Confidence intervals for the ratio of Poisson parameters

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CONFIDENCE INTERVALS FOR THE RATIO OF POISSON PARAMETERS

by

Libo Zhou

Bachelor of Art
Concordia University, Canada
2002

A thesis submitted in partial fulfillment
of the requirements for the

Master of Science in Mathematical Sciences
Department of Mathematical Sciences
College of Sciences

Graduate College
University of Nevada, Las Vegas
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Confidence Intervals for the Ratio of Poisson Parameters

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Examination Committee Chair

Dean of the Graduate College

Examination Committee Member

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ABSTRACT

Confidence Intervals for the Ratio of Poisson Parameters

by

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Dr. Malwane Ananda, Examination Committee Chair
Professor of Statistics
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Let $k_1, k_2$ be non-negative integers with $k_1 + k_2 \geq 2$; let

$X_1, X_2, \ldots, X_{k_1}, Y_1, Y_2, \ldots, Y_{k_2}$ be mutually independent Poisson random variables

with parameters $\lambda_1, \lambda_2, \ldots, \lambda_{k_1}, \mu_1, \mu_2, \ldots, \mu_{k_2}$, respectively. In this article, we

consider constructing confidence intervals for ratio of Poisson

parameters $\theta = \frac{\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdots \lambda_{k_1}}{\mu_1 \cdot \mu_2 \cdot \mu_3 \cdots \mu_{k_2}}$. For the ratio of two Poisson parameters

$\theta = \frac{\lambda_1}{\mu_1}$, an exact formula for confidence interval is given. A numerical

method to obtain confidence interval for $\frac{\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdots \lambda_{k_1}}{\mu_1 \cdot \mu_2 \cdot \mu_3 \cdots \mu_{k_2}}$ is developed.

Examples are given for each of the two cases.
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CHAPTER 1

INTRODUCTION

Let $X$ and $Y$ be two independent Poisson random variables with parameter $\lambda_i$ and $\mu_i$, respectively. Sahai and Khurshid have described a few methods for obtaining a confidence interval for the ratio of two Poisson means $\lambda_i/\mu_i$ such as an exact method, use of F-distribution tables, normal and Poisson approximations, normal approximation using square root transformation, method of maximum likelihood. In present article, we obtain an exact formula for the confidence interval for the ratio of two Poisson parameters. In this case, Suppose

\[ \tilde{X}_i' = (X_{i1}, X_{i2}, \ldots, X_{i,n_i}) \quad \text{and} \quad \tilde{Y}_i' = (Y_{i1}, Y_{i2}, \ldots, Y_{i,n_i}) \]

be random samples from $X$ and $Y$ respectively. Furthermore, suppose $\lambda_i$ and $\mu_i$ follow gamma distributions with parameters $\alpha_i$, $\beta_i$ and $\delta_i$, $\gamma_i$ respectively. In chapter 3, we develop an exact formula for the confidence interval for $\theta = \frac{\lambda_i}{\mu_i}$ using Bayes method. The brief development is as following:

Since $\tilde{X}_i \sim P(\lambda_i)$ and $\pi(\lambda_i) \sim \text{gamma}(\alpha_i, \beta_i)$, we can find that $\lambda_i | \tilde{X} \sim$

\[
\text{gamma} \left( \sum_{i=1}^{n} x_i + \alpha_i, \frac{\beta_i}{1 + n_i \beta_i} \right),
\]

then if we let $\frac{\beta_i}{1 + n_i \beta_i} = \phi_i$ we have $\frac{\lambda_i}{\phi_i} | \tilde{X}_i \sim$
gamma \left( \sum_{i=1}^{n} x_i + \alpha_1, 1 \right). Similarly, with \( \frac{Y_i}{1 + m_i \gamma} = \omega_1 \) we have \( \frac{\mu_1}{\omega_1} \sim \gamma \left( \sum_{i=1}^{m} y_{ii} + \delta_1, 1 \right). \\
\frac{\hat{\lambda}_1}{\hat{\lambda}_1 + \frac{\mu_1}{\omega_1}} \sim \text{Beta} \left( \sum_{i=1}^{n} x_i + \alpha_1, \sum_{i=1}^{m} y_{ii} + \delta_1 \right), \text{after some algebraic operation we can find the confidence interval for} \ \theta = \frac{\hat{\lambda}_1}{\mu_1} \text{is} \ \frac{\beta_1}{\beta_1 + \frac{\mu_1}{\omega_1}} \left( \frac{\phi_1}{1 - \beta_1^{\frac{1}{2}}} \right), \text{where} \ \left( \beta_1, \beta_2 \right) \text{are the critical values of beta distribution with parameters} \ A = \sum_{i=1}^{n} x_i + \alpha_1, \ B = \sum_{i=1}^{m} y_{ii} + \delta_1. \text{An example is also given in this chapter. The method is also suitable for the example in Sahai and Khurshid’s article. Example using the method with data from Sahai and Khurshid’s article is given in chapter 3, example 2.} \\
For the ratio \( \theta = \frac{\hat{\lambda}_1 \cdot \hat{\lambda}_2 \cdot \ldots \cdot \hat{\lambda}_k}{\mu_1 \cdot \mu_2 \cdot \ldots \cdot \mu_k} \), when there are more than one parameters for the numerator or the denominator or both there is no formal distribution for the ratio. Therefore, we develop a numerical method to find its \( (1-\alpha)100\% \) confidence interval. The simple ratio becomes to the following case:
Let \( k_1, k_2 \) be non-negative integers with \( k_1 + k_2 \geq 2 \); let
\[
\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_{k_1}, \tilde{Y}_1, \tilde{Y}_2, \ldots, \tilde{Y}_{k_2}
\]
be mutually independent Poisson random scalars with parameters \( \lambda_1, \lambda_2, \ldots, \lambda_{k_1}; \mu_1, \mu_2, \ldots, \mu_{k_2} \), respectively. In this article, we look at how to construct confidence interval for the ratio
\[
\theta = \frac{\lambda_1 \cdot \lambda_2 \cdots \lambda_{k_1}}{\mu_1 \cdot \mu_2 \cdots \mu_{k_2}}.
\]
Harris has developed a theoretical method to obtain the confidence interval for the quotient \( \frac{\lambda_1 \cdot \lambda_2 \cdots \lambda_{k_1}}{\mu_1 \cdot \mu_2 \cdots \mu_{k_2}} \), where \( \lambda_1, \lambda_2, \ldots, \lambda_{k_1}; \mu_1, \mu_2, \ldots, \mu_{k_2} \) are Poisson parameters for the variables \( X_1, X_2, \ldots, X_{k_1}; Y_1, Y_2, \ldots, Y_{k_2} \). In this article we develop a more general method to calculate the confidence interval for
\[
\theta = \frac{\lambda_1 \cdot \lambda_2 \cdots \lambda_{k_1}}{\mu_1 \cdot \mu_2 \cdots \mu_{k_2}}
\]
in this case: \( X_1, X_2, \ldots, X_{k_1}; Y_1, Y_2, \ldots, Y_{k_2} \) are not \( k_1 + k_2 \) variables but scalars, \( \tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_{k_1}; \tilde{Y}_1, \tilde{Y}_2, \ldots, \tilde{Y}_{k_2} \) each contains random sample of sizes \( n_1, n_2, \ldots, n_{k_1}; m_1, m_2, \ldots, m_{k_2} \), respectively. In chapter 4, we develop a numerical way to calculate a large number of the values for \( \theta \) and then we put these numbers in order so that we could find the \((1-\alpha)100\%\) confidence interval. A calculation procedure of the R program was also given in this chapter. We should note that in particular, for \( k_2 = 0 \) the parameter \( \theta \) is a product of Poisson parameter and the solution can be interpreted as an approximate solution to the corresponding problem of finding confidence intervals for the product of binomial parameters. Estimation of the product of binomial parameters has been investigated by Madansky,
Buehler, and Myhre and Saunders. Tables suitable for use in some problems concerning estimation of binomial parameters have been compiled by Lipow and Riley. Their results have been compared with the result coming from Harris. And all the results will be compared with those of the present article in chapter 4, example 5.
CHAPTER 2

DISTRIBUTIONS CONSIDERED

The Poisson Distribution

The Poisson distribution is a discrete probability distribution belonging to certain random variables N that count, among other things, a number of discrete occurrences that take place during a time-interval of given length. The probability that there are exactly x occurrences (x being a non-negative integer, x = 0, 1, 2, ...) is:

\[ p(N = x) = \frac{e^{-\lambda} \lambda^x}{x!}, \quad x = 0, 1, 2, \ldots \]

where \( \lambda > 0 \).

