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Libo Zhou
University of Nevada, Las Vegas

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CONFIDENCE INTERVALS FOR THE RATIO OF POISSON PARAMETERS

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Libo Zhou

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ABSTRACT

Confidence Intervals for the Ratio of Poisson Parameters

by

Libo Zhou

Dr. Malwane Ananda, Examination Committee Chair
Professor of Statistics
University of Nevada, Las Vegas

Let k_1, k_2 be non-negative integers with $k_1 + k_2 \geq 2$; let $X_1, X_2, \ldots X_{k_1}; Y_1, Y_2, \ldots Y_{k_2}$ be mutually independent Poisson random variables with parameters $\lambda_1, \lambda_2, \ldots \lambda_{k_1}; \mu_1, \mu_2, \ldots \mu_{k_2}$, respectively. In this article, we consider constructing confidence intervals for ratio of Poisson parameters $\theta = \frac{\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdots \lambda_{k_1}}{\mu_1 \cdot \mu_2 \cdot \mu_3 \cdots \mu_{k_2}}$. For the ratio of two Poisson parameters $\theta = \frac{\lambda_1}{\mu_1}$, an exact formula for confidence interval is given. A numerical method to obtain confidence interval for $\theta = \frac{\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdots \lambda_{k_1}}{\mu_1 \cdot \mu_2 \cdot \mu_3 \cdots \mu_{k_2}}$ is developed. Examples are given for each of the two cases.

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CHAPTER 1

INTRODUCTION

Let X and Y be two independent Poisson random variables with parameter λ_1 and μ_1, respectively. Sahai and Khurshid have described a few methods for obtaining a confidence interval for the ratio of two Poisson means λ_1/μ_1 such as an exact method, use of F-distribution tables, normal and Poisson approximations, normal approximation using square root transformation, method of maximum likelihood. In present article, we obtain an exact formula for the confidence interval for the ratio of two Poisson parameters. In this case, Suppose $\tilde{X}_1' = (X_{11}, X_{22}, \ldots. X_{1n_1})$ and $\tilde{Y}_1' = (Y_{11}, Y_{22}, \ldots. Y_{1m_1})$ be ramdom samples from X and Y respectively. Furthermore, suppose λ_1 and μ_1 follow gamma distributions with parameters α_1, β_1 and δ_1, γ_1 respectively. In chapter 3, we develop an exact formula for the confidence interval for $\theta = \dfrac{\lambda_1}{\mu_1}$ using Bayes method. The brief development is as following:

Since $\tilde{X}_1 \sim P(\lambda_1)$ and $\pi(\lambda_1) \sim gamma(\alpha_1, \beta_1)$, we can find that $\lambda_1 | \tilde{X} \sim$ gamma $(\sum_{i=1}^{n_1} x_{1i} + \alpha_1, \dfrac{\beta_1}{1+n_1\beta_1})$, then if we let $\dfrac{\beta_1}{1+n_1\beta_1} = \varphi_1$ we have $\dfrac{\lambda_1}{\varphi_1} | \tilde{X}_1 \sim$

gamma ($\sum_{i=1}^{n_1} x_{1i} + \alpha_1$, 1). Similarly, with $\frac{\gamma_1}{1+m_1\gamma_1} = \omega_1$ we have $\frac{\mu_1}{\omega_1} | \tilde{Y}_1 \sim$ gamma ($\sum_{i=1}^{m_1} y_{1i} + \delta_1$, 1).

Thus, $\dfrac{\frac{\lambda_1}{\varphi_1}}{\frac{\lambda_1}{\varphi_1} + \frac{\mu_1}{\omega_1}} \sim$ Beta ($\sum_{i=1}^{n_1} x_{1i} + \alpha_1, \sum_{i=1}^{m_1} y_{1i} + \delta_1$), after some algebraic operation we can find the confidence interval for $\theta = \frac{\lambda_1}{\mu_1}$ is $\left(\frac{\beta_{\frac{\alpha}{2}}}{1-\beta_{\frac{\alpha}{2}}} \cdot \frac{\varphi_1}{\omega_1}, \frac{\beta_{1-\frac{\alpha}{2}}}{1-\beta_{1-\frac{\alpha}{2}}} \cdot \frac{\varphi_1}{\omega_1}\right)$, where $(\beta_{\frac{\alpha}{2}}, \beta_{1-\frac{\alpha}{2}})$ are the critical values of beta distribution with parameters $A = \sum_{i=1}^{n_1} x_{1i} + \alpha_1$, $B = \sum_{i=1}^{m_1} y_{1i} + \delta_1$. An example is also given in this chapter. The method is also suitable for the example in Sahai and Khurshid's article. Example using the method with data from Sahai and Khurshid's article is given in chapter 3, example 2.

For the ratio $\theta = \frac{\lambda_1 \cdot \lambda_2 \cdots \lambda_{k_1}}{\mu_1 \cdot \mu_2 \cdots \mu_{k_2}}$, when there are more than one parameters for the numerator or the denominator or both there is no formal distribution for the ratio. Therefore, we develop a numerical method to find its $(1-\alpha)100\%$ confidence interval. The simple ratio becomes to the following case:

Let k_1, k_2 be non-negative integers with $k_1 + k_2 \geq 2$; let $\tilde{X}_1, \tilde{X}_2, ...\tilde{X}_{K_1}; \tilde{Y}_1, \tilde{Y}_2, ...\tilde{Y}_{K_2}$ be mutually independent Poisson random scalars with parameters $\lambda_1, \lambda_2, ...\lambda_{k_1}; \mu_1, \mu_2, ...\mu_{k_2}$, respectively. In this article, we look at how to construct confidence interval for the ratio $\theta = \frac{\lambda_1 \cdot \lambda_2 \cdots \lambda_{k_1}}{\mu_1 \cdot \mu_2 \cdots \mu_{k_2}}$.

Harris has developed a theoretical method to obtain the confidence interval for the quotient $\frac{\lambda_1 \lambda_2 \cdots \lambda_{k_1}}{\mu_1 \mu_2 \cdots \mu_3}$, where $\lambda_1, \lambda_2, ...\lambda_{k_1}; \mu_1, \mu_2, ...\mu_{k_2}$ are Poisson parameters for the variables $X_1, X_2, ...X_{k_1}; Y_1, Y_2, ...Y_{k_2}$. In this article we develop a more general method to calculate the confidence interval for $\theta = \frac{\lambda_1 \lambda_2 \cdots \lambda_{k_1}}{\mu_1 \mu_2 \cdots \mu_3}$ in this case: $X_1, X_2, ...X_{k_1}; Y_1, Y_2, ...Y_{k_2}$ are not $k_1 + k_2$ variables but scalars, $\tilde{X}_1, \tilde{X}_2, ...\tilde{X}_{K_1}; \tilde{Y}_1, \tilde{Y}_2, ...\tilde{Y}_{K_2}$ each contains random sample of sizes $n_1, n_2, ...n_{k_1}; m_1, m_2, ...m_{k_2}$, respectively. In chapter 4, we develop a numerical way to calculate a large number of the values for θ and then we put these numbers in order so that we could find the $(1-\alpha)100\%$ confidence interval. A calculation procedure of the R program was also given in this chapter. We should note that in particular, for $k_2 = 0$ the parameter θ is a product of Poisson parameter and the solution can be interpreted as an approximate solution to the corresponding problem of finding confidence intervals for the product of binomial parameters. Estimation of the product of binomial parameters has been investigated by Madansky, Buehler, and Myhre and Saunders. Tables suitable for use in some problems concerning estimation of binomial parameters have been compiled by Lipow and Riley. Their results have been compared with the result coming from Harris. And all the results will be compared with those of the present article in chapter 4, example 5.

