A problem concerning the existence of graphs with certain degrees

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A PROBLEM CONCERNING THE EXISTENCE OF GRAPHS WITH CERTAIN DEGREES

by

Jonathan Seary Summer

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ABSTRACT

A Problem Concerning the Existence of Graphs with Certain Degrees

by

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Let $n \geq 2$ be a positive integer. For a sequence $(S_0, S_1, S_2, \ldots, S_{n-1})$ of nonnegative integers we study the problem of whether there exists a graph containing $\sum_{i=0}^{n-1} S_i$ vertices having exactly $S_i$ vertices whose degrees are congruent to $i$ modulo $n$ for each $i = 0, 1, 2, \ldots, n-1$. When such a graph does exist, the sequence $(S_0, S_1, S_2, \ldots, S_{n-1})$ is said to be realizable. It is known for modulo 2, 3, and 4 that such a sequence is realizable with seventeen exceptions for modulo 3 and twenty-four exceptions for modulo 4. These results are known, but concise proofs of these facts have not appeared until this thesis.

Further it had been conjectured that for each $n \geq 5$ such sequences are realizable with finitely many exceptions. We show that there are finitely many exceptions for modulo 6 and further investigate how our proof technique might be generalized to the modulo $2n$ case.
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CHAPTER 1

INTRODUCTION AND PRELIMINARIES

We begin with several basic definitions for Graph Theory. For additional information about this subject or general concepts presented here, the reader may refer to the standard text by Wilson[3]. A graph $G$ consists of a nonempty finite set $V(G)$ of elements called vertices and a and a set $E(G)$ of unordered pairs of elements of $V(G)$ called edges. For example, Figure 1 shows the graph $G$ with $V(G) = \{a, b, c, d, e, f, g, h, i, j\}$ and $E(G) = \{ab, bc, cd, be, ef, fg, bg, hi, fi, fj\}$. Note for simplicity we use the notation $ab$ for the edge specified by the unordered pair $\{a, b\}$.

![Figure 1.1](image-url)
The degree of a vertex \( v \) of \( G \) is the number of edges incident with \( v \) and is denoted
by \( \text{deg} \ v \). For example in the graph \( G \) of Figure 1, \( \text{deg} \ a = 1 \) and \( \text{deg} \ b = 4 \). One result
from graph theory says that the sum of the degrees of the vertices is equal to two times
the number of edges. A direct result from the previous sentence is the fact that every
graph \( G \) has an even number of vertices of odd degree. A graph is said to be simple if
the graph contains no multiple edges or loops. In this thesis we assume that all graphs are
simple. A complete graph \( K_n \) is a graph having \( n \) vertices, where each pair of distinct
vertices is adjacent. A very common example of a complete is the graph \( K_5 \) pictured in
Figure 1.2 below.

![Figure 1.2](image)

A connected graph in which every vertex has degree 2 is a cycle graph, denoted by \( C_n \).
An example of a cycle graph is the graph \( C_4 \) pictured in the figure below.
The graph obtained from \( C_n \) by removing an edge is the path graph on \( n \) vertices, denoted by \( P_n \). An example of a path is the graph \( P_4 \) pictured in Figure 1.4.

![Figure 1.4](image)

A complete bipartite graph \( K_{m,n} \) is a graph in which \( V(K_{m,n}) = A \cup B \), where \(|A| = m\), \(|B| = n\) and \( E(K_{m,n}) = \{uv \mid u \in A\text{ and } v \in B\} \). \( K_{1,3} \) is an example of a complete bipartite graph pictured in Figure 1.5 below.

![Figure 1.5](image)
The degree sequence of a graph consists of the degrees written in increasing order with repeats where necessary. The degree sequence for the graph $G$ in figure 1 is $(1, 1, 1, 1, 2, 2, 2, 2, 4, 4)$. The degree sequence of $K_{1,3}$ is $(1, 1, 1, 3)$. The frequency sequence modulo $n$ of $G$ is that sequence $(a_0, a_1, \ldots, a_{n-1})$ such that $G$ contains $a_i$ vertices $v$ such that $\deg v \equiv i (\mod n)$. For the graph $G$ of Figure 1, the frequency sequence modulo 2 is $(6, 4)$. The frequency sequence modulo 3 of Figure 1 is $(0, 6, 4)$. Finally the frequency sequence modulo 4 of Figure 1 is $(2, 4, 4, 0)$. Furthermore we define a sequence $(a_0, a_1, \ldots, a_{n-1})$ to be realizable if it is the frequency sequence modulo $n$ of some graph.

In the 1990 spring edition of *The Pi Mu Epsilon Journal*, Dykstra and Schultz[1] investigated the problem of determining which sequences are realizable. They provided three specific theorems dealing with the specific cases of modulo 2, 3, and 4, as well as some general results. The proofs for the special cases of modulo 2, 3, and 4 did not appear in the journal, but the theorems did specifically mention that there were seventeen exceptions (sequences that are not realizable) in modulo 3 and twenty-four exceptions in
modulo 4. The paper went on to conjecture that a graph could be found in modulo $n$ with a finite number of exceptions.

In this thesis, we provide concise proofs for modulo 3 and modulo 4 along with their exceptions. Furthermore, we will provide an alternative proof for modulo 4 showing only that there are finitely many exceptions, and then we extend this to modulo 6.

In the final chapter we indicate how our proofs in chapters 3 and 4 might be extended to solve the conjecture in general.
CHAPTER 2

THE PROOFS AND THE EXCEPTIONS FOR MODULO 3 AND MODULO 4

In the Spring 1990 issue of *The Pi Mu Epsilon Journal*, Dykstra and Schultz posed the following question: given any three nonnegative integers $S_0, S_1, S_2$ such that

$$\sum_{i=0}^{2} S_i > 0,$$

does there exist a graph having $S_0 + S_1 + S_2$ vertices, where exactly $S_i$ vertices have degrees that are congruent to $i$ modulo 3? They state the answer is yes with seventeen exceptions; however, as their main result is in a different direction and the proof involves a tedious analysis of cases, they do not provide a proof of this result nor do they list the seventeen exceptions. In Theorem 1 below, we provide the proof of this fact, along with the seventeen exceptions.

**THEOREM 1** For all triples $(S_0, S_1, S_2)$ of nonnegative integers where $\sum_{i=0}^{2} S_i > 0$, there exists a graph containing exactly $S_i$ vertices whose degrees are congruent to $i$ modulo 3 with seventeen exceptions.

**PROOF** The proof will be done by cases.

**CASE 1** Assume that $S_2 = 0$.

**SUBCASE 1.1** Assume that $S_1 = 2m$, where $m \geq 0$. Then $S_0K_1 \cup mK_2$ is a graph with the desired properties.
SUBCASE 1.2 Assume that $S_i = 1$. There are four sequences that are not realizable. We consider them first.

SUBCASE 1.2.1 Assume that $S_0 \leq 3$. It is not difficult to verify that the sequences $(0, 1, 0), (1, 1, 0), (2, 1, 0), \text{ and } (3, 1, 0)$ are not realizable.

SUBCASE 1.2.2 Assume that $S^q > 4$. Then the graph $G_{K_1} \cup (S_0 - 4)K_1$, where $G_i$ is pictured in the figure below, shows that the sequence $(S_0, 1, 0)$ is realizable.

