, z) = \ln\left(A(t)e^{z(t)}\right) = \ln A(t) + \ln e^{z(t)} = \ln e^{z(t)} = \ln e^{z(t)\ln(1 + e^{r})} = t + z \left(\ln(1 + e^{r})\right) \]

\[= t + z \left(t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots\right) = t + zt - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots. \]

Therefore, \[
\begin{bmatrix}
\dot{\Phi}_1 \\
\dot{\Phi}_2 \\
\dot{\Phi}_3 \\
\dot{\Phi}_4 \\
\end{bmatrix}
= 1 + z \begin{bmatrix}
\dot{\Phi}_1 \\
\dot{\Phi}_2 \\
\dot{\Phi}_3 \\
\dot{\Phi}_4 \\
\end{bmatrix} = \frac{z}{2} \begin{bmatrix}
\dot{\Phi}_2 \\
\dot{\Phi}_3 \\
\dot{\Phi}_4 \\
\end{bmatrix} = \frac{z^2}{3} \begin{bmatrix}
\dot{\Phi}_3 \\
\dot{\Phi}_4 \\
\end{bmatrix} = \frac{z^3}{4} \begin{bmatrix}
\dot{\Phi}_4 \\
\end{bmatrix} = \frac{z^4}{5} \cdots.
\]

Now, by (3.1), \[
\frac{\Phi}{n} = \frac{\Phi(0)}{n!} C_n \left(1, \frac{\dot{\Phi}_1}{1}, \frac{\dot{\Phi}_2}{2}, \frac{\dot{\Phi}_3}{3}, \cdots\right), \quad \text{and}
\]

\[ (PC)_n(z) = \frac{1}{n!} C_n \left(1 + z, \frac{z}{2}, \frac{z^2}{3}, \frac{z^3}{4}, \cdots\right) = \frac{1}{n!} C_n \left(1 + z, z, z, \cdots, (-1)^{n-1} z\right). \quad \square \]
CHAPTER 7

CYCLE INDICATOR REPRESENTATIONS OF
GEGENBAUER SPECIAL POLYNOMIALS

The following special polynomials are $C_n$-representable functions, and their explicit $C_n$-type expressions and recurrence relations are easily obtained via (3.1). (See Table 3 for a list of Gegenbauer-Humbert-type polynomials.)

Example 7.1: (Gould[6], and Hsu[8])

The Gegenbauer-Humbert-type polynomials, $P_n(m, z, y, w)$, have the generating function

$$\Phi(t, z) = (1 - mzt + yt^n) - \sum_{n=0}^{\infty} P_n(m, z, y, w)t^n.$$ 

The case $m = 2, y = 1, w = \lambda$ is called the Gegenbauer special function:

$$P_n(2, z, 1, \lambda) = C_n^{(\lambda)}(z).$$

(For $m = 2, 1 - 2zt + t^2 = [1 - (z + \Delta)t] [1 - (z - \Delta)t], \Delta = \sqrt{z^2 - 1}.$)

It has the $C_n$-type expression

$$C_n^{(\lambda)}(z) = \frac{1}{n!}C_n\left(X_1, X_2, \ldots, X_n\right) = \frac{1}{n!}C_n\left(-2\lambda z, 4\lambda z^2 - 2\lambda, 8\lambda^2 z^3 - 6\lambda z, \ldots\right)$$

where

$$X_j = \lambda\left((z + \Delta)^j + (z - \Delta)^j\right), \text{ and } \Delta = \left(z^2 - 1\right).$$

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Proof:

Let \( \Phi(t,z) = (1 - 2zt + t^2)^{-\lambda} \), so that

\[
\begin{bmatrix}
\Phi \\
n
\end{bmatrix} = C^{(i)}_n(m,z,y,w) = C^{(i)}_n(2,z,1,w) = C^{(i)}(z).
\]

such that \( \Phi(t,z) = \ln \Phi(t,z) = \ln (1 - 2zt + t^2)^{-\lambda} = -\lambda \ln (1 - 2zt + t^2) \).

