Conjugacy numbers for cyclic groups of even order

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CONJUGACY NUMBERS FOR CYCLIC GROUPS OF EVEN ORDER

by

Tanya Jessica Murray

Bachelor of Science
University of Nevada, Las Vegas
2000

A thesis submitted in partial fulfillment of the requirements for the

Masters of Science Degree In Mathematical Sciences
Department of Mathematical Sciences
College of Sciences

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Tanya Jessica Murray

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is approved in partial fulfillment of the requirements for the degree of

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Examination Committee Chair

Dean of the Graduate College

Examination Committee Member

Examination Committee Member

Graduate College Faculty Representative
ABSTRACT

Conjugacy Numbers for Cyclic Groups of Even Order

by

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Let $\Gamma$ be a finite group, let $X$ be a subset of $\Gamma$ where $X^{-1}=X$ and $1 \notin X$. The conjugacy graph $\text{Con}(\Gamma; X)$ has vertex set $\Gamma$ and two vertices $g, h \in \Gamma$ are adjacent if and only if there exists $x \in X$ with $g=xhx^{-1}$. Let $\Omega$ be a group with generating set $\Delta$. The conjugacy number $\text{con}(\Omega; \Delta)$ is defined as the minimum integer $k \geq 2$ for which there exists a nonabelian group $\Gamma$ of order $k|\Omega|$ and a subset $X$ of $\Gamma$ such that $\text{Cay}(\Omega; \Delta)$ is isomorphic to a component of $\text{Con}(\Gamma; X)$. We call this $\Gamma$ a conjugacy group for $\Omega$ and $\Delta$. We will calculate the conjugacy numbers for $C_6$, $C_8$ and $C_{10}$ and identify possible conjugacy groups. Finally we will verify that certain groups of order $4n$ cannot be conjugacy groups for $C_{2n}$. 

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CHAPTER 1

INTRODUCTION

The most widely studied graph associated with groups is the Cayley graph. When solving problems concerning groups, the use of graphs can clarify a presentation and make problems more controllable. Graphs aid in proving general results about groups and particular results about individual groups [3]. Another graph defined to aid in the understanding of group structure is the conjugacy graph. It is necessary to first discuss Cayley graphs and then we will define conjugacy graphs, which were first introduced by Bowman and Schultz in [3].

Cayley Graphs

Arthur Cayley (1821-1895) was the first to represent a group as a network of directed edges, where the vertices correspond to elements and the edges to multiplication by group generators and their inverses [11]. Let \( \Gamma \) be a finite group with a generating set \( \Delta \), where \( \Delta^{-1} = \Delta \) and \( 1 \notin \Delta \). The Cayley graph \( \text{Cay}(\Gamma; \Delta) \) has the vertex set \( \Gamma \), where \( g, h \in \Gamma \) are adjacent if and only if \( h = g\delta \) for some \( \delta \in \Delta \). We present the following examples to provide further understanding of a Cayley graph.

Example 1.1 Let \( \Gamma \) be the group \( S_3 \) and let \( \Delta = \{(12), (123), (132)\} \). To find \( \text{Cay}(\Gamma; \Delta) \) we multiply each element of \( S_3 \) on the right by each element of \( \Delta \) as follows:
\[(1)(12) = (12) \quad (1)(123) = (123) \quad (1)(132) = (132)\]
\[(12)(12) = (1) \quad (12)(123) = (13) \quad (12)(132) = (23)\]
\[(13)(12) = (132) \quad (13)(123) = (23) \quad (13)(132) = (12)\]
\[(23)(12) = (123) \quad (23)(123) = (12) \quad (23)(132) = (13)\]
\[(123)(12) = (23) \quad (123)(123) = (132) \quad (123)(132) = (1)\]
\[(132)(12) = (13) \quad (132)(123) = (1) \quad (132)(132) = (123)\]

The Cayley graph $Cay(S_3; \Delta)$ is shown in Figure 1.A, where $a=(123)$ and $b=(12)$. The convention is the following. The direction of an arrow labeled with $a$ is the multiplication by $a$ on the right, while against the arrow is multiplication by $a^{-1}=(132)$ on the right.

![Figure 1.A The Cayley graph of $S_3$, where $\Delta=\{(12),(123),(132)\}$](image)

The usefulness of Cayley graphs in solving problems involving groups can be further illustrated using Figure 1 by considering the following example as discussed in [3]. If we
were asked to find the element \(aba^2 b\) in \(S_3\), where \(a=(123)\) and \(b=(12)\) we could consider the directed walk starting at 1 and quickly find that \(aba^2 b=a^2\). While graphs of groups do not tell anything about groups which cannot be expressed algebraically, there are still advantages to using graphs over the algebraic approach. One advantage is the speed in which conclusions can be reached using a graph with much less effort than the algebraic method. We include another example of a Cayley graph only this time the Cayley graph is simplified and we omit the directed edges.

**Example 1.2** Let \(\Gamma\) be the group \(C_4 = \{1, x, x^2, x^3\}\) and let \(\Delta = \{x, x^3\}\). To find \(\text{Cay}(\Gamma; \Delta)\) we perform the following calculations similar to example 1.1:

\[
\begin{align*}
1 \cdot x &= x \\
x \cdot x &= x^2 \\
x^2 \cdot x &= x^3 \\
x^3 \cdot x &= 1
\end{align*}
\]

The Cayley graph is shown in Figure 1.B.

![Figure 1.B The Cayley graph of \(C_4\) where \(\Delta = \{x, x^3\}\)]
Example 1.3 Let $\Gamma$ be the group $S_4$ and let $\Delta = \{(12), (13), (14)\}$. Similar to example 1, to find $\text{Cay}(\Gamma; \Delta)$ we multiply each element in $S_4$ on the right by each element in $\Delta$.

The Cayley graph is shown in Figure 1.C.

Figure 1.C The Cayley graph of $S_4$ where $\Delta = \{(12), (13), (14)\}$
Conjugacy Graphs

Now begins our discussion on the concept of conjugacy graphs which was introduced and studied in [3]. For a finite group $\Gamma$, let $X$ be a subset of $\Gamma$ where $X^{-1} = X$ and $1 \not\in X$. The conjugacy graph $\text{Con}(\Gamma; X)$ has vertex set $\Gamma$ and two vertices $g, h \in \Gamma$ are adjacent if and only if there exists $x \in X$ with $g = xhx^{-1}$. If $X$ generates $\Gamma$, then the components of $\text{Con}(\Gamma; X)$ partition the vertices into the conjugacy classes of $\Gamma$. Recall that the conjugacy class for an element $g \in \Gamma$ and is defined as $\text{cl}(g) = \{xgx^{-1} : x \in \Gamma\}$ and is denoted $\text{cl}(g)$. In general the components partition the vertices, or elements of $\Gamma$, into "conjugacy classes" where we are only allowed to do conjugation by elements of $\langle X \rangle$, the subgroup of $\Gamma$ generated by $X$. We provide the following example to illustrate a conjugacy graph.