![Figure 2.1](image)

SUBCASE 1.3 Assume that $S_i = 3$. Then there is one exception to consider.

SUBCASE 1.3.1 Assume that $S_0 = 0$. The sequence $(0, 3, 0)$ is not realizable.

SUBCASE 1.3.2 Assume that $S_0 \geq 1$. Then the graph $K_{t,3} \cup (S_0 - 1)K_1$ shows that $(S_0, 3, 0)$ is realizable.

SUBCASE 1.4 Assume that $S_i = 2m + 1$, where $m \geq 2$. Then the graph $K_1 \cup S_0 K_1 \cup (m - 2)K_2$ shows that $(S_0, S_i, 0)$ is realizable.

CASE 2 Assume that $S_2 = 1$.

SUBCASE 2.1 Assume that $S_1 = 0$. There are four sequences that are not realizable.
SUBCASE 2.1.1 Assume that $S_0 \leq 3$. The sequence $(0, 0, 1)$, is not realizable since it is impossible to construct a graph consisting of one vertex of degree 2. It is also not difficult to see that $(1, 0, 1), (2, 0, 1)$, and $(3, 0, 1)$ are not realizable.

SUBCASE 2.1.2 Assume that $S_0 \geq 4$. Then the graph $G_2 \cup (S_0 - 4)K_1$, where $G_2$ is the graph $K_4$ with one subdivided edge, shows that $(S_0, 0, 1)$ is realizable.

SUBCASE 2.2 Assume that $S_1 = 2m$, where $m \geq 1$. Then the graph $P_3 \cup S_0K_1 \cup (m-1)K_2$ shows that $(S_0, S_1, 1)$ is realizable.

SUBCASE 2.3 Assume that $S_1 = 1$. There are three sequences that are not realizable.

SUBCASE 2.3.1 Assume that $S_0 \leq 2$. The sequences $(0, 1, 1), (1, 1, 1)$, and $(2, 1, 1)$ are not realizable.

SUBCASE 2.3.2 Assume that $S_0 \geq 3$. Then the graph $G_3 \cup (S_0 - 3)K_1$, where $G_3$ is pictured below, shows that $(S_0, 1, 1)$ is realizable.

![Figure 2.2](image)

SUBCASE 2.4 Assume that $S_1 = 3$. There is one sequence that is not realizable.

SUBCASE 2.4.1 Assume that $S_0 = 0$. Then the sequence $(0, 3, 1)$ is not realizable.
SUBCASE 2.4.2 Assume that \( S_0 \geq 1 \). Then the graph \( G_4 \cup (S_0 - 1)K_1 \), where \( G_4 \) is the graph \( K_{1,3} \) with one subdivided edge, shows that the sequence \((S_0, 3, 1)\) is realizable.

SUBCASE 2.5 Assume that \( S_1 = 2m + 1 \), where, \( m \geq 2 \). Then the graph
\[
G_5 \cup S_0K_1 \cup (m - 2)K_2,
\]
where \( G_5 \) is the graph \( K_5 \) with one subdivided edge, shows that the sequence \((S_0, S_1, 1)\) is realizable.

CASE 3 Assume that \( S_2 = 2 \).

SUBCASE 3.1 Assume that \( S_1 = 0 \). There are two sequences that are not realizable

SUBCASE 3.1.1 Assume that \( S_0 \leq 1 \). The sequences \((0, 0, 2)\) and \((1, 0, 2)\) are not realizable.

SUBCASE 3.1.2 Assume that \( S_0 \geq 2 \). The graph \( G_6 \cup (S_0 - 2)K_1 \), where \( G_6 \) is pictured in the figure below, shows that \((S_0, 0, 2)\) is realizable.

\[
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram.png}
\end{array}
\]

Figure 2.3

SUBCASE 3.2 Assume that \( S_1 = 2m \), where \( m \geq 1 \). In this case, the graph
\[
P_4 \cup S_0K_1 \cup (m - 1)K_2
\]
shows that \((S_0, S_1, 2)\) is realizable.

SUBCASE 3.3 Assume that \( S_1 = 1 \). There is one sequence that is not realizable.
SUBCASE 3.3.1 Assume that $S_0 = 0$. Then the sequence $(0,1,2)$ is not realizable.

SUBCASE 3.3.2 Assume that $S_0 \geq 1$. Then the graph $G_y \cup (S_0 - 1)K_1$, where $G_y$ is the graph $K_3$ with one pendant edge added, shows that $(S_0, 1, 2)$ is realizable.

SUBCASE 3.4 Assume that $S_1 = 2m + 1$, where $m \geq 1$. In this case, the graph $G_s \cup (S_0 - 1)K_1 \cup (m - 1)K_2$, where $G_s$ is the graph of $K_{1,3}$ with two subdivided edges, shows that $(S_0, S_1, 2)$ is realizable.

CASE 4 Assume that $S_2 \geq 3$.

SUBCASE 4.1 Assume that $S_1 = 2m$, where $m \geq 0$. The graph $C_{S_1} \cup S_0K_1 \cup mK_2$, shows that the sequence $(S_0, S_1, S_2)$ is realizable.

SUBCASE 4.2 Assume that $S_1 = 1$. There is one sequence that is not realizable.

SUBCASE 4.2.1 Assume that $S_0 = 0$. The sequence $(0,1,3)$ is not realizable.

SUBCASE 4.2.2 Assume that $S_0 \geq 1$. The graph $G_y \cup (S_0 - 1)K_1$, where $G_y$ is the graph $C_{S_1+1}$ with one pendant edge added, shows that the sequence $(S_0, 1, S_2)$ is realizable.

SUBCASE 4.3 Assume that $S_1 = 2m + 1$, where $m \geq 1$. The graph $K_{1,3} \cup (S_0 - 1)K_1 \cup (m - 1)K_2 \cup C_{S_2}$ shows that the sequence $(S_0, S_1, S_2)$ is realizable.

Thus completing the proof of Theorem 1.

In Theorem 2, we shall look at the existence of a graph in modulo 4. Not only will we identify the necessary graphs, but we will provide the twenty-four exceptions. The proof format is similar to the proof of Theorem 1. We note that this result (like the case for
modulo 3) was provided in [1] again without proof and without the explicit list of exceptions.

**Theorem 2** Given a sequence \( S = (S_0, S_1, S_2, S_3) \) of nonnegative integers such that \( S_1 + S_3 \) is even and that \( \sum_{i=0}^{3} S_i > 0 \), there exists a graph containing exactly \( S_i \) vertices whose degrees are congruent to \( i \) modulo 4 with twenty-four exceptions.

**Proof** The proof will be done by cases.

**Case 1** Assume that \( S_2 = 0 \).

**Subcase 1.1** Assume that \( S_2 = 0 (\text{mod } 4) \). Then \( S_3 = 4m \), where \( m \geq 0 \). Then \( S_i \) is necessarily even, say that \( S_i = 2a \), where \( a \geq 0 \). Then the graph \( S_0K_1 \cup aK_2 \cup mK_4 \), has \( S_0 \) vertices are of degree 0, \( S_1 \) vertices are of degree 1, and \( S_3 \) vertices of degree 3.