The Gamma Distribution

The gamma distribution is a continuous probability distribution with the probability density function:

\[ f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}}, \quad x \geq 0. \]

\( \alpha > 0 \) is the shape parameter and \( \beta > 0 \) is the scale parameter.

Related Knowledge about Gamma Distribution

If X is gamma distributed with the parameter \( \alpha \) and \( \beta \), then \( X/\beta \)
will be distributed with the parameters $\alpha$ and 1. This is expressed as:

If $X \sim \text{gamma}(\alpha, \beta)$, then $X/\beta \sim \text{gamma}(\alpha, 1)$.

Proof,

Let $Y = \frac{X}{\beta}$

$$F(Y) = \Pr\left( \frac{X}{\beta} \leq y \right)$$

$$= \Pr(x \leq \beta y)$$

$$= \int_0^{\beta y} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}} dx$$

$$f(Y) = F'(Y)$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} (\beta y)^{\alpha-1} e^{-\frac{\beta y}{\beta}} (\beta y)'$$

$$= \frac{1}{\Gamma(\alpha)} y^{\alpha-1} e^{-y},$$

so $Y \sim \text{gamma}(\alpha, 1)$, that is $X/\beta \sim \text{gamma}(\alpha, 1)$.

We can also prove the above result by using the moment generating function: If $X \sim \text{gamma}(\alpha, \beta)$, then the moment generating function of $X$ is

$$M_x(t) = \frac{1}{(1 - \beta t)^\alpha}$$

then $M_{X/\beta}(t) = M_x\left(\frac{t}{\beta}\right)$
The Beta Distribution

The Beta distribution is a continuous probability distribution with the probability density function defined on the interval [0,1]:

\[ f(x) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \]

where \( \alpha \) and \( \beta \) are parameters that must be greater than zero and \( \Gamma \) is the gamma function.

The expected value and variance of a beta random variable \( X \) with parameters \( \alpha \) and \( \beta \) are given by formulae:

\[ E(X) = \frac{\alpha}{\alpha + \beta} \]

\[ V(X) = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)} \]

Related Knowledge about Beta Distribution

If \( X_1 \) and \( X_2 \) are two independent random variables. And \( X_1 \) is gamma distributed with parameters \( \alpha \) and 1, \( X_2 \) is gamma distributed
with parameters $\beta$ and 1, then $\frac{X_1}{X_1 + X_2}$ will be Beta distributed with parameters $\alpha$ and $\beta$. This can be expressed as: If $X_1$ and $X_2$ are independent, $X_1 \sim \text{gamma} (\alpha, 1)$ and $X_2 \sim \text{gamma} (\beta, 1)$, then $\frac{X_1}{X_1 + X_2} \sim \text{beta} (\alpha, \beta)$.

Proof,

If $X_1 \sim \text{gamma} (\alpha, 1)$ and $X_2 \sim \text{gamma} (\beta, 1)$, then they have the following probability density functions respectively:

$$f(x_1) = \frac{1}{\Gamma(\alpha)} x_1^{\alpha-1} e^{-x_1}, \quad 0 < x_1 < \infty.$$  

$$f(x_2) = \frac{1}{\Gamma(\beta)} x_2^{\beta-1} e^{-x_2}, \quad 0 < x_2 < \infty.$$  

When they are independent they have the joint probability density function:

$$h(x_1, x_2) = f(x_1) \cdot f(x_2)$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} x_1^{\alpha-1} x_2^{\beta-1} e^{-x_1-x_2}, \quad 0 < x_1 < \infty, \quad 0 < x_2 < \infty,$$

zero elsewhere, where $\alpha > 0$, $\beta > 0$. Let $Y_1 = X_1 + X_2$ and $Y_2 = \frac{X_1}{X_1 + X_2}$.

The space $A$ is, exclusive of the points on the coordinate axes, the first quadrant of the $x_1 x_2$-plane. Now

$$y_1 = u_1(x_1, x_2) = x_1 + x_2$$
\[
y_2 = u_2(x_1 + x_2) = \frac{x_1}{x_1 + x_2}
\]

may be written as \( x_1 = y_1 y_2, \ x_2 = y_1 (1 - y_2), \) so

\[
J = \begin{pmatrix} y_2 & y_1 \\ 1 - y_2 & -y_1 \end{pmatrix} = -y_1 \neq 0.
\]

The transaction is one-to-one, and it maps \( A \) to \( B = \{(y_1, y_2) : 0 < y_1 < \infty, 0 < y_2 < 1\} \) in the \( y_1, y_2 \)-plane.

The joint probability density function of \( Y_1 \) and \( Y_2 \) is then

\[
g(y_1, y_2) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} (y_1 y_2)^{\alpha-1} [y_1 (1 - y_2)]^{\beta-1} e^{-\gamma \cdot (y_1)}
\]

\[
= \frac{y_2^{\alpha-1} (1 - y_2)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} y_1^{\alpha+\beta-1} e^{-\gamma \cdot y_1}, \quad 0 < y_1 < \infty, \quad 0 < y_2 < 1,
\]

\[
= 0 \text{ elsewhere.}
\]

The marginal probability density function of \( Y_2 \) is

\[
g_2(y_2) = \frac{y_2^{\alpha-1} (1 - y_2)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \int y_1^{\alpha+\beta-1} e^{-\gamma \cdot y_1} dy_1
\]

\[
= \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} y_2^{\alpha-1} (1 - y_2)^{\beta-1}, \quad 0 < y_2 < 1,
\]

\[
= 0 \text{ elsewhere}.
\]

This is the probability density function of the beta distribution with parameter \( \alpha \) and \( \beta \).

So we have proved that if \( x_1 \) and \( x_2 \) are independent, \( X_1 \sim \text{gamma} (\alpha, 1) \) and \( X_2 \sim \text{gamma} (\beta, 1), \) then \( \frac{X_1}{X_1 + X_2} \sim \text{beta} (\alpha, \beta). \)
Bayes Method

The posterior distribution of $\theta$ given $x$ denoted $\pi(\theta|x)$ is defined to be the conditional distribution of $\theta$ given the sample observation $x$. And with $X$ has marginal (unconditional) density $m(x) = \int_\theta f(x/\theta)\pi(\theta)d\theta$, the posterior is $\pi(\theta|x) = \frac{f(x/\theta)\cdot\pi(\theta)}{m(x)}$ where $\pi(\theta)$ is the prior, usually given or assumed.

With the square loss $L = [\theta - E(\theta)]^2$, the Bayes estimate of $\theta/x$ is just the mean of the posterior distribution.