**Example 1.4** Let $\Gamma$ be the group $S_3$ and let $X = \{(12), (123), (132)\}$. To find $\text{Con}(\Gamma; X)$ we perform the following table of calculations, where in column I each element of $S_3$ is conjugated by (12), in column II each element of $S_3$ is conjugated by (123) and in column III each element of $S_3$ is conjugated by (132).

<table>
<thead>
<tr>
<th>Column I</th>
<th>Column II</th>
<th>Column III</th>
</tr>
</thead>
</table>

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Since $X$ generates $S_3$, we see that the conjugacy classes of $S_3$ are $\{(1)\}, \{(12), (13), (23)\}$ and $\{(123), (132)\}$ and these sets partition the vertices of the conjugacy graph into components, which can be seen in Figure 1.D.

Example 1.5 Let $\Gamma$ be the group $S_4$ and let $X=\{(12), (13), (14)\}$. To find $\text{Con}(S_4; X)$ we perform calculations similar to Example 1.4, each element of $S_4$ is conjugated by each element of $X$. Again $X$ generates $S_4$. It is known that two permutations are in the same conjugacy class if and only if they have the same cycle structure [11]. Therefore the conjugacy classes of $S_4$ are $\{(1)\}, \{(12), (13), (14), (23), (24), (34)\}, \{(12)(34), (13)(24), (14)(23)\}, \{(123), (124), (132), (134), (142), (143), (234), (243)\}$ and $\{(1234), (1243), (1324), (1342), (1423), (1432)\}$. See Figure 1.E for $\text{Con}(S_4; X)$. 

\[
\begin{align*}
(12)(132)(12) &= (123) \\
(123)(132)(132) &= (132) \\
(132)(132)(123) &= (132)
\end{align*}
\]
Figure 1.E The conjugacy graph of $S_4$ where $X = \{(12),(13),(14)\}$
CHAPTER 2

CONJUGACY NUMBERS AND CONJUGACY GROUPS

We move to the notion of a conjugacy number, which was first introduced and studied in [3]. Let $\Omega$ be a group with generating set $\Delta$. The conjugacy number $\text{con}(\Omega;\Delta)$ is defined as the minimum integer $k \geq 2$ for which there exists a nonabelian group $\Gamma$ of order $k|\Omega|$ and a subset $X$ of $\Gamma$ such that $\text{Cay}(\Omega;\Delta)$ is isomorphic to a component of $\text{Con}(\Gamma;X)$. Such a group $\Gamma$ is called a conjugacy group for $\Omega$ and $\Delta$.

For a better understanding of where this conjugacy number comes from we must provide the following information, which is discussed in [3]. Let $\Gamma$ be a group and let $X$ be a generating set such that $X^{-1} = X$ and let $\Omega$ be a subgroup of $\Gamma$. The left Schreier coset digraph $\overline{S}(\Gamma/\Omega;X)$ is the directed graph whose vertex set is the left cosets of $\Omega$ and there is a directed edge from $g\Omega$ to $h\Omega$ if $h\Omega = xg\Omega$ for $x \in X$. Proposition 1 from [3] states that if $G$ is a component of the directed conjugacy graph $\overline{\text{Con}}(\Gamma;X)$ and $g$ is a fixed vertex of $G$ then $G$ is isomorphic to $\overline{S}\left(\langle X \rangle / C_{\langle X \rangle}(g);X \right)$, where $C_{\langle X \rangle}(g)$ is the centralizer of $g$ in $\langle X \rangle$. The proof [3] of Proposition 1 defines $\Psi: V(G) \rightarrow V(\overline{S}\left(\langle X \rangle / C_{\langle X \rangle}(g);X \right))$ by $\Psi(xg^{-1}) = zC_{\langle X \rangle}(g)$. To show that $\Psi$ preserves directed edges, two adjacent vertices in $G$ are considered. Let us call these vertices $z_1g_{z_1}^{-1}$ and $z_2g_{z_2}^{-1}$. Since these vertices are adjacent in $G$ there exists $x \in X$ such that
\[ z_1g_2z_1^{-1} = x(z_2g_2z_2^{-1})^{-1} \] which can be written as 
\[ z_1g = xz_2g_2z_2^{-1}z_1 \text{ or } z_2^{-1}x^{-1}z_1g = g_2z_2^{-1}x^{-1}z_1. \]

Hence \( (z_2^{-1}x^{-1}z_1)g = g(z_2^{-1}x^{-1}z_1) \) and it follows that 
\[ z_2^{-1}x^{-1}z_1 \in C_{\langle x \rangle}(g). \]

Since 
\[ z_2^{-1}x^{-1}z_1 \in C_{\langle x \rangle}(g) \] it follows that 
\[ z_1C_{\langle x \rangle}(g) = xz_2C_{\langle x \rangle}(g). \]

Therefore 
\[ z_1C_{\langle x \rangle}(g) \] is adjacent to 
\[ z_2C_{\langle x \rangle}(g) \] in 
\[ \overrightarrow{S}\left(\langle x \rangle/C_{\langle x \rangle}(g); X\right). \]

Let \( \Omega \) be a finite group and \( \Delta \) be a generating set for \( \Omega \) with \( \Delta^{-1} = \Delta \). Now we consider if there exists a finite group \( \Gamma \) with a subset \( X \) such that \( \text{Cay}(\Omega; \Delta) \) is isomorphic to \( G \) where \( G \) is a component of \( \text{Con}(\Gamma; X) \). By Proposition 1 of [3] we know that 
\[ G \equiv \overrightarrow{S}\left(\langle x \rangle/C_{\langle x \rangle}(g); X\right). \]

For \( G \) to be isomorphic to \( \text{Cay}(\Omega; \Delta) \) then the order of \( G \) must equal the order of \( \Omega \). Since 
\[ G \equiv \overrightarrow{S}\left(\langle x \rangle/C_{\langle x \rangle}(g); X\right) \]
we know that 
\[ |G| = \left|\langle x \rangle/C_{\langle x \rangle}(g)\right|. \]
Therefore 
\[ |\Omega| = \left|\langle x \rangle\right|/\left|C_{\langle x \rangle}(g)\right|. \]

Consider \( |\Gamma| \) which can be written as 
\[ |\Gamma| = \left|C_{\langle x \rangle}(g)\right| \left|\langle x \rangle\right|/\left|C_{\langle x \rangle}(g)\right| \]
\[ \text{or } |\Gamma| = \left|C_{\langle x \rangle}(g)\right| \left|\langle x \rangle\right|/\left|C_{\langle x \rangle}(g)\right|. \]

Since 
\[ \left|C_{\langle x \rangle}(g)\right| \left|\langle x \rangle\right| \]
is just an integer we can call it \( k \). Hence 
\[ |\Gamma| = k |\Omega| \]
for some positive integer \( k \).