So the sequence \((S_0, S_1, 0, S_3)\), is realizable.

**Subcase 1.2** Assume that \( S_2 = 1 (\text{mod } 4) \), say that \( S_3 = 4m + 1 \), where \( m \geq 0 \). In this case, \( S_i \) is necessarily odd.

**Subcase 1.2.1** Assume that \( S_i = 1 \). Then there are four sequences that are not realizable. We consider them first.

**Subcase 1.2.1.1** Assume that \( S_0 \leq 3 \) and \( S_3 = 1 \). Then the sequence \((0, 1, 0, 1)\) is not realizable since it is impossible to construct a graph of only two vertices when one vertex is of degree 3. The sequence \((1, 1, 0, 1)\) is not realizable since it is impossible to construct a graph having three vertices when one vertex is of degree 3. The sequence \((2, 1, 0, 1)\) is not realizable since it is impossible to construct a graph having four vertices in which one has degree 3 necessarily requiring two of degree 4. The sequence \((3, 1, 0, 1)\)
is not realizable since it is impossible to construct a graph having five vertices, one of
degree 3, two or three of degree 4, and one of degree 1.

SUBCASE 1.2.1.2 Assume that $S_0 \geq 4$ or $S_3 \geq 5$. First if $S_0 \geq 4$, then the graph
$G_1 \cup (S_0 - 4)K_1 \cup mK_4$, where $G_1$ is pictured in the figure 2.4.1 below, shows that
$(S_0, 1, 0, S_3)$ is realizable. Second if $S_3 \geq 5$, then the graph $G_2 \cup S_3 K_1 \cup (m - 1)K_4$,
where $G_2$ is pictured in the figure 2.4.2 below, shows that $(S_0, 1, 0, S_3)$ is realizable.

SUBCASE 1.2.2 Assume that $S_1 = 2a + 1$, where $a \geq 1$. Then the graph
$K_{1,3} \cup S_1 K_1 \cup (a - 1)K_2 \cup mK_4$ shows that the sequence $(S_0, S_1, 0, S_3)$ is realizable.

SUBCASE 1.3 Assume that $S_3 = 2(\text{mod } 4)$, say that $S_3 = 4m + 2$, where $m \geq 0$. In this
case $S_1$ is necessarily even.

SUBCASE 1.3.1 Assume that $S_1 = 0$. There are three exceptions to consider.

SUBCASE 1.3.1.1 Assume that $S_0 \leq 2$ and that $S_3 = 2$. The sequences $(0, 0, 0, 2)$,
$(1, 0, 0, 2)$, and $(2, 0, 0, 2)$ are not realizable.
SUBCASE 1.3.1.2 Assume that $S_0 \geq 3$ or $S_3 \geq 6$. First if $S_0 \geq 3$, then the graph $G_3 \cup (S_0 - 3)K_1 \cup mK_4$, where $G_3$ is the graph of $K_3$ with one deleted edge, shows that the sequence $(S_0, 0, 0, S_3)$ is realizable. Second if $S_3 \geq 6$, then the graph $G_4 \cup S_0K_1 \cup (m - 1)K_4$ where $G_4$ is pictured below, shows that the sequence $(S_0, 0, 0, S_3)$ is realizable.

![Figure 2.5](image)

SUBCASE 1.3.2 Assume that $S_1 = 2$. There are two exceptions here.

SUBCASE 1.3.2.1 Assume that $S_0 \leq 1$ and $S_3 = 2$. The sequence $(0, 2, 0, 2)$ is not realizable since it is impossible to construct a graph consisting of four vertices, where two vertices are of degree 3 and two are of degree 1. The sequence $(1, 2, 0, 2)$ is not realizable since there is no graph of five vertices having one of degree 4, two of degree 3, and two of degree 1.

SUBCASE 1.3.2.2 Assume that $S_0 \geq 2$ or $S_3 \geq 6$. If $S_0 \geq 2$, then the graph $G_3 \cup (S_0 - 2)K_1 \cup mK_4$, where $G_3$ is pictured in Figure 2.6.1 below, shows that
$(S_0, 2, 0, S_3)$ is realizable. Next if $S_3 \geq 6$, then the graph $G_a \cup S_0 K_1 \cup (m-1)K_4$, where $G_a$ is pictured in Figure 2.6.2 below, shows that $(S_0, 2, 0, S_3)$ is realizable.

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{figure2.6.1}
\caption{Figure 2.6.1}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{figure2.6.2}
\caption{Figure 2.6.2}
\end{figure}

**SUBCASE 1.3.3** Assume that $S_1 = 2a$, where $a \geq 2$. Then the graph $G_1 \cup S_0 K_1 \cup (a-2)K_2 \cup mK_4$, where $G_1$ is pictured in the figure below, shows that $(S_0, S_1, 0, S_3)$ is realizable.

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{figure2.7}
\caption{Figure 2.7}
\end{figure}
SUBCASE 1.4 Assume that $S_3 = 3 \pmod{4}$, say that $S_3 = 4m + 3$, where $m \geq 0$. In this case $S_3$ is necessarily odd.

SUBCASE 1.4.1 Assume that $S_3 = 1$. There is one exception that must be considered.

SUBCASE 1.4.1.1 Assume that $S_0 = 0$ and $S_3 = 3$. Then the sequence $(0, 1, 0, 3)$ is not realizable since it is impossible to construct a graph consisting of four vertices having three of degree 3 and one of degree 1.

SUBCASE 1.4.1.2 Assume that $S_0 \geq 1$ or $S_3 \geq 7$. First if $S_0 \geq 1$, then the graph $G_s \cup (S_0 - 1)K_1 \cup mK_4$, where $G_s$ is the graph of $K_4$ with one pendant edge added, shows that $(S_0, 1, 0, S_3)$ is realizable. Second if $S_3 \geq 7$, then the graph $G_s \cup S_0K_1 \cup (m - 1)K_4$, where $G_s$ is pictured in the figure below, shows that $(S_0, 1, 0, S_3)$ is realizable.

![Figure 2.8](image-url)
SUBCASE 1.4.2 Assume that $S_i = 2a + 1$, where $a \geq 1$. Then the graph $G_{10} \cup S_0K_1 \cup (a - 1)K_2 \cup mK_4$, where $G_{10}$ is pictured in the figure below, shows that $(S_0, S_i, 0, S_3)$ is realizable.

![Figure 2.9](image-url)

CASE 2 Assume that $S_3 = 1$.

SUBCASE 2.1 Assume that $S_3 \equiv 0 \pmod{4}$, say that $S_3 = 4m$, where $m \geq 0$. In this case $S_i$ is necessarily even.

SUBCASE 2.1.1 Assume that $S_i = 0$. Then there are five exceptions. We consider these first.

SUBCASE 2.1.1.1 Assume that $S_6^0 < 4$ and $S_3 = 0$. The sequences $(0, 0, 1, 0), (1, 0, 1, 0), (2, 0, 1, 0), (3, 0, 1, 0)$ and $(4, 0, 1, 0)$ are not realizable.