CHAPTER 2

DISTRIBUTIONS CONSIDERED

The Poisson Distribution

The Poisson distribution is a discrete probability distribution belonging to certain random variables N that count, among other things, a number of discrete occurrences that take place during a time-interval of given length. The probability that there are exactly x occurrences (x being a non-negative integer, $x=$ 0, 1, 2, ...) is:

$$p(N=x)=\frac{e^{-\lambda}\lambda^{x}}{x!},\quad x=0,1,2,\cdots$$

where $\lambda>0$.

The Gamma Distribution

The gamma distribution is a continuous probability distribution with the probability density function :

$$f(x)=\frac{1}{\Gamma(\alpha)\beta^{\alpha}}x^{\alpha-1}e^{-\frac{x}{\beta}},\text{ where } x\geq 0.$$

$\alpha>$ 0 is the shape parameter and $\beta>$ 0 is the scale parameter.

Related Knowledge about Gamma Distribution

If X is gamma distributed with the parameter α and β, then X/β

will be distributed with the parameters α and 1. This is expressed as:

If X ~gamma(α,β), then X/β ~gamma$(\alpha,1)$.

Proof,

Let $Y=\frac{X}{\beta}$

$$F(Y)=\Pr(\frac{x}{\beta}\le y)$$

$$=\Pr(x\le \beta y)$$

$$=\int_0^{\beta y}\frac{1}{\Gamma(\alpha)\beta^{\alpha}}x^{\alpha-1}e^{-\frac{x}{\beta}}dx$$

$$f(Y)=F'(Y)$$

$$=\frac{1}{\Gamma(\alpha)\beta^{\alpha}}(\beta y)^{\alpha-1}e^{-\frac{\beta y}{\beta}}(\beta y)'$$

$$=\frac{1}{\Gamma(\alpha)}y^{\alpha-1}e^{-y},$$

so Y ~gamma$(\alpha,1)$, that is X/β ~gamma$(\alpha,1)$.

We can also prove the above result by using the moment generating function: If X ~gamma(α,β), then the moment generating function of X is

$$M_x(t)=\frac{1}{(1-\beta t)^{\alpha}}$$

then $M_{\frac{x}{\beta}}(t)=M_x(\frac{t}{\beta})$

$$= \frac{1}{(1-\beta\frac{t}{\beta})^{\alpha}}$$

$$= \frac{1}{(1-t)^{\alpha}}$$

so $X/\beta \sim \text{gamma}(\alpha,1)$.

The Beta Distribution

The Beta distribution is a continuous probability distribution with the probability density function defined on the interval $[0,1]$:

$$f(X) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} X^{\alpha-1}(1-X)^{\beta-1},$$

where α and β are parameters that must be greater than zero and Γ is the gamma function.

The expected value and variance of a beta random variable X with parameters α and β are given by formulae:

$$E(X) = \frac{\alpha}{\alpha+\beta}$$

$$V(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Related Knowledge about Beta Distribution

If X_1 and X_2 are two independent random variables. And X_1 is gamma distributed with parameters α and 1, X_2 is gamma distributed with parameters β and 1, then $\frac{X_1}{X_1+X_2}$ will be Beta distributed with parameters α and β. This can be expressed as: If X_1 and X_2 are independent, $X_1 \sim$ gamma (α, 1) and $X_2 \sim$ gamma (β, 1), then $\frac{X_1}{X_1+X_2} \sim$ beta (α, β).

Proof,

If $X_1 \sim$ gamma (α, 1) and $X_2 \sim$ gamma (β, 1), then they have the following probability density functions respectively:

$$f(x_1) = \frac{1}{\Gamma(\alpha)} x_1^{\alpha-1} e^{-x_1}, \quad 0 < x_1 < \infty.$$

$$f(x_2) = \frac{1}{\Gamma(\beta)} x_2^{\beta-1} e^{-x_2}, \quad 0 < x_2 < \infty.$$

When they are independent they have the joint probability density function:

$$h(x_1, x_2) = f(x_1) \cdot f(x_2)$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} x_1^{\alpha-1} x_2^{\beta-1} e^{-x_1-x_2}, \quad 0 < x_1 < \infty, \; 0 < x_2 < \infty,$$

zero elsewhere, where $\alpha>0$, $\beta>0$. Let $Y_1 = X_1 + X_2$ and $Y_2 = \frac{X_1}{X_1+X_2}$.

The space A is, exclusive of the points on the coordinate axes, the first quadrant of the $x_1 x_2$ - plane. Now

$$y_1 = u_1(x_1, x_2) = x_1 + x_2$$

$$y_2 = u_2(x_1 + x_2) = \frac{x_1}{x_1 + x_2}$$

may be written as $x_1 = y_1 y_2$, $x_2 = y_1(1 - y_2)$, so

$$J = \begin{pmatrix} y_2 & y_1 \\ 1 - y_2 & -y_1 \end{pmatrix} = -y_1 \neq 0 .$$

The transaction is one-to-one, and it maps A to B = $\{(y_1, y_2)$: $0 < y_1 < \infty$,

$0 < y_2 < 1\}$ in the $y_1 y_2$ - plane.

The joint probability density function of Y_1 and Y_2 is then

$$g(y_1, y_2) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} (y_1 y_2)^{\alpha-1} \left[y_1(1 - y_2)\right]^{\beta-1} e^{-y_1} \cdot (y_1)$$

$$= \frac{y_2^{\alpha-1}(1 - y_2)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} y_1^{\alpha+\beta-1} e^{-y_1}, \quad 0 < y_1 < \infty, \quad 0 < y_2 < 1,$$

$= 0$ elsewhere.

The marginal probability density function of Y_2 is

$$g_2(y_2) = \frac{y_2^{\alpha-1}(1 - y_2)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \int y_1^{\alpha+\beta-1} e^{-y_1} dy_1$$

$$= \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} y_2^{\alpha-1}(1 - y_2)^{\beta-1}, \quad 0 < y_2 < 1,$$

$= 0$ elsewhere .

This is the probability density function of the beta distribution with parameter α and β.

So we have proved that if x_1 and x_2 are independent, $X_1 \sim$ gamma (α, 1) and $X_2 \sim$ gamma (β, 1), then $\frac{X_1}{X_1 + X_2} \sim$ beta (α, β).

Bayes Method

The posterior distribution of θ given x denoted $\pi(\theta/x)$ is defined to be the conditional distribution of θ given the sample observation x. And with X has marginal (unconditional) density $m(x) = \int_\theta f(x/\theta)\pi(\theta)d\theta$, the posterior is $\pi(\theta/x) = \frac{f(x/\theta)\cdot\pi(\theta)}{m(x)}$ where $\pi(\theta)$ is the prior, usually given or assumed.

With the square loss $L = [\theta - E(\theta)]^2$, the Bayes estimate of θ/x is just the mean of the posterior distribution.

Generalized Bayes Estimate

For $X = (x_1, x_2, \ldots x_n)$, if each variable is Poisson distributed with parameter λ then we have $X = \sum_{i=1}^{n} x_i \sim Poisson(n\lambda)$. When the prior is a constant prior, which is $f(\lambda) = \begin{cases} c, \lambda \geq 0 \\ 0, o.w. \end{cases}$. The posterior distribution will be

$$f(\lambda \mid X) = \frac{f(\lambda, X)}{f(X)}$$

$$= \frac{f(X \mid \lambda)\cdot f(\lambda)}{\int_0^\infty f(X \mid \lambda)\cdot f(\lambda)d\lambda}$$

$$= \frac{\dfrac{e^{-n\lambda} \cdot \lambda^{\sum_{i=1}^{n} x_i}}{(\sum_{i=1}^{n} x_i)!} \cdot c}{\int_0^\infty \dfrac{e^{-n\lambda} \cdot \lambda^{\sum_{i=1}^{n} x_i}}{(\sum_{i=1}^{n} x_i)!} \cdot c d\lambda}$$

$$= \frac{e^{-n\lambda} \cdot \lambda^{\sum_{i=1}^{n} x_i}}{\int_0^\infty e^{-n\lambda} \cdot \lambda^{\sum_{i=1}^{n} x_i} d\lambda}$$

$$\propto \; e^{-n\lambda} \cdot \lambda^{\sum_{i=1}^{n} x_i}$$

$$\propto \; e^{-\frac{\lambda}{1/n}} \cdot \lambda^{\sum_{i=1}^{n} x_i + 1 - 1}$$

So we have $\lambda \mid X \sim gamma(\sum_{i=1}^{n} x_i + 1, \frac{1}{n})$.