Furthermore we can say that \( k \geq 2 \) since the identity of \( \Gamma \) is always an independent vertex of \( \text{Con}(\Gamma; X) \). Now we consider an example.

**Example 2.1** Let \( \Omega = C_4 \) with \( \Delta = \{x, x^2, x^3\} \). We will verify that 
\( \text{con}(C_4; \Delta) = 3 \)
and a conjugacy group for \( C_4 \) and \( \Delta \) is \( A_4 \). First we show that \( \text{con}(C_4; \Delta) \neq 2 \). For 
\( \text{con}(C_4; \Delta) = 2 \) there must exist a nonabelian group \( \Gamma \) such that 
\[ |\Gamma| = 2 |\Omega| = 8 \]
where \( \Gamma \) has a conjugacy class of order four. There are two nonabelian groups of order eight,
namely the dihedral group $D_4$ and the quaternion group $Q$. It will be shown later that $D_4$ has conjugacy classes of order one and two. We consider $Q$ with presentation
\[ \langle s, t; s^4 = 1, s^2 = t^2, sts = t \rangle \]
and note that it is not difficult to show that its center is $\{1, s^2\}$ [8]. Through straightforward calculations it can be shown that the conjugacy classes of $Q$ are $\{1\}, \{s, s^3\}, \{s^2\}, \{t, s^2 t\}$ and $\{st, s^3 t\}$. Therefore there does not exist a $\Gamma$ of order eight with a conjugacy class of order four. Hence $\text{con}(C_4; \Delta) \neq 2$.

Next we show that $\text{con}(C_4; \Delta) = 3$. By the calculations of Example 1.2 and including multiplying $C_4$ by $x^2$ on the right we get the $\text{Cay}(C_4; \Delta)$ as shown in Figure 2.A.

![Figure 2.A The Cayley graph of $C_4$ where $\Delta=\{x,x^2,x^3\}$](image)

Let $\Gamma$ be $A_4$ with $X=\{(12)(34), (13)(24), (14)(23)\}$. We conjugate each element of $A_4$ by each element of $X$. We find that when conjugating by elements of $\langle X \rangle$ the elements of $A_4$ are partitioned into the following “conjugacy classes”: $\{(1)\}$.  

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\{(12)(34),(13)(24),(14)(23)\}, \{(123),(134),(142),(243)\} and \{(124),(132),(143),(234)\}.

The conjugacy class containing (123) gives the graph shown in Figure 2.B.

![Figure 2.B](image)

Figure 2.B The component of $\text{Con}(A_4 \times X)$ containing $\text{cl}((123))$.

Clearly $\text{Cay}(C_4; \Delta)$ is isomorphic to a component of $\text{Con}(A_4 \times X)$ therefore we can find $\text{con}(C_4;\Delta)$. Using $|\Gamma|=k|\Omega|$ where $|\Gamma|=12$ and $|\Omega|=4$, we get that $k=3$. Hence $\text{con}(C_4;\Delta)=3$ and a conjugacy group for $C_4$ and $\Delta$ is $A_4$. 

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CHAPTER 3

CONJUGACY NUMBERS FOR CYCLIC GROUPS OF SMALL EVEN ORDER

We will calculate $\text{con}(C_n; \Delta)$ for small even $n$. We must first discuss the following results and observations. The first is well known and covered in an elementary abstract algebra course. We include its proof for aid in understanding the concepts.

**Proposition 3.1** Let $a, b \in G$ where $G$ is a group. If $a \neq b$ then either $\text{cl}(a) = \text{cl}(b)$ or $\text{cl}(a) \cap \text{cl}(b) = \emptyset$.

**Proof.** Assume that $a \neq b$ and $\text{cl}(a) \cap \text{cl}(b) \neq \emptyset$. Then there exists $c \in \text{cl}(a)$ and $c \in \text{cl}(b)$. Consider $\text{cl}(a) = \{xax^{-1} : x \in G\}$. Since $c \in \text{cl}(a)$ there exists $g \in G$ such that $gag^{-1} = c$. Since $c \in \text{cl}(b)$ there exists $h \in G$ such that $hbh^{-1} = c$. Let $y \in \text{cl}(a)$. Then there exists $t \in G$ such that $tar^{-1} = y$. Since $a = g^{-1}hbg^{-1}g$ we get $y = tg^{-1}hbg^{-1}g^{-1} = (tg^{-1}h)b(tg^{-1}h)^{-1}$. Therefore $y \in \text{cl}(b)$ and $\text{cl}(a) \subseteq \text{cl}(b)$. Let $y \in \text{cl}(b)$. Then there exists $t \in G$ such that $tbt^{-1} = y$. Since $b = h^{-1}gag^{-1}h$ we get $y = th^{-1}gag^{-1}ht^{-1} = (th^{-1}g)a(th^{-1}g)^{-1}$. Therefore $y \in \text{cl}(a)$ and $\text{cl}(a) \subseteq \text{cl}(b)$. Hence $\text{cl}(a) = \text{cl}(b)$.

Next we present a useful result for the dihedral groups. For an integer $n \geq 3$, define the dihedral group of order $2n$ by the presentation $D_n = \langle x, y : x^n = y^2 = 1, xyx = y \rangle$. Thus the elements of $D_n$ are $1, x, x^2, \ldots, x^{n-1}, y, xy, x^2y, \ldots, x^{n-1}y$. 

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Proposition 3.2 The conjugacy classes of $D_n$ are $\{1\}, \{x^k, x^{-k}\}$ for $k=1,2,\ldots,n-1$ and
$\{y, xy, x^2y, \ldots, x^{n-1}y\}$ when $n$ is odd and the conjugacy classes of $D_n$ are $\{1\}, \{x^k, x^{-k}\}$ for $k=1,2,\ldots,n-1$, $\{y, x^2y, x^4y, \ldots, x^{n-2}y\}$ and $\{xy, x^3y, x^5y, \ldots, x^{n-1}y\}$ when $n$ is even.

Proof. Consider $(x'y')x^k(x'y')^{-1}$ for $0 \leq i \leq n-1, 0 \leq j \leq 1$ and $0 \leq k \leq n-1$.