SUBCASE 2.1.1.2 Assume that $S_6^0 \geq 5$ or $S_3 \geq 4$. If $S_6^0 \geq 5$, then the graph $G_{11} \cup (S_6 - 5)K_1 \cup mK_4$, where $G_{11}$ is the graph $K_3$ with one subdivided edge, shows that $(S_0, 0, 1, S_3)$ is realizable. Next if $S_3 \geq 4$, then the graph $G_{12} \cup S_6K_1 \cup (m - 1)K_4$,
where $G_{12}$ is the graph $K_4$ with one subdivided edge, shows that $(S_0, 0, 1, S_3)$ is realizable.

**SUBCASE 2.1.2** Assume that $S_1 = 2a$, where $a \geq 1$. Then the graph $P_3 \cup S_0K_1 \cup (a-1)K_2 \cup mK_4$, shows that the sequence $(S_0, S_1, 1, S_3)$ is realizable.

**SUBCASE 2.2** Assume that $S_3 \equiv 1 (\text{mod } 4)$, say that $S_3 = 4m + 1$, where $m \geq 0$. In this case $S_1$ is necessarily odd.

**SUBCASE 2.2.1** Assume that $S_1 = 1$. There are three sequences that are not realizable. We shall consider them first.

**SUBCASE 2.2.1.1** Assume that $S_0 \leq 2$ and $S_3 = 1$. The sequences $(0, 1, 1, 1), (1, 1, 1, 1), (2, 1, 1, 1)$ are not realizable.

**SUBCASE 2.2.1.2** Assume that $S_0 \geq 3$ or $S_3 \geq 5$. If $S_0 \geq 3$, then the graph $G_{13} \cup (S_0 - 3)K_1 \cup mK_4$, where $G_{13}$ is pictured in Figure 2.10.1 below, shows that $(S_0, 1, 1, S_3)$ is realizable. Next if $S_3 \geq 5$, then the graph $G_{14} \cup S_0K_1 \cup (m-1)K_4$, where $G_{14}$ is pictured in Figure 2.10.2 below, shows that $(S_0, 1, 1, S_3)$ is realizable.
SUBCASE 2.2.2 Assume that $S_1 = 2a + 1$, where $a \geq 1$. Then the graph
\[ G_{13} \cup S_0K_1 \cup (a-1)K_2 \cup mK_4, \]
where $G_{13}$ is the graph of $K_{1,3}$ with one subdivided edge, shows that $(S_0, S_1, 1, S_3)$ is realizable.

SUBCASE 2.3 Assume that $S_1 \equiv 2 \pmod{4}$, say that $S_1 = 4m + 2$, where $m \geq 0$. In this case $S_1$ is even.

SUBCASE 2.3.1 Assume that $S_1 = 0$. Then there are two exceptions to consider first.

SUBCASE 2.3.1.1 Assume that $S_0 \leq 1$ and $S_3 = 2$. The sequences $(0, 0, 1, 2)$, and $(1, 0, 1, 2)$ are not realizable.

SUBCASE 2.3.1.2 Assume that $S_0 \geq 2$ or $S_3 \geq 6$. Then the graph
\[ G_{16} \cup (S_0 - 2)K_1 \cup mK_4, \]
where $G_{16}$ is pictured in Figure 2.11.1 below, shows that the sequence $(S_0, 0, 1, S_3)$ is realizable. Second if $S_3 \geq 6$, then the graph
\[ G_{17} \cup S_0K_1 \cup (m - 1)K_4, \]
where $G_{17}$ is pictured in Figure 2.11.2 below, shows that the sequence $(S_0, 0, 1, S_3)$ is realizable.

Figure 2.11.1

Figure 2.11.2
**SUBCASE 2.3.2** Assume that $S_1 = 2a$, where $a \geq 1$. Then the graph $G_{18} \cup S_0K_1 \cup (a-1)K_2 \cup mK_4$, where $G_{18}$ is pictured in the figure below, shows that $(S_0, S_1, 1, S_3)$ is realizable.

![Figure 2.12](image)

**SUBCASE 2.4** Assume that $S_3 \equiv 3 \pmod{4}$, say that $S_3 = 4m + 3$, where $m \geq 0$. In this case $S_3$ is necessarily odd. Let $S_3 = 2a + 1$, where $a \geq 0$. Then the graph $G_{19} \cup S_6K_1 \cup aK_2 \cup mK_4$, where $G_{19}$ is pictured in the figure below, shows that the sequence $(S_6, S_3, 1, S_3)$ is realizable.
CASE 3 Assume that $S_2 = 2$.

SUBCASE 3.1 Assume that $S_3 \equiv 0 \pmod{4}$, say that $S_3 = 4m$, where $m \geq 0$. So $S_1$ is even.

SUBCASE 3.1.1 Assume that $S_1 = 0$. Then there are four exceptions to consider.

SUBCASE 3.1.1.1 Assume that $S_0 \leq 3$ and $S_3 = 0$. The sequences $(0, 0, 2, 0)$, $(1, 0, 2, 0)$, $(2, 0, 2, 0)$, $(3, 0, 2, 0)$ are not realizable.

SUBCASE 3.1.1.2 Assume that $S_0 \geq 4$ or $S_3 \geq 4$. First if $S_0 \geq 4$, then the graph $G_{20} \cup (S_0 - 4)K_1 \cup mK_4$, where $G_{20}$ pictured below, shows that the sequence $(S_0, 0, 2, S_3)$ is realizable. Next if $S_3 \geq 4$, then the graph $G_{21} \cup S_0 K_1 \cup (m-1)K_4$, where $G_{21}$ is the graph of $K_4$ with two subdivided edges, shows that the sequence $(S_0, 0, 2, S_3)$ is realizable.
SUBCASE 3.1.2 Assume that $S_1 = 2a$, where $a \geq 1$. Then the graph $P_4 \cup S_0K_1 \cup (a-1)K_2 \cup mK_4$ shows that the sequence $(S_0, S_1, 2, S_3)$ is realizable.

SUBCASE 3.2 Assume that $S_3 \equiv 1 \pmod{4}$, say that $S_3 = 4m + 1$, where $m \geq 0$. Let $S_1 = 2a + 1$, where $a \geq 0$. Then the graph $G_{22} \cup S_0K_1 \cup aK_2 \cup mK_4$, where $G_{22}$ is the graph $K_3$ with one pendant edge added, shows that the sequence $(S_0, S_1, 2, S_3)$ is realizable.

SUBCASE 3.3 Assume that $S_3 \equiv 2 \pmod{4}$, say that $S_3 = 4m + 2$, where $m \geq 0$. Then $S_1$ is even. Let $S_1 = 2a$, where $a \geq 0$. Then the graph $G_{23} \cup S_0K_1 \cup aK_2 \cup mK_4$, where $G_{23}$ is the graph $K_4$ with one deleted edge, shows that the sequence $(S_0, S_1, 2, S_3)$ is realizable.

SUBCASE 3.4 Assume that $S_3 \equiv 3 \pmod{4}$, say that $S_3 = 4m + 3$, where $m \geq 0$. Then $S_1$ is odd. Let $S_1 = 2a + 1$, where $a \geq 0$. Then the $G_{24} \cup S_0K_1 \cup aK_2 \cup mK_4$, where $G_{24}$ is pictured in the figure below, shows that the sequence $(S_0, S_1, 2, S_3)$ is realizable.
CASE 4 Assume that \( S_2 \geq 3 \).