CHAPTER 3

CONFIDENCE INTERVAL FOR A SIMPLE CASE

Theoretical Development

We start from when $k_1 = 1$, $k_2 = 1$. The problem then is simplified as: Let $\tilde{X}'_1 = (x_{11}, x_{12}, \ldots x_{1n_1})$, $\tilde{Y}_1' = (y_{11}, y_{22}, \ldots y_{1m_1})$ be two Poisson random samples with parameters λ_1 and μ_1. With the assumption that the prior distribution $\pi(\lambda_1) \sim gamma(\alpha_1, \beta_1)$, $\pi(\mu_1) \sim gamma(\delta_1, \gamma_1)$ we find an exact formula for $100(1-\alpha)\%$ confidence interval for the parameter $\theta = \frac{\lambda_1}{\mu_1}$. And with the square loss $L = [\theta - E(\theta)]^2$ we also find the Bayes estimate for $\theta = \frac{\lambda_1}{\mu_1}$.

We will use the Bayes' method and the knowledge in chapter1 to obtain a $100(1-\alpha)\%$ confidence interval for the parameter $\theta = \frac{\lambda_1}{\mu_1}$. Since $\tilde{X}'_1 = (x_{11}, x_{12}, \ldots x_{1n_1})$ is a Poisson random sample with parameter λ_1,

$$f(\tilde{X}_1 \mid \lambda_1) = f(X_{11} \mid \lambda_1) \cdot f(X_{12} \mid \lambda_1) \cdots f(X_{1n_1} \mid \lambda_1)$$

$$= \frac{\lambda_1^{x_{11}} \cdot e^{-\lambda_1}}{x_{11}!} \cdot \frac{\lambda_1^{x_{12}} \cdot e^{-\lambda_1}}{x_{12}!} \cdots \frac{\lambda_1^{x_{1n_1}} \cdot e^{-\lambda_1}}{x_{1n_1}!}$$

$$= \frac{\lambda_1^{\sum_{i=1}^{n_1} x_{1i}} \cdot e^{-n_1\lambda_1}}{\prod_{i=1}^{n_1} x_{1i}!} \quad , \qquad x_{1i} = 1,2,\ldots n$$

Since $\pi(\lambda_1) \sim gamma(\alpha_1, \beta_1)$

$\pi(\lambda_1) = \frac{1}{\Gamma(\alpha_1)\beta_1^{\alpha_1}} \lambda_1^{\alpha_1 - 1} e^{-\frac{\lambda_1}{\beta_1}}$, where $\lambda_1 > 0$

then, $f(\lambda_1 \mid \tilde{X}_1) \propto f(\tilde{X}_1 \mid \lambda_1) \cdot \pi(\lambda_1)$

$$\propto \lambda_1^{\sum_{i=1}^{n_1} x_{1i}} \cdot e^{-n_1\lambda_1} \cdot \lambda_1^{\alpha_1 - 1} \cdot e^{-\frac{\lambda_1}{\beta_1}}$$

$$\propto \lambda_1^{\sum_{i=1}^{n_1} x_{1i} + \alpha_1 - 1} \cdot e^{-\lambda_1 (\frac{1+n_1\beta_1}{\beta_1})}$$

so, $\lambda_1 \mid \tilde{X}_1 \sim$ gamma$(\sum_{i=1}^{n_1} x_{1i} + \alpha_1,\ \frac{\beta_1}{1+n_1\beta_1})$,

Similarly, $\mu_1 \mid \tilde{Y}_1 \sim$ gamma$(\sum_{i=1}^{m_1} y_{1i} + \delta_1,\ \frac{\gamma_1}{1+m_1\gamma_1})$.

Let $\frac{\beta_1}{1+n_1\beta_1} = \varphi_1$, $\frac{\gamma_1}{1+m_1\gamma_1} = \omega_1$

Then, $\frac{\lambda_1}{\varphi_1} \mid \tilde{X}_1 \sim$ gamma $(\sum_{i=1}^{n_1} x_{1i} + \alpha_1,\ 1)$, $\frac{\mu_1}{\omega_1} \mid \tilde{Y}_1 \sim$ gamma $(\sum_{i=1}^{m_1} y_{1i} + \delta_1,\ 1)$

So we have $\frac{\frac{\lambda_1}{\varphi_1}}{\frac{\lambda_1}{\varphi_1} + \frac{\mu_1}{\omega_1}} \sim$ Beta $(\sum_{i=1}^{n_1} x_{1i} + \alpha_1, \sum_{i=1}^{m_1} y_{1i} + \delta_1)$,

if $A = \sum_{i=1}^{n_1} x_{1i} + \alpha_1$, $B = \sum_{i=1}^{m_1} y_{1i} + \delta_1$, that is $\frac{1}{1 + (\frac{\varphi_1}{\omega_1}) \cdot (\frac{\lambda_1}{\mu_1})} \sim$ Beta (A, B)

With the given data it is easy to find the value of A and B. We can use Minitab to find a $100(1-\alpha)\%$ confidence interval $(\beta_{\frac{\alpha}{2}}, \beta_{1-\frac{\alpha}{2}})$ for the parameter $\frac{1}{1+(\frac{\varphi_1}{\omega_1})\cdot(\frac{\lambda_1}{\mu_1})}$. This can be written as

$\Pr(\beta_{\frac{\alpha}{2}} < \frac{1}{1+(\frac{\varphi_1}{\omega_1})\cdot(\frac{\lambda_1}{\mu_1})} < \beta_{1-\frac{\alpha}{2}}) = 100(1-\alpha)\%$. After some algebraic calculation

this will be $\Pr(\frac{\beta_{\frac{\alpha}{2}}}{1-\beta_{\frac{\alpha}{2}}} \cdot \frac{\varphi_1}{\omega_1} < \frac{\lambda_1}{\mu_1} < \frac{\beta_{1-\frac{\alpha}{2}}}{1-\beta_{1-\frac{\alpha}{2}}} \cdot \frac{\varphi_1}{\omega_1}) = 100(1-\alpha)\%$.

Thus, the $100(1-\alpha)\%$ confidence interval for the parameter $\theta = \frac{\lambda_1}{\mu_1}$ is

$(\frac{\beta_{\frac{\alpha}{2}}}{1-\beta_{\frac{\alpha}{2}}} \cdot \frac{\varphi_1}{\omega_1} \; , \; \frac{\beta_{1-\frac{\alpha}{2}}}{1-\beta_{1-\frac{\alpha}{2}}} \cdot \frac{\varphi_1}{\omega_1})$.

Under square loss $L = [\theta - E(\theta)]^2$, the Bayes estimate for the parameter $\frac{1}{1+(\frac{\varphi_1}{\omega_1})\cdot(\frac{\lambda_1}{\mu_1})}$ is just the mean of beta distribution with parameters A and B, witch is $\frac{A}{A+B}$. Then we have $\frac{1}{1+(\frac{\varphi_1}{\omega_1})\cdot(\frac{\lambda_1}{\mu_1})} = \frac{A}{A+B}$,

then $\frac{\lambda_1}{\mu_1} = \frac{\varphi_1}{\omega_1} \cdot \frac{A}{B}$, which is just the Bayes estimate for the parameter $\theta = \frac{\lambda_1}{\mu_1}$.

