Cayley maps for certain cyclic groups with odd generators

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CAYLEY MAPS FOR CERTAIN CYCLIC GROUPS WITH ODD GENERATORS

by

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ABSTRACT

Cayley Maps for Certain Cyclic Groups with Odd Generators

by

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For this thesis I plan on using the AMS format. My Thesis Advisor and I will meet regularly to discuss my thesis topic, prove conjectures, write results as we make progress, develop a program for calculation of genus values under certain constraints, and then organize the work for both oral and written presentation.

A Cayley graph provides us with a discrete model for a finite group with specified generating set. It is desirable to represent such structures in their simplest form and also so that certain symmetries are emphasized. By simplest form, we mean to draw these graphs on surfaces so that their edges do not cross (except at their common vertices) and by emphasizing certain symmetries, we mean to impose a local symmetry by insisting that the rotation of generators emanating from each vertex of the given Cayley graph is identical. Such embeddings (or drawings) of Cayley graphs are called Cayley maps. In this thesis we begin the classification of Cayley maps for the cyclic group \( \mathbb{Z}_{2p} \), where \( p \) is prime, with generating set \( \Omega \) consisting of the odd integers.
Due to the local symmetry that is specified at each vertex, it is possible to represent such a Cayley map by an index one voltage graph embedding (a pseudograph with one vertex and \((p - 1)/2\) edges). In this work, we determine the genera for Cayley maps that are covering embeddings of certain planar voltage graphs having simple region structures, namely, those planar voltage graphs consisting of only singleton or stacked loops with at most one region of size greater than 2. In addition to providing results in these cases, we also discuss the more general problem involving any arbitrary planar voltage graph.
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CHAPTER 1

PRELIMINARIES

A graph $G$ is a finite nonempty set of objects called vertices along with a (possibly empty) set of unordered pairs of distinct vertices called edges. The vertex set is denoted by $V(G)$ while the edge set is denoted by $E(G)$. For a graph $G$ the cardinality of $V(G)$ is called the order of $G$ and is commonly denoted $n(G)$, or more simply by $n$ when the graph under consideration is clear. The cardinality of the edge set is called the size of $G$ and is often denoted $m(G)$ or $m$ when the graph under consideration is understood.

A graph $G$ is $k$-partite, $k \geq 1$, if it is possible to partition $V(G)$ into $k$ subsets $V_1, V_2, \ldots, V_k$ (called partite sets) such that every element of $E(G)$ joins a vertex of $V_i$ and vertex of $V_j$, $i \neq j$. Every graph is $n$-partite. The following discussion will focus on bipartite graphs, that is, 2-partite graphs. A complete bipartite graph $G$ is a graph with partite sets $V_1, V_2 \subseteq V(G)$ with the added property that for every $u \in V_1$ and $v \in V_2$, the edge $uv \in E(G)$. If $|V_1| = r$ and $|V_2| = s$ then the complete bipartite graph is denoted by $K_{r,s}$.

Figure 1.1 shows a graph $G$ with $V(G) = \{a, b, c, d, e\}$ and $E(G) = \{\{a, b\}, \{b, c\}, \{c, d\}, \{d, e\}, \{e, a\}, \{b, e\}\}$. The order of $G$ is 5, and the size of $G$ is 6. For simplicity we will denote edges by suppressing the braces and commas. So for example, $E(G) =$

\begin{align*}
\end{align*}
\{ab, bc, cd, de, ea, be\}. Two vertices \(u\) and \(v\) of \(G\) are adjacent if there exists an edge \(e=uv\) in \(E(G)\).

Figure 1.1
SIMPLE GRAPH

![Simple Graph](image)

Figure 1.2 shows a bipartite graph \(G\) having six vertices, three in each partite set, where each vertex from one partite set is adjacent to each vertex from the other partite set. Thus \(G\) is the complete bipartite graph \(K_{3,3}\).

Figure 1.2
COMPLETE BIPARTITE GRAPH

![Complete Bipartite Graph](image)

An important concept is that of a graph being connected. Let \(u\) and \(v\) be two vertices in a graph \(G\), a \(u\)-\(v\) walk of \(G\) is a finite alternating sequence
$u = u_0, e_1, u_1, e_2, \ldots, u_{k-1}, e_k, u_k = v$ of vertices and edges, beginning with vertex $u$ and ending with the vertex $v$, such that $e_i = u_{i-1}u_i \in E(G)$ for $i = 1, 2, \ldots, k$. Often only the vertices of a walk are indicated since the edges present are then evident. The number $k$ is called the length of the walk. A $u$-$v$ path is a $u$-$v$ walk in which no vertex is repeated. A vertex $u$ is said to be connected to a vertex $v$ in a graph $G$ if there is a $u$-$v$ path. A graph $G$ is connected if every two vertices in $V(G)$ are connected.

Another important concept for this work is that of surfaces. A compact orientable 2-manifold is a surface that may be thought of as a sphere on which has been placed a number of ‘handles’ or, equivalently, a sphere in which has been placed a number of ‘holes’. The number of handles (or holes) is referred to as the genus of the surface. A graph $G$ with $n$ vertices and $m$ edges is said to be embeddable on a surface $S$ if it is possible to distinguish a collection of $n$ distinct points of $S$ that correspond to the vertices of $G$ and a collection of $m$ curves, pairwise disjoint except possibly for endpoints, on $S$ that correspond to the edges of $G$ such that if a curve $A$ corresponds to an edge $e = uv$, then only the endpoints of $A$ correspond to vertices of $G$, namely $u$ and $v$. Intuitively, $G$ is embeddable on $S$ if it can be drawn on $S$ so that edges intersect only at vertices. A question that arises is, is it possible to embed (or draw) a graph in the plane so that none of its edges cross? The graphs for which the answer is yes are called planar graphs.

If a planar graph is embedded on the plane then it is called a plane graph. Given a plane graph $G$, a region (typically denoted $R$) of $G$ is a maximal portion of the plane for which any two points may be joined by a curve $A$ such that each point of $A$ neither corresponds to a vertex of $G$ nor lies on any curve corresponding to an edge in $G$. For a plane graph $G$, the boundary of a region $R$ consists of all those points $x$ corresponding to
vertices and edges of $G$ having the property that $x$ can be joined to a point of $R$ by a curve, all of whose points different from $x$ belong to $R$. The size of a region $R$ is the number of edges on the boundary of $R$ and is denoted $|R|$.

Figure 1.3 shows a plane graph with three regions and we see that the region $R_1$ is the dotted region and has four edges on its boundary, $R_2$ has three edges on its boundary, and $R_3$ is the exterior region and has five edges on its boundary.

By the genus of a graph $G$, denoted $\text{gen}(G)$, we mean the smallest genus of all surfaces on which $G$ can be embedded. A region is called 2-cell if any closed curve in that region can be continuously deformed or contracted in that region to a single point. Our primary interest lies with embeddings of graphs that are 2-cell embeddings. It is convenient to denote the surface of genus $k$ by $S_k$. Thus $S_0$ represents the sphere (A graph embedding on the sphere is equivalent to an embedding in the plane through stereographic projection), $S_1$ represents the torus, and $S_2$ represents the double torus (or...
the sphere with two handles). Figure 1.4 shows two examples of the same graph embedded on the torus. The left embedding is not a 2-cell embedding since a closed curve can be drawn in the outer region as shown that cannot be contracted to a point, whereas the right embedding is a 2-cell embedded on the torus.

Figure 1.4

2-CELL EMBEDDINGS

If \( G \) is a connected graph that is 2-cell embedded on some surface, then Euler's identity provides a connection between the number of regions in its 2-cell embedding and the genus of the surface on which \( G \) is embedded.

Theorem 1.2  If \( G \) is a connected graph with order \( n \) and size \( m \) that is 2-cell embedded on \( S_k \) with \( r \) regions, then \( n - m + r = 2 - 2k \).

The neighborhood \( N(v) \) of a vertex \( v \) in a graph \( G \) is the set of all vertices of \( G \) that are adjacent to \( v \). A rotation embedding scheme \( \varphi \) is a collection of cyclic permutations \( \rho_v : N(v) \to N(v) \), one for each \( v \in V(G) \).

Figure 1.5 is an illustration of a graph with five vertices embedded on the torus.
Figure 1.5

ROTATION EMBEDDING SCHEME

Observe that in this embedding the edges incident with $v_i$ are arranged cyclically counterclockwise about $v_i$ in the order $v_i v_2, v_i v_3, v_i v_4, v_i v_5$. This gives a cyclic permutation $\rho_i$ of the subscripts of the vertices adjacent with $v_i$, namely $\rho_1 = (2 \ 3 \ 4 \ 5)$. Similarly for each vertex $v_i (1 \leq i \leq 5)$, one can describe a cyclic permutation $\rho_i$ for $v_i$. In this case, we have

$$\rho_1 = (2345)$$
$$\rho_2 = (1543)$$
$$\rho_3 = (1254)$$
$$\rho_4 = (1325)$$
$$\rho_5 = (1432)$$

With the aid of the cyclic permutations we can trace out the edges of the regions in Figure 1.5 in a clockwise direction. We start from the vertex $v_i$ with the edge $v_i v_2$ which can be seen from $\rho_1$. The edge incident to $v_2$ following $v_2 v_1$ if we proceeded counterclockwise about $v_2$ is $v_2 \rho_2(1) = v_2 v_5$. Similarly the edge following $v_3 v_2$ is
\[ v_2 v_{\rho(2)} = v_2 v_1 \] and the edge following \[ v_3 v_1 \] is \[ v_1 v_{\rho(5)} = v_1 v_2. \] Thus we have traversed the whole region. The three edges on the boundary of the region \( R_f \) in Figure 1.5 are \( v_f v_2, v_2 v_5, v_5 v_1 \).

It is well known (see Edmonds [3]) that the 2-cell embeddings of a connected graph \( G \) are in one-to-one correspondence with the rotation embedding schemes of \( G \).

**Theorem 1.3** Let \( G \) be a connected graph with \( V(G) = \{1, 2, ..., n\} \). If \( G \) is 2-cell embedded on \( S_k \), then this 2-cell embedding uniquely determines a rotation embedding scheme

\[ \varphi = \{ \rho_i : N(i) \to N(i) \mid 1 \leq i \leq n \}. \]

Conversely, such a rotation embedding scheme uniquely determines a 2-cell embedding of \( G \) on some surface.

If \( \varphi \) is a rotation scheme for \( G \), then the ordered pair \( (G, \varphi) \) is called a map and we say that the genus \( \text{gen}(G, \varphi) \) of the map \( (G, \varphi) \) is \( k \) if \( \varphi \) determines a 2-cell embedding of \( G \) on \( S_k \). Furthermore we note that the \( \text{gen}(G) = \min_{\varphi} \text{gen}(G, \varphi) \).

This work focuses on embeddings of a special class of graphs that provide concrete representations of groups. Let \( \Gamma \) be a finite group and \( \Omega \) be a generating set for \( \Gamma \) such that \( \Omega = \Omega^{-1} \), i.e., \( \Omega \) is closed under inverses, and such that the identity \( e \in \Omega \). The **Cayley graph** for \( \Gamma \) and \( \Omega \) is the graph whose vertices are the elements of \( \Gamma \), where two vertices \( g_1 \) and \( g_2 \) of \( G \) are adjacent if and only if \( g_1 = g_2 h \) for some \( h \in \Omega \). Such a Cayley graph will be denoted by \( G_\Omega (\Gamma) \).
Let $\Gamma$ be a group and $\Omega$ be a generating set for $\Gamma$. For a cyclic permutation $\rho : \Omega \to \Omega$, the Cayley map $(\Gamma, \Omega, \rho)$ is the map $(G_\Omega(\Gamma), \varphi)$ where $\varphi = \{\rho_x | x \in \Gamma\}$ is the rotation embedding scheme for $G_\Omega(\Gamma)$ such that $\rho_x(y) = xp(x^{-1}y)$ for each $x \in \Gamma$ and each $y \in N(x)$. In other words, a Cayley map is a 2-cell embedding of a Cayley graph in which each vertex rotation $\rho_x$ is determined by the same cyclic ordering from $\rho$ of the elements of $\Omega$. Figure 1.6 illustrates the vertex rotation at $x$.

Figure 1.6

**ROTATION AROUND A VERTEX**

A significant benefit of Cayley maps is that they provide sufficient symmetry to be able to describe the graph embedding from a different approach. Let $\Gamma$ be a group with generating set $\Omega$ and let $\Delta \subset \Omega$ such that if $x \in \Delta$ then $x^{-1} \notin \Delta$ unless $x^2 = e$. A pseudo graph (a graph that allows directed edges and multiple loops) $K$ of order 1 is called a **voltage graph of index one** corresponding to $\Gamma$ and $\Omega$ if $K$ has size $|\Delta|$ and the directed edges of $K$ are labeled with the distinct elements of $\Delta$. Traversing along the edge in the
direction of the arrow gives the assigned group element while traversing the edge against
the arrow gives the inverse of the assigned group element. Figure 1.7 is an example of a
voltage graph with one directed edge. More specifically if we let $\Gamma = \mathbb{Z}_4$ and $\Omega = \{1, 4\}$,
then Figure 1.8 shows an index one voltage graph for $\Gamma$ and $\Omega$ that lifts to the Cayley
graph embedding shown in Figure 1.6.

Let $K$ be a voltage graph of index one corresponding to a finite group $\Gamma$ and a
generating set $\Omega$, and suppose that $K$ is 2-cell embedded on some surface. Then this
embedding of $K$ determines a 2-cell embedding of the Cayley graph $G_{\alpha}(\Gamma)$ on some
surface. We shall say that $K$ lifts to the covering graph $G_{\alpha}(\Gamma)$.

Let $R$ be a region of an embedding of a voltage graph $K$, and let $h_1, h_2, \ldots, h_k$ denote
the sequence of arcs (elements of $\Omega$) encountered as the boundary of $R$ is traversed in the
clockwise direction (i.e. with the boundary on the left-hand side). Then we say that $R$ is
bounded by an orbit of length $k$, and the boundary element of $R$ is defined to be the
product (or sum in an additive group) \( \prod_{i=1}^{k} h_i^{m_i} \), where \( m_i = 1 \) if \( h_i \) is oriented in the same
direction as the boundary of \( R \) is traversed and \( m_i = -1 \) if \( h_i \) is oriented in the opposite
direction. However, in voltage graphs, a problem arises when generators of order 2 are
used. A directed loop of order two produces a pair of multiple edges in the covering. To
avoid this, in voltage graphs, a generator of order 2 is represented by a *spoke* (or half-
edge).

Figure 1.8 shows a voltage graph where \( \Gamma = \mathbb{Z}_{10} \) and \( \Omega = \{1, 2, 5, 8, 9\} \) so that the
voltage graph for \( \Gamma \) and \( \Omega \) has two directed edges and a spoke. The particular voltage
graph embedding shown in Figure 1.8 lifts to a particular Cayley map for \( \Gamma \) and \( \Omega \). The
element 5 has order 2 and as such is represented by the spoke. A voltage graph
embedding can be used to find the number of regions and the structure of the regions in
the covering embedding.

Figure 1.8

VOLTAGE GRAPH WITH TWO LOOPS AND A SPOKE
The following result is due to Gross and Alpert [4].

Theorem 1.4 For a finite group $\Gamma$ with generating set $\Omega$, let $K$ be a voltage graph of index one corresponding to $\Gamma$ and $\Omega$ that is 2-cell embedded on some surface. Furthermore, let $R$ be a region of the embedding of $K$ that is bounded by an orbit of length $k \ (k \geq 1)$, and let $s$ denote the order of the boundary element of $R$ in $\Gamma$. Then $R$ lifts (or corresponds) to $|\Gamma|/s$ regions in the covering graph embedding, where each such region is bounded by an orbit of length $ks$. Moreover, two distinct regions of $K$ correspond to distinct sets of regions in the covering embedding.

If $R$ is a region whose boundary element is the identity, then $R$ satisfies the Kirchhoff Voltage Law or more simply the KVL. This brings up a special case of Theorem 1.4.

Corollary 1.5 For a finite group $\Gamma$ with generating set $\Omega$, let $K$ be a voltage graph of index one corresponding to $\Gamma$ and $\Omega$ that is 2-cell embedded on some surface. If $R$ is a region of this embedding that is bounded by an orbit of length $k \ (k \geq 1)$ and has the identity of $\Gamma$ as its boundary element, then $R$ lifts to $|\Gamma|$ regions, each of which is bounded by an orbit of length $k$.

Let $p$ be a prime integer. We focus on the additive group of the integers modulo $2p$, denoted $\mathbb{Z}_{2p}$. If the cyclic permutation $\rho$ contains $b,a,a^{-1},c$ (or $b,a,p,a^{-1},c$) where $c \neq b^{-1}$ in this order and the loop corresponding to the generator $a$ in the voltage graph embedding is not contained inside any other loop; then the loop associated with $a$ is called a singleton loop. If the cyclic permutation $\rho$ contains generators $a_1,a_2,...,a_k a_k^{-1},...,a_2^{-1},a_1^{-1}$ in this order and the structure is not contained within any other
loop, then this structure of the index one voltage graph embedding is called a *stacked loop*. See Figure 1.9 for examples. If the spoke corresponding to the prime \( p \) is between any two consecutive generators of a stacked loop, then the structure is still referred to as a stacked loop. Figure 1.9 is an example of a voltage graph with three singleton loops and two stacked loops.

![Figure 1.9

STACKED LOOPS](image)

Theorem 1.6 Let \( p \) be a prime and \( \Gamma = \mathbb{Z}_{2p} \). If \( \Omega = \{1,3,\ldots,2p-1\} \) then \( G_{\Omega}(\Gamma) \) is the complete bipartite graph \( K_{p,p} \).

Proof: Let \( p \) be a prime and \( \Gamma = \mathbb{Z}_{2p} \) with \( \Omega = \{1,3,\ldots,2p-1\} \). Let \( V_1 = \{1,3,\ldots,2p-1\} \) and \( V_2 = \{2,4,\ldots,2p\} \). We will show that \( V_1 \) and \( V_2 \) are the partite sets for \( G_{\Omega}(\Gamma) \). Let \( h \in \Omega \) and \( g_1 \in V_1 \). Then \( g_1+h \) is even (an odd integer plus an odd integer is an even
integer). So no vertex in $V_1$ is adjacent to any other vertex in $V_1$. Let $g_2 \in V_2$. Then $g_2 + h$ is odd (an even integer plus an odd integer is an odd integer). So no vertex in $V_2$ is adjacent to any other vertex in $V_2$. If we add modulo $2p$ each of the integers $1, 2, 3, \ldots, 2p$ to the integer $g_2$, where the addition is taken modulo $2p$, then we obtain all the integers $1, 2, 3, \ldots, 2p$. So if we add each of the elements from $\Omega$ to the integer $g_2$ we would obtain all of the odd integers $1, 3, \ldots, 2p-1$. Thus $g_2$ is adjacent to every element of $V_1$ and since we chose $g_2$ arbitrarily, every element of $V_2$ is adjacent to every element of $V_1$. □

Next we discuss counting with the aid of generating functions as this will be useful later. In general for the infinite sequence $S: h_0, h_1, h_2, \ldots, h_n, \ldots$, the generating function for $S$ is the infinite series $g(x) = h_0 + h_1 x + h_2 x^2 + \ldots + h_n x^n + \ldots$. We can use the same definition for finite sequences by taking any finite sequence $h_0, h_1, h_2, \ldots, h_k$ and appending an infinite sequence of zeros onto the end so that the sequence is $h_0, h_1, h_2, \ldots, h_k, 0, 0, \ldots$ and the generating function is then $g(x) = h_0 + h_1 x + h_2 x^2 + \ldots + h_k x^k$.