SUBCASE 4.1 Assume that \( S_3 \equiv 0 \pmod{4} \), say that \( S_3 = 4m \), where \( m \geq 0 \). Then \( S_1 \) is even. Let \( S_1 = 2a \), where \( a \geq 0 \). Then the graph \( S_0 \cup K_1 \cup aK_2 \cup C_{S_2} \cup mK_4 \) shows that the sequence \( (S_0, S_1, S_2, S_3) \) is realizable.

SUBCASE 4.2 Assume that \( S_3 \equiv 1 \pmod{4} \), say that \( S_3 = 4m + 1 \), where \( m \geq 0 \). Then \( S_1 \) is odd. Let \( S_1 = 2a + 1 \), where \( a \geq 0 \). Then the graph \( G_{25} \cup S_0 \cup K_1 \cup aK_2 \cup mK_4 \) containing \( S_2 - 3 \) additional subdivided edges, where \( G_{25} \) is the cycle \( C_4 \) with one pendant edge added, shows that the sequence \( (S_0, S_1, S_2, S_3) \) is realizable.

SUBCASE 4.3 Assume that \( S_3 \equiv 2 \pmod{4} \), say that \( S_3 = 4m + 2 \), where \( m \geq 0 \). Then \( S_1 \) is even. Let \( S_1 = 2a \), where \( a \geq 0 \). Then the graph \( G_{26} \cup S_0 \cup K_1 \cup aK_2 \cup mK_4 \) containing \( S_2 - 3 \) additional subdivided edges, where \( G_{26} \) is pictured in the figure below, shows that the sequence \( (S_0, S_1, S_2, S_3) \) is realizable.

Figure 2.15

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SUBCASE 4.4 Assume that $S_1 = 3 \pmod{4}$, say that $S_1 = 4m + 3$, where $m \geq 0$. Then $S_1$ is odd. Let $S_1 = 2a + 1$, where $a \geq 0$. Then the graph $G_{27} \cup S_0K_1 \cup aK_2 \cup mK_4$ containing $S_2 - 3$ additional subdivided edges, where $G_{27}$ is pictured in the figure below, shows that the sequence $(S_0, S_1, S_2, S_3)$ is realizable.
We have now provided proofs for modulo 3 and 4 along with their exceptions. While the authors of [1] stated these two results, their general study of the problem went in a different direction. They did conclude their paper with the following conjecture.

**Conjecture** For each integer $n \geq 2$, every sequence $(S_0, S_1, \ldots, S_{n-1})$ of nonnegative integers (with $\sum S_{2i-1}$ even if $n$ is even) is realizable with a finite number of exceptions.

In the next chapter, we revisit the modulo 4 case in order to provide a better approach to solving the conjecture in general. In addition, we use this new method to show that the conjecture is true for modulo 6. We are focusing on the conjecture for $n$ even due to the extra parity condition that seems to aid in handling the numerous cases.
A NEW LOOK AT THE PROOF AND THE EXCEPTIONS FOR MODULO 4

Before we look at a new proof for modulo 4, we state the following lemma, which plays a substantial role in the rest of this thesis. It was originally proved in [2].

**Lemma 3** Let $x, y$ and $r$ be nonnegative integers such that $x + y > 0$ and $r < x + y - 1$. Then there exists a graph of order $x + y$ containing $x$ vertices of degree $r$ and $y$ vertices of degree $(r + 1)$ if and only if $rx + (r + 1)y$ is even.

In the previous chapter we provided the proof for the existence of a graph in modulo 4, and in so doing, we provided the twenty-four exceptions that were not contained in the original paper [1]. In Theorem 4, we provide an alternative proof for modulo 4 that guarantees us a finite number of exceptions. This can be seen as a beginning to the study of the conjecture that concluded the paper [1].

**Theorem 4** Let $(S_0, S_1, S_2, S_3)$ be a sequence of nonnegative integers such that $S_1 + S_3$ is even and $\sum_{i=0}^{1} S_i \geq 10$. Then there exists a graph with exactly $S_i$ vertices whose degrees are congruent to $i$ modulo 4, with a finite number of exceptions.

**Proof** We proceed by cases.

**Case 1** Assume that $S_1 + S_3 > 2$.

**Subcase 1.1** Assume that $S_1 + 2S_2$ is even. Since $S_1 + S_2 > 2$ and $S_1 + 2S_2$ is even,
by Lemma 3, we are assured the existence of a graph $H$, consisting of $S_1$ vertices of degree 1 and $S_2$ vertices of degree 2. Let $S_3 = 4m + r$, where $m \geq 0$ and $0 \leq r \leq 3$. In this subcase, $S_1$ and $S_3$ are necessarily both even. Since $S_3$ is even, either $r = 0$ or $r = 2$.

**SUBCASE 1.1.1** Assume that $r = 0$. The graph $H \cup S_0K_1 \cup mK_4$ is the desired graph.

**SUBCASE 1.1.2** Assume that $r = 2$. By assumption, we know that $S_1 + S_2 \geq 3$. This implies (since $S_1 + 2S_2$ is even) that $S_1 + 2S_2 \geq 4$. Therefore the number of edges in $H$ is

$$\frac{S_1 + 2S_2}{2} \geq 2.$$  

Let $wx$ and $yz$ denote two edges of the graph $H$ (note it may be that $x = y$). We begin the construction of our desired graph in this case by forming a new graph $H'$ obtained from the graph $H$ by adding new vertices $u$ and $v$ and the edge $uv$, deleting the edges $wx$ and $yz$, and adding the edges $wu$, $xu$, $yv$, and $zv$. Then $H' \cup S_0K_1 \cup mK_4$ is the desired graph.

**SUBCASE 1.2** Assume that $S_1 + 2S_2$ is odd. Since $S_1$ is odd, we know $S_3$ is odd, and thus $S_3 \geq 1$. Let $S_3 = 4m + r$, where necessarily $r = 1$ or $r = 3$.

**SUBCASE 1.2.1** Assume that $S_1 + S_2 \geq 3$. Since $S_1 - 1 + 2S_2$ is even and $S_1 - 1 + S_2 \geq 2$, we know by Lemma 3, that there exists a graph $H$ having $S_1 - 1$ vertices of degree 1 and $S_2$ vertices of degree 2. Let $xy$ be an edge of $H$.

**SUBCASE 1.2.1.1** Assume that $r = 1$. In this case, we form a new graph $H'$ from $H$ by adding two vertices $u$ and $v$, deleting the edge $xy$, and adding the edges $uv$, $xu$, and $yu$. After doing this, $u$ has degree 3 and $v$ has degree 1 while the vertices...
\(x\) and \(y\) still have the degree that they had in \(H\). Thus \(H' \cup S_0K_1 \cup mK_4\) is the desired graph.

**SUBCASE 1.2.1.2** Assume that \(r = 3\). We form a new graph \(H'\) from \(H\) by adding four new vertices \(s, t, u, v\), deleting the edge \(xy\), and adding the edges \(st, su, tu, uv, xs,\) and \(yt\). Then \(s, t,\) and \(u\) have degree 3, the vertex \(v\) has degree 1, and \(x\) and \(y\) have the same degree that they had in \(H\). Thus \(H' \cup S_0K_1 \cup mK_4\) is the desired graph.