Now, \( 1 - 2zt + t^2 = [1 - (z + \Delta)t][1 - (z - \Delta)t] \), so

\[
\Phi = \ln \Phi = -\lambda \ln [1 - (z + \Delta)t][1 - (z - \Delta)t] = -\lambda \{ \ln (1 - (z + \Delta)t) + \ln (1 - (z - \Delta)t) \}
\]

\[= -\lambda \sum_{n=1}^{\infty} \frac{(z + \Delta)^n + (z - \Delta)^n}{n} t^n, \text{ since } \ln (1 - x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} \text{ (after expanding)}. \]

Therefore

\[
\begin{bmatrix}
\Phi \\
n
\end{bmatrix} = \frac{\lambda}{n} (z + \Delta + z - \Delta) = 2\lambda \Delta.
\]

\[
\begin{bmatrix}
\Phi \\
n
\end{bmatrix} = \lambda \left[ (z + \Delta)^i + (z + \Delta)^i \right] = \lambda \{ z^2 + 2z\Delta + \Delta^2 + z^2 - 2z\Delta + \Delta^2 \} = \lambda (4z^2 - 2).
\]

Similarly,

\[
\begin{bmatrix}
\Phi \\
n
\end{bmatrix} = \lambda \left[ (z + \Delta)^2 + (z + \Delta)^2 \right] = \lambda (z^3 + 3z^2\Delta + 3z\Delta^2 + \Delta^3 + z^3 - 3z\Delta^2 + 3z\Delta^2 - \Delta^3)
\]

\[= \lambda (z^3 + 3z\Delta^2 + z^3 + 3z\Delta^2) = \lambda \left( z^3 + 3z \left( \sqrt{z^2 - 1} \right)^2 + z^3 + 3z \left( \sqrt{z^2 - 1} \right)^2 \right) = 8\lambda z^3 - 6\lambda z. \]
By (3.1),

$$C_n^{(A)}(z) = \frac{\Phi(0)}{n!} C_n \left( 1 \cdot \frac{\Phi}{1}, 2 \cdot \frac{\Phi}{2}, \ldots, n \cdot \frac{\Phi}{n} \right)$$

$$= \frac{1}{n!} C_n \left( 2\lambda z, 4\lambda z^2 - 2\lambda, 8\lambda z^3 - 6\lambda z, \ldots \right).$$

**Example 7.2**: The Dickson Polynomial of the 2\textsuperscript{nd} Kind (Hsu and Shiue\cite{9})::

The Dickson polynomials, $D_n(z, \alpha)$, are generated by the following function:

$$\Phi(t, z) = \frac{2 - zt}{1 - zt + \alpha t^2} = \sum_{s=0}^{\infty} D_s(z, \alpha) t^s$$

and is expressed using the cycle indicator as

$$D_s(z, \alpha) = \frac{2}{n!} C_n(\delta_1(z), \delta_2(z), \ldots, \delta_s(z)),$$

where $\delta_s(z) = \frac{(z + \Delta)^s + (z - \Delta)^s - z^s}{2^s}$. and

$$\Delta = \sqrt{z^2 - 4\alpha}.$$

**Proof:**

Since $1 - zt + \alpha t^2 = \left(1 - \frac{z + \Delta}{2} t\right) \left(1 - \frac{z - \Delta}{2} t\right)$, we have

$$\Phi(t, z) = \ln \Phi(t, z) = \ln \frac{2 - zt}{1 - zt + \alpha t^2}$$

$$= \ln (2 - zt) - \ln (1 - zt + \alpha t^2) = \ln \left(2 \left(1 - \frac{zt}{\frac{2 z - \Delta}{2}}\right)\right) - \ln \left(1 - \frac{z + \Delta}{2} t\right) - \ln \left(1 - \frac{z - \Delta}{2} t\right)$$

$$= \ln 2 + \ln \left(1 - \frac{z}{2} t\right) - \ln \left(1 - \frac{z + \Delta}{2} t\right) - \ln \left(1 - \frac{z - \Delta}{2} t\right)$$

$$= \ln 2 - \sum_{s=1}^{\infty} \frac{(z/2)^s}{s} t^s - \sum_{s=1}^{\infty} \frac{(z + \Delta/2)^s}{s} t^s - \sum_{s=1}^{\infty} \frac{(z - \Delta/2)^s}{s} t^s.$$
because \( \ln(1-t) = -\sum_{n=1}^{\infty} \frac{t^n}{n} \).