Generalized Bayes Estimate

For $X = (x_1, x_2, \ldots, x_n)$, if each variable is Poisson distributed with parameter $\lambda$ then we have $X = \sum_{i=1}^n x_i \sim \text{Poisson}(n\lambda)$. When the prior is a constant prior, which is $f(\lambda) = \begin{cases} c, & \lambda \geq 0 \\ 0, & \text{o.w.} \end{cases}$. The posterior distribution will be

$$f(\lambda | X) = \frac{f(\lambda, X)}{f(X)}$$

$$= \frac{f(X | \lambda) \cdot f(\lambda)}{\int_0^\infty f(X | \lambda) \cdot f(\lambda)d\lambda}$$

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\[
\frac{e^{-n\lambda} \cdot \sum_{i=1}^{n} x_i^\lambda}{(\sum_{i=1}^{n} x_i)!} \cdot c
= \frac{\int_0^\infty e^{-n\lambda} \cdot \sum_{i=1}^{n} x_i^\lambda}{(\sum_{i=1}^{n} x_i)!} \cdot cd\lambda
\]

\[
= \frac{e^{-n\lambda} \cdot \sum_{i=1}^{n} x_i^\lambda}{\int_0^\infty e^{-n\lambda} \cdot \sum_{i=1}^{n} x_i^\lambda} \cdot d\lambda
\]

\[
\propto e^{-n\lambda} \cdot \sum_{i=1}^{n} x_i^\lambda
\]

\[
\propto e^{-\frac{\lambda}{\sum_{i=1}^{n} x_i + 1}}
\]

So we have \( \lambda | X \sim \text{gamma}(\sum_{i=1}^{n} x_i + 1, \frac{1}{n}) \).
CONFIDENCE INTERVAL FOR A SIMPLE CASE

Theoretical Development

We start from when $k_1 = 1$, $k_2 = 1$. The problem then is simplified as:

Let $\tilde{X}_1 = (x_{11}, x_{12}, \ldots, x_{i_n})$, $\tilde{X}_2 = (y_{11}, y_{22}, \ldots, y_{j_m})$ be two Poisson random samples with parameters $\lambda_i$ and $\mu_i$. With the assumption that the prior distribution $\pi(\lambda_i) \sim \text{gamma}(\alpha_1, \beta_1)$, $\pi(\mu_i) \sim \text{gamma}(\delta_i, \gamma_i)$, we find an exact formula for $100(1-\alpha)\%$ confidence interval for the parameter $\theta = \frac{\lambda_i}{\mu_i}$. And with the square loss $L = [\theta - E(\theta)]^2$ we also find the Bayes estimate for

$\theta = \frac{\lambda_i}{\mu_i}$.

We will use the Bayes' method and the knowledge in chapter 1 to obtain a $100(1-\alpha)\%$ confidence interval for the parameter $\theta = \frac{\lambda_i}{\mu_i}$. Since $\tilde{X}_1 = (x_{i1}, x_{i2}, \ldots, x_{i_n})$ is a Poisson random sample with parameter $\lambda_i$,

$$f(\tilde{X}_1 | \lambda_i) = f(X_{i1} | \lambda_i) \cdot f(X_{i2} | \lambda_i) \cdots f(X_{i_n} | \lambda_i)$$

$$= \frac{\lambda_i^{x_{i1}} \cdot e^{-\lambda_i}}{x_{i1}!} \cdot \frac{\lambda_i^{x_{i2}} \cdot e^{-\lambda_i}}{x_{i2}!} \cdots \frac{\lambda_i^{x_{i_n}} \cdot e^{-\lambda_i}}{x_{i_n}!}$$

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\[
\frac{\lambda^{\sum_{i=1}^{n} x_{ii}} \cdot e^{-n \lambda_i}}{\prod_{i=1}^{n} x_{ii}^{!}}, \quad x_{ii} = 1, 2, \ldots n
\]

Since \( \pi(\lambda_i) \sim \text{gamma}(\alpha_i, \beta_i) \)

\[
\pi(\lambda_i) = \frac{1}{\Gamma(\alpha_i)\beta_i^{\alpha_i}} \lambda_i^{\alpha_i-1} e^{-\lambda_i/\beta_i}, \text{ where } \lambda_i > 0
\]

then, \( f(\lambda_i \mid \tilde{X}_i) \propto f(\tilde{X}_i \mid \lambda_i) \cdot \pi(\lambda_i) \)

\[
\propto \lambda_i^{\sum_{i=1}^{n} x_{ii}} \cdot e^{-n \lambda_i} \cdot \lambda_i^{\alpha_i-1} \cdot e^{-\lambda_i/\beta_i}
\]

\[
\propto \lambda_i^{\sum_{i=1}^{n} x_{ii} + \alpha_i - 1} \cdot e^{-\lambda_i (1 + n \beta_i)} / \beta_i
\]

so, \( \lambda_i \mid \tilde{X}_i \sim \text{gamma}(\sum_{i=1}^{n} x_{ii} + \alpha_i, \frac{\beta_i}{1 + n \beta_i}) \),

Similarly, \( \mu_1 \mid \tilde{Y}_1 \sim \text{gamma}(\sum_{i=1}^{m} y_{ij} + \delta_i, \frac{\gamma_i}{1 + m_1 \gamma_i}) \).

Let \( \frac{\beta_i}{1 + n \beta_i} = \phi_i, \quad \frac{\gamma_i}{1 + m_1 \gamma_i} = \omega_i \)

Then, \( \frac{\lambda_i}{\phi_i} \mid \tilde{X}_i \sim \text{gamma} \left( \sum_{i=1}^{n} x_{ii} + \alpha_i, 1 \right), \quad \frac{\mu_1}{\omega_i} \mid \tilde{Y}_1 \sim \text{gamma} \left( \sum_{i=1}^{m} y_{ij} + \delta_i, 1 \right) \)

\[
\frac{\lambda_i}{\phi_i} \quad \text{Beta} \left( \sum_{i=1}^{n} x_{ii} + \alpha_i, \sum_{i=1}^{m} y_{ij} + \delta_i \right)
\]

So we have \( \frac{\phi_1}{\lambda_i} + \frac{\mu_1}{\omega_i} \sim \text{Beta} \left( \sum_{i=1}^{n} x_{ii} + \alpha_i, \sum_{i=1}^{m} y_{ij} + \delta_i \right) \)

if \( A = \sum_{i=1}^{n} x_{ii} + \alpha_i, \quad B = \sum_{i=1}^{m} y_{ij} + \delta_i \), that is \( \frac{1}{1 + \left( \frac{\phi_1}{\lambda_i} \cdot \frac{\mu_1}{\omega_i} \right)} \sim \text{Beta} \left( A, B \right) \)
With the given data it is easy to find the value of $A$ and $B$. We can use Minitab to find a $100(1-\alpha)\%$ confidence interval $(\beta_{\alpha}, \beta_{1-\alpha})$ for the parameter $\frac{1}{1+\left(\frac{\varphi_1}{\omega_1}\right)\cdot\left(\frac{\lambda}{\mu_1}\right)}$. This can be written as

$$\Pr\left(\frac{\beta_{\alpha}}{1-\beta_{\alpha}} < \frac{\varphi_1}{\omega_1} < \frac{\beta_{1-\alpha}}{1-\beta_{1-\alpha}}\right) = 100(1-\alpha)\%.$$ 

After some algebraic calculation this will be $\Pr\left(\frac{\beta_{\alpha}}{1-\beta_{\alpha}} \cdot \varphi_1 < \frac{\lambda}{\mu_1} < \frac{\beta_{1-\alpha}}{1-\beta_{1-\alpha}} \cdot \varphi_1\right) = 100(1-\alpha)\%$.

Thus, the $100(1-\alpha)\%$ confidence interval for the parameter $\theta = \frac{\lambda}{\mu_1}$ is

$$\left(\frac{\beta_{\alpha}}{1-\beta_{\alpha}} \cdot \varphi_1, \frac{\beta_{1-\alpha}}{1-\beta_{1-\alpha}} \cdot \varphi_1\right).$$

Under square loss $L = [\theta - E(\theta)]^2$, the Bayes estimate for the parameter $\theta = \frac{\lambda}{\mu_1}$ is just the mean of beta distribution with parameters $A$ and $B$, which is $\frac{A}{A+B}$. Then we have $\frac{1}{1+\left(\frac{\varphi_1}{\omega_1}\right)\cdot\left(\frac{\lambda}{\mu_1}\right)} = \frac{A}{A+B}$.