Observe that
$$(x'y')x^k(x'y')^{-1}=(x'y')x^k(y^{-i}x^{-i})=egin{cases} x^k, j=0 \\ x^{-i}, j=1 \end{cases}.$$ 

Therefore the conjugacy class containing $x^k$ is $\{x^k, x^{-k}\}$ for all $0 \leq k \leq n-1$.

Now consider $(x'y')x^k(yx'y')^{-1}$ for $0 \leq i \leq n-1, 0 \leq j \leq 1$ and $0 \leq k \leq n-1$. Observe that
$$(x'y')x^k(yx'y')^{-1}=(x'y')x^k(yx^{-i})=egin{cases} x^k, j=0 \\ x^{-i}, j=1 \end{cases}.$$ 

For $k=0$ we get $cl(y)=\{x^i y\mid 0 \leq i \leq n-1\}$. For $k=1$ we get $cl(xy)=\{x^{2j+1}y\mid 0 \leq j \leq n-1\} \cup \{x^{2j-1}\mid 0 \leq j \leq n-1\} = \{x^{2j+1}y\mid -1 \leq j \leq n-1\}$.

Case 1. Let $n$ be even. We claim that $cl(y) \subseteq cl(xy) = \emptyset$. Assume, to the contrary, that $cl(y) \subseteq cl(xy) \neq \emptyset$. Let $z \in cl(y)$ and $z \in cl(xy)$. For $z \in cl(y)$ we get $z=x^{2i}y$ for some $i$ such that $0 \leq i \leq n-1$. For $z \in cl(xy)$ we get $z=x^{2j+1}y$ for some $j$ with $-1 \leq j \leq n-1$. Then $x^{2j}y=x^{2j+1}y$ or $x^{2i-2j-1}=1$ or $x^{2(i-j)-1}=1$. This implies that $n\mid [2(i-j)-1]$. This is clearly a contradiction since $n$ is even. Therefore $cl(y) \subseteq cl(xy) = \emptyset$. 

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Case 2. Let \( n \) be odd. We claim that \( cl(y) = cl(xy) \). We show that \( xy \in cl(y) \). Since \( \gcd(n, 2) = 1 \), there exists an \( i \) such that \( 2i = 1 \) mod \( n \). Thus \( xy = x^{2i}y \) and \( x^{2i}y \in cl(y) \). Since \( xy = x^{2i}y \in cl(y) \) then \( cl(y) = cl(xy) \).

Hence the conjugacy classes of \( D_n \) when \( n \) is even will be \( \{1\}, \{x^k, x^{-k}\} \) for \( k = 1, 2, \ldots, n-1 \) and \( \{y, x^2y, x^4y, \ldots, x^{n-2}y\} \) and when \( n \) is odd the conjugacy class of \( D_n \) will be \( \{1\}, \{x^k, x^{-k}\} \) for \( k = 1, 2, \ldots, n-1 \) and \( \{y, x^2y, x^3y, x^4y, \ldots, x^{n-2}y, x^{n-1}y\} \).

We can say that the conjugacy classes of \( D_n \) where \( n \) is even will have orders 1, 2 and \( n \), and the conjugacy classes of \( D_n \) where \( n \) is odd will have orders 1, 2 and \( n \).

We now present a useful result for the cross product of an abelian group with a nonabelian group.

**Proposition 3.3** Let \( G \) be a nonabelian group and let \( H \) be an abelian group. For \((g, h) \in G \times H\) we get \( |cl(g, h)| = |cl(g)| \).

**Proof.** We show that \( cl(g, h) = cl(g) \times \{h\} \). Since \( H \) is abelian observe that \((x, y)(g, h)(x^{-1}, y^{-1}) = (xyx^{-1}, h)\) for all \( y \in H \). First let \((x, y) \in cl(g, h)\). Then there exists \((a, b) \in G \times H\) such that \((a, b)(g, h)(a, b)^{-1} = (x, y)\) but \((a, b)(g, h)(a, b)^{-1} = (a, b)(g, h)(a^{-1}, b^{-1}) = (aga^{-1}, bhb^{-1}) = (aga^{-1}, h)\). Since \( aga^{-1} \in cl(g) \) we see that \((x, y) \in cl(g) \times \{h\}\). Now let \((x, y) \in cl(g) \times \{h\}\). Then \( x \in cl(g) \) and \( y = h \).
and there exists an \( a \in G \) such that \( aga^{-1} = x \). Now \((x, y) = (aga^{-1}, h) = (a, 1)(g, h)(a, 1)^{-1} \) and so \((x, y) \in cl(g, h)\). Then \( cl(g, h) = cl(g) \times \{h\} \). Therefore \(|cl(g, h)| = |cl(g)|\).

Next we provide a useful result that was stated and proved in [3]. We include the proof here for better understanding. We remind the reader that the conjugacy number \( con(\Omega; \Delta) \) is defined as the minimum integer \( k \geq 2 \) for which there exists a nonabelian group \( \Gamma \) of order \( k|\Omega| \) and a subset \( X \) of \( \Gamma \) such that Cay(\( \Omega; \Delta \)) is isomorphic to a component of Con(\( \Gamma; X \)).

**Proposition 3.4** Let \( n \geq 6 \) be an even integer. Then \( con(C_n; \Delta) \leq 4 \) for every generating set \( \Delta \) of \( C_n \).

**Proof.** We must show that there exists a nonabelian group \( \Gamma \) of order \( 4n \) and a subset \( X \) of \( \Gamma \) for which Cay(\( \Omega; \Delta \)) is isomorphic to a component of Con(\( \Gamma; X \)). Let \( \Gamma \) be the group \( D_{2n} \) and let \( X = \{x^k | k \in \Delta \} \). Let \( G \) be the component of Con(\( D_{2n}; X \)) containing \( y \). We know from Proposition 3.2 that the vertices of \( G \) are \( \{x^{2i}y | 0 \leq i \leq n - 1\} \). We must show that \( G \) is isomorphic to Cay(\( C_n; \Delta \)). Define a function \( \Psi : V(Cay(C_n; \Delta)) \rightarrow V(G) \) by \( \Psi(i) = x^{2i}y \). Suppose that \( i \) is adjacent to \( j \) in Cay(\( C_n; \Delta \)). Therefore there exists \( k \in \Delta \) such that \( j \equiv (i + k) \mod n \). Since \( j \equiv (i + k) \mod n \) we know \( 2j \equiv 2(i + k) \mod 2n \) and \( x^{2j}y = x^{2(i+k)}y \) in \( D_{2n} \). Consider \( x^{2(i+k)}y = x^{4i+2k}y = x^kx^{2i}y = x^k(x^{2i}y)x^{-k} \). Hence \( x^{2j}y = x^k(x^{2i}y)x^{-k} \) and it follows that \( x^{2j}y \) is adjacent to \( x^{2j}y \) in \( G \) and Cay(\( C_n; \Delta \)) \( \cong G \). Note that \(|\Gamma| = 4n = 4|\Omega|\). Since the conjugacy number is defined as the
minimum integer $k$ for which such a group $\Gamma$ of order $k|\Omega|$ exists, we can conclude that
\[ \text{con}(C_n; \Delta) \leq 4 \] for an even integer $n \geq 6$.