A partition of a positive integer $n$ is a set of $k$ positive integers $i_1, i_2, \ldots, i_k$ such that $n = i_1 + i_2 + \ldots + i_k$. The integers $i_1, i_2, \ldots, i_k$ are called the parts of the partition. For example the partitions of 5 are

5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1.

Thus there are seven partitions of the integer 5. We can count the number of partitions of the integer five using a generating function, namely

$$g(x) = (1 + x + x^2 + x^3 + x^4 + x^5)(1 + x^2 + x^4)(1 + x^3)(1 + x^4)(1 + x^5) =$$

$$x^{21} + x^{20} + 2x^{19} + 3x^{18} + 5x^{17} + 7x^{16} + 7x^{15} + 10x^{14} + 11x^{13} + 13x^{12} + 12x^{11} + 12x^{10} + 13x^9 + 11x^8 + 10x^7 + 7x^6 + 7x^5 + 5x^4 + 3x^3 + 2x^2 + x + 1,$$
where the coefficient of $x^4$ in the polynomial expansion of this generating function is the number of partitions of 5. The exponents of $x$ in the expression $(1 + x + x^2 + x^3 + x^4 + x^5)$ represent the number of times that 1 gets added into the partition, i.e. if we choose $1x^0$ this would represent 1 being a part of the partition zero times, i.e., not being a part of the partition. The exponents of the term $(1 + x^2 + x^4)$ represent the number of times that two gets added into the partition, i.e. if we choose $x^4 = x^{2\cdot2}$ this would represent 2 being a part of the partition twice.

What if we want to have the partitions to be of distinct parts? For example the partitions of 5 with distinct parts are: 5, 4+1, 3+2. Thus there are three partitions of 5 with distinct parts. We can count the number of these partitions of the integer five using a generating function, namely $g(x) = (1 + x)(1 + x^2)(1 + x^3)(1 + x^4)(1 + x^5)$. The coefficient of $x^5$ in the polynomial expansion of this generating function is the number of partitions of 5 with distinct parts. When we multiply the factors of $g(x)$ together we can only get $x^5$ in three ways: $1\cdot1\cdot1\cdot1\cdot x^5$, $1\cdot x^2\cdot x^3\cdot1\cdot1$, and $x^1\cdot1\cdot1\cdot x^4\cdot1$. So the coefficient of $x^5$ in the expansion of $g(x)$ is 3 which is the number of partitions of five with distinct parts. In each factor $(1+x^i)$ of $g(x)$, $1 \leq i \leq 5$, the 1 represents the choice of not using $i$ in the partition of 5, while the $x^i$ represents the choice of using $i$ in the partition.

In general the generating function for the partitions of the integer $n$ into unequal parts no part greater than $m$ is given by $g_m(u) = \prod_{i=1}^m (1 + u^i)$. In this case, the coefficient of $u^n$ is the number of partitions of the integer $n$. We will denote this coefficient by $[u^n] g_m(u)$. The generating function $g_m(u)$ will be used in the proofs of Propositions 2.2,
2.3, 2.4, and 2.5. We can also modify this generating function if we want to specify the parts that are used. For example, we might wish to know how many partitions of 17 there are with unequal parts using only 2, 3, 4, 5, 6, 7, 9, 11, 15. We want the generating function

\[ g(u) = \frac{1}{(1+u^2)(1+u^3)(1+u^4)(1+u^5)} \]

In other words let \( T = \{2,3,4,5,6,7,9,11,15\} \). Then \( g_T(u) = \prod_{i \in T} (1+u^i) \) and since we want the partitions of 17, we want the coefficient of \( u^{17} \) in \( g_T(u) = 1 + u^2 + u^3 + u^4 + 2u^5 + 2u^6 + 3u^7 + 2u^8 + 5u^9 + 3u^{10} + 6u^{11} + 5u^{12} + 6u^{13} + 7u^{14} + 8u^{15} + 9u^{16} + 8u^{17} + 12u^{18} + 8u^{19} + 14u^{20} + 11u^{21} + 14u^{22} + 12u^{23} + 15u^{24} + 14u^{25} + 15u^{26} + 17u^{27} + 13u^{28} + 18u^{29} + 14u^{30} + 18u^{31} + 14u^{32} + 18u^{33} + 13u^{34} + 17u^{35} + 15u^{36} + 14u^{37} + 15u^{38} + 12u^{39} + 14u^{40} + 11u^{41} + 14u^{42} + 8u^{43} + 12u^{44} + 8u^{45} + 9u^{46} + 8u^{47} + 7u^{48} + 6u^{49} + 5u^{50} + 6u^{51} + 3u^{52} + 5u^{53} + 2u^{54} + 3u^{55} + 2u^{56} + 2u^{57} + u^{58} + u^{59} + u^{60} + u^{62} \). We find \( \left[ u^{17} \right] g_T(u) = 8 \). The partitions are \( 2+15, 6+11, 2+4+11, 2+6+9, 3+5+9, 4+6+7, 2+3+5+7, 2+4+5+6 \).

The general problem of interest in this work is to classify (find the genus values) for all of the Cayley maps for \( \mathbb{Z}_{2p} \). Since we are focusing on the group \( \mathbb{Z}_{2p} \) the element of order two is always \( p \). So by always setting \( p \) as the first element in our cyclic permutations we order the rest of the elements in the permutation. To illustrate the general problem let us focus on when \( p = 5 \). This example will have one spoke and two loops.

Figure 1.10 shows the three embeddings of a voltage graph with two loops and a spoke. Figure (a) shows a planar embedding with two regions of size one and one region.
of size three. Figure 1.10(b) shows a planar embedding with one region of size one and two regions of size two. Figure 1.10(c) is an embedding on the torus and has one region of size 5.

Example 1.1: Using the generating set \( \Omega = \{1, 3, 5, 7, 9\} \) for \( \mathbb{Z}_{2p} \) we obtain the Cayley graph \( K_{5,5} \). The spoke 5 will be put first in any cyclic permutation \( \rho \) so that the other elements are then ordered. So there are \( 4! = 24 \) permutations.

We can describe all of the permutations that have the spoke followed by one singleton loop then another (Figure 1.10(a)) by the general permutation \( \rho = (p \ a \ a^{-1} \ b \ b^{-1}) \) where \( \Omega = \{p, a, a^{-1}, b, b^{-1}\} \). When \( a=1 \) and \( b=3 \) we have \( \rho_1 = (5 \ 1 \ 9 \ 3 \ 7) \). So that we find the boundary elements to be 1,3, and 9. Thus for \( \rho_1 \) the order of each of the regions is 10 so that each region lifts to \( 10/10 = 1 \) region in the covering graph embedding. Similarly for the permutations that have the spoke followed by one loop inside the other (Figure 1.10(b)) we have the general permutation \( \rho = (p \ a \ b \ b^{-1} \ a^{-1}) \) where \( \Omega = \{p, a, a^{-1}, b, b^{-1}\} \). The permutations that have the spoke followed by the start of one loop, the start of the other
loop, the end of the first loop, and then the end of the second loop (Figure 1.10 (c)) have the general permutation $\rho = (p \ a \ b \ a^{-1} \ b^{-1})$ where $\Omega = \{p, a, a^{-1}, b, b^{-1}\}$.

There is only one choice for $p$ in the permutation. For $a$ there are four choices but then $a^{-1}$ is determined. Leaving 2 choices for $b$ and then $b^{-1}$ is determined. So each general permutation produces 8 cyclic permutations. The inverse of the cyclic permutation produces a mirror image of the embedding and thus the same number of regions in the covering graph embedding.

Table 1.1 is a summary of the 24 cyclic permutations. The first column and the second column show the two permutations that are inverses of each other. The third column shows how many regions in the covering graph embedding the regions in the voltage graph lift to. The first eight are from the general permutation $\rho = (p \ a \ a^{-1} \ b \ b^{-1})$, the second set of eight are from $\rho = (p \ a \ b \ b^{-1} \ a^{-1})$, the third set of eight are from $\rho = (p \ a \ b \ a^{-1} \ b^{-1})$

This example is nice because it is small. Each of the eight permutations coming from a general permutation lifts to the same number of regions in the covering graph embedding. In general this is not true, take $p = 7$. This example will have one spoke and three loops.

Figure 1.11 shows 15 voltage graphs each of which has 3 loops. Figure 1.11(a) shows a planar voltage graph with three regions of size one and one region of size four. Figure 1.11 (b), (c), and (e) show planar voltage graphs with two regions of size one, one region of size two, and one region of size three. Figure 1.11 (d) shows a planar voltage graph with three regions of size two and one regions of size one. Figure 1.11 (f), (g), (h), (i), and (j) show a voltage graph embedded on a taurus with one region of size six and
one region of size one. Figure 1.11 (k), (l), and (o) shows a voltage graph embedded on a
torus with one region of size five and one region of size two. Figure 1.11 (m) and (n)
shows a voltage graph embedded on the torus with one region of size three and one
region of size four. Figure 1.11 (o) shows a voltage graph embedded on the torus with
one region of size three and one region of size four.

Table 1.1

<table>
<thead>
<tr>
<th>permutation</th>
<th>inverse</th>
<th>Number of regions in the covering graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_1=(5, 1, 9, 3, 7)$</td>
<td>$\rho_8=(5, 7, 3, 9, 1)$</td>
<td>3</td>
</tr>
<tr>
<td>$\rho_2=(5, 1, 9, 7, 3)$</td>
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<td>3</td>
</tr>
<tr>
<td>$\rho_3=(5, 3, 7, 1, 9)$</td>
<td>$\rho_7=(5, 9, 1, 7, 3)$</td>
<td>3</td>
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<tr>
<td>$\rho_4=(5, 7, 3, 1, 9)$</td>
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<td>$\rho_5=(5, 1, 3, 7, 9)$</td>
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<td>5</td>
</tr>
<tr>
<td>$\rho_{10}=(5, 1, 7, 3, 9)$</td>
<td>$\rho_{11}=(5, 9, 3, 7, 1)$</td>
<td>5</td>
</tr>
<tr>
<td>$\rho_{13}=(5, 3, 1, 9, 7)$</td>
<td>$\rho_{16}=(5, 7, 9, 1, 3)$</td>
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</tr>
<tr>
<td>$\rho_{14}=(5, 3, 9, 1, 7)$</td>
<td>$\rho_{15}=(5, 7, 1, 9, 3)$</td>
<td>5</td>
</tr>
<tr>
<td>$\rho_{17}=(5, 1, 3, 9, 7)$</td>
<td>$\rho_{21}=(5, 7, 8, 3, 1)$</td>
<td>5</td>
</tr>
<tr>
<td>$\rho_{18}=(5, 1, 7, 9, 3)$</td>
<td>$\rho_{23}=(5, 3, 9, 7, 1)$</td>
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<tr>
<td>$\rho_{19}=(5, 3, 1, 7, 9)$</td>
<td>$\rho_{22}=(5, 9, 7, 1, 3)$</td>
<td>5</td>
</tr>
<tr>
<td>$\rho_{20}=(5, 3, 9, 7, 1)$</td>
<td>$\rho_{24}=(5, 1, 7, 9, 3)$</td>
<td>5</td>
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</table>
Example 1.2: Using the generating set \( \Omega = \{1, 3, 5, 7, 9, 11, 13\} \) for \( \mathbb{Z}_{14} \) we obtain the Cayley graph \( K_{7,7} \). There are \( 6! = 720 \) permutations. The spoke followed by three loops one after the other gives the general permutation \( \rho = (p \ a \ a^{-1} \ b \ b^{-1} \ c \ c^{-1}) \) where \( \Omega = \{p, a, a^{-1}, b, b^{-1}, c, c^{-1}\} \).

Figure 1.11

FIFTEEN VOLTAGE GRAPHS

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Table 1.2 shows us the 48 permutations that give rise to such a voltage graph. It also gives the number of regions the voltage graph lifts to in the covering graph embedding and the genus of the Cayley map that arise from the cyclic permutations.

Table 1.2

<table>
<thead>
<tr>
<th>permutation</th>
<th>inverse</th>
<th>Number of regions in the covering graph</th>
<th>Genus of $K_{7,7}$</th>
</tr>
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<tbody>
<tr>
<td>$\rho_1=(7, 1, 13, 3, 11, 5, 9)$</td>
<td>$\rho_{25}=(7, 9, 5, 11, 3, 13, 1)$</td>
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<td>16</td>
</tr>
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<td>$\rho_2=(7, 1, 13, 3, 11, 9, 5)$</td>
<td>$\rho_{26}=(7, 9, 5, 11, 3, 13, 1)$</td>
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<td>$\rho_3=(7, 1, 13, 11, 3, 5, 9)$</td>
<td>$\rho_{27}=(7, 9, 5, 3, 11, 13, 1)$</td>
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<tr>
<td>$\rho_4=(7, 1, 13, 11, 3, 9, 5)$</td>
<td>$\rho_{28}=(7, 9, 5, 3, 11, 13, 1)$</td>
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<td>$\rho_7=(7, 1, 13, 9, 5, 3, 11)$</td>
<td>$\rho_{31}=(7, 11, 3, 5, 9, 13, 1)$</td>
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<td>16</td>
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<tr>
<td>$\rho_8=(7, 1, 13, 9, 5, 11, 3)$</td>
<td>$\rho_{32}=(7, 3, 11, 5, 9, 13, 1)$</td>
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<td>$\rho_{14}=(7, 13, 1, 5, 9, 11, 3)$</td>
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<td>$\rho_{15}=(7, 13, 1, 9, 5, 3, 11)$</td>
<td>$\rho_{39}=(7, 11, 3, 5, 9, 1, 13)$</td>
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<td>$\rho_{41}=(7, 9, 5, 13, 1, 11, 3)$</td>
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<td>$\rho_{18}=(7, 3, 11, 1, 13, 9, 5)$</td>
<td>$\rho_{42}=(7, 9, 5, 13, 1, 11, 3)$</td>
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<tr>
<td>$\rho_{19}=(7, 3, 11, 1, 13, 1, 5, 9)$</td>
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<td>10</td>
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<td>$\rho_{20}=(7, 3, 11, 1, 13, 1, 9, 5)$</td>
<td>$\rho_{44}=(7, 9, 5, 1, 13, 11, 3)$</td>
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<td>16</td>
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<tr>
<td>$\rho_{21}=(7, 11, 3, 1, 13, 5, 9)$</td>
<td>$\rho_{45}=(7, 9, 5, 13, 1, 11, 3)$</td>
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<td>16</td>
</tr>
<tr>
<td>$\rho_{22}=(7, 11, 3, 1, 13, 9, 5)$</td>
<td>$\rho_{46}=(7, 9, 5, 13, 1, 11, 3)$</td>
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<td>10</td>
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<tr>
<td>$\rho_{23}=(7, 11, 3, 13, 1, 5, 9)$</td>
<td>$\rho_{47}=(7, 9, 5, 1, 13, 3, 11)$</td>
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<td>16</td>
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<tr>
<td>$\rho_{24}=(7, 11, 3, 13, 1, 9, 5)$</td>
<td>$\rho_{48}=(7, 9, 5, 1, 13, 3, 11)$</td>
<td>5</td>
<td>16</td>
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</tbody>
</table>

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There is only one choice for \( p \). There are 6 choices for \( a \) but then \( a^{-1} \) is determined. Then there are 4 choices for \( b \) and \( b^{-1} \) is determined. Leaving 2 choices for \( c \) and determining \( c^{-1} \). Thus each general permutation produces 48 cyclic permutations. We have three regions of size one and one region of size four that contains the spoke.

Comparing the number of lifts in table 1.2 with the lifts from the first four lines of Table 1.1 we are comparing where there are only singleton loops with the spoke in the largest region. Notice how in example 1 the cyclic permutations form the same general permutation all give rise to the same number of regions in the covering graph embedding. In Example 1.2 the cyclic permutations give rise to different numbers of regions in the covering graph embedding. This is because with regions of size three or greater, not including the spoke, we have the possibility of the region satisfying the KVL or semi KVL. Material for this chapter came from J. Riordan [5], G. Chartrand & L. Lesniak [2], and R. A. Brualdi [1].

We wish to classify Cayley maps for complete bipartite graphs that arise from \( Z_{2p} \) with the generating set \( \Omega = \{1, 3, \ldots, 2p-1\} \). We will in this work classify only those Cayley maps that are generated by the odd elements of \( Z_{2p} \) where \( p \) is a prime and are lifts of certain plane voltage graphs. In Chapter 2 we determine the genera for Cayley maps that are covering embeddings of planar voltage graphs consisting of only singleton loops. In Chapter 3 we determine the genera for Cayley maps that are covering embeddings of planar voltage graphs consisting of stacked loops or singleton loops with at most one region of size greater than 2. In addition to providing results in these cases, we also discuss in Chapter 4 the more general problem involving Cayley maps that are covering graph embeddings of arbitrary plane voltage graphs.
CHAPTER 2

PLANAR VOLTAGE GRAPHS WITH ONLY SINGLETON LOOPS

Figure 2.1 shows two voltage graphs having only singleton loops: 2.1 (a) has the spoke in the largest region and 2.1 (b) has the spoke in a region of order two.

Figure 2.1

SINGLETON LOOPS

(a)  (b)
Section 2.1

General Setting

Let $p \geq 3$ be a prime number. In the cyclic group $Z_{2p}$ we take the generating set $\Omega = \{1, 3, \ldots, 2p - 1\}$, the odd integers in $Z_{2p}$. In this case the Cayley graph $G_{\Omega}(Z_{2p})$ is the complete bipartite graph $K_{p,p}$, which has $2p$ vertices and $p^2$ edges. So if this Cayley graph is 2-cell embedded in the surface $S_k$ then applying Euler’s identity and solving, we obtain the formula $k = 1 + \frac{p^2 - 2p - r}{2}$ for the genus of $K_{p,p}$, where $r$ denotes the number of regions in the Cayley graph embedding. Since the genus $k = 1 + \frac{p^2 - 2p - r}{2}$ of our surface is an integer, it follows that $p^2$ and $r$ have the same parity so that $r$ is an odd number. The elements of $Z_{2p}$ and thus the boundary elements for the regions in the voltage graph embedding have four possible orders, namely 1, 2, $p$, or $2p$. From Theorem 1.4 we see that regions in the voltage graph embedding with these elements lift respectively to 2, $p$, 2, or 1 region in the Cayley graph embedding. If $R$ is a region whose boundary element is $p$, then we say $R$ satisfies the semi $KVL$. If $R$ satisfies the semi $KVL$ then it has a boundary element of order two and $R$ lifts to $p$ regions in the covering graph embedding.