Then $\frac{\lambda}{\mu_1} = \frac{\varphi_1}{\omega_1} \cdot \frac{A}{B}$, which is just the Bayes estimate for the parameter $\theta = \frac{\lambda}{\mu_1}$.
Example 1

With the assumption that the prior distribution
\[ \pi(\lambda_i) \sim \text{gamma}(\alpha_i = 2, \beta_i = 2), \pi(\mu_i) \sim \text{gamma}(\delta_i = 3, \gamma_i = 2), \]
we use Minitab to
generate a random value for \( \lambda_i \) and \( \mu_i \), and we get \( \lambda_i = 5.961133 \),
\( \mu_i = 9.52169 \). Then we use \( \tilde{X}_i \sim \text{P}(\lambda_i = 5.961133) \) to generate 25 data for \( \tilde{X}_i \),
that is \( \tilde{X}_i' = (9, 7, 7, 7, 5, \ldots, 2, 5, 5, 3, 6) \), \( n_i = 25 \); and we use \( \tilde{Y}_i \sim \text{P}(\mu_i = 9.52169) \)
to generate 20 data for \( \tilde{Y}_i \), that is \( \tilde{Y}_i' = (6, 12, 12, 8, \ldots, 5, 8, 11, 6, 9) \), \( m_i = 20 \).

In this example we find a \( 100(1-\alpha)\% = 95\% \) confidence interval for the
parameter \( \theta = \frac{\lambda_i}{\mu_i} \). And with the square loss \( L = [\theta - E(\theta)]^2 \) we find the
Bayes estimate for \( \theta = \frac{\lambda_i}{\mu_i} \).

According to what we have done in Chapter 2 we have
\[ \frac{1}{1 + (\phi_i \cdot \frac{\lambda_i}{\mu_i})} \sim \text{Beta} \left( A, B \right), \text{ where } A = \sum_{i=1}^{n} x_{ii} + \alpha_i, B = \sum_{i=1}^{m} y_{ii} + \delta_i \]
and \( \frac{\beta_i}{1 + n_i \beta_i} = \varphi_i, \frac{\gamma_i}{1 + m_i \gamma_i} = \omega_i \).

Then \( A = \sum_{i=1}^{n} x_{ii} + \alpha_i = 145 + 2 = 147, B = \sum_{i=1}^{m} y_{ii} + \delta_i = 189 + 3 = 192, \)
\[ \varphi_i = \frac{\beta_i}{1 + n_i \beta_i} = \frac{2}{1 + 25 \times 2} = \frac{2}{51}, \omega_i = \frac{\gamma_i}{1 + m_i \gamma_i} = \frac{2}{1 + 20 \times 2} = \frac{2}{41}, \]
Since \( \Pr(\beta_i < \frac{1}{\frac{\varphi_i}{1 - \frac{\gamma_i}{\omega_i}} < \beta_i \frac{1}{2}}) = 100(1-\alpha)\% = 95\% \) that is
Pr\left( \frac{1}{1 - \beta^a_2 \cdot \omega_i} < \frac{\hat{\lambda}_i}{\mu_i} < \frac{1}{1 - \beta^{1-a}_2 \cdot \omega_i} \right) = 95\%, \text{ we can use Minitab to get}
\beta^a_2 = 0.381356, \beta^{1-a}_2 = 0.486642. \text{ Thus the } (1-\alpha)\% = 95\% \text{ confidence interval for the parameter } \theta = \frac{\hat{\lambda}_i}{\mu_i} \text{ is } \left( \frac{\beta^a_2 \cdot \phi_1}{1 - \beta^a_2 \cdot \omega_i}, \frac{\beta^{1-a}_2 \cdot \phi_1}{1 - \beta^{1-a}_2 \cdot \omega_i} \right), \text{ which is (0.4955683, 0.7620842).}

Under square loss \( L = (\theta - E(\theta))^2 \), the Bayes estimate for the parameter \( \frac{1}{1 + \left( \frac{\phi_1}{\omega_i} \right) \cdot \left( \frac{\hat{\lambda}_i}{\mu_i} \right)} \) is just the mean of beta distribution with parameters \( A \) and \( B \), which is \( \frac{A}{A+B} \). Then we have \( \frac{1}{1 + \left( \frac{\phi_1}{\omega_i} \right) \cdot \left( \frac{\hat{\lambda}_i}{\mu_i} \right)} = \frac{A}{A+B} \),
then \( \frac{\hat{\lambda}_i}{\mu_i} = \frac{\phi_1 \cdot A}{\omega_i \cdot B} = \frac{41}{51} = 0.615502 \), which is just the Bayes estimate for the parameter \( \theta = \frac{\hat{\lambda}_i}{\mu_i} \), this estimate value lies in the 100(1-\alpha)\% = 95\% confidence interval (0.4955683, 0.7620842), and it is also close to the value of \( \frac{\hat{\lambda}_i}{\mu_i} = 0.6260583 \).

**Example 2**

In Sahai and Khurshid's article, they calculated the confidence interval using a few method with the data \( x_1 = 12, y_1 = 13 \). With the same
data, we will use our method to calculate the confidence interval for
\[ \theta = \frac{\lambda}{\mu} \]
and compare the result with that of the exact method from Sahai and Khurshid's article.

Since there is no prior assumption in the exact method from Sahai and Khurshid's article, we have to make our prior assumption non-informative. For the prior \( \pi(\lambda_i) \sim \text{gamma}(\alpha_i, \beta_i) \), \( \pi(\mu_i) \sim \text{gamma}(\delta_i, \gamma_i) \), we should take \( \alpha_i \), equal to 1 and \( \beta_i, \gamma_i \) equal to a big number to make the prior non-informative. With such prior assumption, the posterior will be \( \lambda_i \mid X_i \sim \text{gamma}(13,1) \) and \( \mu_i \mid Y_i \sim \text{gamma}(14,1) \). Then we have the critical values for \( \text{Beta}(13,14) \) are \( \beta_{1/2} = 0.299272 \), \( \beta_{1-0.2} = 0.666292 \). The confidence interval is \((0.4271, 1.9967)\). This is close to the confidence interval \((0.3860, 2.1990)\) in Sahai and Khurshid's article.

If we take the prior as a generalized constant prior, we will get the same posterior and the same result for this example.
CHAPTER 4

CONFIDENCE INTERVAL FOR A COMPLICATED CASE

Illustration and procedure of R program

We have developed an exact formula in chapter 2 for the simple case when \( k_1 = 1, k_2 = 1 \). When \( k_1 \) and \( k_2 \) are not both one at the same time, which means for the ratio \( \theta = \frac{\lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdots \lambda_{k_1}}{\mu_1 \cdot \mu_2 \cdot \mu_3 \cdots \mu_{k_2}} \) when there are more than one parameters for the numerator or denominator or both, the case could become much more complicated and we can not find a formal distribution for the ratio.

In this chapter, we will develop a numerical way to find the confidence interval for this case. And an R program is developed according to the following procedure.

1. We assume the prior for \( \lambda_1, \lambda_2, \ldots, \lambda_{k_1}, \mu_1, \mu_2, \ldots, \mu_{k_2} \), this can be expressed as following:

\[
\lambda_1 \sim \text{gamma}(\alpha_1, \beta_1) \\
\lambda_2 \sim \text{gamma}(\alpha_2, \beta_2) \\
\vdots
\]

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Here, the values of $\alpha's, \beta's, \delta's, \gamma's$ will be assumed for each prior. And random values for $\lambda's$ and $\mu's$ will be generated.