We now calculate $\text{con}(C_n; \Delta)$ for small even $n$. By Proposition 3.4 we know that $\text{con}(C_n; \Delta) \leq 4$ for even $n \geq 6$. In the following results we consider $\text{con}(C_n; \Delta)$ for $n = 6, 8$ and $10$. We begin with $n = 6$.

**Proposition 3.5** The conjugacy number $\text{con}(C_6; \Delta) = 3$ if and only if $\Delta = \{2, 3, 4\}$ or $\Delta = \{1, 2, 4, 5\}$ or $\Delta = \{1, 2, 3, 4, 5\}$.

Proof. Figure 3.A shows three possible Cayley graphs for $C_6$. If $\text{con}(C_6; \Delta) = 2$, then there is a nonabelian group $\Gamma$ with $|\Gamma| = 12$ and a subset $X$ of $\Gamma$ such that a component of the conjugacy graph $\text{Con}(\Gamma; X)$ is isomorphic to the Cayley graph $\text{Cay}(C_6; \Delta)$. Note that $\Gamma$ must contain a conjugacy class containing at least 6 elements. There are three nonabelian groups of order 12, namely $A_4$, $D_6$ and $T$, where $T$ is the semidirect product of $C_3$ by $C_4$ [5]. For $T$ we use the presentation $\langle s, t; s^6 = 1, s^3 = t^2, sts = t \rangle$ [10]. Through straightforward calculations, we find that $A_4$ has conjugacy classes of orders 1, 3 and 4. By Proposition 3.2 we know that $D_6$ has conjugacy classes of orders 1, 2 and 3. We can also show that $T$ has conjugacy classes of orders 1, 2 and 3. Therefore we conclude that $(C_6; \Delta) \neq 2$. We now check nonabelian groups of order 18 to verify $\text{con}(C_6; \Delta) = 3$ for a given $\Delta$. 
Cay\((C_6; \Delta)\) with \(\Delta = \{1, 5\}\)

Cay\((C_6; \Delta)\) with \(\Delta = \{1, 3, 5\}\)

Cay\((C_6; \Delta)\) with \(\Delta = \{2, 3, 4\}\)

Figure 3.A Three possible Cayley graphs for \(C_6\)

The nonabelian groups of order 18 are \(D_6\), \(S_3 \times C_3\) and the semidirect product of \(C_3 \times C_3\) and \(C_2\) [5]. We check these groups for conjugacy classes of order at least 6. By

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Proposition 3.2 we know that \( D_9 \) has conjugacy classes of orders 1, 2 and 9 and the conjugacy class of order 9 contains the elements \( \{ y, xy, x^2y, \ldots, x^8y \} \). It can be shown through tedious calculation that when the elements of \( \text{cl}(y) \) are conjugated by any \( x' \) it produces a 9-cycle which clearly cannot give a Cayley graph on six vertices. In addition it can be shown that when the elements of \( \text{cl}(y) \) are conjugated by any \( x'y \) the result is one independent vertex and four independent edges. Finally it can be shown that if the elements of \( \text{cl}(y) \) are conjugated by any two \( x'y \) the result is a path on nine vertices and it is again impossible to obtain a Cayley graph on six vertices. Therefore \( D_9 \) is not a conjugacy group for \( C_6 \). Next we look at \( S_3 \times C_3 \). Since \( C_3 \) is abelian we know by Proposition 3.3 that the orders of the conjugacy classes of \( S_3 \times C_3 \) will be the orders of the conjugacy classes found in \( S_3 \). Since \( S_3 \) has order 6 and the identity is always in its own conjugacy class then \( S_3 \) cannot have a conjugacy class of order 6. Therefore, by Proposition 3.3, \( S_3 \times C_3 \) cannot have a conjugacy class of order 6.

Finally we consider the semidirect product of \( C_3 \times C_3 \) and \( C_2 \). We use \( G = \langle x, y, z : x^2 = y^3 = z^3 = 1, yz = zy, yxy = x, zxz = x \rangle \) as our presentation for the semidirect product of \( C_3 \times C_3 \) and \( C_2 \) [8]. Then \( G \) has the elements \( 1, x, y, y^2, z, z^2, xy, xy^2, xz, xz^2, yz, yz^2, y^2z, y^2z^2, xyz, xy^2z, xy^2z^2, xy^2z \). It can be shown that the conjugacy classes of \( G \) are \( \{ 1 \}, \{ y, y^2 \}, \{ z, z^2 \}, \{ yz, y^2z \}, \{ yz^2, y^2z \} \) and \( \{ x, xy, xy^2, xz, xz^2, xyz, xyz^2, xy^2z, xy^2z^2 \} \). Consider the \( \text{cl}(x) \) which has order 9. Let
\[ X = \{xz, y, y^2\}. \] If cl(x) is conjugated by \( X \) we obtain the following graph, shown in Figure 3.B

![Figure 3.B Components of Con(G; X) containing cl(x)](image)

Clearly the component of Con(G; X) containing cl(x), shown in Figure 3.B, has a component which is isomorphic to the Cayley graph \( \text{Cay}(C_6; \Delta) \), where \( \Delta = \{2, 3, 4\} \), shown in Figure 3.A. For \( \Delta = \{1, 2, 4, 5\} \) and for \( \Delta = \{1, 2, 3, 4, 5\} \) let \( X = \{xz, y, y^2, xyz\} \) and let \( X = \{xz, y, y^2, xyz, xy^2z\} \), respectively, to obtain \( \text{con}(C_6;\Delta) = 3 \). Therefore the semidirect product of \( C_3 \times C_3 \) and \( C_2 \) is the unique conjugacy group of \( C_6 \) and we can conclude that \( \text{con}(C_6;\Delta) = 3 \) for \( \Delta = \{2, 3, 4\} \), \( \Delta = \{1, 2, 4, 5\} \) and \( \Delta = \{1, 2, 3, 4, 5\} \). There are two remaining generating sets, namely \( \{1, 5\} \) and \( \{1, 3, 5\} \). It is not difficult to show that there is no subset \( X \) of \( G \) for which a component of Con(G; X) is isomorphic to \( \text{Cay}(C_6; \Delta) \), where \( \Delta = \{1, 5\} \) or \( \Delta = \{1, 3, 5\} \). \( \Box \)
Proposition 3.6  The conjugacy number $\text{con}(C_8; \Delta)=3$ if and only if $\Delta=\{1,2,6,7\}$ or $\Delta=\{1,3,5,7\}$ or $\Delta=\{2,3,5,6\}$ or $\Delta=\{1,2,4,6,7\}$ or $\Delta=\{1,3,4,5,7\}$ or $\Delta=\{1,2,3,5,6,7\}$ or $\Delta=\{1,2,3,4,5,6,7\}$.