Example 1

With the assumption that the prior distribution

$\pi(\lambda_1) \sim gamma(\alpha_1 = 2, \beta_1 = 2)$, $\pi(\mu_1) \sim gamma(\delta_1 = 3, \gamma_1 = 2)$, we use Minitab to generate a random value for λ_1 and μ_1, and we get $\lambda_1 = 5.961133$, $\mu_1 = 9.52169$. Then we use $\tilde{X}_1 \sim \mathrm{P}(\lambda_1 = 5.961133)$ to generate 25 data for $\tilde{X}_1$, that is $\tilde{X}_1' = (9,7,7,7,5,...,2,5,5,3,6,6)$, $n_1 = 25$; and we use $\tilde{Y}_1 \sim \mathrm{P}(\mu_1 = 9.52169)$ to generate 20 data for $\tilde{Y}_1$, that is $\tilde{Y}'_1 = (6,12,11,12,8,...,5,8,11,8,6,9)$, $m_1 = 20$. In this example we find a $100(1-\alpha)\% = 95\%$ confidence interval for the parameter $\theta = \frac{\lambda_1}{\mu_1}$. And with the square loss $L = [\theta - E(\theta)]^2$ we find the Bayes estimate for $\theta = \frac{\lambda_1}{\mu_1}$.

According to what we have done in Chapter 2 we have

$\frac{1}{1+(\frac{\varphi_1}{\omega_1})\cdot(\frac{\lambda_1}{\mu_1})} \sim$ Beta (A, B), where $A = \sum_{i=1}^{n_1} x_{1i} + \alpha_1$, $B = \sum_{i=1}^{m_1} y_{1i} + \delta_1$

and $\frac{\beta_1}{1+n_1\beta_1} = \varphi_1$, $\frac{\gamma_1}{1+m_1\gamma_1} = \omega_1$.

Then $A = \sum_{i=1}^{n_1} x_{1i} + \alpha_1 = 145 + 2 = 147$, $B = \sum_{i=1}^{m_1} y_{1i} + \delta_1 = 189 + 3 = 192$,

$\varphi_1 = \frac{\beta_1}{1+n_1\beta_1} = \frac{2}{1+25\times 2} = \frac{2}{51}$, $\omega_1 = \frac{\gamma_1}{1+m_1\gamma_1} = \frac{2}{1+20\times 2} = \frac{2}{41}$,

Since $\Pr(\beta_{\frac{\alpha}{2}} < \frac{1}{1+(\frac{\varphi_1}{\omega_1})\cdot(\frac{\lambda_1}{\mu_1})} < \beta_{1-\frac{\alpha}{2}}) = 100(1-\alpha)\% = 95\%$ that is

$\Pr(\frac{\beta_{\frac{\alpha}{2}}}{1-\beta_{\frac{\alpha}{2}}}\cdot\frac{\varphi_1}{\omega_1}<\frac{\lambda_1}{\mu_1}<\frac{\beta_{1-\frac{\alpha}{2}}}{1-\beta_{1-\frac{\alpha}{2}}}\cdot\frac{\varphi_1}{\omega_1})=95\%$, we can use Minitab to get

$\beta_{\frac{\alpha}{2}}=0.381356$, $\beta_{1-\frac{\alpha}{2}}=0.486642$. Thus the $(1-\alpha)\%=95\%$ confidence interval

for the parameter $\theta=\frac{\lambda_1}{\mu_1}$ is $(\frac{\beta_{\frac{\alpha}{2}}}{1-\beta_{\frac{\alpha}{2}}}\cdot\frac{\varphi_1}{\omega_1}, \frac{\beta_{1-\frac{\alpha}{2}}}{1-\beta_{1-\frac{\alpha}{2}}}\cdot\frac{\varphi_1}{\omega_1})$, which is

$(0.4955683, 0.7620842)$.

Under square loss $L=[\theta-E(\theta)]^2$, the Bayes estimate for the

parameter $\frac{1}{1+(\frac{\varphi_1}{\omega_1})\cdot(\frac{\lambda_1}{\mu_1})}$ is just the mean of beta distribution with

parameters A and B, witch is $\frac{A}{A+B}$. Then we have $\frac{1}{1+(\frac{\varphi_1}{\omega_1})\cdot(\frac{\lambda_1}{\mu_1})}=\frac{A}{A+B}$,

then $\frac{\lambda_1}{\mu_1}=\frac{\varphi_1}{\omega_1}\cdot\frac{A}{B}=\frac{41}{51}\cdot\frac{147}{192}=0.615502$, which is just the Bayes estimate for

the parameter $\theta=\frac{\lambda_1}{\mu_1}$, this estimate value lies in the $100(1-\alpha)\%=95\%$

confidence interval $(0.4955683, 0.7620842)$, and it is also close to the value

of $\frac{\lambda_1}{\mu_1}=0.6260583$.

Example 2

In Sahai and Khurshid' s article, they calculated the confidence interval using a few method with the data $x_1=12$, $y_1=13$. With the same data, we will use our method to calculate the confidence interval for $\theta = \lambda / \mu$ and compare the result with that of the exact method from Sahai and Khurshid' s article.

Since there is no prior assumption in the exact method from Sahai and Khurshid' s article, we have to make our prior assumption non-informative. For the prior $\pi(\lambda_1) \sim gamma(\alpha_1, \beta_1)$, $\pi(\mu_1) \sim gamma(\delta_1, \gamma_1)$, we should take α_1, equal to 1 and β_1, γ_1 equal to a big number to make the prior non-informative. With such prior assumption, the posterior will be $\lambda_1 \mid X_1 \sim gamma(13,1)$ and $\mu_1 \mid Y_1 \sim gamma(14,1)$. Then we have the critical values for $Beta(13,14)$ are $\beta_{\frac{\alpha}{2}} = 0.299272$, $\beta_{1-\frac{\alpha}{2}} = 0.666292$. The confidence interval is $(0.4271, 1.9967)$. This is close to the confidence interval $(0.3860, 2.1990)$ in Sahai and Khurshid's article.

If we take the prior as a generalized constant prior, we will get the same posterior and the same result for this example.

CHAPTER 4

CONFIDENCE INTERVAL FOR A COMPLICATED CASE

Illustration and procedure of R program

We have developed an exact formula in chapter 2 for the simple case when $k_1 = 1$, $k_2 = 1$. When k_1 and k_2 are not both one at the same time, which means for the ratio $\theta = \dfrac{\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdots \lambda_{k_1}}{\mu_1 \cdot \mu_2 \cdot \mu_3 \cdots \mu_{k_2}}$ when there are more than one parameters for the numerator or denominator or both, the case could become much more complicated and we can not find a formal distribution for the ratio.

In this chapter, we will develop a numerical way to find the confidence interval for this case. And an R program is developed according to the following procedure.

1. We assume the prior for $\lambda_1, \lambda_2, \ldots \lambda_{k_1}; \mu_1, \mu_2, \ldots \mu_{k_2}$, this can be expressed as following:

   $\lambda_1 \sim gamma(\alpha_1, \beta_1)$

   $\lambda_2 \sim gamma(\alpha_2, \beta_2)$

   .

   .

.

$\lambda_{k_1} \sim gamma(\alpha_{k_1}, \beta_{k_2})$

$\mu_1 \sim \text{gamma}(\delta_1, \gamma_1)$

$\mu_2 \sim \text{gamma}(\delta_2, \gamma_2)$

.

.

.

$\mu_{k_2} \sim \text{gamma}(\delta_{k_2}, \gamma_{k_2})$

Here, the values of $\alpha's, \beta's, \delta's, \gamma's$ will be assumed for each prior. And random values for $\lambda's$ and $\mu's$ will be generated.

2. We use the generated values for $\lambda's$ and $\mu's$ to generate the data vectors. This is expressed as following:

$\tilde{X}_1' = (x_{11}, x_{12}, ..., x_{1n_1}) \sim \text{P}(\lambda_1)$

$\tilde{X}_2' = (x_{21}, x_{22}, ..., x_{2n_2}) \sim \text{P}(\lambda_2)$

.

.

.