**SUBCASE 1.2.2** Assume that \(s_1 + s_2 = 3\). Since \(s_1\) is odd, either \(s_1 = 1\) and \(s_2 = 2\) or \(s_1 = 3\) and \(s_2 = 0\). So there are a total of four cases depending on whether

- \(r = 1\) or \(r = 3\). See Figure 1 for the graphs \(H_i\) \((1 \leq i \leq 4)\). If \(s_1 = 1, s_2 = 2\) and \(r = 1\), then \(H_1 \cup S_0K_1 \cup mK_4\) is the desired graph. If \(s_1 = 1, s_2 = 2\), and \(r = 3\), then \(H_2 \cup S_0K_1 \cup mK_4\) is the desired graph. If \(s_1 = 3, s_2 = 0\), and \(r = 1\), then \(H_3 \cup S_0K_1 \cup mK_4\) is the desired graph. If \(s_1 = 3, s_2 = 0\), and \(r = 3\), then \(H_4 \cup S_0K_1 \cup mK_4\) is the desired graph.

**CASE 2** Assume that \(s_2 + s_3 > 3\).

**SUBCASE 2.1** Assume that \(2s_2 + 3s_3\) is even. Since \(s_2 + s_3 > 3\) and \(2s_2 + 3s_3\) is even, by Lemma 3 we are guaranteed the existence of a graph \(H\) consisting of \(s_2\) vertices of degree 2 and \(s_3\) vertices of degree 3. Since \(s_3\) is even, \(s_1\) is even. So the graph \(H \cup S_0K_1 \cup \frac{s_1}{2}K_2\) is the desired graph.
**SUBCASE 2.2** Assume that $2S_2 + 3S_3$ is odd. Since $S_3$ is odd, we know $S_3 \geq 1$. Thus by Lemma 3, there exists a graph $H$ consisting of $S_2 + 1$ vertices of degree 2 and $S_3 - 1$ vertices of degree 3. Necessarily, $S_1$ is odd and $S_1 \geq 1$. Let $u$ be a vertex of degree 2 in $H$. Form $H'$ from $H$ by adding a new vertex $v$ and the edge $uv$. Then the graph $G = H' \cup S_0 K_1 \cup \frac{S_1 - 1}{2} K_2$ is the desired graph.

![Graphs H1, H2, H3, H4](image)

**Figure 3.1**

**CASE 3** Assume that $S_1 + S_2 \leq 2$ and $S_2 + S_3 \leq 3$. From these inequalities, there are ten distinct cases to consider as shown in Table 1. The graph $F$ in column $H$ for subcase 3.7 is $K_3$ with one pendant edge added to it ($H_1$ from Figure 1). Further, since $S_1 + S_2 + S_3 \leq 5$; it follows that $S_0 \geq 5$. 

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Table 1

<table>
<thead>
<tr>
<th>Subcase</th>
<th>$S_1$</th>
<th>$S_2$</th>
<th>$S_3$</th>
<th>$H$</th>
<th>Number of edges deleted from $K_5$ to form $H'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>3.2</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>$K_2$</td>
<td>2</td>
</tr>
<tr>
<td>3.3</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$K_1$</td>
<td>1</td>
</tr>
<tr>
<td>3.4</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>$K_3$</td>
<td>1</td>
</tr>
<tr>
<td>3.5</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>$K_2$</td>
<td>1</td>
</tr>
<tr>
<td>3.6</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>$K_2$</td>
<td>1</td>
</tr>
<tr>
<td>3.7</td>
<td>1</td>
<td>0</td>
<td>3</td>
<td>$F$</td>
<td>1</td>
</tr>
<tr>
<td>3.8</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$P_3$</td>
<td>1</td>
</tr>
<tr>
<td>3.9</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>$K_2$</td>
<td>0</td>
</tr>
<tr>
<td>3.10</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>$P_4$</td>
<td>1</td>
</tr>
</tbody>
</table>

In subcase 3.1, the desired graph is $S_0 K_1$, while in subcase 3.9, the desired graph is $S_0 K_1 \cup K_2$. To complete the construction of the desired graph for the remaining subcases we construct $H'$ from $H \cup K_5$ by deleting at most two independent edges from $K_5$ and adding the appropriate edges and adding up with a graph where the $S_1 + S_2 + S_3$ vertices of $H$ have degrees 1, 2, or 3 and the vertices of the graph $K_5$ remain 4. Finally the graph $H' \cup (S_0 - 5)K_1$ is a graph containing exactly $S_i (i \leq 0 \leq 3)$ vertices whose degrees are congruent to $i$ modulo 4.

We have now provided a new proof for modulo 4 consisting of fewer cases than the proof provided in Theorem 2. Though this proof fails to identify the twenty-four
exceptions, it still guarantees us that there are a finite number of exceptions because there
is a finite number of sequences \((S_0, S_1, S_2, S_3)\) satisfying \(\sum_{i=0}^{3} S_i < 10\). This completes the
proof. We next aim to extend this method to modulo 6.
CHAPTER 4

THE CONSTRUCTION OF A GRAPH IN MODULO 6

In Chapter 3, we provided a new proof for the modulo 4 case that was far simpler and less cumbersome than the proof provided in Theorem 2. In Theorem 5 we provide an extension of our work from modulo 4 to modulo 6.

THEOREM 5 Let \((S_0, S_1, S_2, S_3, S_4, S_5)\) be a sequence of nonnegative integers such that \(S_1 + S_3 + S_5\) is even and \(\sum_{i=0}^{5} S_i \geq 30\). Then there exists a graph with exactly \(S_i\) \((0 \leq i \leq 5)\) vertices whose degrees are congruent to \(i\) modulo 6, with a finite number of exceptions.

PROOF Suppose that \((S_0, S_1, S_2, S_3, S_4, S_5)\) is a sequence of nonnegative integers such that \(S_1 + S_3 + S_5\) is even and \(\sum_{i=0}^{5} S_i \geq 30\).

CASE 1 Assume that \(\sum_{i=0}^{3} S_i \geq 10\).

SUBCASE 1.1 Assume that \(S_1 + S_3\) is even. Then \(S_5\) is even. Also note that \(4S_2 + 5S_5\) is even. Since \(S_1 + S_3\) is even and \(\sum_{i=0}^{3} S_i \geq 10\), we know by Theorem 4 that there exists a graph \(H_1\) consisting of \(S_i\) \((0 \leq i \leq 3)\) vertices whose degrees are congruent to \(i\) modulo 6.
4. Indeed from the constructive proof of Theorem 4, if \( S_1 + S_2 > 2 \) or \( S_2 + S_3 > 3 \), the 
\( H_1 \) has \( S_i \) vertices of degree \( i \) for each \( i = 0, 1, 2, 3 \); while if \( S_1 + S_2 \leq 2 \) and 
\( S_2 + S_3 \leq 3 \), then \( H_1 \) has five vertices of degree 4, \( S_0 - 5 \) vertices of degree 0, and 
\( S_i (1 \leq i \leq 3) \) vertices of degree \( i \).