Therefore

\[
\left[ \frac{\Phi}{n} \right] = \frac{1}{2^n} n \left[ (z + \Delta)^n + (z + \Delta)^n - z^n \right].
\]

so that by Theorem 3.1

\[
D_n(z, \alpha) = \frac{\Phi(0)}{n!} C_n \left( \frac{\delta_1(z)}{1}, \frac{\delta_2(z)}{2}, \ldots, \frac{\delta_n(z)}{n} \right)
\]

\[
= \frac{2}{n!} C_n (\delta_1(z), \delta_2(z), \ldots, \delta_n(z)). \quad \Box
\]

To illustrate:

\[
D_1(z, a) = \frac{1}{1!} C_1 \left( \frac{\delta_1}{1} \right) = \delta_1(t) = t. \quad \Box
\]
The following result is well known (Harary and Palmer [7]).

**Lemma 8.1:**

For the cycle indicators, the following recurrence relations hold:

\[ C_{n+1}(t_1, t_2, \ldots, t_n) = \sum_{j=0}^{n} (n)_{j} t_{j+1} C_{n-j}(t_1, \ldots, t_{n-j}), \quad (8.1) \]

where \( C_0(\cdot) = 1, C_1(t_1) = t_1 \), and \( (n)_j = n(n-1)\cdots(n-j+1) \).

This follows from the recurrence relations for Bell polynomials (Harary & Palmer [7], Riordan [14]).

There is a useful re-statement of (8.1) as follows:

**Proposition 8.1:** (Hsu and Shiue [10])

For the sequence of special functions \( \{ \rho_n(z) \} \) defined by

\[ \rho_n(z) := \left[ \begin{array}{c} \Phi \\ t^n \end{array} \right] = \left[ \begin{array}{c} t^n \\ \Phi(t, z) \end{array} \right], \quad (8.2) \]

where \( \Phi(t) = \Phi(t, z) \) is a formal power series in \( t \) with coefficients containing \( z \) as a real or complex parameter, and \( \Phi(0) = \Phi(0, z) \) is a positive constant not depending on \( z \), the following difference equations or recurrence relations exist:
\[(n-1)p_{n-1}(z) = \sum_{j=0}^{\infty} \lambda_{j+1}(z)p_{n-j}(z), \quad (8.3)\]

where the coefficients are given by
\[
\lambda_{j+1}(z) = (j+1)\left[t^{j+1}\right] \ln \Phi(t, z). \quad (8.4)
\]

**Proof:**

These are obtained via Leibniz \(n\)th derivative formula to the left hand side of the expression:
\[
\Phi(t, z) \frac{\partial}{\partial t} \ln \Phi(t, z) = \frac{\partial}{\partial t} \Phi(t, z). \quad \square
\]

**Example 8.1:** (Hsu and Shiue[10])

An application of \((8.3)\) and \((8.4)\) to \((4.4)\), \(H_n(z) = C_n(2z, -2, 0, \ldots, 0)\), yields the recurrence formula for the Hermite polynomials:
\[
H_{n+1}(z) = 2zH_n(z) - 2nH_{n-1}(z). \quad \square
\]