Proposition 2.1 Let $p$ be an odd prime and let $\Gamma = Z_{2p}$ with generating set $\Omega = \{1, 3, \ldots, 2p - 1\}$, the odd integers. Let $K$ be a plane voltage graph of index one corresponding to $\Gamma$ and $\Omega$. If $R$ is a region of $K$ with size one then $R$ lifts to one region in the covering graph embedding.
Proof: Let $R$ be a region of size one in the voltage graph $K$. Assigning one generator from $\Omega$ to the boundary of $R$ gives an odd integer other than $p$ for the boundary element. The order of an odd boundary (besides $p$) element is $2p$ so that the region lifts to one region in the covering graph embedding. \( \square \)

For the following discussion it is convenient to think of the generators in the form $\Omega = \{ p-(p-1), p-(p-3), \ldots, p-4, p-2, p, p+2, p+4, \ldots, p+(p-1), p+(p-3) \}$. This allows each generator to be thought of as being a distance away from $p$. Notice that each element $x \in \Omega$ can be written as $x = p + 2i$, where $i \in \left\{ 0, \pm 1, \pm 2, \ldots, \pm \frac{p-1}{2} \right\}$. We call $i$ the $p$-distance of $x$. So the $p$-distance of $x$ is the number of integers $x$ is away from $p$. For example the element $p + 2$ is the first element to the right of $p$ so it has a $p$-distance of 1. The element $p - 2$ is the first element to the left of $p$ so it has a $p$-distance of $-1$, the negative telling us that the element lies to the left of $p$. We denote the set of positive $p$-distances by $P^+$.

The remainder of this chapter contains theorems and proofs for voltage graphs having only singleton loops. In particular we will provide a formula for the number of Cayley maps having a certain genus value when the underlying voltage graph has an odd number of loops with the spoke in the largest region or the spoke in a region of size two. We will also give a formula for the number of Cayley maps with a certain genus when the voltage graph has an even number of loops with the spoke in the largest region or the spoke in a region of size two. Relevant examples will be provided for clarification.
Section 2.2

The Spoke in the Largest Region

Figure 2.2 shows a voltage graph with five loops and one spoke without labeled edges. The spoke is in position one in the figure. If the spoke was in position two, three, four, or five the picture would look essentially the same.

Example 2.1: Using the generating set $\Omega = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21\}$ from $\mathbb{Z}_{22}$ we obtain the Cayley graph $K_{11,11}$. There are a total of $10! = 3,628,800$ cyclic orderings of our generating set. The spoke followed by five loops one after the other gives the general permutation $\rho = (p \ a \ a^{-1} \ b \ b^{-1} \ c \ c^{-1} \ d \ d^{-1} \ e \ e^{-1})$ where $\Omega = \{p, a, a^{-1}, b, b^{-1}, c, c^{-1}, d, d^{-1}, e, e^{-1}\}$. There is only one choice for $p$. There are 10 choices of elements from $\Omega$ for $a$ but then $a^{-1}$ is determined. Then there are 8 choices for $b$ and $b^{-1}$ is determined. Next there are 6 choices for $c$ determining $c^{-1}$ and 4 choices for $d$ determining $d^{-1}$, leaving 2 choices for $e$ and determining $e^{-1}$. Thus each general permutation produces 3840 cyclic permutations.

We have five regions of size one and one region of size six that contains the spoke. There
are 22 vertices and 121 edges. Plugging into Euler's equation we get \( k = \frac{101 - r}{2} \) for the genus of a Cayley map for \( \mathbb{Z}_{22} \) with generating set \( \Omega \), where \( r \) denotes the number of regions in the Cayley map. The genus must be an integer so that the number of regions \( r \) is odd.

In assigning elements from \( \Omega \) to the boundary of the largest region, \( R \), we are assigning an element to every loop of the voltage graph. Since we add the generators to calculate the boundary element there is a certain amount of symmetry. The following table shows all the possible combinations of elements that could be assigned to the boundary of \( R \). There is no need to look at all 3840 permutations as Table 2.1 gives all the necessary information. The first five columns represent the elements \( a, b, c, d, e \in \Omega \) on the boundary of \( R \).

The five regions of size 1 in the voltage graph all have order 22 and each lifts to one region in the covering graph embedding. The one region of size 6 has two possibilities: it can have a boundary element of order 1 and lift to 22 regions in the covering graph embedding or it may have a boundary element of order 11 and lift to 2 regions in the covering graph embedding.

Permuting the five columns shows us that there are 5! permutations for each row in the table. For each cyclic permutation its inverse generates a mirror image that has the same boundary elements. Thus there are 240 permutations that produce the genus 37 and 3600 permutations that produce the genus 47, giving the total of 3840 permutations.
In Example 2.1 we were lucky because of the structure of this case allows us to use a table that is relatively small to examine the exterior region boundary element. This table grows exponentially as $p$ increases so we need a much better method of counting the number of Cayley maps with a specific genus.

Example 2.1 had an odd number of loops. The following discussion is for voltage graphs that have an even number of loops and contain only singleton loops with the spoke in the largest region. In Proposition 2.3 we discuss voltage graphs that have an odd

### Table 2.1

<table>
<thead>
<tr>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
<th>$e$</th>
<th>regions with the boundary element (a+b+c+d+e+11)</th>
<th>genus</th>
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number of loops and contain only singleton loops with the spoke in the largest region.

The proofs are similar and a lot of the same details apply.

Proposition 2.2 Let $p$ be an odd prime such that $m = \frac{p-1}{2}$ is even, and let $\Gamma = \mathbb{Z}_2p$ with generating set $\Omega = \{1, 3, \ldots, 2p-1\}$. Let $S = 1 + 2 + \ldots + (p-1)/2$. Then of the Cayley maps that lift from plane voltage graphs consisting of only singleton loops and containing exactly one region of size greater than one, there are exactly

$$\left(\frac{p-1}{2}\right)! \sum_{n=S}^{\frac{S}{p}} \left[ \frac{p^{-S}p^{-1}}{16} \right] g_n(u)$$

Cayley maps $M = CM(\Gamma, \Omega, p)$ such that

$$\text{gen}(M) = \frac{2p^2 - 7p + 5}{4}$$

and

$$\left(\frac{p-1}{2}\right)! \left(\frac{p^{-1}}{2} - \sum_{n=S}^{\frac{S}{p}} \left[ \frac{p^{-S}p^{-1}}{16} \right] g_n(u)\right)$$

such that

$$\text{gen}(M) = \frac{2p^2 - 5p + 3}{4}.$$

Proof: Let $K$ be a plane voltage graph of index one corresponding to $\Gamma$ and $\Omega$ such that $K$ consists only of singleton loops and has exactly one region of size greater than one. There are only singleton loops and the one spoke for the generator $p$. In fact there is one singleton loop for each generator $1, 3, \ldots, p-1$. Thus there are $\frac{p-1}{2}$ regions of size one.

For each loop we choose one generator (different from $p$) from the list above and a direction for that loop. Thus there are a total of $2^{\frac{p-1}{2}} \left(\frac{p-1}{2}\right)!$ distinct voltage graphs $K$.

By Proposition 2.1 each region of size one lifts to one region in the covering graph embedding so that there are $\frac{p-1}{2}$ regions produced by the regions of size one. Let $r$ be
the number of regions in the Cayley map that lifts from $K$. Recall that the genus of the
Cayley map for $\mathbb{Z}_{2p}$ with the generating set $\Omega$, is $k = 1 + \frac{p^2 - 2p - r}{2}$. Since $p^2$ and $r$
have the same parity and we have an even number of regions being produced by the
regions of size one, the one region $R$ of size $\frac{p+1}{2}$ must lift to an odd number of regions.
Thus $R$ lifts to either 1 or $p$ regions in the Cayley map.

Next we determine how many voltage graph embeddings have $R$ lifting to $p$
regions in the Cayley map. Examining the boundary element of $R$, note that since $K$ is
planar, if $x$ is an element of $\Omega$ then either $x$ or $x^{-1}$ (and not both) is on the boundary of
the region $R$. Thus if all of the elements of $\Omega$ that are larger than $p$ are on the boundary,
then the boundary element for $R$ is $p + (p + 2) + \ldots + (p + p - 1) =\n\left(\frac{p+1}{2}\right) + 2 + 4 + \ldots + p - 1 = \left(\frac{p+1}{2}\right) + 2\left(1 + 2 + \ldots + \frac{p-1}{2}\right)$. Since $\frac{p+1}{2}$ is odd we
know $\left(\frac{p+1}{2}\right) \equiv p \mod 2p$. Thus the order of the boundary element of $R$ is 2 in $\mathbb{Z}_{2p}$ if
and only if $1 + 2 + \ldots + (p-1)/2$ is a multiple of $p$. In other words, $R$ (with boundary
element $p + (p + 2) + \ldots + (p + p - 1)$) lifts to $p$ regions if and only if $1 + 2 + \ldots + (p-1)/2$ is a
multiple of $p$.

Let $S = 1 + 2 + \ldots + (p-1)/2$. In a sense, we can think of this sum as representing an
assignment of the generators $p, (p + 2), \ldots, (p + p - 1)$ to the boundary of $R$. Now we
wish to consider an arbitrary assignment of generators to the boundary of $R$. Such an
assignment must contain each generator or its inverse (but not both). We will describe
next how the boundary element of $R$ for such an arbitrary assignment is related to the sum $S$.

Let us begin with an assignment of the generators greater than or equal to $p$ on the boundary of $R$. We investigate the boundary element of $R$ when this assignment is modified in order to represent an arbitrary assignment of generators to the boundary of $R$. In replacing a generator on the boundary of $R$ with its inverse, we first remove the $p$-distance of that generator from $S$ and then add in the $p$-distance of its inverse to $S$. Thus doing this for more than one generator, we see that $S$ changes by a decrease of $2X$, where $X$ is the sum of the $p$-distances of the generators being replaced by their inverses. So the boundary element for $R$ is $p^\left(\frac{p+1}{2}\right)+2(S-2X)$. Thus $R$ lifts to $p$ regions if and only if $S-2X$ is a multiple of $p$, thus the equation $S-2X= pn$ follows. Since $0 \leq X \leq S$, we have $\frac{-S}{p} \leq n \leq \frac{S}{p}$. Since $n$ is an integer, we see that $\left\lfloor \frac{-S}{p} \right\rfloor \leq n \leq \left\lfloor \frac{S}{p} \right\rfloor$. Now solving for $X$ in the equation $S-2X= pn$, we get $X = \frac{p^2-8pn-1}{16}$. So we must count the number of distinct subsets of $\left\{1,2,\ldots,\frac{p-1}{2}\right\}$ that sum to an integer $X$ that will satisfy $S-2X= pn$.

Recall that $m=\frac{p-1}{2}$. For $n$ with $\left\lfloor \frac{-S}{p} \right\rfloor \leq n \leq \left\lfloor \frac{S}{p} \right\rfloor$, the number of partitions of $\frac{p^2-8pn-1}{16}$ with distinct parts, no part greater than $m$, is given by the generating function $g_m(u) = \prod_{j=1}^{m} \left(1+u^j\right)$. Thus the number of times $S-2X$ is a multiple of $p$ is the
sum of the coefficients of $u^X$, where $X = \frac{p^2 - 8pn - 1}{16}$ and $\left\lfloor \frac{-S}{p} \right\rfloor \leq n \leq \left\lceil \frac{S}{p} \right\rceil$ in the expansion of $g_m(u)$. Thus $\sum_{n=\left\lfloor \frac{-S}{p} \right\rfloor}^{\left\lceil \frac{S}{p} \right\rceil} \left[ \frac{p^2 - 8pn - 1}{16} \right] g_m(u)$ gives the number of distinct subsets of $\left\{ 1, 2, \ldots, \frac{p-1}{2} \right\}$ that sum to an integer that will satisfy $S - 2X = pn$ and thus the number of (unordered) sets of generators for the boundary of $R$ that sum to a multiple of $p$. For one such set, there are $\left( \frac{p-1}{2} \right)!$ ways to arrange the generators and therefore the total number of voltage graphs with $R$ lifting to $p$ regions is $\left( \frac{p-1}{2} \right)! \sum_{n=\left\lfloor \frac{-S}{p} \right\rfloor}^{\left\lceil \frac{S}{p} \right\rceil} \left[ \frac{p^2 - 8pn - 1}{16} \right] g_m(u)$. Thus there are $\left( \frac{p-1}{2} \right)! \sum_{n=\left\lfloor \frac{-S}{p} \right\rfloor}^{\left\lceil \frac{S}{p} \right\rceil} \left[ \frac{p^2 - 8pn - 1}{16} \right] g_m(u)$ Cayley maps $M = CM(\Gamma, \Omega, p)$ such that $r = \frac{p-1}{2} + p = \frac{3p-1}{2}$, and the genus of such a Cayley map is $\text{gen}(M) = 1 + \frac{p^2 - 2p - \frac{3p-1}{2}}{2} = \frac{2p^2 - 7p + 5}{4}$. Since the rest of the voltage graphs have $R$ lifting to one region, there are $\left( \frac{p-1}{2} \right)! \left( \frac{p-1}{2} \right) \left[ \frac{p^2 - 8pn - 1}{16} \right] g_m(u)$ Cayley maps such that $r = \frac{p-1}{2} + 1 = \frac{p+1}{2}$ and the genus of such a Cayley map is $\text{gen}(M) = 1 + \frac{p^2 - 2p - \frac{p+1}{2}}{2} = \frac{2p^2 - 5p + 3}{4}$. \(\square\)
Since Proposition 2.2 covers when there are an even number of loops in our voltage graphs we must also consider what occurs when there are an odd number of loops in our voltage graphs. The following result is for voltage graphs that have an odd number of loops and contain only singleton loops with the spoke in the largest region.

Proposition 2.3 Let $p$ be an odd prime such that $m = \frac{p-1}{2}$ is an odd number, and $\Gamma = \mathbb{Z}_{2p}$ with generating set $\Omega = \{1, 3, ..., 2p-1\}$. Let $S = 1 + 3 + ... + (p-1)/2$. Then of the Cayley maps that lift from planar voltage graphs consisting only of singleton loops and containing exactly one region of size greater than one there are

$$\left(\frac{p-1}{2}\right) \cdot \left(\frac{p^3 - 8p - 1}{16}\right)^{\frac{5}{p}} g_m(u)$$

Cayley maps $M = \text{CM}(\Gamma, \Omega, p)$ such that

$$\text{gen}(M) = \frac{2p^2 - 9p + 5}{4} \text{ and } \left(\frac{p-1}{2}\right) \cdot \left(\frac{p^3 - 8p - 1}{16}\right)^{\frac{5}{p}} g_m(u)$$

maps $M$ such that

$$\text{gen}(M) = \frac{2p^2 - 5p + 1}{4}.$$

The proof of this follows the same pattern as the proof of Proposition 2.2. The major difference comes from the fact that $m = \frac{p-1}{2}$ is odd so that there are an odd number of regions of size one, each of which lifts to one region in the covering graph embedding.

Since $p^2$ and $r$ have the same parity the one region $R$ of size $\frac{p+1}{2}$ must lift to an even number of regions. Thus $R$ lifts to either 2 or $2p$ regions in the covering graph embedding. If all the generators assigned to the loops of the boundary of $R$ are larger than
$p$ we have the boundary element of $R$ is $p\left(\frac{P+1}{2}\right) + 2\left(1 + 2 + \ldots + \frac{p-1}{2}\right)$. Since $\frac{p+1}{2}$ is even, we know $p\left(\frac{P+1}{2}\right) \equiv 2p \mod 2p$. We will modify this sum to represent an arbitrary sum in the same manner as in the proof of Proposition 2.2. Thus we can count the number of voltage graphs in which $R$ lifts to $2p$ regions in the exact same manner as we counted $R$ lifting to $p$ regions in the proof of Proposition 2.2. Thus there are $\left(\frac{p-1}{2}\right)! \sum_{n=\frac{-p+1}{2}}^{\frac{p}{2}} \left[ \begin{array}{c} \frac{p^2-8p-1}{16} \\ u \end{array} \right] g_m(u)$ maps $M$ such that $r = \frac{p-1}{2} + 2p = \frac{5p-1}{2}$ and the genus of such a Cayley map is $\text{gen}(M) = 1 + \frac{p^2 - 2p - \frac{5p-1}{2}}{2} = \frac{2p^2 - 9p + 5}{4}$. There are $2^{p-1} \left(\frac{p-1}{2}\right)! \sum_{n=\frac{-p+1}{2}}^{\frac{p}{2}} \left[ \begin{array}{c} \frac{p^2-8p-1}{16} \\ u \end{array} \right] g_m(u)$ maps $M$ such that $r = \frac{p-1}{2} + 2 = \frac{p+3}{2}$ and the genus of such a Cayley map is $\text{gen}(M) = 1 + \frac{p^2 - 2p - \frac{p+3}{2}}{2} = \frac{2p^2 - 5p + 1}{4}$.

Example 2.2: Let us revisit the group and generating set from Example 2.1. Using the generating set $\Omega = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21\}$ from $\mathbb{Z}_{22}$, we can generate $K_{11,11}$. Now we will apply Proposition 2.3. Our prime number for this calculation $p=11$. So there are $\left(\frac{p-1}{2}\right)! \sum_{n=\frac{-p+1}{2}}^{\frac{p}{2}} \left[ \begin{array}{c} \frac{p^2-8p-1}{16} \\ u \end{array} \right] g_m(u)$ Cayley maps $M = \text{CM} (\Gamma, \Omega, \rho)$ such that $\text{gen}(M) = \frac{2p^2 - 9p + 5}{4}$. Now we take the set of positive $p$-distances $\{1, 2, 3, 4, 5\}$ and sum.
the elements, so that \( S = 15 \). We want the sum

\[
\sum_{\substack{n = -15 \\ n \geq -11}}^{15} \binom{15}{n} \frac{11^{2n-8(n+1)-1}}{16} g_5(u).
\]

This means that \( n \) ranges from \(-2\) to 2, giving that we are looking for the coefficient of the powers 24.75, 13, 7.5, 2, and -3.5 in the expansion of:

\[
(1 + u)(1 + u^2)(1 + u^3)(1 + u^4)(1 + u^3) = u^{15} + u^{14} + u^{13} + 2u^{12} + 2u^{11} + 3u^{10} + 3u^9 + 3u^8 + 3u^7 + 3u^6 + 3u^5 + 2u^4 + 2u^3 + u^2 + u + 1.
\]

The coefficients of fractional powers are zero so we add the coefficients of \( u^2 \) and \( u^{13} \). Thus the sum is 2. Multiplying this by 5! We get 240 Cayley maps \( M = CM(\Gamma, \Omega, \rho) \) with \( \text{gen}(M) = \frac{2(11^2) - 99 + 5}{4} = 37 \) and \( 5!(2^5 - 2) = 120(30) = 3600 \) Cayley maps \( M = CM(\Gamma, \Omega, \rho) \) with \( \text{gen}(M) = \frac{2(11^2) - 55 + 1}{4} = 47 \).