2. We use the generated values for $\lambda's$ and $\mu's$ to generate the data vectors. This is expressed as following:

\[ \tilde{X}_j = (x_{j1}, x_{j2}, \ldots, x_{jn_j}) \sim P(\lambda_j) \]
\[ \tilde{X}_2 = (x_{21}, x_{22}, \ldots, x_{2n_2}) \sim P(\lambda_2) \]
\[ \vdots \]
\[ \tilde{X}_{k_i} = (x_{ki1}, x_{ki2}, \ldots, x_{k_in_{ki}}) \sim P(\lambda_{k_i}) \]
\[ \tilde{Y}_1 = (y_{11}, y_{12}, \ldots, y_{1m_1}) \sim P(\mu_1) \]
\[ \tilde{Y}_2 = (y_{21}, y_{22}, \ldots, y_{2m_2}) \sim P(\mu_2) \]
\[ \vdots \]
3. As we have proved, the conditional parameters for $\lambda \mid \tilde{X}$ and $\mu \mid \tilde{Y}$ will also be gamma distributed, the parameters of these gamma distributions are known so we can generate the random values (a large number, say 1000000) for $\lambda \mid \tilde{X}$’s and $\mu \mid \tilde{Y}$’s. This is expressed as following:

$$\lambda_1 \mid \tilde{X}_1 \sim \text{gamma}\left(\sum_{i=1}^{n_1} x_{1i} + \alpha_1, \frac{\beta_1}{1 + n_1 \beta_1}\right)$$

$$\lambda_2 \mid \tilde{X}_2 \sim \text{gamma}\left(\sum_{i=1}^{n_2} x_{2i} + \alpha_2, \frac{\beta_2}{1 + n_2 \beta_2}\right)$$

$$\lambda_i \mid \tilde{X}_i \sim \text{gamma}\left(\sum_{i=1}^{n_i} x_{ki} + \alpha_k, \frac{\beta_k}{1 + n_k \beta_k}\right)$$

$$\mu_1 \mid \tilde{Y}_1 \sim \text{gamma}\left(\sum_{i=1}^{m_1} y_{1i} + \delta_1, \frac{\gamma_1}{1 + m_1 \gamma_1}\right)$$

$$\mu_2 \mid \tilde{Y}_2 \sim \text{gamma}\left(\sum_{i=1}^{m_2} y_{2i} + \delta_2, \frac{\gamma_2}{1 + m_2 \gamma_2}\right)$$
\[ \mu_{k_1} \mid \tilde{y}_{k_1} \sim \text{gamma}\left(\sum_{i=1}^{m_{k_1}} y_{k_1}, \frac{y_{k_1}}{1 + m_{k_1} y_{k_1}}\right) \]

4. Using the generated values in step 3 we calculate \( \theta = \frac{\lambda_1 \cdot \lambda_2 \cdots \lambda_{k_i}}{\mu_1 \cdot \mu_2 \cdots \mu_{k_i}} \) and we will get a large number (1000000) of values for this ratio, let's say they are 1000000 q's.

5. We just need to put these 1000000 q's in order and it will be easy to get a 100(1-\( \alpha \))% confidence interval.

**Example 1**

Since the R program is developed for any non-negative number \( k_1 \) and \( k_2 \). In this example we will calculate the 100(1-\( \alpha \))% = 95% confidence interval for the ratio \( \theta = \frac{\lambda_1}{\mu_1} \), this is a simple case when \( k_1 = 1, k_2 = 1 \). The following is the procedure and results:

1. We assume the parameters of the priors for \( \lambda_1 \sim \text{gamma}(\alpha_1, \beta_1) \)
   and \( \mu_1 \sim \text{gamma}(\delta_1, \gamma_1) \). Let \( \alpha_1 = 2, \beta_1 = 2 \) and \( \delta_1 = 3, \gamma_1 = 2 \).

2. A random value for \( \lambda_1 \) and \( \mu_1 \) will be generated. \( \lambda_1 = 5.961133 \), \( \mu_1 = 9.52169 \).

3. Then we use \( \lambda_1 = 5.961133 \) as the Poisson parameter to generate 25 data for \( \tilde{X}_1 \), that is \( \tilde{X}_1 = (9, 7, 7, 7, 5, \ldots, 2, 5, 5, 3, 6, 6) \), \( n_1 = 25 \); and we use
\( \mu_1 = 9.52169 \) as the Poisson parameter to generate 20 data for \( \tilde{Y}_1 \), that is \( \tilde{Y}_1 = (6, 12, 11, 8, \ldots, 5, 8, 11, 8, 6, 9) \), \( m_1 = 20 \).

4. The sum of Poisson data of each Poisson distribution will be calculated and will get the values of \( \sum_{i=1}^{\tilde{m}_1} x_{ii} + \alpha_i \) and \( \sum_{i=1}^{\tilde{m}_1} y_{ii} + \delta_i \).

5. We use \( \lambda_1 \sim \text{gamma}(\sum_{i=1}^{n_1} x_{ii} + \alpha_1, \frac{\beta_1}{1+n_1 \beta_1}) \) and \( \mu_1 \sim \text{gamma}(\sum_{i=1}^{\tilde{m}_1} y_{ii} + \delta_1, \frac{\gamma_1}{1+m_1 \gamma_1}) \) to generate 1000000 values for \( \lambda_1 \sim \tilde{X}_1 \) and \( \mu_1 \sim \tilde{Y}_1 \).

6. The ratio \( \theta = \frac{\lambda_1}{\mu_1} \) will be calculated and we will get 1000000 values of \( \theta = \frac{\lambda_1}{\mu_1} \). We put them in order and it is easy to get 100(1-\( \alpha \))\% = 95\% confidence interval \((0.4957661, 0.7623171)\).

Example 2

In this example we will do the case when \( k_1 = 2 \) and \( k_2 = 2 \). The problem becomes to how to calculate the 100(1-\( \alpha \))\% confidence interval for \( \theta = \frac{\lambda_1 \cdot \lambda_2}{\mu_1 \cdot \mu_2} \). We will use the R program to do this and the procedure is as the following:

1. Assume the parameters for the priors:
   \( \lambda_1 \sim \text{gamma}(\alpha_1 = 2, \beta_1 = 2) \)
\( \lambda_2 \sim \text{gamma}(\alpha_2 = 2, \beta_2 = 3) \)

\( \mu_1 \sim \text{gamma}(\delta_1 = 3, \gamma_1 = 3) \)

\( \mu_2 \sim \text{gamma}(\delta_2 = 4, \gamma_2 = 3) \)

2. A random value for \( \lambda_1, \lambda_2, \mu_1 \text{and} \mu_2 \) will be generated:

\( \lambda_1 = 1.028019 \)

\( \lambda_2 = 3.547733 \)

\( \mu_1 = 4.95997 \)

\( \mu_2 = 8.370036 \)

3. With the input data numbers for each Poisson distribution the Poisson distributions are generated:

\( \tilde{X}_1' = (2, 2, 0, 0, 4, \ldots, 1, 0, 2, 0, 1, 0) \sim \text{P}(\lambda_1 = 1.028019), \ n_1 = 20 \)

\( \tilde{X}_2' = (4, 2, 2, 0, \ldots, 4, 3, 4, 6) \sim \text{P}(\lambda_2 = 3.547733), \ n_2 = 15 \)

\( \tilde{Y}_1' = (4, 3, 6, 6, 9, \ldots, 2, 7, 7, 8, 7) \sim \text{P}(\mu_1 = 4.95997), \ m_1 = 20 \)

\( \tilde{Y}_2' = (3, 13, 9, 5, 4, \ldots, 10, 12, 5, 9, 9, 8) \sim \text{P}(\mu_2 = 8.370036), \ m_2 = 25 \)

4. With the above assumption and Poisson data, 1000000 posterior random variables will be generated according to the following:

\[ \lambda_1 | \tilde{X}_1 \sim \text{gamma}\left(\sum_{i=1}^{n_1} x_{1i} + \alpha_1, \frac{\beta_1}{1 + n_1 \beta_1}\right) \]

\[ \lambda_2 | \tilde{X}_2 \sim \text{gamma}\left(\sum_{i=1}^{n_2} x_{2i} + \alpha_2, \frac{\beta_2}{1 + n_2 \beta_2}\right) \]

\[ \mu_1 | \tilde{Y}_1 \sim \text{gamma}\left(\sum_{i=1}^{m_1} y_{1i} + \delta_1, \frac{\gamma_1}{1 + m_1 \gamma_1}\right) \]
\[ \mu_2 | \bar{Y}_2 \sim \text{gamma}(\sum_{i=1}^{m_2} y_{2i} + \delta_2, \frac{r_2}{1 + m_2 r_2}) \]

5. 1000000 values for the ratio \( \theta = \frac{\lambda_1 \lambda_2}{\mu_1 \mu_2} \) will be calculated.