**Example 8.2:** (Andrews[1], and Stanley[19])

\(C_{\mu}\) - type representations and recurrence relations for partition functions can be derived. Consider a formal power series given by the function
\[
\Phi(t) = \prod_{i=1}^{\infty} (1-t^i)^{-\lambda(i)} = 1 + \sum_{n=1}^{\infty} r(n) t^n \quad \text{where } \{\lambda(i)\} \text{ is a sequence of non-negative real numbers.} \quad \Phi(t) \text{ is a } C_{\mu} - \text{representable function, so that a kind of recurrence relation should be obtainable via (8.1), (8.2).}
\]

**Proof:**

Since \(\ln(1-t^i) = \sum_{d|i} \left(-\frac{t^d}{d}\right)\),
\[\Phi(t) = \ln \Phi(t) = \sum_{i=1}^{\infty} (-\lambda(i)) \ln(1 - t^i) = \sum_{i=1}^{\infty} \sum_{d=1}^{\infty} \lambda(i) \frac{t^d}{d} = \sum_{n=1}^{\infty} \left( \sum_{d|n} \lambda(n) \frac{t^n}{d} \right) = \sum_{n=1}^{\infty} \left( \sum_{i|n} \lambda(i) \frac{t^n}{n} \right) = \sum_{n=1}^{\infty} \frac{D_n}{n} t^n \quad \text{where} \quad D_n = \sum_{i|n} \lambda(i) \quad \text{(substituting} \quad i = \frac{n}{d} ).

From (3.2),

\[
\begin{bmatrix} \Phi \\ n \end{bmatrix} = \Phi(t) \left[ t^n \right] = \sum_{n=1}^{\infty} r(n) t^n = r(n) = \frac{\Phi(0)}{n!} C_n \left( \begin{bmatrix} \Phi \\ 1 \end{bmatrix} + \frac{\Phi(2)}{2!} + \frac{\Phi(3)}{3!} + \cdots + \frac{\Phi(n)}{n!} \right)
\]

\[
= \frac{\Phi(0)}{n!} C_n \left( 1, \frac{D_1}{1}, \frac{D_2}{2}, \ldots, \frac{D_n}{n} \right) = \frac{1}{n!} C_n \left( D_1, D_2, \ldots, D_n \right).
\]

By (8.2), \((n+1)p_{n+1}(z) = \sum_{j=0}^{\infty} \lambda_{j+1}(z)p_{n-j}(z).
\ where \ p_n(z) = \begin{bmatrix} \Phi \\ n \end{bmatrix} = r(n). \ and \ by \ (8.3),
\]

\[
 \lambda_{j+1}(z) = (j+1)! t^{j+1} |\ln \Phi(t, z).
\]

Rewriting (8.3),

\[
nr(n) = \sum_{j=0}^{\infty} \lambda_{j+1}(z) r(n-1-j) = \sum_{k=1}^{\infty} \lambda_k(z) r(n-k) = \sum_{k=1}^{\infty} k \begin{bmatrix} \Phi \\ k \end{bmatrix} r(n-k) = \sum_{k=1}^{\infty} k \frac{D_k}{k} r(n-k)
\]

\[
= \sum_{n=1}^{\infty} D_n r(n-k)
\]

So, \[\lambda_k(z) = j \left[ \begin{bmatrix} \Phi \\ j \end{bmatrix} \right].
\]

In particular let \(\lambda(i) = 1, \ D_n = \sum_{d|n} d, \ r(n) = \frac{1}{n!} C_n \left( D_1, D_2, \ldots, D_n \right), \) then

\[
nr(n) = \sum_{k=1}^{\infty} D_k r(n-k), \quad \text{where} \quad r(n) \quad \text{is a partition function representing the number of partitions of} \quad k. \quad \Box
\]
As another example, let \( \lambda(i) = i \), so \( D_\alpha = \sum_{\lambda | n} d^2 \), \( r(n) = \frac{1}{n!} C_\alpha (D_1, D_2, \ldots, D_n) \), thus
\[
nr(n) = \sum_{k=1}^n D_k r(n-k),
\]
where now \( r(n) \) is a partition function representing the number of plane partitions of \( n \), respectively. These examples have the above recurrence relations (Andrews [1], Stanley[19]). □