This is a vast improvement in the method of calculation. Recall that in Example 2.1 we calculated the number of Cayley maps using a table and now we do not need a table but can find the number of Cayley maps by using the formula that matches with the constraints of the propositions of this chapter.

Section 2.3

The Spoke in a Region of Size Two

Figure 2.3 (a) is a representation of a voltage graph embedding with 6 singleton loops with the spoke in a region of size two. In Figure 2.3 (a) the voltage graph is embedded on the plane, which is equivalent to it being embedded on the sphere. Thinking
of it as being embedded on the sphere we can slide the loop that contains the spoke around the sphere so that the regions are the same but the image would be Figure 2.3 (b). These two figures are representations of the exact same voltage graph embedding.

In this section we repeat the process from Section 2 except that now, in our voltage graphs, the spoke is in a region of size two, i.e., the spoke is inside one of the loops. The following discussion is for voltage graphs that have an odd number of loops and contain only singleton loops with the spoke in a region of size two.

Proposition 2.4 Let \( p \) be an odd prime such that \( m = \frac{p-1}{2} \) is odd, and let \( \Gamma = \mathbb{Z}_2^p \) with generating set \( \Omega = \{1, 3, \ldots, p\} \). Let \( S = 1 + 2 + \cdots + (p-1)/2 \). Then of the Cayley maps that lift from plane voltage graphs consisting of only singleton loops with at exactly one region of size greater than two and exactly one region of size two, there are exactly
\[
\left(\frac{p-1}{2}\right)! \left\{ \sum_{n=\frac{5}{p}}^{ \frac{5}{p} } \left[ \frac{p^\gamma - 8pn - 1}{16} \right] g_m(u) \right\} \text{ Cayley maps } M = CM(\Gamma, \Omega, \rho) \text{ such that }
\]

\[
\text{gen}(M) = \frac{2p^2 - 7p + 3}{4} \quad \text{and} \quad \left(\frac{p-1}{2}\right)! \left\{ \frac{p^\gamma - 8pn - 1}{16} \right\} g_m(u) \text{ Cayley maps } M
\]

such that \( \text{gen}(M) = \frac{2p^2 - 5p + 1}{4} \).

Proof: Let \( K \) be a plane voltage graph of index one corresponding to \( \Gamma \) and \( \Omega \) such that \( K \) consists of exactly one region of size greater than two and exactly one region of size two. First we count the number of distinct plane voltage graphs \( K \). There are \( \frac{p-1}{2} \) loops in the voltage graph embedding. For each loop we choose a generator and a direction for that loop. Thus there are a total of \( 2^\frac{p-1}{2} \left(\frac{p-1}{2}\right)! \) voltage graph embeddings in this case.

There are \( \frac{p-3}{2} \) regions of size one, each of which by Proposition 2.1, produces one region in the covering graph embedding. Since \( \frac{p-3}{2} \) is even; the regions of size one produce an even number of regions in the covering graph embedding. The region of size two has \( p \) and one other odd element on its boundary, thus the boundary element is even. However, the boundary element of the region of size two cannot be \( 2p \) since that would require \( p \) being on the boundary twice. So the region of size two has an even boundary element, which has order \( p \) in \( \mathbb{Z}_{2p} \) and so produces two regions in the covering graph embedding. Since \( p^2 \) and \( r \) have the same parity and there are an even number of regions
of size one and one region of size two the remaining region \(R\) of size \(\frac{p-1}{2}\) must lift to an odd number of regions in the Cayley map. Thus \(R\) lifts to either 1 or \(p\) regions in the covering graph embedding.

Examining the order of the boundary element of \(R\) note that if \(x\) is an element of \(\Omega\) then either \(x\) or \(x^{-1}\) (and not both) is on the boundary of \(R\). Thus if all of the elements of \(\Omega\) that are larger than \(p\) are on the boundary of \(R\), then the boundary element for \(R\) is

\[
(p+2) + \ldots + (p+p-1) = p \left( \frac{p-1}{2} \right) + 2 + 4 + \ldots + p-1 = p \left( \frac{p-1}{2} \right) + 2 \left( 1 + 2 + \ldots + \frac{p-1}{2} \right).
\]

Observe that \(p \left( \frac{p-1}{2} \right) = p \text{ mod } 2p\). Thus only when \(1 + 2 + \ldots + \frac{p-1}{2}\) is a multiple of \(p\) will the order of the boundary element be 2. Therefore, by following the same arguments for modifying the sum \(S\) to represent an arbitrary sum as in the proof of Proposition 2.2, we can count the number of Cayley maps in the same manner as in Proposition 2.2.

Thus there are

\[
\left( \frac{p-1}{2} \right)! \sum_{n=-\frac{5}{p}}^{\frac{5}{p}} \left[ \frac{p^2 - 8pn - 1}{16} \right] g_m(u) \text{ Cayley maps } M = CM(\Gamma, \Omega, \rho)
\]

such that \(r = \frac{p^2 - 2p - 3p + 1}{2}\), and the genus of such a Cayley map is

\[
\text{gen}(M) = 1 + \frac{p^2 - 2p - 3p + 1}{2} \text{ so that by simplifying } \text{gen}(M) = \frac{2p^2 - 7p + 3}{4}. \text{ Since the rest of the voltage graphs have } R \text{ lifting to one region, there are}
\]

\[
\left( \frac{p-1}{2} \right)! \left( \frac{p-1}{2} \right) - \sum_{n=-\frac{5}{p}}^{\frac{5}{p}} \left[ \frac{p^2 - 8pn - 1}{16} \right] g_m(u) \text{ Cayley maps } M \text{ such that}
\]
\[ r = \frac{p-3}{2} + 2 + 1 = \frac{p+3}{2} \quad \text{and the genus of such a Cayley map is} \]

\[ \text{gen}(M) = 1 + \frac{p^2 - 2p - \frac{p+3}{2}}{2} = \frac{2p^2 - 5p + 1}{4}. \]

Proposition 2.4 covers when the voltage graph embedding has an odd number of loops so that now we will examine when there are an even number of loops. The following result is for planar voltage graph embeddings that have an even number of loops and contain only singleton loops with the spoke in a region of size two.

Proposition 2.5 Let \( p \) be an odd prime such that \( m = \frac{p-1}{2} \) is an even number and \( \Gamma = \mathbb{Z}_2 \), with generating set \( \Omega = \{1, 3, \ldots, p\} \). \( \Omega = \{1, 3, \ldots, 2p-1\} \). Let \( S = 1 + 2 + \ldots + (p-1)/2 \).

Then of the Cayley maps that lift from plane voltage graphs consisting of only singleton loops and containing exactly one region of size greater than two and exactly one region of size two, there are exactly \( \left( \frac{p-1}{2} \right)! \left( \frac{p^2 - 8p - 1}{16} \right) g_m(u) \) Cayley maps \( M = CM(\Gamma, \Omega, \rho) \) such that \( \text{gen}(M) = \frac{2p^2 - 9p + 3}{4} \) and

\[ \left( \frac{p-1}{2} \right)! \left( \frac{2}{2} \right)^{\frac{p-1}{2}} g_m(u) \] such that \( \text{gen}(M) = \frac{2p^2 - 5p - 1}{4} \).

Proof: Let \( K \) be a plane voltage graph of index one corresponding to \( \Gamma \) and \( \Omega \) such that \( K \) has exactly one region of size greater than two and exactly one region of size two.

Since \( \frac{p-3}{2} \) is an odd integer there will be an odd number of regions of size one in \( K \),
which will by Proposition 2.1 produce an odd number of regions in the corresponding Cayley map. The region of size two produces two regions in the covering graph embedding. Therefore the region, \( R \), of size \( \frac{p-1}{2} \) must lift to an even number of regions in the Cayley map. Thus \( R \) lifts to either 2 or \( 2p \) regions in the covering graph embedding.

If all of the elements of \( \Omega \) that are larger than \( p \) are on the boundary of \( R \), then the boundary element for \( R \) is \((p + 2) + \ldots + (p + p - 1) = p \left( \frac{p-1}{2} \right) + 2 + 4 + \ldots + p - 1 = p \left( \frac{p-1}{2} \right) + 2 \left( 1 + 2 + \ldots + \frac{p-1}{2} \right) \). Observe that \( p \left( \frac{p-1}{2} \right) \equiv 0 \mod 2p \). Thus only when \( 1 + 2 + \ldots + \frac{p-1}{2} \) is a multiple of \( p \) will the boundary element of \( R \) be zero in \( \mathbb{Z}_{2p} \).

Therefore, by following the same arguments for modifying this sum to represent a general sum as in the proof of Proposition 2.2, we can count the number of Cayley maps in the same manner as in Proposition 2.2.

Thus there are \( \binom{p-1}{2} \frac{p-3}{2} + 2 + 2p = \frac{5p+1}{2} \) Cayley maps \( M = CM(\Gamma, \Omega, \rho) \) such that \( r = \frac{p-3}{2} + 2 + 2p = \frac{5p+1}{2} \) and the genus of the Cayley maps is
\[
\text{gen}(M) = 1 + \frac{p^2 - 2p - \frac{5p+1}{2}}{2} = 2p^2 - 9p + 3.
\]
There are \( \left( \frac{p-1}{2} \right)! \left( \frac{p-1}{2} - \sum_{n=-5}^{5} \binom{p^2 - 8pn - 1}{u} \right) \) Cayley maps such that

\[
r = \frac{p-3}{2} + 2 + 2 = \frac{p+5}{2} \quad \text{and the genus of the Cayley graph is}
\]

\[
\text{gen}(M) = 1 + \frac{p^2 - 2p - p + 5}{2} = \frac{2p^2 - 5p - 1}{4}.
\]

In this chapter we have classified Cayley maps for \( \mathbb{Z}_{2p} \) with odd integer generators that arise as lifts of certain planar voltage graph embeddings. We have shown a formula for calculating the number of Cayley maps and their genus values when the Cayley map is a lift of a plane voltage graph of index one that fits certain constraints. This formula only depends on the prime integer \( p \) of the group \( \mathbb{Z}_{2p} \). The first type of voltage graph we used in this chapter consisted of only singleton loops and contained exactly one region of size two or greater. The second type of voltage graph we used in this chapter consisted of only singleton loops and contained exactly one region of size two and exactly one region of size greater than two.
CHAPTER 3

PLANAR VOLTAGE GRAPHS WITH ONLY SINGLETON LOOPS AND STACKED LOOPS

Section 3.1

Stacked Loops

In Chapter 2 we studied planar voltage graphs with only singleton loops. Clearly there are more complicated region structures for planar voltage graph embeddings. In this chapter we investigate a particular type of Voltage graph embedding.

Consider a cyclic permutation $\rho$ containing the generators $a_1, a_2, \ldots, a_k, a_k^{-1}, \ldots, a_2^{-1}, a_1^{-1}$ in this order such that this structure is not contained within any other loop. This structure of the index one voltage graph embedding is called a stacked loop. If the spoke corresponding to the prime $p$ is between any two consecutive generators of a stacked loop, then the structure is still referred to as a stacked loop.

The voltage graph in Figure 3.1 (a) consists of only singleton loops and stacked loops while the voltage graph in Figure 3.1 (b) is an example of a voltage graph embedding that will not be allowed in this chapter.
The difference between this chapter and Chapter 2 is that we allow certain regions of size two, i.e., those that occur within stacked loops, along with the regions of size one and the one region of size greater than one. This makes the calculation of how many regions the region of size greater than two lifts to much more difficult.

Proposition 3.1 Let $p$ be an odd prime and let $\Gamma = \mathbb{Z}_{2p}$ with generating set $\Omega = \{1, 3, \ldots, 2p-1\}$, the odd integers. Let $K$ be a planar voltage graph of index one corresponding to $\Gamma$ and $\Omega$. If $R$ is a region of $K$ with size two then $R$ lifts to two regions in the covering graph embedding.

Proof Two distinct generators $x$ and $y$, that are not inverses of each other, lie on the boundary of $R$. Each of the elements of $\Omega$ are odd and the sum of two odd integers is even. Since $x$ and $y$ are not inverses of each other, their sum is not 0 modulo $2p$. Thus the greatest common divisor between the boundary element of $R$ and $2p$ is 2, and therefore
the order of the boundary element of $R$ is $\frac{2p}{2} = p$. Lastly by Theorem 1.4, $R$ lifts to two regions in the covering graph embedding. □

The regions in any stacked loop consist of one region of size one and some number of regions of size two. The size $j$ of a stacked loop is the number of loops in the stack. So a stacked loop of size $j$ has $j-1$ regions of size two. A plane voltage graph having only singleton loops and stacked loops with the spoke in the largest region or with the spoke in a singleton loop has three possible sizes of regions: regions of size 1, regions of size 2, and possibly one region of size greater than 2.

Calculating the number of Cayley maps that arise from certain voltage graphs in this chapter we must know how many regions of size one and two there are in the voltage graph. Let $K$ be a plane voltage graph that consists only of singleton loops and stacked loops. Let $R$ denote the largest region in $K$ and for $i=1, 2$, let $r_i$ denote the number of regions of size $i$ in $K$. For the remainder of this chapter $r_1$ and $r_2$ will be used in describing the voltage graph embeddings of significance. A region of size three that contains the spoke is also of interest.

Proposition 3.2 Let $p$ be an odd prime and let $\Gamma = \mathbb{Z}_{2p}$ with generating set $\Omega = \{1, 3, \ldots, 2p-1\}$, the odd integers. Let $K$ be a plane voltage graph of index one corresponding to $\Gamma$ and $\Omega$. If $R$ is a region of $K$ with size three containing the spoke, then $R$ lifts to one region in the covering graph embedding.

Proof: The region $R$ of size three with two directed edges and the spoke has on the boundary two generators from $\Omega$ that are not inverses of one another and the spoke $p$. The two generators sum to an even element that is not congruent to 0 modulo $2p$. So by
adding the spoke the boundary element is odd but not a multiple of \( p \). Thus this one region of size three lifts to one region in the covering graph embedding. □

Now we wish to discuss certain voltage graph embeddings that have only regions of size one, two, or a region of size three that contains the spoke. Using Propositions 2.1, 3.1, and 3.2 it is not difficult to calculate genus of a Cayley map arising from such a voltage graph embedding. Since the region sizes are so limited, each voltage graph embedding basically consists of at most two stacked loops (perhaps one or both is a singleton loop). Hence it is easy to count the number of Cayley maps that arise from such a voltage graph embedding.

A plane voltage graph having only singleton loops and stacked loops with the spoke inside a region of size 2 has the following possible sizes of regions: regions of size 1, regions of size 2, and two possible regions of size greater than two (if there are two then one will have size 3). In sections 3.1, 3.2, 3.3, we consider such voltage graph embeddings where the spoke is in the largest region, the spoke is in a singleton loop (region of size two), the spoke is in a region of size 3, respectively. Table 3.1 shows how the placement of the spoke and number of regions of size one affect the region \( R \).

Throughout this entire chapter, we will use the following notation. For a subset \( T \) of \( P^+ \) let the sum of the elements of \( T \) be denoted by \( S_T = \sum_{i \in T} i \), the generating function for the partitions of integers whose unequal parts come from \( T \) be denoted by \( g_T(u) = \prod_{i \in T} (1 + u^i) \), and the sum of certain coefficients of \( g_T(u) \) be denoted by

\[
C_T = \sum_{n=-S_T/p}^{S_T/p} \binom{S_T}{n} \frac{S_T-np}{2} g_T(u).
\]
Table 3.1

<table>
<thead>
<tr>
<th>Proposition</th>
<th>Odd</th>
<th>Even</th>
<th>$R$ satisfies</th>
<th>Placement of Spoke</th>
<th>boundary element of $R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.3</td>
<td></td>
<td></td>
<td>$KVL$</td>
<td>in $R$</td>
<td>0</td>
</tr>
<tr>
<td>3.4</td>
<td></td>
<td></td>
<td>nothing</td>
<td>in $R$</td>
<td>even but not 0 modulo 2p</td>
</tr>
<tr>
<td>3.5</td>
<td></td>
<td></td>
<td>semi $KVL$</td>
<td>in $R$</td>
<td>$p$</td>
</tr>
<tr>
<td>3.6</td>
<td></td>
<td></td>
<td>nothing</td>
<td>in $R$</td>
<td>odd but not $p$ modulo 2p</td>
</tr>
<tr>
<td>3.7</td>
<td></td>
<td></td>
<td>$KVL$</td>
<td>in a region of size 2</td>
<td>0</td>
</tr>
<tr>
<td>3.8</td>
<td></td>
<td></td>
<td>nothing</td>
<td>in a region of size 2</td>
<td>even but not 0 modulo 2p</td>
</tr>
<tr>
<td>3.9</td>
<td></td>
<td></td>
<td>semi $KVL$</td>
<td>in a region of size 2</td>
<td>$p$</td>
</tr>
<tr>
<td>3.10</td>
<td></td>
<td></td>
<td>nothing</td>
<td>in a region of size 2</td>
<td>odd but not $p$ modulo 2p</td>
</tr>
<tr>
<td>3.11</td>
<td></td>
<td></td>
<td>semi $KVL$</td>
<td>in a region of size 3</td>
<td>$p$</td>
</tr>
<tr>
<td>3.12</td>
<td></td>
<td></td>
<td>nothing</td>
<td>in a region of size 3</td>
<td>odd but not $p$ modulo 2p</td>
</tr>
<tr>
<td>3.13</td>
<td></td>
<td></td>
<td>$KVL$</td>
<td>in a region of size 3</td>
<td>0</td>
</tr>
<tr>
<td>3.14</td>
<td></td>
<td></td>
<td>nothing</td>
<td>in a region of size 3</td>
<td>even but not 0 modulo 2p</td>
</tr>
</tbody>
</table>

Example 3.1: Let $T = \{1, 2, 3\}$ For $T$ we have $S_T = 6$ and $\left\lfloor \frac{6}{11} \right\rfloor = 0, \left\lfloor \frac{-6}{11} \right\rfloor = 0$ so that in calculating the $C_T$ the exponent we are looking for is $(S_T - np)/2 = (6-11(0))/2 = 3$. So in the expansion of $g_T(u) = (1+u)(1+u^2)(1+u^3) = 1+u+u^2+2u^3+u^4+u^5+u^6$ we are interested in the coefficient of $u^3$ and thus $C_T = 2$.