6. We just need to put these 1000000 values of \( \theta \) in order, and with the assumption \( \alpha = 0.05 \) it’s easy to get 100(1-\( \alpha \))% = 95% confidence interval for \( \theta \), which is (0.04432612, 0.1341951).

Example 3

In this example we will do the case when \( k_1 = 3 \) and \( k_2 = 1 \). The problem becomes to how to calculate the 100(1-\( \alpha \))% confidence interval for \( \theta = \frac{\lambda_1 \lambda_2 \lambda_3}{\mu_1} \). We will use the R program to do this and the procedure is as the following:

1. Assume the parameters for the priors:

   \( \lambda_1 \sim \text{gamma}(\alpha_1 = 2, \beta_1 = 1) \)

   \( \lambda_2 \sim \text{gamma}(\alpha_2 = 2, \beta_2 = 2) \)

   \( \lambda_3 \sim \text{gamma}(\alpha_3 = 3, \beta_3 = 1) \)

   \( \mu_1 \sim \text{gamma}(\delta_1 = 4, \gamma_1 = 2) \)

2. A random value for \( \lambda_1, \lambda_2, \lambda_3 \text{ and } \mu_1 \) will be generated:

   \( \lambda_1 = 1.529129 \)

   \( \lambda_2 = 3.956051 \)
\[ \lambda_3 = 1.258633 \]
\[ \mu_1 = 14.86363 \]

3. With the input data numbers for each Poisson distribution the Poisson distributions are generated:

\[ \tilde{X}_1' = (0,1,1,0,4,\ldots,1,3,0,2,2,1) \sim P(\lambda_1 = 1.529129), \ n_1 = 25 \]
\[ \tilde{X}_2' = (3,7,5,4,3,\ldots,6,3,2,1,6) \sim P(\lambda_2 = 3.956051), \ n_2 = 15 \]
\[ \tilde{X}_3' = (1,3,0,2,0,1,0,2,0) \sim P(\lambda_3 = 1.258633), \ n_3 = 10 \]
\[ \tilde{Y}_1' = (18,10,10,15,\ldots,11,10,17,15) \sim P(\mu_1 = 14.86363), \ m_1 = 25 \]

4. With the above assumption and Poisson data 1000000 posterior random variables will be generated according to the following:

\[
\lambda_1 | \bar{X}_1 \sim \text{gamma} \left( \sum_{i=1}^{n_1} x_{1i} + \alpha_1, \frac{\beta_1}{1 + n_1 \beta_1} \right)
\]
\[
\lambda_2 | \bar{X}_2 \sim \text{gamma} \left( \sum_{i=1}^{n_2} x_{2i} + \alpha_2, \frac{\beta_2}{1 + n_2 \beta_2} \right)
\]
\[
\lambda_3 | \bar{X}_3 \sim \text{gamma} \left( \sum_{i=1}^{n_3} x_{3i} + \alpha_3, \frac{\beta_3}{1 + n_3 \beta_3} \right)
\]
\[
\mu_1 | \bar{Y}_1 \sim \text{gamma} \left( \sum_{i=1}^{m_1} y_{1i} + \delta_1, \frac{\gamma_1}{1 + m_1 \gamma_1} \right)
\]

5. 1000000 values for the ratio \[ \theta = \frac{\lambda_1 \cdot \lambda_2 \cdot \lambda_3}{\mu_1} \] will be calculated.

6. We just need to put these 1000000 values of \( \theta \) in order, and with the assumption \( \alpha = 0.05 \) it's easy to get 100(1-\( \alpha \))% = 95% confidence interval for \( \theta \), which is (0.1925228, 0.8255976).

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Example 4

In this example we will do the case when $k_1 = 2$ and $k_2 = 3$. The problem becomes to how to calculate the $100(1-\alpha)\%$ confidence interval for $\theta = \frac{\lambda_1 \cdot \lambda_2}{\mu_1 \cdot \mu_2 \cdot \mu_3}$. We will use the R program to do this and the procedure is as the following:

1. Assume the parameters for the priors:
   
   $\lambda_1 \sim \text{gamma}(\alpha_1 = 2, \beta_1 = 2)$
   
   $\lambda_2 \sim \text{gamma}(\alpha_2 = 3, \beta_2 = 3)$
   
   $\mu_1 \sim \text{gamma}(\delta_1 = 3, \gamma_1 = 2)$
   
   $\mu_2 \sim \text{gamma}(\delta_2 = 4, \gamma_2 = 2)$
   
   $\mu_3 \sim \text{gamma}(\delta_3 = 5, \gamma_3 = 1)$

2. A random value for $\lambda_1, \lambda_2, \mu_1, \mu_2, and \mu_3$ will be generated:
   
   $\lambda_1 = 2.060402$
   
   $\lambda_2 = 8.268878$
   
   $\mu_1 = 1.069710$
   
   $\mu_2 = 4.241796$
   
   $\mu_3 = 4.004304$

3. With the input data numbers for each Poisson distribution the Poisson distributions are generated:

   $\bar{X}_1' = (3, 3, 1, 3, 1, ..., 1, 3, 3, 3, 5) \sim P(\lambda_1 = 2.060402), \ n_1 = 25$
\[ \tilde{X}_2' = (8,12,15,9,8,\ldots,9,11,8,12,11) \sim P(\lambda_2 = 8.268878), \quad n_2 = 20 \]

\[ \tilde{Y}_1' = (1,0,3,0,1,\ldots,3,2,0,1,1,0) \sim P(\mu_1 = 1.069710), \quad m_1 = 15 \]

\[ \tilde{Y}_2' = (2,7,5,0,2,\ldots,5,2,3,6,5) \sim P(\mu_2 = 4.241796), \quad m_2 = 15 \]

\[ \tilde{Y}_3' = (2,2,5,3,2,\ldots,4,3,5,3,6,2) \sim P(\mu_3 = 4.004304), \quad m_3 = 25 \]

4. With the above assumption and Poisson data, 1000000 posterior random variables will be generated according to the following:

\[ \lambda_1 | \tilde{X}_1 \sim \text{gamma}(\sum_{i=1}^{n} x_{1i} + \alpha_1, \frac{\beta_1}{1 + n_1 \beta_1}) \]

\[ \lambda_2 | \tilde{X}_2 \sim \text{gamma}(\sum_{i=1}^{n} x_{2i} + \alpha_2, \frac{\beta_2}{1 + n_2 \beta_2}) \]

\[ \mu_1 | \tilde{Y}_1 \sim \text{gamma}(\sum_{i=1}^{m} y_{1i} + \delta_1, \frac{\gamma_1}{1 + m_1 \gamma_1}) \]

\[ \mu_2 | \tilde{Y}_2 \sim \text{gamma}(\sum_{i=1}^{m} y_{2i} + \delta_2, \frac{\gamma_2}{1 + m_2 \gamma_2}) \]

\[ \mu_3 | \tilde{Y}_3 \sim \text{gamma}(\sum_{i=1}^{m} y_{3i} + \delta_3, \frac{\gamma_3}{1 + m_3 \gamma_3}) \]

5. 1000000 values for the ratio \( \theta = \frac{\tilde{\lambda}_1 \cdot \tilde{\lambda}_2}{\mu_1 \cdot \mu_2 \cdot \mu_3} \) will be calculated.

6. We just need to put these 1000000 values of \( \theta \) in order, and with the assumption \( \alpha = 0.05 \) it's easy to get 100(1-\( \alpha \))% = 95% confidence interval for \( \theta \), which is (0.5111273,1.820691).