Proof. Figure 3.C shows three possible Cayley graphs for $C_8$.

$\text{Cay}(C_8; \Delta)$ with $\Delta=\{1,7\}$ or $\Delta=\{3,5\}$

$\text{Cay}(C_8; \Delta)$ with $\Delta=\{1,4,7\}$ or $\Delta=\{3,4,5\}$

$\text{Cay}(C_8; \Delta)$ with $\Delta=\{1,3,5,7\}$

Figure 3.C  Three possible Cayley graphs for $C_8$
If \( \text{con}(C_8; \Delta) = 2 \), then there exists a nonabelian group \( \Gamma \) of order 16 which contains a conjugacy class of order at least 8 and a subset \( X \) of \( \Gamma \) such that a component of the conjugacy graph \( \text{Con}(\Gamma; X) \) is isomorphic to the Cayley graph \( \text{Cay}(C_8; \Delta) \). There are nine nonabelian groups of order 16 [5]. We begin with \( D_8 \). We know that \( D_8 \) has conjugacy classes of order 1, 2 and 4. Therefore the conjugacy graph of \( D_8 \) does not contain a component which is isomorphic to a Cayley graph of \( C_8 \).

Next we consider \( D_4 \times C_2 \) and \( Q \times C_2 \). Notice that \( C_2 \) is an abelian group, therefore \( D_4 \times C_2 \) and \( Q \times C_2 \) will have conjugacy classes with orders as found in \( D_4 \) and \( Q \), respectively. We know that \( D_4 \) has conjugacy classes of orders 1 and 2. Therefore \( D_4 \times C_2 \) does not contain a conjugacy class of order 8. The quaternion group, \( Q \) has order 8 and therefore it cannot contain a conjugacy class of order 8.

We now look at the quasihedral group of order 16 with presentation \( \left\langle s, t ; s^8 = t^2 = 1, st = ts^3 \right\rangle \) [8]. It can be shown through straightforward calculations that the conjugacy classes of this group are \( \{1\}, \{s, s^3\}, \{s^2, s^6\}, \{s^4\}, \{s^5\}, \{t, s^2t, s^4t, s^6t\} \) and \( \{st, s^3t, s^4t, s^7t\} \). Clearly there are no conjugacy classes of order 8.

Now we look at the modular group of order 16 with presentation \( \left\langle s, t ; s^8 = t^2 = 1, st = ts^3 \right\rangle \) [8]. It can be shown that the conjugacy classes of this group are \( \{1\}, \{s, s^3\}, \{s^2, s^7\}, \{s^4\}, \{s^6\}, \{t, s^4t\}, \{s^2t, s^3t\} \) and \( \{s^5, s^7t\} \). Therefore there is no conjugacy class of order 8.
Next we look at three of the remaining four nonabelian groups of order 16 with the presentations \( \langle s, t; s^4 = t^4 = 1, st = ts^3 \rangle \) and \( \langle a, b, c; a^4 = b^2 = c^2 = 1, cbca^2b = 1, bab = ab, cac = a \rangle [8] \). It can be shown that these groups only have conjugacy classes of orders 1 and 2.

Finally we look at the group with presentation \( \langle s, t; s^8 = t^2, sts = t \rangle [8] \). The conjugacy classes of this group are \( \{1\}, \{t, s^2, t, s^4, t, s^6, t\}, \{s, s^7\}, \{s^2, s^6\}, \{s^3, s^5\} \) and \( \{s^4\} \). Hence there are no conjugacy classes of order 8. We conclude that \( \text{con}(C_8; \Delta) \neq 2 \).

Next we show that \( \text{con}(C_8; \Delta) = 3 \) for a given \( \Delta \). To verify that \( \text{con}(C_8; \Delta) = 3 \), we must show that there exists a nonabelian group of order 24 with a conjugacy class of order at least 8 with a component of the conjugacy graph \( \text{Con}(\Gamma; X) \) isomorphic to the Cayley graph \( \text{Cay}(C_8; \Delta) \). There are 12 nonabelian groups of order 24, we show that \( S_4 \) works. The conjugacy classes of \( S_4 \) are \( \{(1)\}, \{(12), (13), (14), (23), (24), (34)\}, \{(12)(34), (13)(24), (14)(23)\}, \{123, (124), (132), (134), (142), (143), (234), (243)\} \) and \( \{(1234), (1243), (1324), (1342), (1423), (1432)\} \). We have a conjugacy class of order 8. We must verify that there does exist an \( X \) which will give a component of \( \text{Con}(S_4; X) \) that is isomorphic to \( \text{Cay}(C_8; \Delta) \). Let \( X = \{(12)(34), (14)(23), (1234), (1432)\} \). The component of the conjugacy graph, \( \text{Con}(S_4; X) \), containing the element (123) is shown in Figure 3.D.
Clearly $\text{Con}(S_4; X)$ has a component which is isomorphic to the Cayley graph of $C_8$ where $\Delta = \{1, 3, 5, 7\}$ which is shown in Figure 3.C. Therefore $\text{con}(C_8; \Delta) = 3$ for $\Delta = \{1, 3, 5, 7\}$. For $\Delta = \{1, 2, 6, 7\}$ and $\Delta = \{2, 3, 5, 6\}$ let $X = \{(24), (12)(34), (1234), (1432)\}$. For $\Delta = \{1, 2, 4, 6, 7\}$ and $\Delta = \{2, 3, 4, 5, 6\}$ let $X = \{(24), (12)(34), (13)(24), (1234), (1432)\}$. For $\Delta = \{1, 3, 4, 5, 7\}$ let $X = \{(24), (13)(24), (1234), (1432)\}$. For $\Delta = \{1, 2, 3, 4, 5, 6, 7\}$ let $X = \{(13), (24), (12)(34), (14)(23), (1234), (1432)\}$. Therefore we conclude that $\text{con}(C_8; \Delta) = 3$ for the given $\Delta$ and a conjugacy group for $C_8$ and for the given $\Delta$ is $S_4$. There are four remaining generating sets, namely $\Delta = \{1, 7\}$, $\Delta = \{3, 5\}$, $\Delta = \{1, 4, 7\}$ and $\Delta = \{3, 4, 5\}$. For these generating sets it is not difficult to show that there does not exist a subset $X$ of $S_4$ for which a component of $\text{Con}(S_4; X)$ is isomorphic to $\text{Cay}(C_8; \Delta)$. For thoroughness we check the remaining 11 nonabelian groups of order 24 and show that none of these satisfy the required conditions [5].