$\tilde{X}_{k_1}' = (x_{k_1 1}, x_{k_1 2}, ..., x_{k_1 n_{k_1}}) \sim \text{P}(\lambda_{k_1})$

$\tilde{Y}_1' = (\text{y}_{11}, y_{12}, ..., y_{1m_1}) \sim \text{P}(\mu_1)$

$\tilde{Y}_2' = (\text{y}_{21}, y_{22}, ..., y_{2m_2}) \sim \text{P}(\mu_2)$

.

.

.

$\tilde{Y}_{k_2}' = (y_{k_2 1}, y_{k_2 2}, \ldots, y_{k_2 m_{k_2}}) \sim P(\mu_{k_2})$

3. As we have proved, the conditional parameters for $\lambda \mid \tilde{X}$ and $\mu \mid \tilde{Y}$ will also be gamma distributed, the parameters of these gamma distributions are known so we can generate the random values(a large number, say 1000000) for $\lambda \mid \tilde{X}$'s and $\mu \mid \tilde{Y}$'s. This is expressed as following:

$$\lambda_1 \mid \tilde{X}_1 \sim \text{gamma}(\sum_{i=1}^{n_1} x_{1i} + \alpha_1, \frac{\beta_1}{1+n_1\beta_1})$$

$$\lambda_2 \mid \tilde{X}_2 \sim \text{gamma}(\sum_{i=1}^{n_2} x_{2i} + \alpha_2, \frac{\beta_2}{1+n_2\beta_2})$$

.

.

.

$$\lambda_{k_1} \mid \tilde{X}_{k_1} \sim \text{gamma}(\sum_{i=1}^{n_{k_1}} x_{k_1 i} + \alpha_{k_1}, \frac{\beta_{k_1}}{1+n_{k_1}\beta_{k_1}})$$

$$\mu_1 \mid \tilde{Y}_1 \sim \text{gamma}(\sum_{i=1}^{m_1} y_{1i} + \delta_1, \frac{\gamma_1}{1+m_1\gamma_1})$$

$$\mu_2 \mid \tilde{Y}_2 \sim \text{gamma}(\sum_{i=1}^{m_2} y_{2i} + \delta_2, \frac{\gamma_2}{1+m_2\gamma_2})$$

.

.

$$\mu_{k_2} \mid \tilde{Y}_{k_2} \sim \text{gamma}(\sum_{i=1}^{m_{k_2}} y_{k_2 i} + \delta_{k_2}, \frac{\gamma_{k_2}}{1 + m_{k_2}\gamma_{k_2}})$$

4. Using the generated values in step 3 we calculate $\theta = \frac{\lambda_1 \cdot \lambda_2 \cdots \lambda_{k_1}}{\mu_1 \cdot \mu_2 \cdots \mu_{k_2}}$ and we will get a large number (1000000) of values for this ratio, let's say they are 1000000 q's.
5. We just need to put these 1000000 q's in order and it will be easy to get a $100(1-\alpha)\%$ confidence interval.

Example 1

Since the R program is developed for any non-negative number k_1 and k_2. In this example we will calculate the $100(1\text{-}\alpha)\%$ =95% confidence interval for the ratio $\theta = \frac{\lambda_1}{\mu_1}$, this is a simple case when $k_1 = 1$, $k_2 = 1$. The following is the procedure and results:

1. We assume the parameters of the priors for $\lambda_1 \sim gamma(\alpha_1, \beta_1)$ and $\mu_1 \sim \text{gamma}(\delta_1, \gamma_1)$. Let $\alpha_1 = 2$, $\beta_1 = 2$ and $\delta_1 = 3$, $\gamma_1 = 2$.
2. A random value for λ_1 and μ_1 will be generated. $\lambda_1 = 5.961133$, $\mu_1 = 9.52169$.
3. Then we use $\lambda_1 = 5.961133$ as the Poisson parameter to generate 25 data for $\tilde{X}_1$, that is $\tilde{X}_1' = (9,7,7,7,5,...,2,5,5,3,6,6)$, $n_1 = 25$; and we use

$\mu_1 = 9.52169$ as the Poisson parameter to generate 20 data for $\tilde{Y}_1$, that is $\tilde{Y}'_1 = (6,12,11,12,8,...,5,8,11,8,6,9)$, $m_1 = 20$.

4. The sum of Poisson data of each Poisson distribution will be calculated and will get the values of $\sum_{i=1}^{n_1} x_{1i} + \alpha_1$ and $\sum_{i=1}^{m_1} y_{1i} + \delta_1$.

5. We use $\lambda_1 | \tilde{X}_1 \sim \text{gamma}(\sum_{i=1}^{n_1} x_{1i} + \alpha_1, \frac{\beta_1}{1+n_1\beta_1})$ and $\mu_1 | \tilde{Y}_1 \sim \text{gamma}(\sum_{i=1}^{m_1} y_{1i} + \delta_1, \frac{\gamma_1}{1+m_1\gamma_1})$ to generate 1000000 values for $\lambda_1 | \tilde{X}_1$ and $\mu_1 | \tilde{Y}_1$.

6. The ratio $\theta = \frac{\lambda_1}{\mu_1}$ will be calculated and we will get 1000000 values of $\theta = \frac{\lambda_1}{\mu_1}$. We put them in order and it is easy to get $100(1\text{-}\alpha)\%=95\%$ confidence interval $(0.4957661, 0.7623171)$.

Example 2

In this example we will do the case when $k_1 = 2$ and $k_2 = 2$. The problem becomes to how to calculate the $100(1\text{-}\alpha)\%$ confidence interval for $\theta = \frac{\lambda_1 \cdot \lambda_2}{\mu_1 \cdot \mu_2}$. We will use the R program to do this and the procedure is as the following:

1. Assume the parameters for the priors:

   $\lambda_1 \sim gamma(\alpha_1 = 2, \beta_1 = 2)$

$\lambda_2 \sim gamma(\alpha_2 = 2, \beta_2 = 3)$

$\mu_1 \sim \text{gamma}(\delta_1 = 3, \gamma_1 = 3)$

$\mu_2 \sim \text{gamma}(\delta_2 = 4, \gamma_2 = 3)$

2. A random value for $\lambda_1, \lambda_2, \mu_1 and \mu_2$ will be generated:

$\lambda_1 = 1.028019$

$\lambda_2 = 3.547733$

$\mu_1 = 4.95997$

$\mu_2 = 8.370036$

3. With the input data numbers for each Poisson distribution the Poisson distributions are generated:

$\tilde{X}_1' = (2,2,0,0,4,...,1,0,2,0,1,0) \sim \text{P}(\lambda_1 = 1.028019)$, $n_1 = 20$

$\tilde{X}_2' = (4,2,2,0,...,4,3,4,6) \sim \text{P}(\lambda_2 = 3.547733)$, $n_2 = 15$

$\tilde{Y}_1' = (4,3,6,6,9,...,2,7,7,8,7) \sim \text{P}(\mu_1 = 4.95997)$, $m_1 = 20$

$\tilde{Y}_2' = (3,13,9,5,4,...,10,12,5,9,9,8) \sim \text{P}(\mu_2 = 8.370036)$, $m_2 = 25$

4. With the above assumption and Poisson data, 1000000 posterior random variables will be generated according to the following:

$$\lambda_1 \mid \tilde{X}_1 \sim \text{gamma}\left(\sum_{i=1}^{n_1} x_{1i} + \alpha_1, \frac{\beta_1}{1 + n_1\beta_1}\right)$$

$$\lambda_2 \mid \tilde{X}_2 \sim \text{gamma}\left(\sum_{i=1}^{n_2} x_{2i} + \alpha_2, \frac{\beta_2}{1 + n_2\beta_2}\right)$$

$$\mu_1 \mid \tilde{\text{Y}}_1 \sim \text{gamma}\left(\sum_{i=1}^{m_1} y_{1i} + \delta_1, \frac{\gamma_1}{1 + m_1\gamma_1}\right)$$

$$\mu_2 | \tilde{Y}_2 \sim \text{gamma}(\sum_{i=1}^{m_2} y_{2i} + \delta_2, \frac{\gamma_2}{1+m_2\gamma_2})$$

5. 1000000 values for the ratio $\theta = \frac{\lambda_1\lambda_2}{\mu_1\mu_2}$ will be calculated.