Section 3.2

The Spoke in the Largest Region

Figure 3.2 shows a voltage graph embedding with three singleton loops and two stacked loops. There are five regions of size one, six regions of size two, and the largest region of size 6, which contains the spoke.
Let $K$ be a plane voltage graph for $\Gamma = \mathbb{Z}_{2p}$ and $\Omega = \{1, 3, \ldots, 2p-1\}$ that consists only of singleton loops and stacked loops with the spoke in the largest region. We first consider when $r_1$ is odd. In this case we will see that due to parity constraints, the region $R$ must lift to an even number of regions. Whether the region $R$ lifts to 2 or $2p$ regions will be handled separately in Proposition 3.3 and Corollary 3.4. A major difference between this chapter and Chapter 2 is that the largest region does not have all of the elements of the generating set along its boundary. Instead we must take subsets of the generating set that have the size equal to the number of loops (generators) on the boundary of $R$.

Proposition 3.3 Let $p$ be an odd prime and let $\Gamma = \mathbb{Z}_{2p}$ with generating set $\Omega = \{1, 3, \ldots, 2p-1\}$. Let $K$ be a plane voltage graph of index one corresponding to $\Gamma$ and $\Omega$ such that $K$ consists only of singleton loops and stacked loops with the spoke in the...
largest region. Let $K$ consist of $h$ stacked loops, where $h = h_1 + h_2 + ... + h_t$, and $h_i$ is the number of stacked loops of size $j_i$ for $i = 1, 2, ..., t$. If $r_i$ is odd then the number of Cayley maps $M = CM(\Gamma, \Omega, \rho)$ with $\text{gen}(M) = \frac{p^2 - 4p - r_i - 2r_2}{2} + 1$ that lift from such planar voltage graphs is

$$\left( \begin{array}{c} r_1 \\ h_1 h_2 ... h_t \end{array} \right) \frac{r_1! r_2! 2^{r_2}}{\sum_{|T| + r_2}} C_T.$$

Proof: Due to the stacked loops in $K$ we know that there are $r_1 = \frac{p-1}{2} - \sum_{i=1}^t h_i j_i + h$ regions of size one and $r_2 = \sum_{i=1}^t h_i (j_i - 1)$ regions of size two. Recall that the genus of a Cayley map for $\mathbb{Z}_{2p}$ with the generating set $\Omega$ is $k = 1 + \frac{p^2 - 2p - r}{2}$, where $r$ denotes the number of regions in the Cayley map. Since $k$ is an integer, $p^2$ and $r$ have the same parity so that $r$ is odd. By Proposition 2.1 each of the $r_1$ regions of size one lifts to one region in the covering graph embedding. By Proposition 3.1 each of the $r_2$ regions of size two lifts to two regions in the covering graph embedding. Thus there is an even number of regions in the covering graph embedding coming from the regions of size two and because $r_1$ is odd, there is an odd number of regions in the covering graph embedding coming from the regions of size one. Thus the one region $R$ of size greater than two lifts to an even number of regions in the covering graph embedding. We will count the number of times $R$ lifts to $2p$ regions and thus the number of Cayley maps $M$ with $\text{gen}(M)$

$$= \frac{p^2 - 4p - r_i - 2r_2}{2} + 1.$$
In order to count the number of times $R$ lifts to $2p$ regions, we will make a similar argument to that used in the proof of Proposition 2.2. Propositions 2.1 and 3.1 tell us that regions of size one and two, in a plane voltage graph corresponding to $\Gamma$ and $\Omega$, lift to 1 and 2 regions, respectively, in the covering graph embedding no matter what generators are assigned to them. Thus we only need to concern ourselves with assigning generators to the boundary of $R$. More specifically we determine when $R$ satisfies the $KVL$ and thus lifts to $2p$ regions in the covering graph embedding. The difference between counting here and in Proposition 2.2 is that all of the generators do not lie on the boundary of $R$ here as they did in Proposition 2.2. Recall that we may express the elements of $\Omega$ in terms of their $p$-distances. Let $T = \{ij|1 \leq j \leq r_1\}$ be a subset of the positive $p$-distances. Suppose that the generators $p_1(p+2i_1), (p+2i_2), ..., (p+2i_n)$ occur on the boundary of $R$ so that $p+(p+2i_1)+(p+2i_2)+...+(p+2i_n)$ is the boundary element of $R$. By factoring this boundary element is $(n+1)p+2(i_1+i_2+...+i_n)$. By the hypothesis we know $n+1$ is even so that this boundary element is 0 modulo $2p$ if and only if $i_1+i_2+...+i_n$ is a multiple of $p$. Thus $R$ lifts to $2p$ regions if and only if $i_1+i_2+...+i_n$ is a multiple of $p$.

Let $S_T = \sum_{i \in T} i$. This sum is the largest possible sum of $p$-distances we can get from the boundary of $R$ using elements of $\Omega$ that correspond to the $p$-distance for the generators in $T$. In replacing some elements with their inverses, first remove the $p$-distance of the element and then add in the $p$-distance of its inverse. Thus $S_T$ changes by a decrease of $2X$, where $X$ is the sum of the $p$-distances of the elements being replaced by their inverses. The region $R$ will satisfy the $KVL$ if and only if $S_T-2X=pn$ for some
integer $n$. Since we declared that $n_1$ is odd and the spoke is on the boundary of $R$ it follows that $|R|$ is even. So that there are an even number of elements on the border of $R$ and thus we can only get an even boundary element of which we will count when the boundary element is an even multiple of $p$. Since $0 \leq X \leq S_T$, we have $\frac{-S_T}{p} \leq n \leq \frac{S_T}{p}$.

Since $n$ is an integer we see that $\left\lfloor \frac{-S_T}{p} \right\rfloor \leq n \leq \left\lfloor \frac{S_T}{p} \right\rfloor$. Now solving for $X$ in the equation $S_T - 2X = pn$, we obtain $X = \frac{S_T - pn}{2}$.

The number of partitions of $X = \frac{S_T - pn}{2}$ using unequal parts from $T$ is given by the generating function $g_T(u) = \prod_{i\in T} (1 + u^i)$. Thus the number of times $S_T - 2X$ is a multiple of $p$ is the sum of the coefficients of $u^x$, where $X = \frac{S_T - pn}{2}$ and

$$\left\lfloor \frac{-S_T}{p} \right\rfloor \leq n \leq \left\lfloor \frac{S_T}{p} \right\rfloor$$

in the expansion of $g_T(u)$. Thus $C_T = \sum_{n=\lfloor -\frac{S_T}{p} \rfloor}^{\lfloor \frac{S_T}{p} \rfloor} \left[ u^{\frac{S_T - np}{2}} \right] g_T(u)$ gives the number of distinct subsets of $T$ that sum to an integer that will satisfy $S_T - 2X = pn$ and thus the number of sets of generators represented by the $p$-distances in $T$ or their inverses for the boundary of $R$ that sum to a multiple of $p$. Now we sum the $C_T$'s for every set $T \subseteq P^*$ with $|T| = r_1$ and we have the number of sets of generators for the boundary of $R$ that satisfy the KVL.

For one such set $T$ we wish to determine how many ways the generators corresponding to the elements of $T$ can be arranged on the boundary of $R$, and then how
many ways we can arrange the remaining elements of \( \Omega \) on the edges of the voltage graph that do not lie on the boundary of \( R \). In the cyclic permutation \( \rho \) we have the singleton loops and the stacked loops to arrange; there are \( r_1 \) of these. We will always place the spoke as the first element in the cyclic permutation so that the rest of the elements form an ordered permutation. So we think of \( \rho \) as \( \rho = (p, \beta_1, \beta_2, \ldots, \beta_{r_1}) \), where each \( \beta_i \) \((1 \leq i \leq r_1)\) is either a singleton loop or a stacked loop. If we choose where the stacked loops are in the permutation then the singleton loops are naturally assigned everywhere else. We have to choose \( h_1 \) positions for the stacked loops of size \( j_1 \), \( h_2 \) positions for the stacked loops of size \( j_2 \), \ldots, and \( h_{r_1} \) positions for the stacked loops of size \( j_{r_1} \). Thus there are \( \binom{r_1}{h_1 h_2 \ldots h_{r_1}} \) ways to select the positions of the stacked loops and singleton loops in the permutation \( \rho \). On the boundary of \( R \) there are generators corresponding to elements from the set \( T \) or their inverses. Since we determined which generators are on the boundary of \( R \) the direction for those generators assigned to the loops that form the boundary of \( R \) are determined. However we can arrange the elements of the loops in \( r_1! \) ways. The singleton loops and one loop from each stacked loop are on the border of \( R \). So there are \( \sum_{i=1}^{r_1} h_i (j_i - 1) = r_2 \) loops left for us to put elements on. There are \( r_2! \) ways to assign the elements to the loops and there are two possible directions for each element so that there are \( 2^r r_2! \) ways to assign generators to the loops not on the boundary of \( R \). Thus the number of Cayley maps \( M=CM(\Gamma, \Omega, \rho) \) such that \( \text{gen}(M) = \)
\[ \frac{p^2 - 4p - r_1 - 2r_2}{2} + 1 \text{ is } \left( \frac{r_1}{h_1 h_2 \ldots h_t} \right)^{r_1} r_1! 2^{r_1} \left( \sum_{T \subseteq P^* \setminus T_{\neq n}} C_T \right), \text{ where for each } T \text{ with } T \subseteq P^* \text{ and } \]

\[ |T|=r_1, S_T = \sum_{i \in T} g_T(u) = \prod_{i \in T} (1+u'), \text{ and } C_T = \sum_{m=1}^{S_T} \left( \frac{S_T-mp}{p} \right)^2 g_T(u). \]

As a direct consequence of this we can find, under the same conditions, the number of Cayley Maps that have the region of size greater than two lifting to two regions in the covering graph embedding.

Corollary 3.4 Let \( p \) be an odd prime and let \( \Gamma = \mathbb{Z}_{2p} \) with generating set \( \Omega = \{1, 3, \ldots, 2p-1\} \). Let \( K \) be a plane voltage graph of index one corresponding to \( \Gamma \) and \( \Omega \) such that \( K \) consists only of singleton loops and stacked loops with the spoke in the largest region. Let \( h \) be the number of stacked loops in \( K \), where \( h = h_1 + h_2 + \ldots + h_t \), and \( h_i \) is the number of stacked loops of size \( j_i \) for \( i = 1, 2, \ldots, t \). If \( r_1 \) is odd then the number of Cayley maps \( M = CM(\Gamma, \Omega, \rho) \) with \( \text{gen}(M) = \frac{p^2 - 2p - r_1 - 2r_2}{2} \) that occur as the lift of such a plane voltage graph \( K \) is

\[ \left( \frac{r_1}{h_1 h_2 \ldots h_t} \right) \left( \frac{p-1}{2} \right)^{r_1} 2^{r_1} \left( \sum_{T \subseteq P^* \setminus T_{\neq n}} C_T \right). \]

Proof: Due to the stacked loops in \( K \) we know that there are \( r_1 = \frac{p-1}{2} - \sum_{i=1}^{t} h_i j_i + h \)

regions of size one and \( r_2 = \sum_{i=1}^{t} h_i (j_i - 1) \) regions of size two. Since \( r_1 \) is odd there are an odd number of regions in the covering graph embedding coming from the regions of size one. Each region of size two lifts to two regions in the covering graph embedding. Thus
the one region of size greater than two, call it $R$, lifts to an even number of regions in the covering graph embedding. More specifically we want to count the Cayley maps in which $R$ lifts to two regions in the covering graph embedding. The genus of such a Cayley map is $\text{gen}(M) = \frac{p^2 - 2p - r_1 - 2r_2}{2}$.

We counted the number of times $R$ lifts to $2p$ regions in the proof of Proposition 3.3. To find the number of times that $R$ lifts to two regions we will subtract the number of times it lifts to $2p$ regions from the total number of ways to arrange the elements on our voltage graph.

Using the same method as in the previous result there are $\left(\begin{array}{c} r_1 \\ h_1 h_2 \ldots h_i \end{array} \right)$ ways to arrange the stacked loops and singleton loops in the permutation $p$. There are $\frac{p-1}{2}$ generators and there are $\left(\frac{p-1}{2}\right)!$ ways to assign them to the loops. There is a choice of two directions for each element so we have a total of $\left(\frac{p-1}{2}\right)!2^{\frac{p-1}{2}}$ ways to assign the generators and directions to the loops. Thus there are $\left(\begin{array}{c} r_1 \\ h_1 h_2 \ldots h_i \end{array} \right)\left(\frac{p-1}{2}\right)!2^{\frac{p-1}{2}}$ total permutations of the generators. □

The following result is for voltage graphs that have an even number of regions of size one and consist only of singleton loops and stacked loops with the spoke in the largest region. Let $K$ be a plane voltage graph that consists only of singleton loops and stacked loops with the spoke in the largest region. Let $R$ denote the largest region in $K$. We now consider when $r_1$ is even. In this case we will see that due to parity constraints,
the region $R$ must lift to an odd number of regions and whether $R$ lifts to 1 or $p$ regions will be handled separately in Proposition 3.5 and the Corollary 3.6.

Proposition 3.5 Let $p$ be an odd prime and let $\Gamma = \mathbb{Z}_{2p}$ with generating set $\Omega = \{1, 3, \ldots, 2p-1\}$. Let $K$ be a plane voltage graph of index one corresponding to $\Gamma$ and $\Omega$ such that $K$ consists only of singleton loops and stacked loops with the spoke in the largest region. Furthermore, let $K$ have $h_i$ stacked loops of size $j_i$ for $i = 1, 2, \ldots, t$ where $h = h_1 + h_2 + \ldots + h_t$. If $r_1$ is even then the number of Cayley maps $M = CM(\Gamma, \Omega, \rho)$ with

$$\text{gen}(M) = \frac{p^2 - 3p - r_1 - 2r_2}{2} + 1$$

that occur as the lift of such a plane voltage graph $K$ is

$$\left( \begin{array}{c} r_1 \\ h_1, h_2, \ldots, h_t \end{array} \right) r_1! r_2! 2^h \left( \sum_{\sum h_i = r_1} C_{r_1} \right).$$

Proof: Due to the stacked loops in $K$ we know that there are $r_1 = \frac{p-1}{2} - \sum_{i=1}^{t} h_i j_i + h$ regions of size one and $r_2 = \sum_{i=1}^{t} h_i (j_i - 1)$ regions of size two. Since $r_1$ is even and there are an even number of regions in the covering graph embedding coming from the regions of size one. Each of the regions of size two lifts to two regions in the covering graph embedding. Thus the one region of size greater than two, call it $R$, lifts to an odd number of regions in the covering graph embedding. We will count the number of times $R$ lifts to $p$ regions and thus the number of Cayley maps $M = CM(\Gamma, \Omega, \rho)$ with

$$\text{gen}(M) = \frac{p^2 - 3p - r_1 - 2r_2}{2} + 1.$$ We then appeal to the same way of counting the number of Cayley maps as Proposition 3.3, except since we declared that $r_1$ is even it means that there are
an odd number of elements on the boundary of $R$. So that there are an even number of elements from $\Omega-\{p\}$ on the boundary of $R$ and the spoke is on the boundary of $R$. Thus we can only get an odd multiple of $p$ for the boundary element of $R$. So there are

$$\left( \frac{r_1}{h_1 h_2 \ldots h_t} \right) r_1! \cdot r_2! \cdot 2^s \left( \sum_{\tau \in \rho^* \at \eta} C_T \right)$$

Cayley maps with $\text{gen}(M) = \frac{p^2 - 3p - r_1 - 2r_2}{2} + 1$, where

$$S_T = \sum_{i \in \tau} i, \quad g_T(u) = \prod_{\iota \in \tau} (1 + u' \iota), \quad C_T = \sum_{n \in \frac{S_T}{p}} \left[ \frac{S_T - np}{2} \right] g_T(u).$$

As a direct consequence of this we can find, under the same conditions, the number of Cayley maps that have the region of size greater than two lifting to one region in the covering graph embedding.

Corollary 3.6 Let $p$ be an odd prime and let $\Gamma = \mathbb{Z}_{2p}$ with generating set $\Omega = \{1, 3, \ldots, 2p - 1\}$. Let $K$ be a plane voltage graph of index one corresponding to $\Gamma$ and $\Omega$ such that $K$ consists only of singleton loops and stacked loops with the spoke in the largest region. Let $h$ be the number of stacked loops in $K$, where $h = h_1 + h_2 + \ldots + h_t$ and $h_i$ is the number of stacked loops of size $j_i$ for $i = 1, 2, \ldots, t$. If $r_i$ is even then the number of Cayley maps $M = CM(\Gamma, \Omega, \rho)$ with $\text{gen}(M) = \frac{p^2 - 2p - r_1 - 2r_2 + 1}{2}$ that occur as the lift of such a plane voltage graph $K$ is

$$\left( \frac{r_1}{h_1 h_2 \ldots h_t} \right) \left( \frac{p - 1}{2} \right)^{\frac{p - 1}{2}} - r_1! r_2! \cdot 2^s \left( \sum_{\tau \in \rho^* \at \eta} C_T \right)$$.
Proof: Due to the stacked loops in $K$ we know that there are $r_i = \frac{p-1}{2} - \sum_{i=1}^{t} h_i j_i + h$ regions of size one and $r_2 = \sum_{i=1}^{t} h_i (j_i - 1)$ regions of size two. Since $r_i$ is even there are an even number of regions in the covering graph embedding coming from the regions of size one. Thus the one region of size greater than two, call it $R$, lifts to an odd number of regions in the covering graph embedding. We counted the number of times $R$ lifts to $p$ regions. So to find the number of times that $R$ lifts to 1 region we will subtract the number of times it lifts to $p$ regions from the total number of ways to arrange the generators on the loops of our voltage graph. The genus of the Cayley map $M = CM(\Gamma, \Omega, \rho)$ with $R$ lifting to 1 region in the covering graph embedding is $\text{gen}(M) = \frac{p^2 - 2p - r_i - 2r_2 + 1}{2}$. We then appeal to the same way of counting as in Corollary 3.4 to show that there are $\binom{r_i}{h_1 h_2 \ldots h_t} \left( \frac{p-1}{2} \right) ! \frac{p-1}{2}$ total Cayley maps.$\square$

Section 3.3

The Spoke in a Region of Size Two

Figure 3.3 (a) is a voltage graph that has two stacked loops and four singleton loops with the spoke in a singleton loop. Figure 3.3 (b) has three stacked loops and three singleton loops with the spoke in a stacked loop. In both voltage graphs the spoke is in a region of size two.
Figure 3.3
SINGLETON LOOPS AND STACKED LOOPS WITH THE SPOKE IN A REGION
OF SIZE TWO

We begin by considering plane voltage graphs having an odd number of regions
of size one and consisting only of singleton loops and stacked loops with the spoke in a
region of size two. This means that the spoke can be inside of a singleton loop or in a
region of size one in one of the stacked loops. Let \( K \) be a plane voltage graph that
consists only of singleton loops and stacked loops with the spoke in a region of size two.
Let \( R \) denote the largest region in \( K \) and for \( i = 1, 2 \), let \( r_i \) denote the number of regions of
size in \( K \) having size \( i \). We consider when \( r_1 \) is odd. In this case we will see that due to
parity constraints, the region \( R \) must lift to an even number of regions. Whether \( R \) lifts to
2 or \( 2p \) regions will be handled separately in Proposition 3.7 and Corollary 3.8.