![Figure 3.D](image)

Figure 3.D The component of $\text{Con}(S_4; X)$ containing $\text{cl}((123))$
For the groups $S_3 \times C_4$ and $S_3 \times C_2 \times C_2$, we observe that since $C_4$ and $C_2 \times C_2$ are abelian we know by Proposition 3.3 that both groups will have the conjugacy classes as found in $S_3$. Since $S_3$ has order 6 a conjugacy class of order 8 is impossible.

Now we look at $A_4 \times C_2$. Since $C_2$ is abelian we consider the conjugacy classes of $A_4$. We know $A_4$ has conjugacy classes of orders 1, 3 and 4. Therefore $A_4 \times C_2$ cannot have a conjugacy class of order 8. Similarly we know that $T \times C_2$ will have conjugacy classes of orders 1, 2 and 3.

Next we consider $D_4 \times C_3$ and $Q \times C_3$. Since $C_3$ is abelian we look at $D_4$ and $Q$ which both have order 8 and therefore cannot contain a conjugacy class of order 8. The remaining five groups are found by taking the direct product of any group of order 8 and a group of order 3 [3]. Since there are no nonabelian groups of order 3 and any group of order 8 will not have a conjugacy class of order 8 we conclude that these groups will not give a conjugacy number of three. Therefore we conclude that $\text{con}(C_8; \Delta)=3$ with the given $\Delta$ and the unique $\Gamma$ which gives $\text{con}(C_8; \Delta)=3$ is the permutation group $S_4$. W

**Proposition 3.7** The conjugacy number $\text{con}(C_{10}; \Delta)=4$ for every generating set $\Delta$.

Proof. To see that $\text{con}(C_{10}; \Delta) \neq 2$, we note that if $\text{con}(C_{10}; \Delta)=2$ then there exists a nonabelian group of order 20 with a conjugacy class of order 10 those graph is isomorphic to the Cayley graph of $C_{10}$. There are three nonabelian groups to check [5]. We know that $D_{10}$ has conjugacy classes of orders 1, 2 and 5. It can be shown that the semidirect product of $C_5$ and $C_4$ has conjugacy classes of orders 1, 2 and 5. Finally we
consider the Frobenius of order 20 with presentation \( \langle s, t : s^4 = t^5 = 1, ts = st^2 \rangle \) [8]. It can also be shown that the Frobenius group of order 20 has conjugacy classes of orders 1, 4 and 5. Therefore \( \text{con}(C_{10} ; \Delta) \neq 2 \).

Now we show that \( \text{con}(C_{10} ; \Delta) \neq 3 \). If \( \text{con}(C_{10} ; \Delta) = 3 \), then there exists a nonabelian group of order 30 with a conjugacy class of order 10. There are three nonabelian groups of order 30 [5]. We know that \( D_{15} \) has conjugacy classes of orders 1, 2 and 15. Similar to the conjugacy class of order 9 of \( D_5 \), we know that the conjugacy class of order 15 of \( D_{15} \) contains the elements \( \{y, xy, x^2y, \ldots, x^{14}y\} \). It can be shown through tedious calculation that when the elements of \( \text{cl}(y) \) are conjugated by any \( x^i \) it produces a 15-cycle which clearly cannot give a Cayley graph on ten vertices. In addition it can be shown that when the elements of \( \text{cl}(y) \) are conjugated by any \( x'y \) the result is one independent vertex and seven independent edges. Finally it can be shown that if the elements of \( \text{cl}(y) \) are conjugated by any two \( x'y \) the result is a path on 15 vertices and is again impossible to obtain a Cayley graph on ten vertices. Therefore \( D_{15} \) is not a conjugacy group for \( C_{10} \). Then we check \( D_5 \times C_3 \) and \( D_3 \times C_5 \). The group \( D_5 \times C_3 \) will have conjugacy classes as found in \( D_5 \) and \( D_3 \times C_5 \) will have conjugacy classes as found in \( D_3 \), neither of which contains a conjugacy class of order 10. Therefore \( \text{con}(C_{10} ; \Delta) \neq 3 \), and by Proposition 3.4 we can conclude that \( \text{con}(C_{10} ; \Delta) = 4 \). W
CHAPTER 4

GROUPS THAT CANNOT BE CONJUGACY GROUPS FOR CYCLIC GROUPS OF EVEN ORDER

Now we eliminate certain groups of order $4n$ as possible conjugacy groups for $C_{2n}$. The following results will show that the dihedral group $D_{2n}$, any group of order $4n$ which is a cross product of an abelian group with a nonabelian group, the symmetric group of order $4n$ as well as the alternating groups of order $4n$ are not conjugacy groups for $C_{2n}$. We begin with the dihedral groups, $D_{2n}$.

**Proposition 4.1** Let $\Gamma$ be the dihedral group, $D_{2n}$, $n \geq 3$ with generating set $X$ and let $\Omega$ be the group $C_{2n}$ with generating set $\Delta$. Then $\text{Con}(D_{2n}; X)$ does not contain a component which is isomorphic to Cay($C_{2n}; \Delta$).

Proof. Assume, to the contrary, that $\text{Con}(D_{2n}; X)$ does contain a component which is isomorphic to Cay($C_{2n}; \Delta$). Then $\text{con}(C_{2n}; \Delta)=2$ and $D_{2n}$ contains a conjugacy class of order $2n$. This is a contradiction since $D_{2n}$ has conjugacy classes of order $1, 2$ and $n$. Hence $\text{Con}(D_{2n}; X)$ does not contain a component which is isomorphic to Cay($C_{2n}; \Delta$). W
Furthermore we can conclude that $D_{2n}$ is not a conjugacy group for $C_{2n}$ which will give a conjugacy number of two. Now we consider groups that are the cross product of an abelian and nonabelian group. We present the following result.

**Proposition 4.2** Let $n \geq 3$ be an integer. Let $\Gamma$ be a group of order $4n$ such that $\Gamma$ is isomorphic to $G \times H$, where $G$ is a nonabelian group and $H$ is a nontrivial abelian group. Then $\Gamma$ is not a conjugacy group for $C_{2n}$.

Proof. Assume, to the contrary, that $\Gamma$ is a conjugacy group for $C_{2n}$. Then the conjugacy number of $C_{2n}$ is two and $\Gamma$ contains a conjugacy class of order at least $2n$.