**SUBCASE 1.1.1** Assume that \( S_4 + S_5 > 5 \). By Lemma 3, since \( 4S_4 + 5S_5 \) is even, we 
are assured the existence of a graph \( H_2 \) consisting of \( S_4 \) vertices of degree 4 and \( S_5 \) 
vertices of degree 5. If \( H_1 \) has \( S_0 \) vertices of degree 0, then \( H = H_1 \cup H_2 \) is the 
desired graph. If \( H_1 \) has five vertices of degree 4, then we form a new graph \( H \) from 
\( H_1 \cup H_2 \) by the following: for each such vertex \( x \) of degree 4 in \( H_1 \) we delete one edge 
\( u \nu \) of \( H_2 \) and add the edges \( ux \) and \( vx \) thus increasing the degree of \( x \) to 6 while 
keeping the degrees of \( u \) and \( v \) the same as they were in \( H_2 \). (Since \( S_4 + S_5 > 5 \), it 
follows that \( H_2 \) has at least five edges to accomplish this.) Now \( H \) is the desired graph 
containing \( S_i \) vertices whose degrees are congruent to \( i \) modulo 6 for each \( i = 0, 1, 2, 3, 
4, 5 \).

**SUBCASE 1.1.2** Assume that \( S_4 + S_5 \leq 5 \). From this inequality, there are twelve distinct 
cases to consider as shown in Table 2.

**SUBCASE 1.1.2.1** Assume that \( S_9 \geq 7 \). If \( H_1 \) contains five vertices of degree 4, then 
let \( u_1, u_2, u_3, u_4, u_5 \) denote these vertices and construct \( H \) from \( H_1 \) as follows: first take 
two vertices \( v_1, v_2 \) of degree 0 in \( H_1 \) and add the edges of \( K_{5,2} \) using \( \{u_1, u_2, u_3, u_4, u_5\} \) 
and \( \{v_1, v_2\} \) as the partite sets and next add the edge \( v_1v_2 \). At this point, each vertex
\(u_i (1 \leq i \leq 5)\) has degree 6 as does each of \(v_1\) and \(v_2\). Thus \(H\) is a graph containing \(S_i (0 \leq i \leq 3)\) vertices whose degrees are congruent to \(i\) modulo 6.

On the other hand, if \(H_1\) has no vertices of degree 4, then \(H_1\) (by the construction given in the proof of Theorem 4) contains \(s_n K_1\) as a subgraph. Now form the graph \(H\) from \(H_1\) by adding edges of \(K_1\) to seven vertices of degree 0 in \(H_1\) thus obtaining \(K_1 \cup (S_6 - 7)K_1\) from \(s_n K_1\). And in this case, again \(H\) has \(S_i (0 \leq i \leq 3)\) vertices whose degrees are congruent to \(i\) modulo 6.

Finally, to complete the construction of the desired graph in each of the subcases I-XII we now form \(H'\) from \(H \cup F\) (using \(F\) from Table 2) by deleting at most four edges (independent edges or edges of a subgraph isomorphic to \(P_3\)) from \(H\) and adding appropriate edges to end up with a graph where the \(S_4 + S_5\) vertices of \(F\) have degree 4 or 5 and the vertices of \(H\) maintain the same degree. For example in subcase V of Table 2, the desired graph to complete the proof is shown in Figure 4.1, where the white vertices represent those vertices of \(F\) and the black vertices represent those vertices of \(P_3\) from which we have deleted edges of \(H\). The resulting graph \(H'\) has \(S_i (0 \leq i \leq 5)\) vertices whose degrees are congruent to \(i\) modulo 6 as desired.

**SUBCASE 1.1.2.2** Assume that \(S_6 < 7\). Since \(\sum_{i=0}^{5} S_i \geq 30\) and \(S_6 < 7\), it follows that \(S_1 + S_2 + S_3 \geq 19\). This implies from the constructive proof of Theorem 4, that \(H_1\) has \(S_i (0 \leq i \leq 3)\) vertices of degree \(i\). Furthermore since \(S_1 + S_2 + S_3 \geq 19\), the construction of \(H_1\) will have either four independent edges or a subgraph isomorphic to \(P_3\). And
now we construct $H$ from $H_1 \cup F$ in the same way in which $H'$ was constructed from $H \cup F$ in Subcase 1.1.2.1.

**SUBCASE 1.2** Assume that $S_4 + S_5$ is odd. Thus $S_5$ is odd. By Theorem 4 and its constructive proof, there exists a graph $H_1$ consisting of $S_4 + 1$ vertices of degree 1 and $S_1 + 1$ vertices whose degrees are congruent to $i$ modulo 4.

**SUBCASE 1.2.1** Assume that $S_4 + S_5 > 6$. Note $4S_4 + 5(S_5 - 1)$ is even. Then by Lemma 3, there exists a graph $H_2$ consisting of $S_4$ vertices of degree 4 and $S_5 - 1$ vertices of degree 5. Let $u$ be a vertex in $H_1$ of degree 1. Now form the graph $H'$ from $H_1 \cup H_2$ by deleting two independent edges $wx$ and $yz$ from $H_2$ and adding the edges $wu, xu, yu$ and $zu$ giving us one less vertex of degree 1 and one additional vertex of degree 5. Thus we have obtained the desired graph $H'$ containing exactly $S_i$ ($0 \leq i \leq 5$) vertices whose degrees are congruent to $i$ modulo 6.

**SUBCASE 1.2.2** Assume that $S_4 + S_5 \leq 6$. From this inequality, there are twelve distinct subcases as shown in Table 3 below.

**SUBCASE 1.2.2.1** Assume that $S_6 \geq 7$. The proof is very similar to the one presented in Subcase 1.1.2.1. If $H_1$ contains five vertices of degree 4, then let $u_1, u_2, u_3, u_4, u_5$ denote these vertices and construct $H$ from $H_1$ by taking two vertices $v_1, v_2$ of degree 0 in $H_1$, and add the edges of $K_{5,2}$ using $\{u_1, u_2, u_3, u_4, u_5\}$ and $\{v_1, v_2\}$ as the partite sets. We also add the edge $v_1v_2$. At this point, each vertex $u_i$ ($1 \leq i \leq 5$) has degree 6 as does each of $v_1$ and $v_2$. 

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If $H_1$ has no vertices of degree 4, then $H_1$ contains $S_oK_1$ as a subgraph. Now form the graph $H$ from $H_1$ by adding edges to obtain $K_1 \cup (S_o - 7)K_1$ from $S_oK_1$. Thus in either case, $H$ has $S_1 + 1$ vertices of degree 1 and $S_i (i = 0, 2, 3)$ vertices whose degrees are congruent to $i$ modulo 6.

Now let $u$ be a vertex of degree 1 in $H$. Next we construct $H''$ from $H$ by deleting two independent edges $wx$ and $yz$ from $H$ and adding the edges $uw, ux, uy,$ and $uz$, giving us one less vertex of degree 1 and one additional vertex of degree 5. At this point $H''$ is a graph containing exactly $S_i (0 \leq i \leq 5)$ vertices whose degrees are congruent to $i$ modulo 6.