Proposition 3.7 Let \( p \) be an odd prime and let \( \Gamma = \mathbb{Z}_{2p} \) with generating set
\( \Omega = \{1, 3, \ldots, 2p - 1 \} \). Let \( K \) be a plane voltage graph of index one corresponding to \( \Gamma \) and
\( \Omega \) such that \( K \) consists only of singleton loops and stacked loops with the spoke in a
region of size two. Furthermore, let \( K \) consist of \( h_i \) stacked loops of size \( j_i \) for \( i = 1, 2, \ldots, t \), where \( h = h_1 + h_2 + \ldots + h_t \). If \( r_i \) is odd then the number of Cayley maps \( M = CM(\Gamma,\Omega, p) \) with \( \text{gen}(M) = \frac{p^2 - 4p - r_1 - 2r_2 + 1}{2} \) that occur as the lift of such a plane voltage graph \( K \) is

\[
\left( \frac{r_1 + 1}{h_1 h_2 \ldots h_t} \right) (r_1 + 1)! \left( \frac{r_2 - 1}{2} \right)! 2^{r_1 - 1} \left( \sum_{T \in P^r} C_T \right).
\]

Proof: Due to the stacked loops in \( K \) we know that there are \( \frac{p-1}{2} - \sum h_i j_i + h - 1 \) regions of size one and \( r_2 = \left( \sum h_i (j_i - 1) \right) + 1 \) regions of size two. This proof is similar to the proof of Proposition 3.2. Recall that the genus of a Cayley map for \( \mathbb{Z}_{2p} \) with the generating set \( \Omega \) is \( k = 1 + \frac{p^2 - 2p - r}{2} \), where \( r \) denotes the number of regions in the Cayley map. Since \( k \) is an integer, \( p^2 \) and \( r \) have the same parity and so \( r \) is odd. There are an even number of regions in the covering graph embedding coming from the regions of size two, and because \( r_i \) is odd, there are an odd number of regions in the covering graph embedding coming from the regions of size one. Thus the one region \( R \) of size greater than two lifts to an even number of regions in the covering graph embedding. We will count the number of times \( R \) lifts to \( 2p \) regions and thus the number of Cayley maps \( M \) with 

\[
\text{gen}(M) = \frac{p^2 - 4p - r_1 - 2r_2 + 1}{2}.
\]

To count the number of Cayley maps we will use a similar method to the proof of Proposition 3.3. Let \( T = \{ij|1 \leq j \leq r+1\} \) be a subset of the positive \( p \)-distances. Suppose
that the generators \((p+2i_1), (p+2i_2), \ldots, (p+2i_{n+1})\) occur on the boundary of \(R\) so that 
\[
(p+2i_1)+(p+2i_2)+\ldots+(p+2i_{n+1})
\]
is the boundary element of \(R\). By factoring this boundary element is \((r_1+1)p+2(i_1+i_2+\ldots+i_{n+1})\). By the hypothesis we know \(r_1+1\) is even so that this boundary element is 0 modulo 2\(p\) if and only if \(i_1+i_2+\ldots+i_{n+1}\) is a multiple of \(p\). Thus \(R\) lifts to 2\(p\) regions if and only if \(i_1+i_2+\ldots+i_{n+1}\) is a multiple of \(p\).

Let \(S_T = \sum_{i \in T} i\). This sum is the largest possible sum of \(p\)-distances we can get from the boundary of \(R\) using elements of \(\Omega\) that correspond to the \(p\)-distance for the generators in \(T\). In replacing some elements with their inverses, first remove the \(p\)-distance of the element and then add in the \(p\)-distance of its inverse. Thus \(S_T\) changes by a decrease of \(2X\), where \(X\) is the sum of the \(p\)-distances of the elements being replaced by their inverses. The region \(R\) will satisfy the \(KVL\) if and only if \(S_T-2X = pn\) for some integer \(n\). Since \(n\) is an integer we see that \[
\left\lceil \frac{-S_T}{p} \right\rceil \leq n \leq \left\lfloor \frac{S_T}{p} \right\rfloor.
\]
Now solving for \(X\) in the equation \(S_T-2X = pn\), we get \(X = \frac{S_T-pn}{2}\). The number of ways we can partition \(X = \frac{S_T-pn}{2}\) using unequal parts from \(T\) is given by the generating function \(g_T(u) = \prod_{i \in T} (1+u^i)\). Thus \(C_T = \sum_{n=\frac{-S_T}{p}}^{\frac{S_T}{p}} \left\lceil \frac{S_T-pn}{2} \right\rceil g_T(u)\) gives the number of distinct subsets of \(T\) that sum to an integer that will satisfy \(S_T-2X = pn\) and thus the number of sets of generators represented by the \(p\)-distances in \(T\) or their inverses for the boundary of
$R$ that sum to a multiple of $p$. Now we sum the $C_T$'s for every set $T \subseteq P^*$ of size $r_1+1$ and we have the number of sets of generators that satisfy the $KVL$ for $R$.

For one such set $T$ we wish to know in how many ways the generators corresponding to the elements of $T$ can be arranged on the boundary of $R$, and then in how many ways we can arrange the remaining elements of $\Omega$ on the edges of the voltage graph that do not lie on the boundary of $R$. In the cyclic permutation $\rho$ we have the singleton loops and the stacked loops there are $r_1+1$ of these. We will always place the spoke as the first element in the cyclic permutation so that the rest of the elements form an ordered permutation. So we think of $\rho$ as $\rho = (\rho_1, \beta_1, \beta_2, \ldots, \beta_{r_1+1})$, where each $\beta_i (1 \leq i \leq r_1+1)$ is either a singleton loop or a stacked loop. If we choose where the stacked loops are in the permutation then the singleton loops are naturally assigned everywhere else.

We have to choose $h_i$ for the loops of size $j_1$, $h_2$ for the loops of size $j_2$, ..., $h_i$ for the loops of size $j_i$. Thus there are \[ \binom{r_1+1}{h_1 h_2 \ldots h_i} \] ways to select the positions of the stacked loops and singleton loops in the permutation $\rho$. On the boundary of $R$ there are generators corresponding to elements from the set $T$ or their inverses. Since we determined which generators are on the boundary of $R$ the direction for those generators assigned to the loops that form the boundary of $R$ are determined. However we can arrange the elements of the set $T$ in $r_1+1!$ ways. Each singleton loop is on the border of $R$, and one loop from each stacked loop is on the border of $R$ so there are \[ \left( \sum_{j=1}^{i} h_i j_i - 1 \right) = r_2 - 1 \] loops left for us to assign generators to. There are $(r_2-1)!$ ways to assign the generators to the loops and there are two possible directions for each element. So that there are $2^{r_2-1} (r_2-1)!$
ways to label the remaining loops with generators. Thus the number of Cayley maps \( M = CM(\Gamma, \Omega, \rho) \) such that \( \text{gen}(M) = \frac{p^2 - 4p - r_i - 2r_i + 1}{2} \) is

\[
\left( \frac{r_i + 1}{h_i h_2 \ldots h_i} \right) (r_1 + 1)! (r_2 - 1)! 2^{n-1} \left( \sum_{\substack{T \subseteq P^r \atop \|T\| = r_i + 1}} C_T \right),
\]

where for each \( T \subseteq P^r \) with \( |T| = r_i + 1 \),

\[
S_r = \sum_{i \in T} i, \quad g_r(u) = \prod_{i \in T} (1 + u^i), \quad \text{and} \quad C_T = \sum_{\substack{s \geq p \atop p \mid s}} \left\lfloor \frac{s}{p} \right\rfloor \frac{S-rp}{u^2} g_r(u).
\]

As a direct consequence of this we can find, under the same conditions, the number of Cayley maps that have the region of size greater than two lifting to two regions in the covering graph embedding. Since the proof of Corollary 3.8 is similar to the proof of Corollary 3.4, we omit it.

Corollary 3.8 Let \( p \) be an odd prime and let \( \Gamma = \mathbb{Z}_{2p} \) with generating set \( \Omega = \{1, 3, \ldots, 2p - 1\} \). Let \( K \) be a plane voltage graph of index one corresponding to \( \Gamma \) and \( \Omega \) such that \( K \) consists only of singleton loops and stacked loops with the spoke in a region of size two. Let \( K \) have \( h \) stacked loops, where \( h = h_1 + h_2 + \ldots + h_i \), and where \( h_i \) is the number of stacked loops of size \( j_i \) for each \( i = 1, 2, \ldots, \ell \). If \( r_i \) is odd then the number of Cayley maps \( M = CM(\Gamma, \Omega, \rho) \) with \( \text{gen}(M) = \frac{p^2 - 2p - r_i - 2r_i}{2} \) is

\[
\left( \frac{r_i + 1}{h_i h_2 \ldots h_i} \right) \left( \frac{p-1}{2} \right)^{p-1} - (r_1 + 1)!(r_2 - 1)! 2^{n-1} \left( \sum_{\substack{T \subseteq P^r \atop \|T\| = r_i + 1}} C_T \right).
\]
Next we consider plane voltage graphs with an even number of regions of size one that consist only of singleton loops and stacked loops with the spoke in a region of size two. Let $K$ be a plane voltage graph that consists only of singleton loops and stacked loops with the spoke in a region of size two. Let $R$ denote the largest region in $K$. Now we consider when $r_1$ is even. In this case we will see that due to parity constraints, the region $R$ must lift to an odd number of regions and whether $R$ lifts to 1 or $p$ regions will be handled separately in Proposition 3.9 and Corollary 3.10.

Proposition 3.9 Let $p$ be an odd prime and let $\Gamma = \mathbb{Z}_{2p}$ with generating set $\Omega = \{1,3,\ldots,2p-1\}$. Let $K$ be a plane voltage graph of index one corresponding to $\Gamma$ and $\Omega$ such that $K$ consists only of singleton loops and stacked loops with the spoke in a region of size two. Let $K$ have $h_i$ stacked loops of size $j_i$ for each $i = 1, 2, \ldots, t$, where $h = h_1 + h_2 + \ldots + h_t$. If $r_1$ is even then the number of Cayley maps $M = CM(\Gamma, \Omega, \rho)$ with

$$\text{gen}(M) = \frac{p^2 - 3p - r_1 - 2r_2 + 1}{2}$$

that lift from such a plane voltage graph $K$ is

$$\binom{r_1 + 1}{h_1, h_2, \ldots, h_t} (r_1 + 1)! (r_2 - 1)! 2^{r_2 - 1} \left( \sum_{T \subseteq \mathbb{Z}_{2p}} C_T \right).$$

Proof: Due to the stacked loops we know that $K$ has $r_1 = \frac{p-1}{2} - \sum_{i=1}^t h_i j_i + h - 1$ regions of size one and $r_2 = \left( \sum_{i=1}^t h_i (j_i - 1) \right) + 1$ regions of size two. This proof is similar to the proof of Proposition 3.7. Recall that the genus of a Cayley map for $\mathbb{Z}_{2p}$ with the generating set $\Omega$ is $k = 1 + \frac{p^2 - 2p - r}{2}$, where $r$ denotes the number of regions in the Cayley map.
embedding. Since $k$ is an integer, $p^2$ and $r$ have the same parity so that $r$ is odd. There are an even number of regions in the covering graph embedding coming from the regions of size two, and because $r_1$ is even, there are an even number of regions in the covering graph embedding coming from the regions of size one. Thus the one region $R$ of size greater than two lifts to an odd number of regions in the covering graph embedding. We will count the number of times $R$ lifts to $p$ regions and thus the number of Cayley maps $M$ with $\text{gen}(M) = \frac{p^2 - 3p - r_1 - 2r_2}{2} + 1$.

To count the number of Cayley maps we will use a similar method to the proof of Proposition 3.9. Let $T = \{i|1 \leq j \leq r_1 + 1\}$ be a subset of the positive $p$-distances. Suppose that the generators $(p + 2i_1)(p + 2i_2), \ldots, (p + 2i_{r_1})$ occur on the boundary of $R$ so that $(p + 2i_1) + (p + 2i_2) + \ldots + (p + 2i_{r_1})$ is the boundary element of $R$. By factoring this boundary element is $(r_1 + 1)p + 2(i_1 + i_2 + \ldots + i_{r_1})$. By the hypothesis we know $r_1 + 1$ is odd so that this boundary element is $p$ modulo $2p$ if and only if $i_1 + i_2 + \ldots + i_{r_1}$ is a multiple of $p$. Thus $R$ lifts to $p$ regions if and only if $i_1 + i_2 + \ldots + i_{r_1}$ is a multiple of $p$.

Let $S_T = \sum_{i \in T} i$. In replacing some elements with their inverses, first remove the $p$-distance of the element and then add in the $p$-distance of its inverse. Thus $S_T$ changes by a decrease of $2X$, where $X$ is the sum of the $p$-distances of the elements being replaced by their inverses. The region $R$ will satisfy the semi $KVL$ if and only if $S_T - 2X = pn$ for some integer $n$. Since $n$ is an integer we see that \( \left\lfloor \frac{-S_T}{p} \right\rfloor \leq n \leq \left\lceil \frac{S_T}{p} \right\rceil \). The number of ways...
we can partition $X = \frac{S_T - pn}{2}$ using unequal parts from $T$ is given by the generating

function $g_T(u) = \prod_{i \in T} (1 + u^i)$. Thus $C_T = \sum_{n \mid -S_T} \left[ \frac{S_T - np}{u^2} \right] g_T(u)$ gives the number of distinct

subsets of $T$ that sum to an integer that will satisfy $S_T - 2X = pn$ and thus the number of

sets of generators represented by the $p$-distances in $T$ or their inverses for the boundary of

$R$ that sum to a multiple of $p$. Now we sum the $C_T$'s for every set $T \subseteq P^*$ of size $r_1+1$ and we have the number of sets of generators that satisfy the KVL for $R$.

For counting the number of ways to arrange the generators we will appeal to the
counting procedure in Proposition 3.9. Thus the number of Cayley maps with gen($M$) =

$$\frac{p^2 - 3p - r_1 - 2r_2 + 1}{2} \times \left( \frac{r_1 + 1}{h_1 h_2 \ldots h_i} \right) \left( r_1 + 1 \right)! \left( r_2 - 1 \right)! 2^{n-1} \left( \sum_{T \subseteq P^*} C_T \right)$$

where each $T \subseteq P^*$

with $|T|=r_1+1$, $S_T = \sum_{i \in T} t$, $g_T(u) = \prod_{i \in T} (1 + u^i)$, and $C_T = \sum_{n \mid -S_T} \left[ \frac{S_T - np}{u^2} \right] g_T(u)$. □

As a direct consequence of this we can find, under the same conditions, the
number of Cayley maps that have the region of size greater than two lifting to one region
in the covering graph embedding. Since the proof of Corollary 3.10 is similar to the proof
of Corollary 3.4, we omit it.

Corollary 3.10 Let $p$ be an odd prime and let $\Gamma = \mathbb{Z}_{2p}$ with generating set
$\Omega = \{1, 3, \ldots, 2p - 1\}$. Let $K$ be a plane voltage graph of index one corresponding to $\Gamma$ and $\Omega$, such that $K$ consists only of singleton loops and stacked loops with the spoke in a
region of size two. Furthermore, let $K$ have $h_i$ stacked loops of size $j_i$ for $i = 1, 2, \ldots, t$, where $h = h_1 + h_2 + \ldots + h_t$. If $r_i$ is even then the number of Cayley maps $M = CM(\Gamma, \Omega, \rho)$ with $\text{gen}(M) = \frac{p^2 - 2p - r_i - 2r_j + 1}{2}$ that lift from such a plane voltage graph $K$ is

$$
\left( \frac{r_j + 1}{h_1 h_2 \ldots h_t} \right) \left( \frac{p - 1}{2} \right)! 2^{\frac{p-1}{2}} - (r_1 + 1)!(r_j - 1)! 2^{r_j - 1} \sum_{\tau \in \mathbb{F}^*} C_{\tau}.
$$

Section 3.4

The Spoke in a Region of Size Three

Figure 3.4 is a voltage graph that has three stacked loops and three singleton loops with the spoke in a region of size three.

The following result is for plane voltage graphs that have an odd number of regions of size one and that consist only of singleton loops and stacked loops with the spoke in a region of size three. Let $K$ be a plane voltage graph that consists only of singleton loops and stacked loops with the spoke in a region of size three. Let $R$ denote the largest region without the spoke in $K$. Now we consider when $r_i$ is odd. In this case we will see that due to parity constraints, the region $R$ must lift to an odd number of regions and whether $R$ lifts to $p$ or $1$ regions will be handled separately in Proposition 3.11 and Corollary 3.12.
Proposition 3.11 Let $p$ be an odd prime and let $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_p$ with generating set $\Omega = \{1, 3, ..., 2p-1\}$. Let $K$ be a planar voltage graph of index one corresponding to $\Gamma$ and $\Omega$, such that $K$ consists only of singleton loops and stacked loops, with the spoke in a region of size three. Furthermore, let $K$ have $h_i$ stacked loops of size $j_i$ for $i = 1, 2, ..., t$ where $h = h_1 + h_2 + ... + h_t$. If $r_i$ is odd then the number of Cayley maps $M = CM(\Gamma, \Omega, \rho)$ with $\text{gen}(M) = \frac{p^2 - 3p - r_i - 2r_t + 1}{2}$ that lift from such a voltage graph $K$ is

$$\binom{r_i}{h_1 h_2 ... h_t} r_1! (r_2 + 1)! 2^{\gamma^*} \left( \sum_{T \subseteq P^* \atop |T| = r_t} C_T \right).$$
Proof: Due to the stacked loops in $K$ we know that there are $r_1 = \frac{p-1}{2} - \sum_{i=1}^{t} h_i j_i + h$
regions of size one and $r_2 = \left(\sum_{i=1}^{t} h_i (j_i - 1)\right) - 1$ regions of size two. This proof is similar to
the proof of Proposition 3.7. Recall that the genus of a Cayley map for $\mathbb{Z}_{2p}$ with the
generating set $\Omega$ is $k = 1 + \frac{p^2 - 2p - r}{2}$, where $r$ denotes the number of regions in the
Cayley map embedding. Since $k$ is an integer, $p^2$ and $r$ have the same parity so that $r$ is
odd. By Proposition 2.1 each of the $r_1$ regions of size one lifts to one region in the
covering graph embedding. By Proposition 3.1 each of the $r_2$ regions of size two lifts to
two regions in the covering graph embedding. Thus there is an even number of regions in
the covering graph embedding coming from the regions of size two, and because $r_1$ is
odd, there is an odd number of regions in the covering graph embedding coming from the
regions of size one. By Proposition 3.2 the region of size three lifts to one region in the
covering graph embedding. Thus the one region $R$ of size greater than two lifts to an odd
number of regions in the covering graph embedding. We will count the number of times
$R$ lifts to $p$ regions and thus the number of Cayley maps $M$ with $\text{gen}(M) = \frac{p^2 - 3p - r_1 - 2r_2 + 1}{2}$. We now appeal to the proof of Proposition 3.2 for the number of
Cayley maps with this genus. Thus the number of Cayley maps with $\text{gen}(M) =$
\[ \frac{p^2 - 3p - r_1 - 2r_2 + 1}{2} \text{ is } \left( \frac{r_1}{h_1 h_2 ... h_t} \right) (r_2 - 1)! 2^{r_1} \left( \sum_{\mathcal{T} \subseteq \mathcal{P}'} C_\mathcal{T} \right) \text{ where each } \mathcal{T} \subseteq \mathcal{P}' \text{ with } \] 

\[ |T|=r_1, \ S_\mathcal{T} = \sum_{i \in \mathcal{T}} i, \ g_\mathcal{T}(u) = \prod_{i \in \mathcal{T}} (1+u^i), \text{ and } C_\mathcal{T} = \sum_{n=-s_{Q_{\mathcal{T}}} \over p} {s_{Q_{\mathcal{T}}} \choose u^2} g_\mathcal{T}(u). \quad \square \]

As a direct consequence of this we can find, under the same conditions, the number of Cayley maps that have the region of size greater than two not containing the spoke lifting to one region in the covering graph embedding. Since the proof of Corollary 3.12 is similar to the proof of Corollary 3.4, we omit it.