6. We just need to put these 1000000 values of θ in order, and with the assumption $\alpha = 0.05$ it's easy to get 100(1-α)%=95% confidence interval for θ, which is $(0.04432612, 0.1341951)$.

Example 3

In this example we will do the case when $k_1 = 3$ and $k_2 = 1$. The problem becomes to how to calculate the 100(1-α)% confidence interval for $\theta = \frac{\lambda_1 \cdot \lambda_2 \cdot \lambda_3}{\mu_1}$. We will use the R program to do this and the procedure is as the following:

1. Assume the parameters for the priors:

$$\lambda_1 \sim gamma(\alpha_1 = 2, \beta_1 = 1)$$

$$\lambda_2 \sim gamma(\alpha_2 = 2, \beta_2 = 2)$$

$$\lambda_3 \sim \text{gamma}(\alpha_3 = 3, \beta_3 = 1)$$

$$\mu_1 \sim \text{gamma}(\delta_1 = 4, \gamma_1 = 2)$$

2. A random value for $\lambda_1, \lambda_2, \lambda_3 and\, \mu_1$ will be generated:

$$\lambda_1 = 1.529129$$

$$\lambda_2 = 3.956051$$

$\lambda_3 = 1.258633$

$\mu_1 = 14.86363$

3. With the input data numbers for each Poisson distribution the Poisson distributions are generated:

$\tilde{X}_1' = (0,1,1,0,4,...,1,3,0,2,2,1) \sim P(\lambda_1 = 1.529129)$, $n_1 = 25$

$\tilde{X}_2' = (3,7,5,4,3,...,6,3,2,1,6) \sim P(\lambda_2 = 3.956051)$, $n_2 = 15$

$\tilde{X}_3' = (1,3,0,0,2,0,1,0,2,0) \sim P(\lambda_3 = 1.258633)$, $n_3 = 10$

$\tilde{Y}_1' = (18,10,10,15,...,11,10,17,15) \sim P(\mu_1 = 14.86363)$, $m_1 = 25$

4. With the above assumption and Poisson data 1000000 posterior random variables will be generated according to the following :

$$\lambda_1 \mid \tilde{X}_1 \sim \text{gamma}(\sum_{i=1}^{n_1} x_{1i} + \alpha_1, \frac{\beta_1}{1+n_1\beta_1})$$

$$\lambda_2 \mid \tilde{X}_2 \sim \text{gamma}(\sum_{i=1}^{n_2} x_{2i} + \alpha_2, \frac{\beta_2}{1+n_2\beta_2})$$

$$\lambda_3 \mid \tilde{X}_3 \sim \text{gamma}(\sum_{i=1}^{n_3} x_{3i} + \alpha_3, \frac{\beta_3}{1+n_3\beta_3})$$

$$\mu_1 \mid \tilde{Y}_1 \sim \text{gamma}(\sum_{i=1}^{m_1} y_{1i} + \delta_1, \frac{\gamma_1}{1+m_1\gamma_1})$$

5. 1000000 values for the ratio $\theta = \frac{\lambda_1 \cdot \lambda_2 \cdot \lambda_3}{\mu_1}$ will be calculated.

6. We just need to put these 1000000 values of θ in order, and with the assumption $\alpha = 0.05$ it's easy to get $100(1\text{-}\alpha)\%{=}95\%$ confidence interval for θ, which is $(0.1925228, 0,8255976)$.

Example 4

In this example we will do the case when $k_1 = 2$ and $k_2 = 3$. The problem becomes to how to calculate the 100(1-α)% confidence interval for $\theta = \frac{\lambda_1 \cdot \lambda_2}{\mu_1 \cdot \mu_2 \cdot \mu_3}$. We will use the R program to do this and the procedure is as the following:

1. Assume the parameters for the priors:

$\lambda_1 \sim gamma(\alpha_1 = 2, \beta_1 = 2)$

$\lambda_2 \sim gamma(\alpha_2 = 3, \beta_2 = 3)$

$\mu_1 \sim \text{gamma}(\delta_1 = 3, \gamma_1 = 2)$

$\mu_2 \sim \text{gamma}(\delta_2 = 4, \gamma_2 = 2)$

$\mu_3 \sim \text{gamma}(\delta_3 = 5, \gamma_3 = 1)$

2. A random value for $\lambda_1, \lambda_2, \mu_1, \mu_2, and\, \mu_3$ will be generated:

$\lambda_1 = 2.060402$

$\lambda_2 = 8.268878$

$\mu_1 = 1.069710$

$\mu_2 = 4.241796$

$\mu_3 = 4.004304$

3. With the input data numbers for each Poisson distribution the Poisson distributions are generated:

$\tilde{X}_1' = (3,3,1,3,1,...,1,3,3,3,5) \sim \text{P}(\lambda_1 = 2.060402)$, $n_1 = 25$

$\tilde{X}_2' = (8,12,15,9,8,...,9,11,8,12,11)$~$P(\lambda_2 = 8.268878)$, $n_2 = 20$

$\tilde{Y}_1' = (1,0,3,0,1,...,3,2,0,1,1,0)$~$P(\mu_1 = 1.069710)$, $m_1 = 15$

$\tilde{Y}_2' = (2,7,5,0,2,...,5,2,3,6,5)$~$P(\mu_2 = 4.241796)$, $m_2 = 15$

$\tilde{Y}_3' = (2,2,5,3,2,...,4,3,5,3,6,2)$~$P(\mu_3 = 4.004304)$, $m_3 = 25$

4. With the above assumption and Poisson data, 1000000 posterior random variables will be generated according to the following:

$$\lambda_1 \mid \tilde{X}_1 \sim \text{gamma}(\sum_{i=1}^{n_1} x_{1i} + \alpha_1, \frac{\beta_1}{1+n_1\beta_1})$$

$$\lambda_2 \mid \tilde{X}_2 \sim \text{gamma}(\sum_{i=1}^{n_2} x_{2i} + \alpha_2, \frac{\beta_2}{1+n_2\beta_2})$$

$$\mu_1 \mid \tilde{Y}_1 \sim \text{gamma}(\sum_{i=1}^{m_1} y_{1i} + \delta_1, \frac{\gamma_1}{1+m_1\gamma_1})$$

$$\mu_2 \mid \tilde{Y}_2 \sim \text{gamma}(\sum_{i=1}^{m_2} y_{2i} + \delta_2, \frac{\gamma_2}{1+m_2\gamma_2})$$

$$\mu_3 \mid \tilde{Y}_3 \sim \text{gamma}(\sum_{i=1}^{m_3} y_{3i} + \delta_2, \frac{\gamma_3}{1+m_3\gamma_3})$$

5. 1000000 values for the ratio $\theta = \frac{\lambda_1 \cdot \lambda_2}{\mu_1 \cdot \mu_2 \cdot \mu_3}$ will be calculated.

6. We just need to put these 1000000 values of θ in order, and with the assumption $\alpha = 0.05$ it's easy to get 100(1-α)%=95% confidence interval for θ, which is $(0.5111273, 1.820691)$.

Example 5

We will calculate the confidence upper limits according to the data from table 1 and table 2 in Harris' article. Since there is no prior assumption in Sahai and Khurshid's article, we have to make our prior assumption informative. For the prior $\pi(\lambda_1) \sim gamma(\alpha_1, \beta_1)$, $\pi(\lambda_2) \sim gamma(\alpha_2, \beta_2)$, we should take α_1, α_2 equal to 1 and β_1, β_2 equal to a big number which is close to infinitive to make the prior informative.