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Example 5

We will calculate the confidence upper limits according to the data from table 1 and table 2 in Harris' article. Since there is no prior assumption in Sahai and Khurshid’s article, we have to make our prior assumption informative. For the prior $\pi(\lambda_1) \sim \text{gamma}(\alpha_1, \beta_1)$, $\pi(\lambda_2) \sim \text{gamma}(\alpha_2, \beta_2)$, we should take $\alpha_1, \alpha_2$ equal to 1 and $\beta_1, \beta_2$ equal to a big number which is close to infinitive to make the prior informative.

With data $X_1=3, X_2=5$ and because there is only one variable for each sample, we have $n_1=1, n_2=1$ it is easy to get the posterior $\lambda_1/X_1 \sim \text{gamma}(4,1), \lambda_2/X_2 \sim \text{gamma}(6,1)$. We generate 1000000 values of $\lambda_1/X_1$ and $\lambda_2/X_2$ to get 1000000 values of $\lambda_1\lambda_2$ and we will get the 0.90 upper confidence limit. Divide this limit by 10000 which is just the product of sample size of $n_1=100$ and $n_2=100$ we will get the upper confidence limit for $p_1 \cdot p_2$. The result is 0.00449. The upper limits for $p_1 \cdot p_2$ when $X_1$ and $X_2$ take other values are calculated similarly, and we also use the same method to calculated the upper limits for $p_1 \cdot p_2 \cdot p_3$. The results are given in the following two tables.

If we take the prior as a generalized constant prior, we will get the same posterior and the same result for this example.
Table 1  APPROXIMATE UPPER CONFIDENCE LIMITS FOR $p_1, p_2$ CONFIDENCE COEFFICIENT $\alpha = .90$

<table>
<thead>
<tr>
<th>Sample sizes $n_1, n_2$</th>
<th>Observed values $x_1, x_2$</th>
<th>Buehler's limit</th>
<th>Madansky Likelihood ratio</th>
<th>Madansky linearized</th>
<th>Harris Limit based on (8)</th>
<th>Harris &quot;Randomized&quot; limit based on (8)</th>
<th>Our method</th>
</tr>
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<tr>
<td>100, 100</td>
<td>3, 5</td>
<td>.00412</td>
<td>.00433</td>
<td>.00164</td>
<td>.00486</td>
<td>.00416</td>
<td>.00449</td>
</tr>
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<td>1, 4</td>
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<td>.00182</td>
<td>.00097</td>
<td>.00235</td>
<td>.00184</td>
<td>.00210</td>
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<td>.00167</td>
<td>.00091</td>
<td>.00211</td>
<td>.00170</td>
<td>.00187</td>
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<tr>
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<td>3, 3</td>
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<td>.00133</td>
<td>.00074</td>
<td>.00153</td>
<td>.00128</td>
<td>.00139</td>
</tr>
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<td>1.33/n^2</td>
<td>.458/n^3</td>
<td>0</td>
<td>3.72/n^2</td>
<td>2.29/n^2</td>
<td>2.59/n^3</td>
</tr>
</tbody>
</table>

Table 2  APPROXIMATE UPPER CONFIDENCE LIMITS FOR $p_1, p_2, p_3$ CONFIDENCE COEFFICIENT $\alpha = .90$

<table>
<thead>
<tr>
<th>Simple sizes $n_1, n_2, n_3$</th>
<th>Observed values $x_1, x_2, x_3$</th>
<th>Madansky's likelihood ratio</th>
<th>Harris Limit based on (8)</th>
<th>Harris &quot;Randomized&quot; limit based on (8)</th>
<th>Our method</th>
</tr>
</thead>
<tbody>
<tr>
<td>100, 100, 100</td>
<td>1, 2, 1</td>
<td>.000019</td>
<td>.000019</td>
<td>.000027</td>
<td>.000029</td>
</tr>
<tr>
<td>100, 100, 100</td>
<td>2, 3, 5</td>
<td>.000133</td>
<td>.000186</td>
<td>.000145</td>
<td>.000153</td>
</tr>
</tbody>
</table>

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CHAPTER 5

R PROGRAM AND RESULTS

Example 1

```r
> alph[c(2,2)
> peta[c(2,3)
> a2=c(3,4)
> p2=c(2,3)
> n=c(25,15)
> m=c(20,45)
> lambda=0
> l=0
> num=1
> den=1
> for (i in 1:1) {
+ lambda[i]=rgamma(1,shape=alph[i],scale=peta[i])
+ num=num*lambda[i]
>
+ for (j in 1:1) {
+ l[j]=rgamma(1,shape=a2[j],scale=p2[j])
+ den=den*l[j]
>
+ ratio=num/den
>
+ x1=rpois(n[1],lambda[1])
+ y1=rpois(m[1],l[1])
>
+ a=0
+ p=0
+ q=0
+ b=0
>
+ a[1]=sum(x1)+alph[1] #a records the alpha values of the numerator
+ posterior distn
>
```
> p[1]=peta[1]/(1+n[1]*peta[1]) # p records the peta values of the numerator posterior distn
>
> b[1]=sum(y1)+a2[1]
> # b records the alpha values of the denominator posterior distn
> q[1]=p2[1]/(1+m[1]*p2[1])
>
> # q records the peta values of the denominator posterior distn
>
> lx1=rgamma(1000000,shape=a[1],scale=p[1])
>
> my1=rgamma(1000000,shape=b[1],scale=q[1])
>
> lxm=lx1/my1
> s=order(lxm)
> c=rbind(lxm[s])
> lcl=c[25000]
> ucl=c[975000]
> lambda[1]
> [1] 5.961133
> [1] 9.52169
> lcl
> [1] 0.4957661
> ucl
> [1] 0.7623171
> ratio
> [1] 0.6260583
> x1
> [1] 9 7 7 7 5 1 5 5 5 6 9 9 5 5 6 5 2 13 6 6 2 5 5 3 6 6
> y1
> [1] 6 12 11 11 12 8 9 10 13 9 7 14 9 11 11 8 11 8 6 9
> a[1]
> [1] 147
> p[1]
> [1] 0.03921569
> b[1]
> [1] 192
> q[1]
> [1] 0.04878049

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Example 2

```r
> alph=c(2,2)
> peta=c(2,3)
> a2=c(3,4)
> p2=c(3,3)
> n=c(20,15)
> m=c(20,25)
> lambda=0
> l=0
> num=1
> den=1
> for ( i in 1:2) {
+ lambda[i]=rgamma(1,shape=alph[i],scale=peta[i])
+ num=num*lambda[i]}
>
> for ( j in 1:2) {
+ l[j]=rgamma(1,shape=a2[j],scale=p2[j])
+ den=den*l[j]}
>
> ratio=num/den

> x1=rpois(n[1],lambda[1])
> x2=rpois(n[2],lambda[2])
>
> y1=rpois(m[1],l[1])
> y2=rpois(m[2],l[2])
>
> a=0
> p=0
> q=0
> b=0
>
> a[1]=sum(x1)+alph[1] #a records the alpha values of the numerator posterior distn
> a[2]=sum(x2)+alph[2]
> p[1]=peta[1]/(1+n[1]*peta[1])#p records the peta values of the numerator posterior distn
> b[1]=sum(y1)+a2[1]
> b[2]=sum(y2)+a2[2]#b records the alpha values of the denominator posterior distn
> q[1]=p2[1]/(1+m[1]*p2[1])
>
> #q records the peta values of the denominator posterior distn
```
> lx1=rgamma(1000000,shape=a[1],scale=p[1])
> lx2=rgamma(1000000,shape=a[2],scale=p[2])
> my1=rgamma(1000000,shape=b[1],scale=q[1])
> my2=rgamma(1000000,shape=b[2],scale=q[2])
> lxm=lx1*lx2/(my1*my2)
> s=order(lxm)
> c=rbind(lxm[s])
> lcl=c[25000]
> ucl=c[975000]
> lcl
[1] 0.04432612
> ucl
[1] 0.1341951
> ratio
[1] 0.0878508
> lambda[1]
[1] 1.028019
> x1
[1] 2 2 0 0 4 1 3 0 1 3 0 1 0 0 1 0 2 0 1 0
> lambda[2]
[1] 3.547733
> x2
[1] 4 2 2 0 2 5 2 4 2 2 4 4 3 4 6
> l[1]
[1] 4.95997
> y1
[1] 4 3 6 6 9 4 6 3 8 4 1 3 5 3 5 2 7 7 8 7
> l[2]
[1] 8.370036
> y2
[1] 3 1 3 9 5 4 8 12 10 12 5 12 10 12 8 7 12 7 5 6 10 12 5 9 9 8