**Corollary 3.12** Let \( p \) be an odd prime and let \( \Gamma = \mathbb{Z}_{2p} \) with generating set \( \Omega = \{1,3,...,2p-1\} \). Let \( K \) be a plane voltage graph of index one corresponding to \( \Gamma \) and \( \Omega \) such that \( K \) consists only of singleton loops and stacked loops with the spoke in a region of size three. Furthermore, let \( K \) have \( h_i \) stacked loops of size \( j_i \), for \( i = 1, 2, ..., t \) where \( h = h_1 + h_2 + ... + h_t \). If \( r_i \) is odd then the number of Cayley maps \( M = \text{CM}(\Gamma,\Omega,p) \) with \( \text{gen}(M) = \frac{p^2 - 3p - r_1 - 2r_2 + 1}{2} \) that are the lift of such a voltage graph \( K \) is

\[ \left( \frac{r_1}{h_1 h_2 ... h_t} \right) \left( \frac{p-1}{2} \right)! 2^{r_1} - r_1! (r_2 + 1)! 2^{r_1} \left( \sum_{\mathcal{T} \subseteq \mathcal{P}'} C_\mathcal{T} \right). \]

The following result is for voltage graphs that have an even number of regions of size one and that consist only of singleton loops and stacked loops with the spoke in a region of size three. Let \( K \) be a plane voltage graph that consists only of singleton loops and stacked loops with the spoke in a region of size three. Let \( R \) denote the largest region
in $K$ not containing the spoke. Now we consider when $r_1$ is even. In this case we will see that due to parity constraints, the region $R$ must lift to an even number of regions and whether $R$ lifts to $2$ or $2p$ regions will be handled separately in Proposition 3.13 and Corollary 3.14.

Proposition 3.13 Let $p$ be an odd prime and let $\Gamma = \mathbb{Z}_{2p}$ with generating set $\Omega = \{1, 3, \ldots, 2p - 1\}$. Let $K$ be a plane voltage graph of index one corresponding to $\Gamma$ and $\Omega$, such that $K$ consists only of singleton loops and stacked loops with the spoke in a region of size three. Furthermore, let $K$ have $h_j$ stacked loops of size $j_i$ for $i = 1, 2, \ldots, t$ where $h = h_1 + h_2 + \ldots + h_t$. If $r_1$ is even then the number of Cayley maps $M = CM(\Gamma, \Omega, \rho)$ with $\text{gen}(M) = \frac{p^2 - 4p - r - 2r_2 + 1}{2}$ that arise as the lift of such a voltage graph $K$ is

$$\left(\begin{array}{c}
\frac{r_1}{h_1, h_2, \ldots, h_t} \\
\frac{r_1}{h_1, h_2, \ldots, h_t}
\end{array}\right) r! (2^{r_2} + 1) \left(\frac{\sum C_r}{\prod_{r\geq p}^r}\right).$$

Proof: Due to the stacked loops in $K$ we know that there are $r_1 = \frac{p - 1}{2} - \sum_{i=1}^{t} \frac{h_i j_i + h}{2}$ regions of size one, $r_2 = \left(\sum_{i=1}^{t} h_i (j_i - 1)\right) - 1$ regions of size two. This proof is similar to the proof of Proposition 3.9. Recall that the genus of a Cayley map for $\mathbb{Z}_{2p}$ with the generating set $\Omega$ is $k = 1 + \frac{p^2 - 2p - r}{2}$, where $r$ denotes the number of regions in the Cayley map embedding. Since $k$ is an integer, $p^2$ and $r$ have the same parity so that $r$ is odd. There are an even number of regions in the covering graph embedding coming from the regions of size two, and because $r_1$ is even, there are an even number of regions in the
covering graph embedding coming from the regions of size one. The one region of size three containing the spoke lifts to one region in the covering graph embedding (for the same reason as it did in proposition 3.11). Thus the one region $R$ of size greater than two not containing the spoke lifts to an even number of regions in the covering graph embedding. We will count the number of times $R$ lifts to $2p$ regions and thus the number of Cayley maps $M$ with $\text{gen}(M) = \frac{p^2 - 4p - r_1 - 2r_2 + 1}{2}$. We now appeal to the proof of Proposition 3.3 for the number of Cayley maps with this genus. Thus the number of Cayley maps with $\text{gen}(M) = \frac{p^2 - 4p - r_1 - 2r_2 + 1}{2}$ is

\[
\left( \begin{array}{c} r_1 \\ h_1, h_2, \ldots, h_j \end{array} \right) \frac{r_1!}{(r_2-1)!} 2^{n-1} \left( \sum_{T \subseteq P^+} C_T \right) \quad \text{where for each } T \subseteq P^+ \text{ with } |T|=r_1, \quad S_T = \sum_{i \in T} i,
\]

\[
g_T(u) = \prod_{i \in T} (1 + u^i), \quad \text{and } C_T = \sum_{m=\frac{S_T}{p}}^{\left\lfloor \frac{S_T}{p} \right\rfloor} \left[ \frac{S_T - mg_T}{2} \right] g_T(u).
\]

As a direct consequence of this we can find, under the same conditions, the number of Cayley maps that have the region of size greater than two without the spoke lifting to two regions in the covering graph embedding. Since the proof of Corollary 3.14 is similar to the proof of Corollary 3.4, we omit it.

Corollary 3.14 Let $p$ be an odd prime and let $\Gamma = \mathbb{Z}_{2p}$ with generating set $\Omega = \{1, 3, \ldots, 2p-1\}$. Let $K$ be a plane voltage graph of index one corresponding to $\Gamma$ and $\Omega$, such that $K$ consists only of singleton loops and stacked loops, with the spoke in a region of size three. Furthermore, let $K$ have $h_i$ stacked loops of size $j_i$ for $i = 1, 2, \ldots, t$.
where \( h = h_1 + h_2 + \ldots + h_i \). If \( r_1 \) is even then the number of Cayley maps \( M = CM(\Gamma, \Omega, \rho) \) with \( \text{gen}(M) = \frac{p^2 - 2p - r_1 - 2r_2}{2} \) that are lifts of such a voltage graph \( K \) is

\[
\left( \frac{r_1}{h_1 h_2 \ldots h_i} \right) \left( \frac{p - 1}{2} \right) \left( \frac{p - 1}{2} \right) - r_1 ! \left( r_2 + 1 \right) ! 2^{s+1} \sum_{\frac{F}{F} \in \mathbb{F}} C_T.
\]

This chapter has given us formulas to calculate the number of Cayley maps \( M \) that have certain genus values and are lifts of plane voltage graphs under given conditions. In particular when a Cayley map is a lift of a plane voltage graph that has only singleton loops and stacked loops we have a formula to calculate the genus value and number of Cayley maps that have that genus value. The formulas depend on whether \( r_1 \) is odd or even and on the size of the region containing the spoke.
CHAPTER 4

CONCLUSION

Let $p$ be an odd prime and let $\Gamma = \mathbb{Z}_{2p}$ with generating set $\Omega = \{1, 3, \ldots, 2p - 1\}$. Let $K$ be a plane voltage graph of index one corresponding to $\Gamma$ and $\Omega$ such that $K$ consists only of singleton loops and stacked loops. Let $K$ consist of $h$ stacked loops, where $h = h_1 + h_2 + \ldots + h_i$, and $h_i$ is the number of stacked loops of size $j_i$ for $i = 1, 2, \ldots, r$. In this thesis we have found results for the number of Cayley maps $M = CM(\Gamma, \Omega, \rho)$ with the genus of $M$, dependent on the size of the region the spoke is in and whether the number of regions of size one (which we will continue to denote by $r_1$) is odd or even. For such a voltage graph $K$ we found that a certain region $R$ of size greater than two could satisfy the KVL or not (or the region $R$ could satisfy the semi KVL or not). Knowing whether $r_1$ is odd or even and what size region contains the spoke allowed us to calculate the genus of the Cayley maps that lifted from such a $K$. To count the Cayley maps of the appropriate genus we found the number of Cayley maps that lifted from such a $K$ in which a certain region $R$ of $K$ satisfied the KVL (or semi KVL). Then we subtracted this from the total number of Cayley maps that were lifts of such a $K$ to find the number of times that the region $R$ of $K$ did not satisfy the KVL or semi KVL.
Figure 4.1 shows a voltage graph with two stacked loops of size two and one singleton loop with the spoke in the largest region.

Using the generating set $\Omega = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21\}$ for $\mathbb{Z}_{22}$ we can obtain the Cayley graph $K_{11,11}$. We use the voltage graph $K$ (of Figure 4.1) that has two stacked loops of size two, i.e. $h=h_1=2$ and $j_1=2$. Then $K$ has $\ell = \frac{p-1}{2} - \sum_{i=1}^{r} h_i j_i + h = 3$ regions of size one and $r_2 = \sum_{i=1}^{r} h_i (j_i - 1) = 2$ regions of size two. We now apply Proposition 3.3. We know that there are three regions of size one, two regions of size two, one region of size four, and the spoke is in the largest region.

So the number of Cayley maps $M = CM(\Gamma, \Omega, \rho)$ lifting from such a plane voltage graph $K$ with $\text{gen}(M) = \frac{p^2 - 4p - r_1 - 2r_2}{2} + 1 = \frac{121 - 44 - 3 - 4}{2} + 1 = 36$ is found.
using the formula \( \binom{r_i}{h_1, h_2, \ldots, h_t} r_i! r_j! 2^n \left( \sum_{T \subseteq P^+} \binom{|T|=r_j}{S_T} \right) \), where for each \( T \subseteq P^+ \) with \(|T|=r_i\), we have \( S_T = \sum_{a \in T} a \), \( g_T(u) = \prod_{a \in T} (1 + u^a) \), and \( C_T = \sum_{m \leq \frac{S_T-np}{2}} \binom{S_T}{m} g_T(u) \). There are five elements in \( T \) and we are choosing three of them for the boundary of \( R \) so there are \( \binom{5}{3} = 10 \) subsets \( T \). Also \( \binom{r_i}{2} = \binom{3}{2} = 3, r_i! = 6, r_j! = 2! = 2, \) and \( 2^n = 2^2 = 4 \).

Now let us concern ourselves with the calculation of the sum of the \( C_T \)'s. We are taking the sum over all possible subsets of size 3 of the positive \( p \)-distances. These possible sets are \( T_1 = \{1, 2, 3\} \), \( T_2 = \{1, 2, 4\} \), \( T_3 = \{1, 2, 5\} \), \( T_4 = \{1, 3, 4\} \), \( T_5 = \{1, 3, 5\} \), \( T_6 = \{1, 4, 5\} \), \( T_7 = \{2, 3, 4\} \), \( T_8 = \{2, 3, 5\} \), \( T_9 = \{2, 4, 5\} \), \( T_{10} = \{3, 4, 5\} \). For \( T_1 \) we have \( S_{T_1} = 6 \) and \( \left[ \frac{6}{11} \right] = 0, \left[ -\frac{6}{11} \right] = 0 \) so that the exponent coming from \( \frac{S_{T_1} - np}{2} \) is \((6-11(0))/2 = 3\). So in the expansion of \( g_{T_1}(u) = (1+u)(1+u^2)(1+u^3) = 1+u+u^2+2u^3+u^4+u^5+u^6 \) we are looking for the coefficient of \( u^3 \), which is 2 so that \( C_{T_1} = 2 \). For \( T_2 \) we have \( S_{T_2} = 7 \) and \( \left[ \frac{7}{11} \right] = 0, \left[ -\frac{7}{11} \right] = 0 \) so that \((7-11(0))/2 = 7/2\). We do not have to worry about the expansion since the coefficient of \( u^{7/2} \) in \( g_{T_2}(u) \) is 0. For \( T_3 \) we have \( S_{T_3} = 8 \) and \( \left[ \frac{8}{11} \right] = 0, \left[ -\frac{8}{11} \right] = 0 \) so that the exponent coming from \( \frac{S_{T_3} - np}{2} \) is \((8-11(0))/2 = 4\). So in the expansion of \( g_{T_3}(u) = (1+u)(1+u^2)(1+u^5) = 1+u+u^2+u^3+u^6+u^7+u^8 \) we are looking for
the coefficient of \( u^4 \), which is 0 so that \( C_{r_4} = 0 \). For \( T_4 \) we have \( S_{r_4} = 8 \) and

\[
\left\lfloor \frac{8}{11} \right\rfloor = 0, \left\lfloor \frac{-8}{11} \right\rfloor = 0 \text{ so that the exponent coming from } \frac{S_{r_4} - np}{2} \text{ is } (8-11(0))/2 = 4. \text{ So in the expansion of } g_{r_4}(u) = (1+u)(1+u^3)(1+u^4) = 1+u+u^3+2u^4+u^5+u^7+u^8 \text{ we are looking for the coefficient of } u^4, \text{ which is 2 so that } C_{r_4} = 2. \text{ For } T_5 \text{ we have } S_{r_5} = 9 \text{ and }
\]

\[
\left\lfloor \frac{9}{11} \right\rfloor = 0, \left\lfloor \frac{-9}{11} \right\rfloor = 0 \text{ so that the exponent coming from } \frac{S_{r_5} - np}{2} \text{ is } (9-11(0))/2 = 9/2. \text{ The coefficient of a fractional exponent in the expansion of } g_{r_5}(u) \text{ is 0. For } T_6 \text{ we have } S_{r_6} = 10 \text{ and }
\]

\[
\left\lfloor \frac{10}{11} \right\rfloor = 0, \left\lfloor \frac{-10}{11} \right\rfloor = 0 \text{ so that the exponent coming from } \frac{S_{r_6} - np}{2} \text{ is } (10-11(0))/2 = 5. \text{ So in the expansion of } g_{r_6}(u) = (1+u)(1+u^3)(1+u^5) = 1+u+u^3+2u^4+u^5+u^6+u^9+u^{10} \text{ we are looking for the coefficient of } u^5, \text{ which is 2 so that } C_{r_6} = 2. \text{ For } T_7 \text{ we have } S_{r_7} = 9 \text{ and }
\]

\[
\left\lfloor \frac{9}{11} \right\rfloor = 0, \left\lfloor \frac{-9}{11} \right\rfloor = 0 \text{ so that the exponent coming from } \frac{S_{r_7} - np}{2} \text{ is } (9-11(0))/2 = 9/2. \text{ The coefficient of a fractional exponent in the expansion of } g_{r_7}(u) \text{ is 0. For } T_8 \text{ we have } S_{r_8} = 10 \text{ and }
\]

\[
\left\lfloor \frac{10}{11} \right\rfloor = 0, \left\lfloor \frac{-10}{11} \right\rfloor = 0 \text{ so that the exponent coming from } \frac{S_{r_8} - np}{2} \text{ is } (10-11(0))/2 = 5. \text{ So in the expansion of } g_{r_8}(u) = (1+u^2)(1+u^3)(1+u^5) = 1+u^2+u^3+2u^4+u^5+u^7+u^8+u^{10} \text{ we are looking for the coefficient of } u^5, \text{ which is 2 so that } C_{r_8} = 2. \text{ For } T_9 \text{ we have } S_{r_9} = 11 \text{ and }
\]

\[
\left\lfloor \frac{11}{11} \right\rfloor = 1, \left\lfloor \frac{-11}{11} \right\rfloor = -1 \text{ so that the exponents coming from } \frac{S_{r_9} - np}{2} \text{ are } (11-11(-1))/2 = 11, \text{ (11-11(0))/2 = 11/2, and (11-11(1))/2 = 0. So in the expansion of } g_{r_9}(u) = (1+u^2)(1+u^4)(1+u^5) = 1+u^2+u^4+u^5+u^6+u^7+u^9+u^{11} \text{ we are looking for the}
coefficients of $u^{11}$ and $u^{9}$ which are both 1 so that $C_{r_9} = 2$. For $T_{10}$ we have $S_{r_9} = 12$ and

\[ \frac{12}{11} = 1, \quad \frac{-12}{11} = -1 \]

so that the exponents coming from $\frac{S_{r_9} - np}{2}$ are $(12-11(-1))/2 = 23/2$, $(12-11(0))/2 = 6$, and $(12-11(1))/2 = 1/2$. So in the expansion of $g_{T_{10}}(u) = (1+u^3)(1+u^4)(1+u^5) = 1+u^3+u^4+u^5+u^7+u^8+u^9+u^{12}$ we are looking for the coefficient of $u^6$, which is 0 so that $C_{r_9} = 0$.

Thus the sum of the coefficients is $\sum_{i=1}^{10} C_{r_i} = 10$. Multiplying this by 3, 6, 2, and 4 from the previous parts of the formula we have 1440 Cayley maps $M = CM(\Gamma, \Omega, \rho)$ such that $\text{gen}(M) = 36$. Furthermore by Corollary 3.4 the number of Cayley maps $M = CM(\Gamma, \Omega, \rho)$ such that $\text{gen}(M) = 46$ is $11520 - 1440 = 10080$.