By Proposition 3.3 we know that $\Gamma$ will have the conjugacy classes of orders as those found in $G$. Since $|G| = \frac{4n}{|H|}$, the largest $|G|$ can be occurs when $|H| = 2$, and here we get $|G| = \frac{4n}{2} = 2n$. This is a contradiction since if $|G| = 2n$, it is impossible for $G$ and hence for $\Gamma$, to contain a conjugacy class of order at least $2n$. Thus $\Gamma$ is not a conjugacy group for $C_{2n}$.

Next we will discuss the symmetric groups of order $4n$ but we must first provide the following result from [11]. Let $\pi$ be a permutation of $S_m$. A cycle containing $k$ elements is called a $k$-cycle. The cycle type of $\pi$ is an expression of the form $(1^{n_1}, 2^{n_2}, \ldots, m^{n_m})$ where $n_k$ is the number of cycles of length $k$ in $\pi$. We know that for
any group $G$, where $g \in G$ and $Z_g = \{ h \in G : hgh^{-1} = g \}$, that $|cl(g)| = \frac{|G|}{|Z_g|}$. Now let $G = S_n$. Proposition 1.1.1 of [6] says that if $m = n_1 + 2n_2 + \ldots + mn_m$ and $\pi \in S_n$, then $|Z_\pi|$ depends only on $n_1, n_2, \ldots, n_m$ and $|Z_\pi| = 1^{n_1} 2^{n_2} \ldots m^{n_m} n_m!$. The proof is straightforward. It states that any $h \in Z_\pi$ can either permute the cycles of length $i$ among themselves in $n_i!$ ways or perform a cyclic rotation on each of the individual cycles in $i^n$ ways. Now the equation $|cl(\pi)| = \frac{|G|}{|Z_\pi|}$, can be specialized for the symmetric group $S_n$ to $|cl(\pi)| = \frac{m!}{1^{n_1} 2^{n_2} \ldots m^{n_m} n_m!}$ [7], [9]. The following proposition shows that a conjugacy number of two cannot be obtained by considering the symmetric group of order $4n$ as a possible conjugacy group for $C_{2n}$.

**Proposition 4.3** The symmetric group, $S_m$ $m>3$, of order $4n$ is not a conjugacy group for $C_{2n}$.

**Proof.** Assume, to the contrary, that $S_m$ is a conjugacy group for $C_{2n}$. Then $S_m$ contains a conjugacy class of order at least $2n$. We know that $|S_m| = 4n$ and $|C_{2n}| = 2n$. Since $m! = 4n$ we get that $\frac{1}{2} m! = 2n$ or $|C_{2n}| = \frac{1}{2} m!$. It can be shown that the $(m-1)$-cycles form the conjugacy class of largest order. We know that $|cl(\pi)| = \frac{m!}{1^{n_1} 2^{n_2} \ldots m^{n_m} n_m!}$. For the $(m-1)$-cycles, we get
\[ |cl(x)| = \frac{m!}{1!1!2!0!...(m-2)!0!(m-1)!1!m^00!} = \frac{m!}{m-1} = (m-2)!m. \] Therefore the order of the conjugacy class containing the \((m-1)\)-cycles is \((m-2)!m\). Now, we know \(m > 3\) so \(m-1 > 2\)

\[ \frac{1}{2}(m-1) > 1 \]

\[ \frac{1}{2}(m-1)(m-2)!m > (m-2)!m \]

\[ \frac{1}{2}m! > (m-2)!m \]

This is a contradiction. The largest conjugacy class is too small to give the Cayley graph for \(C_{2n}\). Since \(\frac{1}{2}m! > (m-2)!m\), \(S_m\) cannot contain a conjugacy class of order at least \(\frac{1}{2}m!\).

Finally we consider the alternating group \(A_m\), of order \(4n\). It will be shown that a conjugacy number of two cannot be obtained by considering the alternating group of order \(4n\) as a possible conjugacy group for \(C_{2n}\). For \(m \leq 3\), \(A_m\) cannot have order \(4n\) and we can easily verify that \(A_4\) and \(A_5\) do not work. We consider \(m > 5\).

**Proposition 4.4** Let \(m > 5\) be an integer. The alternating group \(A_m\) of order \(4n\) is not a conjugacy group for \(C_{2n}\).
Proof. We know that $|A_m| = 4n = \frac{1}{2} m!$ and $|C_{2n}| = 2n = \frac{1}{4} m!$. For $A_m$ to be a conjugacy group for $C_{2n}$, $A_m$ must contain a conjugacy class of order at least $2n$. We know that the orders of the conjugacy classes of $A_m$ will be less than or equal to the orders of the conjugacy classes as found in $S_m$. Since the conjugacy class of largest order in $S_m$ is the conjugacy class containing the $(m-1)$-cycles, we must show that the order of the $(m-1)$-cycles is less than the order of $C_{2n}$. We know that the order of the conjugacy class containing the $(m-1)$-cycles is $(m-2)!m$ in $S_m$. We know $m > 5$, so $m-1 > 4$

$$(m-2)!(m-1)m > (m-2)!4m$$

$$\frac{1}{4} m! > (m-2)!m.$$  

Hence it is impossible for $A_m$ to contain a conjugacy class of order greater than or equal to the order of $C_{2n}$ and $A_m$ with order $4n$ is not a conjugacy group for $C_{2n}$. W
CHAPTER 5

CONCLUDING REMARKS AND OPEN QUESTIONS

The initial research on conjugacy graphs and conjugacy numbers was done in [2]. The primary motivation of [2] was the problem of determining which Cayley graphs can be realized as conjugacy graphs. Our research here began by studying cyclic groups of small even order to find any emerging patterns with what we defined as possible conjugacy groups. As we tested all appropriate possible conjugacy groups for these cyclic groups we noticed that we never found a group which gave a conjugacy number of two. This led us to believe that Proposition 3.4 which was initially proved in [2] could eventually be modified to eliminate two as a possible conjugacy number, leaving three and four as the only possible conjugacy number for cyclic groups of even order. This is left as an open question since there are still classifications of groups to be eliminated. Here we have eliminated the dihedral group $D_{2n}$, any group which is a cross product of an abelian group with a nonabelian group, the symmetric group as well as the alternating groups, all of order $4n$, as possible conjugacy groups for $C_{2n}$.

Continued research could discover the pattern for which conjugacy groups gives a conjugacy number of three versus a conjugacy number of four for cyclic groups of even order. Here we only considered conjugacy graphs and conjugacy numbers for cyclic groups of even order. There is still plenty of information awaiting discovery.
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