To complete the construction of the desired graph in each of the subcases I-XII we now form $H'$ from $H'' \cup F$ (using $F$ from Table 3) by deleting at most four edges (independent edges or edges of a subgraph isomorphic to $P_3$ in $H''$) from the graph $H''$ and adding appropriate edges to end up with a graph where the $S_4 + S_5 - 1$ vertices of $F$ have degree 4 or 5 and the vertices of $H''$ have the same degree. For example in subcase X of Table 3, the desired graph to complete the proof is shown in Figure 4.2, where the white vertices represent the vertices of $F$ and the black vertices represent the vertices from which we have deleted an edge of $H''$. Thus $H'$ has $S_i (0 \leq i \leq 5)$ vertices whose degrees are congruent to $i$ modulo 6 as desired.

**SUBCASE 1.2.2.2** Assume that $S_o < 7$. Since $\sum_{i=0}^{5} S_i \geq 30$ and $S_o < 7$, it follows that $S_1 + S_2 + S_3 \geq 19$. By the constructive proof of Theorem 4, the graph $H_1$ consists of $S_1 + 1$ vertices of degree 1 and $S_i (i = 0, 2, 3)$ vertices of degree $i$. Furthermore, $H_1$
contains four independent edges or a subgraph isomorphic to \( P_2 \). Finally we construct \( H \) from \( H_1 \cup F \) in the same way in which \( H' \) was constructed from \( H'' \cup F \) in subcase 1.2.2.1.

**CASE 2** Assume that \( \sum_{i=0}^{1} S_i < 10 \). From this it follows that \( S_4 + S_5 \geq 21 \).

**SUBCASE 2.1** Assume that \( S_2 \) is even. Therefore \( 4S_4 + 5S_5 \) is even. By Lemma 3 there exists a graph \( H_1 \) consisting of \( S_4 \) vertices of degree 4 and \( S_5 \) vertices of degree 5. Since \( S_2 \) is even, \( S_1 + S_2 \) is also even.

**SUBCASE 2.1.1** Assume that \( S_1 \) and \( S_2 \) are both even. There are fifteen distinct subcases where \( S_1 \) and \( S_2 \) are both even and these are shown in Table 4 by deleting at most two independent edges of \( H_1 \) and adding appropriate edges to end up with a graph where the \( S_1 + S_2 \) vertices of \( H_2 \) have degree 1 or 3. We will demonstrate the construction of a new graph \( H' \) from \( H_1 \cup H_2 \), where \( H_2 \) is provided in Table 4. We demonstrate the construction of graph \( H' \) in subcase II by letting \( w \) and \( x \) be the vertices of \( K_2 \) in \( H_2 \). First delete two independent edges \( ab, cd \) from the graph \( H_1 \). Now add four edges \( aw, bw, cx, dx \) giving us the desired graph of \( H' \). Next let \( H \) be the graph obtained from \( H' \) by subdividing \( S_2 \) edges. Now \( H \cup S_nK_1 \) is a graph containing exactly \( S_i (0 \leq i \leq 5) \) vertices of degree \( i \).

**SUBCASE 2.1.2** Assume that both \( S_1 \) and \( S_2 \) are both odd. There are ten distinct subcases to consider as shown in Table 5. The graphs \( F_1, F_2, \) and \( F_3 \) in Table 5 are shown in Figure 4.3. In each particular subcase, we form \( H' \) from \( H_1 \cup H_2 \), where \( H_2 \)
is given in Table 5, by deleting at most one edge from $H_1$ and adding appropriate edges so that the $S_1 + S_3$ vertices of $H_2$ have degree 1 or 3 and the vertices of $H_1$ maintain the same degree. We will demonstrate the construction of the graph $H'$ in subcase I, by letting $u$ be a vertex of degree 1 in $H_2$. Now delete one edge $ab$ from $H_1$ and add the edges $au, bu$ to $H_2$ giving us one vertex of degree 1, one vertex of degree 3 and thus the desired graph $H'$. Finally to complete the construction of the desired graph in this subcase, we form $H''$ from $H'$ by subdividing $S_2$ edges of $H'$. Now $H'' \cup S_0 K_1$ is a graph containing exactly $S_i (0 \leq i \leq 5)$ vertices whose degrees are congruent to $i$ modulo 6.

**SUBCASE 2.2** Assume that $S_5$ is odd. By Lemma 3 there exists a graph $H_1$ consisting of $S_4$ vertices of degree 4 and $S_5 - 1$ vertices of degree 5. There are thirty subcases to consider and these are provided in Table 6, where the graphs $F_1, F_2, \ldots, F_7$ are shown in Figure 4.4. In each subcase I-XXX, we form $H'$ from $H_1 \cup H_2$ by deleting at most three independent edges from $H_1$ and adding appropriate edges, so that one vertex of $H_2$ has degree 5, the remaining $S_1 + S_3$ vertices of $H_2$ have degree 1 or 3, and the vertices of $H_1$ maintain the same degree. We will demonstrate the construction of the graph $H'$ in Subcase XXII by letting $u$ be a vertex of degree 1 in $H_2$. Now delete two independent edges $ab$ and $cd$ from $H_1$ and add the edges $au, bu, cu, du$ giving us the desired graph $H'$. Next let $H$ be the graph obtained from $H'$ by subdividing $S_2$ edges. Now $H \cup S_0 K_1$ is a graph containing exactly $S_i (0 \leq i \leq 5)$ vertices whose degrees are congruent to $i$ modulo 6.
We have now proven that when \( S_i + S_{i+1} + S_{i+2} \) is even and when \( \sum_{i=0}^{5} S_i \geq 30 \), that there exists a graph, consisting of \( S_i \) vertices whose degrees are congruent to \( i \) modulo 6.

Not only does a graph exist under these conditions, but there is also a finite number ways to write the sequence \( (S_0, S_1, S_2, S_3, S_4, S_5) \), when \( \sum_{i=0}^{5} S_i < 30 \). Therefore, we can conclude that there are a finite number exceptions in modulo 6. This concludes the proof.

Table 2

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38

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Figure 4.4

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CHAPTER 5

SOME CLOSING THOUGHTS

In the last two chapters we gave detailed proofs for the existence of a graph in modulo 4 and 6 and why we can conclude there are a finite number of exceptions when the parity is satisfied and when our total number of vertices are large enough. The main part left unanswered is the conjecture posed in this thesis and *The Pi Mu Epsilon Journal*. Though we have not supplied a proof of the conjecture, it seems as though our proofs provided in the last two chapters may actually give an outline to the proof of the conjecture.

In the previous two chapters we showed that when the sum of our vertices was large enough, and when the parity was satisfied, that a graph existed in modulo 4 and 6. In both cases, the sum of the vertices was arbitrarily chosen. It is fair to assume that in the general case provided that the sum of the vertices is large enough and that the parity is satisfied, that a graph will exist. Furthermore it may be easier to show that a graph exists when the modulo is even because of our proofs provided in chapters 3 and 4 deal with an even modulo. Though we have not provided a proof of the conjecture, maybe some of our ideas can be extended to solve it.
BIBLIOGRAPHY


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University of Pittsburgh

Thesis Title:
A Problem Concerning the Existence of Graphs with Certain Degrees

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Committee Member, Dr. Xin Li, Ph. D.
Graduate Faculty Representative, Dr. Rodney Metcalf, Ph. D.