In this thesis we have concerned ourselves only with what happens when we have one region $R$ with a size greater than or equal to two not containing the spoke. We did this so that we could count how many Cayley maps lifting from such a voltage graph had the region $R$ satisfying the KVL (or the semi KVL depending on its size). To further this work we would need to consider voltage graphs that have more than one region of size greater than or equal to two each not containing the spoke. Allowing for two regions of size greater than or equal to two each not containing the spoke would still be quite restrictive so we prefer to allow for an arbitrary number of these regions. We desire a general method that will extend the processes we have developed in this thesis. There are several difficulties that need to be addressed in order to find such a method. One of the difficulties is that once we use a set of positive $p$-distances that satisfy the KVL or the semi KVL for one region, we must make sure not to assign any of these elements to any
other regions independent of the first at the same time. Another problem is to determine what happens when one region is dependent upon the assignment of elements in another region. A third setback, possibly the greatest, is in counting the number of ways to arrange the loops in order to satisfy the given region structure. Fortunately a portion of this work has been done already. Reiper enumerates the embeddings of index one voltage graphs (without spokes) according to region structures. So it may be possible through modifying his techniques to similarly enumerate our voltage graphs by region structures. A fourth difficulty is to calculate the different genus values that arise from the voltage graph embeddings, and this depends on how many regions in the voltage graph embedding satisfy the KVL or the semi KVL. In particular, we believe that each different pairing of numbers satisfying the KVL and semi KVL gives rise to a different genus value.

Let \( p \) be a prime and let \( K \) be a plane voltage graph embedding for the group \( Z_{2p} \) with the generating set \( \Omega = \{ 1, 3, \ldots, 2p-1 \} \). Let \( r_i \) be the number of regions of size \( i \), \( R_O \) be the number of regions with odd size (odd regions), and \( R_E \) be the number of regions with even size (even regions). The genus of the Cayley map that is the lift of \( K \) depends upon how many regions in \( K \) satisfy the KVL and semi KVL. Let \( (k, l) \) be an ordered pair such that \( k \) is the number of odd regions satisfying the semi KVL and \( l \) is the number of even regions satisfying the KVL.

Figure 4.2 shows two planar voltage graph embeddings with six regions of size greater than two and the same region structure. Now let us go over an example to illustrate these problems.
Consider the cyclic group $\Gamma = \mathbb{Z}_{74}$ with the generators $\Omega = \{1, 3, \ldots, 73\}$. Let's say we have a voltage graph with two regions of size three, two regions of size four, and two regions of size five, with the spoke in a region of size five. Let us say that we have the voltage graph embedding in Figure 4.1 (a). There are 13 regions of size one so $r_1 = 13$, there are no regions of size two so $r_2 = 0$, there are three regions of size three, four, and five so $r_3 = 2$, $r_4 = 2$, and $r_5 = 2$. So the possible ordered pairs of odd and even regions that satisfy the KVL, semi KVL are $(0, 0)$, $(1, 0)$, $(1, 1)$, $(1, 2)$, $(2, 0)$, $(2, 1)$, $(2, 2)$, $(3, 0)$, $(3, 1)$, $(3, 2)$, $(4, 0)$, $(4, 1)$, $(4, 2)$. We have $R_O = 17$ and $R_E = 2$. We illustrate what is involved in the classification of these Cayley maps $M = CM(\Gamma, \Omega, p)$ when $(k, l)$ is either $(1, 0)$ or $(1, 1)$.

Case 1: Looking at the genus value for the ordered pair $(1, 0)$, we have one odd region satisfying the semi KVL. So the one odd region lifts to $p = 37$ regions in the covering graph embedding. Each of the other odd regions lifts to one region in the covering graph embedding, and each of the even regions lifts to two regions in the embedding.
covering graph embedding. Thus \( \text{gen}(M) = \left( (p^2 - 2p - p - (17 - 1) - 2(2))/2 \right) + 1 = ((1369 - 74 - 37 - 16 - 4)/2) + 1 = 620 \). This case has one region of odd size satisfying the semi \( KVL \). So for each odd region; \( R_1, R_2, R_5, R_6 \), we must find the number of times the region satisfies the semi \( KVL \) and the number of times that all the other regions of size greater than two do not satisfy the semi \( KVL \) or the \( KVL \). We do so with sub-cases.

Sub case 1.1: We will count the number of times \( R_1 \) satisfies the semi \( KVL \). Since the size of \( R_1 \) is three, we would take each subset of size three, from the positive \( p \)-distances, and find the number of times that a combination of the corresponding generators or their inverses satisfies the semi \( KVL \). For each subset of generators that does satisfy the semi \( KVL \) we then find the number of times the other 15 elements or their inverses can be assigned to the loops of the voltage graph embedding so that no region other than \( R_1 \) satisfies the semi \( KVL \) or the \( KVL \).

Sub case 1.2: Here we count the number of times \( R_2 \) satisfies the semi \( KVL \). Since the size of \( R_2 \) is also three just like \( R_1 \), we repeat Sub-case 1.1.

Sub case 1.3: We count the number of times \( R_5 \) satisfies the semi \( KVL \). Since the size of \( R_5 \) is five, we take each subset of size five, from the positive \( p \)-distances, and find the number of times that a combination of the corresponding generators or their inverses satisfies the semi \( KVL \). For each such subset of generators that does satisfy the semi \( KVL \), we have to find the number of times the other 13 elements or their inverses can be assigned to the loops of the voltage graph embedding so that \( R_5 \) is the only region satisfying the semi \( KVL \) or the \( KVL \).

Sub case 1.4: We count the number of times \( R_6 \) satisfies the semi \( KVL \). The size of \( R_6 \) is five but this region has the spoke in it. So we take each subset of size four, from the
positive $p$-distances, and find the number of times that a combination of the corresponding generators or their inverses sum to 0 modulo $2p$. If all the boundary elements (excluding the spoke $p$) of $R_5$ sum to a multiple of $2p$, then when we add the spoke in and obtain a multiple of $3p$, which satisfies the semi KVL. Finally for each subset of four generators that satisfies the KVL, we have to find the number of times the other 14 generators or their inverses can be assigned to the loops of the voltage graph embedding so that $R_5$ is the only region satisfying the semi KVL or the KVL.

Case 2: We begin by calculating the genus value for the ordered pair (1,1). There is one odd region satisfying the semi KVL. So this one odd region lifts to $p = 37$ regions in the covering graph embedding. All of the other odd regions lift to one region in the covering graph embedding. There is one even region satisfying the KVL so this one even region lifts to $2p = 74$ regions in the covering graph embedding, the other even region lifts to two regions in the covering graph embedding. Thus the genus value in this case is

$$\text{gen}(M) = ((p^2 - 2p - p - 2p = (17 - 1) - 2(2 - 1))/2) + 1 = ((1369 - 74 - 74 - 16 - 2)/2) + 1 = 584.$$ 

This case has one region of odd size satisfying the semi KVL and one even region satisfying the KVL. The pairs of regions that are possible for this are $(R_1, R_3)$, $(R_1, R_4)$, $(R_2, R_3)$, $(R_2, R_4)$, $(R_5, R_3)$, $(R_5, R_4)$, $(R_6, R_3)$, $(R_6, R_4)$. For each of these pairs, we must find the number of times the even region satisfies the KVL and the odd region satisfies the semi KVL. Also we have to make sure that the other regions of size greater than two do not satisfy the semi KVL or the KVL.

Sub case 2.1: The number of times $R_1$ satisfies the semi KVL and $R_3$ satisfies the KVL. Since $R_1$ and $R_3$ do not share an edge, we do not have to worry about one region
depending on the elements we assign to the other region. So since $R_1$ has size three, for
each subset of generators of size three that satisfies the semi $KVL$, we must find all the
subsets of generators of size four from the remaining 15 generators that satisfy the $KVL$.
Also for each of these two subsets of size three and four we must find the number of
times the remaining 11 elements or their inverses can be assigned to the loops of the
voltage graph so that no regions except $R_1$ and $R_3$ satisfy the semi $KVL$ or the $KVL$.

Sub case 2.2: Since $R_1$ and $R_4$ do not share an edge and have the same sizes
respectively, as $R_1$ and $R_3$ we can repeat Sub case 2.1.

Sub-case 2.3: Since $R_2$ and $R_3$ do not share an edge and have the same sizes
respectively, as $R_1$ and $R_3$ we can repeat Sub case 2.1.

Sub case 2.4: We count the number of times $R_2$ satisfies the semi $KVL$ and $R_4$
satisfies the $KVL$. Since $R_2$ and $R_4$ share an edge, we must now take this into account. So
since $R_2$ has size three, we consider each subset of generators of size three that satisfies
the semi $KVL$ and then we take each element one at a time and assign its inverse to the
shared edge with $R_4$. Then we must find all the subsets of size three from the remaining
elements that together with the inverse of the element on the edge shared with $R_2$ satisfy
the $KVL$. Also for each such way of assigning generators to the loops on the boundaries
of $R_2$ and $R_4$ we must find the number of times the remaining 12 elements can be
assigned to the other loops of the voltage graph so that no regions except $R_2$ and $R_4$ satisfy
the semi $KVL$ or the $KVL$.

Subcase 2.5: We count the number of times $R_5$ satisfies the semi $KVL$ and $R_3$
satisfies the $KVL$. Since $R_5$ and $R_3$ do not share an edge, we do not have to worry about
one region depending on the elements we assign to the other region. So since $R_5$ has size

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five, we consider each subset of generators of size five that satisfies the semi $KVL$ and then we find all the subsets of size four from the remaining 13 elements that satisfy the $KVL$. Finally for each such way of assigning generators to the loops on the boundaries of $R_5$ and $R_3$ we must find the number of times the remaining 9 elements can be assigned to the voltage graph so that no regions except $R_5$ and $R_3$ satisfy the semi $KVL$ or the $KVL$.

Sub case 2.6: We count the number of times $R_5$ satisfies the semi $KVL$ and $R_4$ satisfies the $KVL$. Since $R_5$ and $R_4$ share an edge we must take this into account. So since $R_5$ has size five we consider each subset of generators of size five that satisfies the semi $KVL$ and using each element of such a subset, one at a time, we assign its inverse to the shared edge with $R_4$. Then we find each subset of size four from the remaining elements that together with the inverse of the element on the edge shared with $R_5$ satisfies the $KVL$. Finally for each such way of assigning generators to the loops on the boundaries of $R_5$ and $R_4$ we must find the number of times the remaining 10 elements can be assigned to the voltage graph so that no regions except $R_5$ and $R_4$ satisfy the semi $KVL$ or the $KVL$.

Sub case 2.7: We count the number of times $R_6$ satisfies the semi $KVL$ and $R_3$ satisfies the $KVL$. Since $R_6$ and $R_3$ share an edge we take this into account and proceed in a manner similar to sub case 2.6.

Sub case 2.8: We count the number of times $R_6$ satisfies the semi $KVL$ and $R_4$ satisfies the $KVL$. Since $R_6$ and $R_4$ do not share an edge, we do not have to worry about one region depending on the elements we assign to the other region. Thus we proceed in a manner similar to sub case 2.5.

We must repeat this process for each of the ordered pairs. This shows us the similarities to the work in Chapter 3 and how we might proceed to classify Cayley maps.
that are lifts of planar voltage graphs with a given region structure. Clearly this method is not ideal since the cases with \((k, l)\), where \(k\) and \(l\) are relatively large will be extremely tedious. Perhaps through modification of our maple program, these calculations can be done with ease.

Figure 4.3 shows a voltage graph embedding that satisfies the conditions of case 2 sub-case 2.1 of the previous example. The region \(R_1\) one satisfies the semi \(KVL\) and the region \(R_3\) satisfies the \(KVL\). Table 4.1 shows the boundary element of each region, the order of each of these boundary elements, how many regions in the covering graph each region of the voltage graph embedding lifts to, and which regions satisfy which properties.
Table 4.1

BOUNDARY ELEMENTS

<table>
<thead>
<tr>
<th>region</th>
<th>Boundary element</th>
<th>order</th>
<th>Lifts to</th>
<th>KVL or semi?</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1</td>
<td>19+7+11 = 37</td>
<td>2</td>
<td>37</td>
<td>Semi KVL</td>
</tr>
<tr>
<td>R2</td>
<td>1+3+9 = 13</td>
<td>74</td>
<td>1</td>
<td>None</td>
</tr>
<tr>
<td>R3</td>
<td>5+13+25+31 = 74</td>
<td>1</td>
<td>74</td>
<td>KVL</td>
</tr>
<tr>
<td>R4</td>
<td>1+15+17-21 = 12</td>
<td>37</td>
<td>2</td>
<td>None</td>
</tr>
<tr>
<td>R5</td>
<td>11+27+29+23+21 = 65</td>
<td>74</td>
<td>1</td>
<td>None</td>
</tr>
<tr>
<td>R6</td>
<td>33+37+35+23-31 = 23</td>
<td>74</td>
<td>1</td>
<td>None</td>
</tr>
</tbody>
</table>

To calculate the genus value for a Cayley map \( M = CM(\Gamma, \Omega, p) \) that covers a voltage graph with a given region structure and a particular ordered pair \((k, l)\) we must count the number of regions in the covering graph embedding. There are \( k \) regions with order two lifting to \( p \) regions in the covering graph embedding. There are \( l \) regions with order one lifting to \( 2p \) regions in the covering graph embedding. There are \( R_{O-l} \) regions that each lift to one region in the covering graph embedding. There are \( R_{E-k} \) regions that each lift to two regions in the covering graph embedding. Thus the genus of \( M \) is given by

\[
\text{gen}(M) = \left\lfloor \frac{p^2 - 2p - p(k) - 2p(l) - (R_{O-l}) - 2(R_{E-k})}{2} \right\rfloor + 1
\]

Using this formula for Sub case 2.1 we find

\[
\text{gen}(M) = \left\lfloor \frac{((37^2 - 2(37) - 37 - 2(37) - (17 - 1) - 2(2 - 1))/2) + 1 = ((1369 - 74 - 37 - 74 - 16 - 2)/2) + 1 = 584.}
\]
Thus we have begun the classification of Cayley maps for $\mathbb{Z}_{2p}$ with generating set $\Omega = \{1, 3, \ldots, 2p-1\}$ by determining the genera of Cayley maps that arise as lifts of certain plane voltage graphs, namely those with only singleton loops, stacked loops and at most one region (excluding the spoke) of size greater than two. We have also indicated where the research may proceed next in order to obtain the complete classification of the Cayley maps that arise as lifts of plane voltage graphs. I believe that classifying the general case is possible once the planar problem is solved. Once this is done the remaining hurdle is to allow for an element and its inverse to be on the same region boundary.
APPENDIX I

A PROGRAM TO SUM THE COEFFICIENTS
This function takes two arrays and sums the multiplication of the corresponding places in the array's. Its output is the sum. This is the calculation of how many loops are used in the stacks

\[
\text{sumniji} := \text{proc}(X::\text{array}, Y::\text{array}, t::\text{integer}) \\
\text{local } i, \text{ back, sum; } \\
i := 1; \\
\text{sum} := 0; \\
\text{back} := t; \\
\text{while } (i \leq \text{back}) \text{ do} \\
\quad \text{sum} := \text{sum} + (X[i] \times Y[i]); \\
\quad i := i + 1; \\
\text{od; } \\
\text{sum; } \\
\text{end;}
\]

This is the calculation of the regions of size two.

\[
\text{Sumreg2} := \text{proc}(X::\text{array}, Y::\text{array}, t::\text{integer}) \\
\text{local } i, \text{ back, sum; } \\
i := 1; \\
\text{sum} := 0; \\
\text{back} := t; \\
\text{while } (i \leq \text{back}) \text{ do} \\
\quad \text{sum} := \text{sum} + (X[i] \times (Y[i] - 1)); \\
\quad i := i + 1; \\
\text{od; } \\
\text{sum; } \\
\text{end;}
\]

This is the calculation of the regions of size one.

\[
\text{Sumarray} := \text{proc}(X::\text{array}, t::\text{integer}) \\
\text{local } i, \text{ SumX, back; } \\
i := 1; \\
\text{SumX} := 0; \\
\text{back} := t; \\
\text{while } (i \leq \text{back}) \text{ do} \\
\quad \text{SumX} := \text{SumX} + X[i]; \\
\quad i := i + 1; \\
\text{od; } \\
\text{SumX; }
\]
This is the calculation of the sum of the coefficients. It takes as input two arrays, there are \( h_i \) stacked loops of size \( j_i \), the first array is the \( h_i \)'s the second the \( j_i \)'s. It also takes two integers as input the first is the length of the arrays the second is the prime for our group.

\[
\text{Setup}:=\text{proc}(X::\text{array}, Y::\text{array}, t::\text{integer}, a::\text{integer})
\]

\[
\text{local } n, \text{ sigma}, p, r1, r2, S, C, \text{allp}, i, T, s, q, k, m, g, b, \text{Final};
\]

\[
n:=\text{Sumarray}(X, t);
\]

\[
\text{sigma}:=\text{sumniji}(X, Y, t);
\]

\[
p:=a;
\]

\[
r1:=(p-1)/2-\text{sigma}+n;
\]

\[
r2:=\text{Sumreg2}(X, Y, t);
\]

\[
S:=\text{array}(1..r1);
\]

\[
\text{with}((\text{combinat, numbcomb})\text{):}
\]

\[
C:=\text{numbcomb}((p-1)/2, r1);
\]

\[
\text{with}((\text{combi}stuct})\text{):}
\]

\[
b:=\text{array}(1..C);
\]

\[
\text{allp} := \text{iterstructs}(\text{Combina}tion((p-1)/2), \text{size}=r1);
\]

\[
\text{for } i \text{ to } C \text{ do } b[i]:=\text{nextstruct(allp) od;}
\]

\[
T:=0;
\]

\[
\text{for } i \text{ to } C \text{ do}
\]

\[
\text{mul}(1+u^j, j=t[i]);
\]

\[
\text{expand}((\text{mul}(1+u^j, j=t[i]));
\]

\[
s:=\text{add}(j, j=b[i]);
\]

\[
q:=\text{iquest}(s, p);
\]

\[
h:=\text{iquest}(s, p);
\]

\[
\text{for } k \text{ to } 2h+1 \text{ do}
\]

\[
m:=q+k-1;
\]

\[
\text{if irem}(s-p*m, 2)=0 \text{ then}
\]

\[
\text{readlib(coeftayl)}:
\]

\[
g:=\text{expand}((\text{mul}(1+u^j, j=b[i])));
\]

\[
\text{print}(b[i], \text{ expand}((\text{mul}(1+u^j, j=b[i])));
\]

\[
T:=T+\text{coeftayl}(g, u=0, (s-p*m)/2);
\]

\[
\text{fi};
\]

\[
\text{od;}
\]
od;

print("Sum of Coefficients"=T);

end;

This would be the calculation for a voltage graph with only singleton loops.

\begin{verbatim}
  h := array(1..1);
  h[1]:=0;
  h(1);

  j := array(1..1);
  j[1]:=0;
  j(1);

  Setup(h, j, 1, 11);
\end{verbatim}

Here is an example of the setup and the execution of the program for \( p=17 \) and having two stacked loops of size two and one stacked loop of size three.

\begin{verbatim}
  h := array(1..2);
  h[1]:=2;
  h(1);
  h[2]:=1;
  h(1);

  j := array(1..2);
  j[1]:=2;
  j(1);
  j[2]:=3;
  j(2);

  Setup(h, j, 2, 17);
\end{verbatim}
BIBLIOGRAPHY


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