With data $X_1 = 3, X_2 = 5$ and because there is only one variable for each sample, we have $n_1 = 1, n_2 = 1$ it is easy to get the posterior $\lambda_1 / X_1 \sim gamma(4,1)$, $\lambda_2 / X_2 \sim gamma(6,1)$. We generate 1000000 values of λ_1 / X_1 and λ_2 / X_2 to get 1000000 values of $\lambda_1 \lambda_2$ and we will get the 0.90 upper confidence limit. Divide this limit by 10000 which is just the product of sample size of $n_1 = 100$ and $n_2 = 100$ we will get the upper confidence limit for $p_1 \cdot p_2$. The result is 0.00449. The upper limits for $p_1 \cdot p_2$ when X_1 and X_2 take other values are calculated similarly, and we also use the same method to calculated the upper limits for $p_1 \cdot p_2 \cdot p_3$. The results are given in the following two tables.

If we take the prior as a generalized constant prior, we will get the same posterior and the same result for this example.

Table 1 APPROXIMATE UPPER CONFIDENCE LIMITS FOR

$p_1 p_2$ CONFIDENCE COEFFICIENT α = .90

Sample sizes n_1, n_2	Observed values x_1, x_2	Buehler's limit	Madansky Likelihood ratio	Madansky linearized	Harris Limit based on (8)	Harris "Randomized" limit based on (8)	Our method
100, 100	3, 5	.00412	.00433	.00164	.00486	.00416	.00449
100, 100	1, 4	.00188	.00182	.00097	.00235	.00184	.00210
100, 100	2, 2	.00168	.00167	.00091	.00211	.00170	.00187
150, 150	3, 3	.00128	.00133	.00074	.00153	.00128	.00139
n, n	0, 0	$1.33/n^2$	$.458/n^2$	0	$3.72/n^2$	$2.29/n^2$	$2.59/n^2$

Table 2 APPROXIMATE UPPER CONFIDENCE LIMITS FOR

$p_1 p_2 p_3$ CONFIDENCE COEFFICIENT α = .90

Simple sizes n_1, n_2, n_3	Observed values x_1, x_2, x_3	Madansky's likelihood ratio	Harris Limit based on (8)	Harris "Randomized" limit based on (8)	Our method
100, 100, 100	1, 2, 1	.000019	.000019	.000027	.000029
100, 100, 100	2, 3, 5	.000133	.000186	.000145	.000153

CHAPTER 5

R PROGRAM AND RESULTS

Example 1

```
> alph=c(2,2)
> peta=c(2,3)
> a2=c(3,4)
> p2=c(2,3)
> n=c(25,15)
> m=c(20,45)
> lambda=0
> l=0
> num=1
> den=1
> for ( i in 1:1) {
+ lambda[i]=rgamma(1,shape=alph[i],scale=peta[i])
+ num=num*lambda[i]}
>
> for ( j in 1:1) {
+ l[j]=rgamma(1,shape=a2[j],scale=p2[j])
+ den=den*l[j]}
>
> ratio=num/den
>
> x1=rpois(n[1],lambda[1])
> y1=rpois(m[1],l[1])
>
> a=0
> p=0
> q=0
> b=0
>
> a[1]=sum(x1)+alph[1] #a records the alpha values of the numerator
posterial distrn
>
```

```
> p[1]=peta[1]/(1+n[1]*peta[1])#p records the peta values of the
numerator posterial distrn
>
> b[1]=sum(y1)+a2[1]
> #b records the alpha values of the denominator posterial distrn
> q[1]=p2[1]/(1+m[1]*p2[1])
>
>
> #q records the peta values of the denominator posterial distrn
>
> lx1=rgamma(1000000,shape=a[1],scale=p[1])
>
> my1=rgamma(1000000,shape=b[1],scale=q[1])
>
>
> lxm=lx1/my1
> s=order(lxm)
> c=rbind(lxm[s])
> lcl=c[25000]
> ucl=c[975000]
> lambda[1]
[1] 5.961133
> l[1]
[1] 9.52169
> lcl
[1] 0.4957661
> ucl
[1] 0.7623171
> ratio
[1] 0.6260583
> x1
 [1]  9  7  7  7  5  1  5  5  5  6  9  9  5  6  5  2 13  6  6  2  5  5  3  6  6
> y1
 [1]  6 12 11 11 12  8  9 10 13  9  7 14  9 11  5  8 11  8  6  9
> a[1]
[1] 147
> p[1]
[1] 0.03921569
> b[1]
[1] 192
> q[1]
[1] 0.04878049
```

Example 2

```
> alph=c(2,2)
> peta=c(2,3)
> a2=c(3,4)
> p2=c(3,3)
> n=c(20,15)
> m=c(20,25)
> lambda=0
> l=0
> num=1
> den=1
> for ( i in 1:2) {
+ lambda[i]=rgamma(1,shape=alph[i],scale=peta[i])
+ num=num*lambda[i]}
>
> for ( j in 1:2) {
+ l[j]=rgamma(1,shape=a2[j],scale=p2[j])
+ den=den*l[j]}
>
> ratio=num/den
>
> x1=rpois(n[1],lambda[1])
> x2=rpois(n[2],lambda[2])
>
> y1=rpois(m[1],l[1])
> y2=rpois(m[2],l[2])
>
> a=0
> p=0
> q=0
> b=0
>
> a[1]=sum(x1)+alph[1] #a records the alpha values of the numerator
posterial distrn
> a[2]=sum(x2)+alph[2]
> p[1]=peta[1]/(1+n[1]*peta[1])#p records the peta values of the
numerator posterial distrn
> p[2]=peta[2]/(1+n[2]*peta[2])
> b[1]=sum(y1)+a2[1]
> b[2]=sum(y2)+a2[2]#b records the alpha values of the denominator
posterial distrn
> q[1]=p2[1]/(1+m[1]*p2[1])
> q[2]=p2[2]/(1+m[2]*p2[2])
>
> #q records the peta values of the denominator posterial distrn
```

```
>
> lx1=rgamma(1000000,shape=a[1],scale=p[1])
> lx2=rgamma(1000000,shape=a[2],scale=p[2])
> my1=rgamma(1000000,shape=b[1],scale=q[1])
> my2=rgamma(1000000,shape=b[2],scale=q[2])
>
> lxm=lx1*lx2/(my1*my2)
> s=order(lxm)
> c=rbind(lxm[s])
> lcl=c[25000]
> ucl=c[975000]
> lcl
[1] 0.04432612
> ucl
[1] 0.1341951
> ratio
[1] 0.0878508
> lambda[1]
[1] 1.028019
> x1
 [1] 2 2 0 0 4 1 3 0 1 3 0 1 0 0 1 0 2 0 1 0
> lambda[2]
[1] 3.547733
> x2
 [1] 4 2 2 0 2 5 2 4 2 2 4 4 3 4 6
> l[1]
[1] 4.95997
> y1
 [1] 4 3 6 6 9 4 6 3 8 4 1 3 5 3 5 2 7 7 8 7
> l[2]
[1] 8.370036
> y2
 [1]  3 13  9  5  4  8 12 10 12  5 12 10 12  8  7 12  7  5  6 10 12  5  9  9
8
```