Example 3

> alph=c(2,2,3)
> peta=c(1,2,1)
> a2=c(4,4)
> p2=c(2,3)
> n=c(20,15,10)
> m=c(25,45)
> lambda=0
> l=0
num=1
den=1
for (i in 1:3) {
    lambda[i]=rgamma(1,shape=alph[i],scale=peta[i])
    num=num*lambda[i]
}
for (j in 1:1) {
    l[j]=rgamma(1,shape=a2[j],scale=p2[j])
    den=den*l[j]
}
ratio=num/den
x1=rpois(n[1],lambda[1])
x2=rpois(n[2],lambda[2])
x3=rpois(n[3],lambda[3])
y1=rpois(m[1],l[1])
a=0
p=0
q=0
b=0
a[1]=sum(x1)+alph[1] #a records the alpha values of the numerator posterior distn
a[2]=sum(x2)+alph[2]
a[3]=sum(x3)+alph[3]
p[1]=peta[1]/(1+n[1]*peta[1])#p records the peta values of the numerator posterior distn
p[3]=peta[3]/(1+n[3]*peta[3])
b[1]=sum(y1)+a2[1] #b records the alpha values of the denominator posterior distn
q[1]=p2[1]/(1+m[1]*p2[1])

lx1=rgamma(1000000,shape=a[1],scale=p[1])
lx2=rgamma(1000000,shape=a[2],scale=p[2])
 lx3=rgamma(1000000,shape=a[3],scale=p[3])
```r
> my1=rgamma(1000000,shape=b[1],scale=q[1])
> lx1=lx1*lx2*lx3/my1
> s=order(lx1)
> c=rbind(lx1[s])
> l1=c[25000]
> u1=c[975000]
> l1
[1] 0.1925228
> u1
[1] 0.8255976
> ratio
[1] 0.5122483
> lambda[1]
[1] 1.529129
> x1
[1] 0 1 1 0 4 2 0 2 3 2 2 4 0 1 1 3 0 2 2 1
> lambda[2]
[1] 3.956051
> x2
[1] 3 7 5 4 3 4 2 4 5 1 6 3 2 1 6
> lambda[3]
[1] 1.258633
> x3
[1] 1 3 0 0 2 0 1 0 2 0
> l[1]
[1] 14.86363
> y1
[1] 18 10 10 15 12 17 26 11 15 7 12 24 16 23 16 18 11 16 10 15 14 11 10 17 15

Example 4

> alph=c(2,3)
> peta=c(2,3)
> a2=c(3,4,5)
> p2=c(2,2,1)
> n=c(25,20)
> m=c(15,15,25)
> lambda=0
> l=0
> num=1
> den=1
> for (i in 1:2) {
```

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+ lambda[i]=rgamma(1,shape=alph[i],scale=peta[i])
+ num=num*lambda[i]
> for (j in 1:3) {
+ l[j]=rgamma(1,shape=a2[j],scale=p2[j])
+ den=den*l[j]}
> ratio=num/den
> x1=rpois(n[1],lambda[1])
> x2=rpois(n[2],lambda[2])
> y1=rpois(m[1],l[1])
> y2=rpois(m[2],l[2])
> y3=rpois(m[3],l[3])
> a=0
> p=0
> q=0
> b=0
> a[1]=sum(x1)+alph[1] #a records the alpha values of the numerator posterior distn
> a[2]=sum(x2)+alph[2]
> p[1]=peta[1]/(1+n[1]*peta[1])#p records the peta values of the numerator posterior distn
> b[1]=sum(y1)+a2[1]
> b[3]=sum(y3)+a2[3]#b records the alpha values of the denominator posterior distn
> q[1]=p2[1]/(1+m[1]*p2[1])
> q[3]=p2[3]/(1+m[3]*p2[3])
> #q records the peta values of the denominator posterior distn
> lx1=rgamma(1000000,shape=a[1],scale=p[1])
> lx2=rgamma(1000000,shape=a[2],scale=p[2])
> my1=rgamma(1000000,shape=b[1],scale=q[1])
> my2=rgamma(1000000,shape=b[2],scale=q[2])
> my3=rgamma(1000000,shape=b[3],scale=q[3])
> lxm=lx1*lx2/(my1*my2*my3)
> s=order(lxm)
> c=cbind(lxm[s])
> lcl=c[25000]
> ucl=c[975000]
> lcl
[1] 0.5111273
> ucl
[1] 1.820691
> lambda[1]
[1] 2.060402
Example 5

```r
> x1
[1] 3 3 1 3 1 2 1 4 0 3 2 1 1 3 0 0 3 0 1 4 1 3 3 5
> lambda[2]
[1] 8.268878
> x2
[1] 8 12 15 9 8 6 7 8 13 6 10 9 11 14 8 9 11 8 12 11
> l[1]
[1] 1.069710
> y1
[1] 1 0 3 0 1 3 0 0 0 3 2 0 1 1 0
> l[2]
[1] 4.241796
> y2
[1] 2 7 5 0 2 5 9 3 9 6 5 2 3 6 5
> l[3]
[1] 4.004304
> y3
[1] 2 2 5 3 2 3 6 6 5 5 5 6 3 2 7 5 2 5 2 4 3 5 3 6 2
> ratio
[1] 0.9376818
```

```r
> lx1=rgamma(1000000,4,1)
> lx2=rgamma(1000000,6,1)
>
> lxm=lx1*lx2
> s=order(lxm)
> c=rbind(lxm[s])
> lcl=c[50000]
> ucl=c[900000]
> ucl
[1] 44.89672
> lx1=rgamma(1000000,2,1)
> lx2=rgamma(1000000,5,1)
>
> lxm=lx1*lx2
> s=order(lxm)
> c=rbind(lxm[s])
> lcl=c[50000]
> ucl=c[900000]
> ucl
[1] 21.04216
> lx1=rgamma(1000000,3,1)
```

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> lx2 = rgamma(1000000, 3, 1)
> lx = lx1 * lx2
> s = order(lx)
> c = rbind(lx[s])
> lcl = c[50000]
> ucl = c[90000]
> ucl
[1] 18.72712
> lx1 = rgamma(1000000, 1, 1)
> lx2 = rgamma(1000000, 4, 1)
> lx3 = rgamma(1000000, 6, 1)
> lx = lx1 * lx2 * lx3
> s = order(lx)
> c = rbind(lx[s])
> lcl = c[50000]
> ucl = c[90000]
> ucl
[1] 2.588229

> lx1 = rgamma(1000000, 2, 1)
> lx2 = rgamma(1000000, 3, 1)
> lx3 = rgamma(1000000, 2, 1)
> lx = lx1 * lx2 * lx3
> s = order(lx)
> c = rbind(lx[s])
> lcl = c[50000]
> ucl = c[90000]
> ucl
[1] 28.63170

> lx1 = rgamma(1000000, 3, 1)
> lx2 = rgamma(1000000, 4, 1)
> lx3 = rgamma(1000000, 6, 1)
> lx = lx1 * lx2 * lx3
> s = order(lxm)
> c = rbind(lxm[s])
> lcl = c[50000]
> ucl = c[900000]
> ucl
[1] 153.4817
BIBLIOGRAPHY


Harris, Bernard Hypothesis testing and confidence intervals for products and quotients of Poisson parameters with applications to reliability. *J. Amer. Statist. Assoc.* 66 (1971), 609--613. (Reviewer: A. P. Basu) 62N05 (62F05)


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