**Example 8.3:** (Hsu and Shiue[10])

The last example can be generalized as follows:

Let \( H \) be a given set of positive integers, and consider the following function as a generating function for \( p(H.n) \) (Andrews[1]) with \( p(H.0) = 1 \):
\[
\Phi(t) = \prod_{i \in H} (1 - t^{\lambda(i)}) = \sum_{p(H.n)} p(H.n) t^n.
\]

**Proof:**

Clearly, for the same reasons as Example 8.2,
\[
\Phi = \ln \Phi(t) = \sum_{i \in H} (-\lambda(i)) \ln(1 - t^{\lambda(i)}) = \sum_{i \in H} \sum_{d|\lambda} t^{\lambda/d} = \sum_{n \in H \mod d} \left( \sum_{d \mid n} \left( \frac{n}{d} \right) \lambda\left( \frac{n}{d} \right) \right) t^n
\]
\[
= \sum_{n \in H \mod d} \sigma(H.n) \frac{t^n}{n}, \text{ where } \sigma(H.n) = \sum_{d \mid n, d \in H} d \lambda(d).
\]

By 3.1, 8.2, 8.3, and vs in Example 8.2
\[
r(n) = \frac{1}{n!} C_\alpha (\sigma(H.1), \sigma(H.2), \ldots, \sigma(H.n)),
\]
\[
nr(n) = \sum_{k=1}^n \lambda_k(z) r(n-k),
\]
and letting \( \lambda(i) = 1 \), \( \lambda(i) = i \), the following is obtained:
\[ p(H,n) = \frac{1}{n!} C_n(\sigma(H,1),\sigma(H,2),\ldots,\sigma(H,n)), \text{ and} \]

\[ np(H,n) = \sum_{k=1}^{n} \sigma(H,n) p(H,n-k). \quad (9.5) \]

The recurrence relation (8.4) has been proved as Theorem 14.8 in Apostol [2], in which the generating function \( \Phi(t) \) is assumed to be analytic in the unit disk \(|t| < 1\). When \( \lambda(i) = 1 \) and \( H \) represents the set of odd natural numbers, (8.4) reduces to Theorem 1 of Robbins' result (Robbins [15]).

**Example 8.4:** (Chen [4], Zeilberger [20])

Consider some basic symmetric functions on several variables:

\[ h_n(x_1, x_2, \ldots, x_m) = \sum_{1 \leq i_1 < i_2 < \cdots < i_s \leq m} x_{i_1} x_{i_2} \cdots x_{i_s} \]

\[ e_n(x_1, x_2, \ldots, x_m) = \sum_{1 \leq i_1 < i_2 < \cdots < i_s \leq m} x_{i_1} x_{i_2} \cdots x_{i_s} . \]

By Waring's formulas, the generating functions \( \Phi_1(t), \Phi_2(t), \ldots, \Phi_n(t) \) of \( \{h_n, e_n\} \) are as follows:

\[ \Phi_1(t) = \sum_{n=0}^{\infty} h_n t^n = \exp \left( p_1 t + p_2 \frac{t^2}{2} + p_3 \frac{t^3}{3} + \cdots \right) \], and

\[ \Phi_2(t) = \sum_{n=0}^{\infty} e_n t^n = \exp \left( p_1 t - p_2 t^2 + p_3 \frac{t^3}{3} - \cdots \right) \], where

\[ p_n(x_1, x_2, \ldots, x_m) = x_1^n + x_2^n + \cdots + x_m^n, \quad n = 1, 2, \ldots \]

Thus, the following is obtained:

\[ \Phi_1(t) = \ln \Phi_1(t) = \sum_{n=1}^{\infty} \frac{p_n}{n} t^n = \sum_{n=1}^{\infty} \left[ \Phi_1 \right]_n t^n , \quad \text{and} \]

