5-1-2014

Lattice Methods For The Valuation of Options with Regime Switching

Atul Sancheti
University of Nevada, Las Vegas

LATTICE METHODS FOR THE VALUATION OF OPTIONS WITH REGIME SWITCHING

by

Atul Sancheti

Bachelor of Technology, Electronics & Communication Engineering
Indian Institute of Technology Guwahati
2010

A thesis submitted in partial fulfullment of
the requirements for the

Master of Science - Mathematical Sciences

Department of Mathematical Sciences
College of Sciences
The Graduate College

University of Nevada, Las Vegas
May 2014

UNIVERSITY OF NEVADA LAS VEGAS

THE GRADUATE COLLEGE

We recommend the thesis prepared under our supervision by

Atul Sancheti

entitled

Lattice Methods for the Valuation of Options with Regime Switching

is approved in partial fulfillment of the requirements for the degree of

Master of Science - Mathematical Sciences

Department of Mathematical Sciences

Hongtao Yang, Ph.D., Committee Chair

Amei Amei, Ph.D., Committee Member

Xin Li, Ph.D., Committee Member

Pushkin Kachroo, Ph.D., Graduate College Representative

Kathryn Hausbeck Korgan, Ph.D., Interim Dean of the Graduate College

May 2014

ABSTRACT

LATTICE METHODS FOR THE VALUATION OF OPTIONS WITH REGIME SWITCHING

by

Atul Sancheti

Hongtao Yang, Examination Committee Chair
Associate Professor of Mathematical Science
University of Nevada, Las Vegas

In this thesis, we have developed two numerical methods for evaluating option prices under the regime switching model of stock price processes: the Finite Difference lattice method and the Monte Carlo lattice method.

The Finite Difference lattice method is based on the explicit finite difference scheme for parabolic problems. The Monte Carlo lattice method is based on the simulation of the Markov chain. The advantage of these methods is their flexibility to compute the option prices for any given stock price at any given time. Numerical examples are presented to examine these methods. It has been shown that the proposed methods provides fast and accurate approximations of option prices. Hence they should be helpful for practitioners working in this field.

ACKNOWLEDGEMENTS

I would like to express my sincere thanks and gratitude to my advisor Dr. Hongtao Yang for inspiring, motivating and guiding me througout my research. I am also grateful to Dr. Pushkin Kachroo for his constant unconditional support and motivation. I am also like to extend my sincere gratitude to my committee members Dr. Amei Amei and Dr. Xin Li for their insightful remarks which helped me in improving my work.

I would also like to take this opportunity to show my gratitude and appreciation towards my parents Mr. Pramod Kumar Sancheti, Mrs. Sunita Sancheti, my brother Ashwin Sancheti and my entire family for being there and showing their love and support throughout the course of my research. Their constant enthusiasm and quest to know about my research motivated me to work more effectively and efficiently towards my goal.

I am also grateful to the support and motivation provided by my friends Anuj Nayyar, Himanshu Verma, Pratik Verma, Romesh Khaddar, Sergio Contreras, Shaurya Agarwal and Sourabh Sriom during this journey.

TABLE OF CONTENTS

LIST OF TABLES

CHAPTER 1

INTRODUCTION

1.1 Options

An option is a contract that gives the holder the right - but not the obligation, to buy or sell an underlying asset at a contractually specified strike price on a range of future dates. There are two different types of options namely Call Options and Put Options. Call Options give the right to buy the underlying asset, whereas Put Options give the right to sell. The price is known as the strike price or exercise price and the date is known as the expiration date or maturity. There are two major styles of options that are traded at exchanges: the European and American options.

The European options can only be exercised at the end of its life or at the expiration date of the contract. These options stop trading a day before than the third friday of the expiration month. In addition, it is not easy to learn the official closing price or the settlement price for the expiration period for European-style options. Moreover, the settlement price is not published until hours after the market opens for trading. Also, European options sometimes trade at a discount rate than its comparable American Option.

The right to exercise is one of the key differences that set apart American options from European Options. These options can be exercised anytime before the option expires. This allows investors more opportunities to exercise the contract and therefore provides a relatively highly price than European Options. It was also interesting to note that a majority of stocks, options and exchange traded funds (ETFs) have American-style options. The trading for American options cease at the close of business on the third Friday of the expiration month. Also, the settlement price with American options is the regular closing price, or the last trade before the market closes on the third Friday.

There are the over-the-counter traded options such as Asian Options, Bermuda options, and look-back options, which are referred as the exotic options.