Example 3

```
> alph=c(2,2,3)
> peta=c(1,2,1)
> a2=c(4,4)
> p2=c(2,3)
> n=c(20,15,10)
> m=c(25,45)
> lambda=0
> l=0
```

```
> num=1
> den=1
> for ( i in 1:3) {
+ lambda[i]=rgamma(1,shape=alph[i],scale=peta[i])
+ num=num*lambda[i]}
>
> for ( j in 1:1) {
+ l[j]=rgamma(1,shape=a2[j],scale=p2[j])
+ den=den*l[j]}
>
> ratio=num/den
>
> x1=rpois(n[1],lambda[1])
> x2=rpois(n[2],lambda[2])
>
> x3=rpois(n[3],lambda[3])
>
> y1=rpois(m[1],l[1])
>
>
> a=0
> p=0
> q=0
> b=0
>
> a[1]=sum(x1)+alph[1] #a records the alpha values of the numerator
posterial distrn
> a[2]=sum(x2)+alph[2]
> a[3]=sum(x3)+alph[3]
>
>
> p[1]=peta[1]/(1+n[1]*peta[1])#p records the peta values of the
numerator posterial distrn
> p[2]=peta[2]/(1+n[2]*peta[2])
> p[3]=peta[3]/(1+n[3]*peta[3])
>
> b[1]=sum(y1)+a2[1]
> #b records the alpha values of the denominator posterial distrn
> q[1]=p2[1]/(1+m[1]*p2[1])
>
>
> #q records the peta values of the denominator posterial distrn
>
> lx1=rgamma(1000000,shape=a[1],scale=p[1])
> lx2=rgamma(1000000,shape=a[2],scale=p[2])
> lx3=rgamma(1000000,shape=a[3],scale=p[3])
```

```
>
> my1=rgamma(1000000,shape=b[1],scale=q[1])
>
> lxm=lx1*lx2*lx3/my1
> s=order(lxm)
> c=rbind(lxm[s])
> lcl=c[25000]
> ucl=c[975000]
> lcl
[1] 0.1925228
> ucl
[1] 0.8255976
> ratio
[1] 0.5122483
> lambda[1]
[1] 1.529129
> x1
 [1] 0 1 1 0 4 2 0 2 3 2 2 4 0 1 1 3 0 2 2 1
> lambda[2]
[1] 3.956051
> x2
 [1] 3 7 5 4 3 4 2 4 5 1 6 3 2 1 6
> lambda[3]
[1] 1.258633
> x3
 [1] 1 3 0 0 2 0 1 0 2 0
> l[1]
[1] 14.86363
> y1
 [1] 18 10 10 15 12 17 26 11 15  7 12 24 16 23 16 18 11 16 10 15 14 11
10 17 15
```

Example 4

```
> alph=c(2,3)
> peta=c(2,3)
> a2=c(3,4,5)
> p2=c(2,2,1)
> n=c(25,20)
> m=c(15,15,25)
> lambda=0
> l=0
> num=1
> den=1
> for ( i in 1:2) {
```

```
+ lambda[i]=rgamma(1,shape=alph[i],scale=peta[i])
+ num=num*lambda[i]}
> for ( j in 1:3) {
+ l[j]=rgamma(1,shape=a2[j],scale=p2[j])
+ den=den*l[j]}
> ratio=num/den
> x1=rpois(n[1],lambda[1])
> x2=rpois(n[2],lambda[2])
> y1=rpois(m[1],l[1])
> y2=rpois(m[2],l[2])
> y3=rpois(m[3],l[3])
> a=0
> p=0
> q=0
> b=0
> a[1]=sum(x1)+alph[1] #a records the alpha values of the numerator
posterial distrn
> a[2]=sum(x2)+alph[2]
> p[1]=peta[1]/(1+n[1]*peta[1])#p records the peta values of the
numerator posterial distrn
> p[2]=peta[2]/(1+n[2]*peta[2])
> b[1]=sum(y1)+a2[1]
> b[2]=sum(y2)+a2[2]
> b[3]=sum(y3)+a2[3]#b records the alpha values of the denominator
posterial distrn
> q[1]=p2[1]/(1+m[1]*p2[1])
> q[2]=p2[2]/(1+m[2]*p2[2])
> q[3]=p2[3]/(1+m[3]*p2[3])
> #q records the peta values of the denominator posterial distrn
>
> lx1=rgamma(1000000,shape=a[1],scale=p[1])
> lx2=rgamma(1000000,shape=a[2],scale=p[2])
> my1=rgamma(1000000,shape=b[1],scale=q[1])
> my2=rgamma(1000000,shape=b[2],scale=q[2])
> my3=rgamma(1000000,shape=b[3],scale=q[3])
> lxm=lx1*lx2/(my1*my2*my3)
> s=order(lxm)
> c=rbind(lxm[s])
> lcl=c[25000]
> ucl=c[975000]
> lcl
[1] 0.5111273
> ucl
[1] 1.820691
> lambda[1]
[1] 2.060402
```

```
> x1
 [1] 3 3 1 3 1 2 1 4 0 3 2 1 1 3 0 0 3 0 1 4 1 3 3 3 5
> lambda[2]
[1] 8.268878
> x2
 [1]  8 12 15  9  8  6  7  8 13  6 10  9 11 14  8  9 11  8 12 11
> l[1]
[1] 1.069710
> y1
 [1] 1 0 3 0 1 3 0 0 0 3 2 0 1 1 0
> l[2]
[1] 4.241796
> y2
 [1] 2 7 5 0 2 5 9 3 9 6 5 2 3 6 5
> l[3]
[1] 4.004304
> y3
 [1] 2 2 5 3 2 3 6 6 5 5 5 6 3 2 7 5 2 5 2 4 3 5 3 6 2
> ratio
[1] 0.9376818
```

Example 5

```
> lx1=rgamma(1000000,4,1)
> lx2=rgamma(1000000,6,1)
>
> lxm=lx1*lx2
> s=order(lxm)
> c=rbind(lxm[s])
> lcl=c[50000]
> ucl=c[900000]
> ucl
[1] 44.89672
> lx1=rgamma(1000000,2,1)
> lx2=rgamma(1000000,5,1)
>
> lxm=lx1*lx2
> s=order(lxm)
> c=rbind(lxm[s])
> lcl=c[50000]
> ucl=c[900000]
>
> ucl
[1] 21.04216
> lx1=rgamma(1000000,3,1)
```

```
> lx2=rgamma(1000000,3,1)
>
> lxm=lx1*lx2
> s=order(lxm)
> c=rbind(lxm[s])
> lcl=c[50000]
> ucl=c[900000]
> ucl
[1] 18.72712
> lx1=rgamma(1000000,4,1)
> lx2=rgamma(1000000,4,1)
>
> lxm=lx1*lx2
> s=order(lxm)
> c=rbind(lxm[s])
> lcl=c[50000]
> ucl=c[900000]
> ucl
[1] 31.17266
> lx1=rgamma(1000000,1,1)
> lx2=rgamma(1000000,1,1)
>
> lxm=lx1*lx2
> s=order(lxm)
> c=rbind(lxm[s])
> lcl=c[50000]
> ucl=c[900000]
> ucl
[1] 2.588229


> lx1=rgamma(1000000,2,1)
> lx2=rgamma(1000000,3,1)
> lx3=rgamma(1000000,2,1)
> lxm=lx1*lx2*lx3
> s=order(lxm)
> c=rbind(lxm[s])
> lcl=c[50000]
> ucl=c[900000]
> ucl
[1] 28.63170

> lx1=rgamma(1000000,3,1)
> lx2=rgamma(1000000,4,1)
> lx3=rgamma(1000000,6,1)
> lxm=lx1*lx2*lx3
```

```
> s=order(lxm)
> c=rbind(lxm[s])
> lcl=c[50000]
> ucl=c[900000]
> ucl
[1] 153.4817
```

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VITA

Graduate College
University of Nevada, Las Vegas

Libo Zhou

Local Address:
1600 E. University Ave. #230
Las Vegas, NV 89119

Home Address:
7431 Kingsley #508
Cote-St-Luc Quebec
Canada
H4W 1P1

Degrees:
Bachelor of Art, Actuarial Mathematics, 2002
Concordia University, Canada

Thesis Title: Confidence Intervals for the Ratio of Poisson Parameters

Thesis Examination Committee:
Chairperson, Dr. Malwane Ananda, Ph. D.
Committee Member, Dr. Chih-Hsiang Ho, Ph. D.
Committee Member, Dr. Hokwon Cho, Ph. D.
Graduate Faculty Representative, Dr. Yitung Chen, Ph. D.