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\[ \Phi_2(t) = \ln \Phi_2(t) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{p_n t^n}{n} = \sum_{n=1}^{\infty} \left[ \frac{\Phi_2}{n} \right] t^n. \]

\[ n \left[ \frac{\Phi_2}{n} \right] = n(-1)^{n-1} \left( \frac{p_n}{n} \right) = (-1)^{n-2} p_n. \]

Using (3.1), the \( C_n \)-type expressions are obtained:

\[ h_n = \frac{1}{n!} C_n \left( p_1, p_2, \ldots, p_n \right), \text{ and } e_n = \frac{1}{n!} C_n \left( p_1, -p_2, \ldots, (-1)^{n-1} p_n \right). \]

Also, by using (8.4), \( C_{n-1} \left( t_1, \ldots, t_{n-1} \right) = \sum_{i=0}^{n} (n), t_{r-i} C_{n-i} \left( t_1, \ldots, t_{n-r} \right) \).

Newton's formulas were obtained (Boas and Buck[3], Chen[4]) as follows:

\[ (n+1)h_{n+1} = \sum_{i=0}^{n} p_i h_{n-i}, \text{ and } (n+1)e_{n+1} = \sum_{i=0}^{n} (-1)^i p_i e_{n-i}. \]
## Table 1: Polynomials generated by Di Bruno’s formula

\[
Y_1 = f_1 \, g_1 \\
Y_2 = f_1 \, g_2 + f_2 \, g_1 \\
Y_3 = f_1 \, g_3 + f_2 \, (3g_2 \, g_1) + f_3 \, g_1 \\
Y_4 = f_1 \, g_4 + f_2 \, (4g_3 \, g_1 + 3g_2^2) + f_3 \, (6g_2 \, g_1^2) + f_4 \, g_1^3 \\
Y_5 = f_1 \, g_5 + f_2 \, (5g_4 \, g_1^3 + 10g_3 \, g_2) + f_3 \, (10g_3 \, g_1^2) + f_4 \, g_1^3 \\
\]

\[
Y_6 = + f_1 \, (15g_4 \, g_1^3 + 60g_3 \, g_2 \, g_1 + 15g_2^3) \\
+ f_2 \, (20g_3 \, g_1^2 + 45g_2^2 \, g_1^2) + f_3 \, (15g_2 \, g_1^3) + f_4 \, g_1^3 \\
Y_7 = + f_1 \, (21g_5 \, g_1^4 + 105g_4 \, g_2 \, g_1 + 70g_3 \, g_4 \, g_1 + 105g_3 \, g_2^2) \\
+ f_2 \, (35g_4 \, g_1^3 + 210g_3 \, g_2 \, g_1^2 + 105g_2^3 \, g_1) \\
+ f_3 \, (35g_3 \, g_1^4 + 105g_2^2 \, g_1^3) + f_4 \, g_1^5 \\
\]

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\[
\begin{align*}
&f_1 g_6 + f_2 (8g_7 g_1 + 28g_6 g_2 + 56g_5 g_3 + 35g_4^2) \\
&+ f_3 (28g_4 g_1^2 + 168g_3 g_2 g_1 + 280g_2 g_3 g_1 + 210g_1 g_4^2 + 280g_2^2 g_1) \\
&Y_s = + f_4 (56g_5 g_1^3 + 420g_4 g_2 g_1^2 + 280g_3 g_2^2 g_1 + 840g_2 g_3 g_2 g_1 + 105g_1 g_4^3 g_1) \\
&+ f_5 (70g_4 g_1^4 + 560g_3 g_2 g_1^3 + 420g_2^2 g_1^2) \\
&+ f_6 (56g_5 g_1^5 + 210g_4^2 g_1^4) + f_7 (28g_3 g_1^6) + f_8 g_1^8
\end{align*}
\]