The valuation and optimal exercise of derivatives with American-style exercise features is one of the most important and challenging problems in option pricing theory. These types of derivatives are found in all major financial markets including the equity, commodity, foreign exchange, insurance, energy, sovereign, agency, municipal, mortgage, credit, real estate, convertible, swap, and emerging markets. In spite of the recent developments made in this emerging field, the valuation and optimal exercise of American options remains one of the most difficult problems in derivatives finance. This can be mainly attributed to the fact that finite difference and binomial techniques become impractical when considering multiple factor models which provides a better and more detailed description of practical financial problems [1, 2, 3].

1.2 Problems

Besides the classic Black-Scholes model for the underlying assets, various other models have been proposed, for example, jump diffusion models, regime switching models, and stochastic volatility models (see [4, 5, 6] and references cited therein). As in [6], we suppose that the underlying economy switches among n states $\{1, 2, \ldots, n\}$, which is modeled by a finite Markov chain $X(t)$ with the rate matrix $Q = (q_{ij})$. Let constant r_i be the interest rate when the economy is in the i-th state at time t, that is, $X(t) = i$. Assume that the stock pays the continuous dividend at constant rate d. The stock price process $S(t)$ is modeled by the following stochastic differential equation (SDE):

$$dS(t) = S(t)\left(\mu_{X(t)}dt + \sigma_{X(t)}dW(t)\right), \quad t > 0, \tag{1.1}$$

where $W(t)$ is a standard Brownian motion under the risk neutral measure, $\mu_i = r_i - d$, and constant σ_i is the stock volatility in the i-th state of economy.

Consider an American call option with strike price \$$K$ and expiry date T years. Denote by $C_i(S,t)$ the call price in the i-th state. Let $C(S,t) = (C_1(S,t), \ldots, C_n(S,t)^T)$. As usual, we have the following variational inequality problem:

$$C_{i,t}(S,t) + \mathcal{A}_i C(S,t) \le 0, \quad S > 0, \quad 0 < t \le T, \tag{1.2}$$

$$C_i(S,t) \ge (S-K)^+, \quad S \ge 0, \quad 0 \le t \le T, \tag{1.3}$$

$$\left(C_{i,t}(S,t) + \mathcal{A}_i C(S,t)\right)\left(C_i(S,t) - (S-K)^+\right) = 0, \quad S \ge 0, 0 < t \le T, \tag{1.4}$$

$$C_{i,t}(0,t) = 0, \quad 0 \le t \le T, \tag{1.5}$$

$$C_i(S,T) = (S-K)^+, \quad S \ge 0, \tag{1.6}$$

for $i = 1, \ldots, n$, where $\langle \cdot, \cdot \rangle$ is the usual inner product on R^2, and

$$\mathcal{A}_i C(S,t) = \frac{1}{2}\sigma_i^2 S^2 C_{i,SS}(S,t) + \mu_i S C_{i,S}(S,t) - r_i C_i(S,t) + \langle QC(S,t), e_i \rangle.$$

Similarly, we have the variational inequality problem for the American put option:

$$P_{i,t}(S,t) + \mathcal{A}_i C(S,t) \le 0, \quad S > 0, \quad 0 < t \le T, \tag{1.7}$$

$$P_i(S,t) \ge (K-S)^+, \quad S \ge 0, \quad 0 \le t \le T, \tag{1.8}$$

$$\left(P_{i,t}(S,t) + \mathcal{A}_i C(S,t)\right)\left(P_i(S,t) - (K-S)^+\right) = 0, \quad S \geq 0, 0 < t \leq T, \quad (1.9)$$

$$P_{i,t}(0,t) = 0, \quad 0 \leq t \leq T, \quad (1.10)$$

$$P_i(S,T) = (K-S)^+, \quad S \geq 0, \quad (1.11)$$

for $i = 1, \ldots, n$.

1.3 Thesis Structure

Numerical methods have been extensively investigated for valuation of American options and other path-dependent financial derivatives for more than three decades (see [7], [8], [9], and references cited therein). In this thesis, we shall develop two lattice methods for the above variational inequality problems. One is the generalization of the lattice method proposed in [12] when there are only two states of economy. The other is a lattice method based on the Monte Carlo simulation of the Markov chain. Lattice methods are more attractive to practitioners since they can be easily implemented. Moreover, it is more flexible to compute option prices and hedge ratios at any given point. A favorable feature of our methods is that there is only one set of nodes for stock price for all states.