**Table 2:** Sheffer-type polynomials

<table>
<thead>
<tr>
<th>Polynomial</th>
<th>Expression</th>
<th>Directly</th>
<th>Classification</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e^t$</td>
<td>$\ln(1+t)$</td>
<td>$(PC)_n(z)$</td>
<td>Poisson-Charlier</td>
</tr>
<tr>
<td>$e^{-\alpha t}$ ($\alpha \neq 0$)</td>
<td>$\ln(1+t)$</td>
<td>$C_n^{(\alpha)}(z)$</td>
<td>Charlier</td>
</tr>
<tr>
<td>$1$</td>
<td>$\ln \frac{1+t}{1-t}$</td>
<td>$(ML)_n(z)$</td>
<td>Mittag-Leffler</td>
</tr>
<tr>
<td>$(1-t)^{-1}$</td>
<td>$\ln \frac{1+t}{1-t}$</td>
<td>$p_n(z)$</td>
<td>Pidduck</td>
</tr>
<tr>
<td>$(1-t)^{-\rho}$ ($\rho &gt; 0$)</td>
<td>$t/(t-1)$</td>
<td>$E_n^{(-\rho)}(z)$</td>
<td>Laguerre</td>
</tr>
<tr>
<td>$e^{\lambda t}$ ($\lambda \neq 0$)</td>
<td>$1 - e^{-t}$</td>
<td>$(Tos)_n^{(\lambda)}(z)$</td>
<td>Toscano</td>
</tr>
<tr>
<td>$1$</td>
<td>$e^{-t} - 1$</td>
<td>$\tau_n(z)$</td>
<td>Touchard</td>
</tr>
<tr>
<td>$1/(1+t)$</td>
<td>$t/(t-1)$</td>
<td>$\ln(z)$</td>
<td>Angelescu</td>
</tr>
<tr>
<td>$(1-t)/(1+t)^2$</td>
<td>$t/(t-1)$</td>
<td>$(De)_n(z)$</td>
<td>Denisyuk</td>
</tr>
<tr>
<td>$(1-3t)/(1+2t)$</td>
<td>$t/(t-1)$</td>
<td>$(Ad)_n(z)$</td>
<td>Adhoc</td>
</tr>
<tr>
<td>$(1-t)^{-\rho}$ ($\rho &gt; 0$)</td>
<td>$e^{-t} - 1$</td>
<td>$T_n^{(-\rho)}(z)$</td>
<td>weighted-Touchard</td>
</tr>
</tbody>
</table>
Table 3: Gegenbauer-Humbert-type Polynomials

\[ P_n \left( 2 \cdot \frac{z}{1.4} \right) = \phi_n (z) \quad \text{(Legendre)} \]

\[ P_n \left( 2 \cdot \frac{z}{1.1} \right) = U_n (z) \quad \text{(Chebyshev, 1st kind)} \]

\[ P_n \left( 2 \cdot \frac{z}{1.1} \right) = C_n^1 (z) \quad \text{(Gegenbauer)} \]

\[ P_n \left( 3 \cdot \frac{z}{1.4} \right) = P_n^* (z) \quad \text{(Pincherle)} \]

\[ (-x)^{\frac{n}{2}} P_n \left( 2 \cdot \sqrt{-x} \cdot \frac{m}{1.6} \right) = F_n^{(m)} (x) \quad \text{(Dilcher)} \]

\[ P_n \left( 2 \cdot \frac{z}{1.6} \right) = P_n^{(m)} (z) \quad \text{(Horadam-Mahon)} \]

\[ P_n \left( 2 \cdot \frac{z}{1.1} \right) = P_{n-1} (z) \quad \text{(Pell)} \]

\[ P_n \left( 2 \cdot \frac{z}{2.1} \right) = F_{n-1} (z) \quad \text{(Fibonacci)} \]

\[ P_n \left( 2 \cdot \frac{z}{2.1} \right) = \Phi_{n-1} (z) \quad \text{(Fermat, 1st kind)} \]

\[ P_n \left( 2 \cdot \frac{z}{2.1} + 1.1 \right) = B_n (z) \quad \text{(Morgan-Voyce)} \]

\[ P_n \left( 2 \cdot \frac{z}{2.1} \cdot a \right) = E_n (z \cdot a) \quad \text{(Dickson, 1st kind)} \]
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