The remaining of the thesis is outlined as follows. In Chapter 2, we shall review basic theory about linear complementary problems (LCP) since these problems are formed by discretizing variational inequality problems (1.2)–(1.6) and (1.7)–(1.11). Especially, two pivoting algorithms: Chandrasekaran and Lemke methods are described for LCPs with M-matrices. In Chapter 3, a Finite Difference lattice method is proposed to compute the option prices for the given stock prices at given times. In Chaper 4, we shall develop a lattice method based on Monte Carlo simulation of Markov Chain. Numerical examples are presented in Chapter 5 to examine our new methods. The Conclusion remarks are given in the last chapter, Chapter 6.

CHAPTER 2

LINEAR COMPLEMENTARY PROBLEMS

2.1 Introduction

Let M be a $n \times n$ matrix in $\mathbb{R}^{n \times n}$ and q a column vector in $\mathbb{R}^n$. Then the linear complementary problem, denoted by LCP(q, M), is to find $w,\ z \in \mathbb{R}^n$ such that

$$w - Mz = q, \tag{2.1}$$

$$w^T z = 0, \tag{2.2}$$

$$w \geq 0, \quad z \geq 0, \tag{2.3}$$

where w^T is the transpose of w and $w \geq 0$ means that every component of w is nonnegative.

The linear complementarity problems (LCPs) can be considered as a more general case for linear, quadratic and bimatrix problems. The study of a LCP has led to development of several highly effective algorithms which aids in solving the highly complex problems in an efficient manner. In this Chapter, we introduce the complementary pivot algorithm for solving LCPs, in particular, the Lemke method.

2.2 Solution Existence and Uniqueness

A matrix is called a P-matrix if its principal minors are positive. In other words, a matrix is a P-matrix if and only if the real eigen values of the principal submatrices of M are positive. Thus positive definite matrices are P-matrices. Concerning the solution existence and uniqueness of the LCP (2.1)–(2.3), we have the following result ([10]).

Theorem 2.1. *(Samelson, Thrall and Wesler) The LCP (q, M) has a unique solution for every q if and only M is a P-matrix.*

A matrix B is called nonnegative (write $B \geq 0$) if all element of B are nonnegative numbers. We say that a matrix A is an M-matrix if there is a positive number s and a nonnegative matrix B such that $A = sI - B$ and $s > \rho(A)$. A matrix of form $sI - B$ is an M-matix if and only if all principal minors of A are positive. Hence, we have the following corollary.

Corollary 2.1. *If M is an M-matrix, then LCP (q, M) has a unique solution for every q.*

It should be pointed out that the corresponding matrix M is an M-matrix for our lattice method for the regime-switching problems.

2.3 An Augmented LCP

Let $d \in \mathbb{R}^n$ be a positive vector and s be a positive number. For the LCP (q, M), the corresponding augmented LCP, denoted by ALCP $(q, M; d, s)$ is LCP $\left(\widetilde{q}, \widetilde{M}\right)$, where

$$\widetilde{q} = \begin{bmatrix} s \\ q \end{bmatrix}, \quad \widetilde{M} = \begin{bmatrix} 0 & -d^T \\ d & M \end{bmatrix}.$$

Here d is called the covering vector. The LCP $\left(\widetilde{q}, \widetilde{M}\right)$ reads as follows: Find $z \in \mathbb{R}^n$ and $t \in \mathbb{R}$ such that

$$\sigma = s + 0t - d^T z \geq 0, \qquad t \geq 0, \quad t\sigma \geq 0,$$

$$w = q + Mz + td \geq 0, \qquad z \geq 0, \quad z^T w = 0.$$

We can see that a solution (t, z) of the ALCP $(q, M; d, s)$ with $t = 0$ provides a solution z of the LCP (q, M). Furthermore, we have the following results ([10]).

Theorem 2.2. *(a) For every given $d > 0$ and $s > 0$, the ALCP$(q, M; d, s)$ has a solution.*

(b) Suppose that there is a positive number k such that if $x \geq 0$ and $e^T x = k$ then $x^T(q + Mx) \geq 0$, where $e = (1, \ldots, 1)^T$. Let (t, z) be a solution of the ALCP$(q, M; e, k)$. Then $t = 0$ and thus z is a solution of the LCP(q, M).

2.4 Pivoting Methods for the LCP (q, M)

From now on, we shall assume that M is an P-matrix. Let $w = (w_1, \ldots, w_n)^T$ and $z = (z_1, \ldots, z_n)^T$ be a solution of the LCP (q, M). Notice that equation (2.3) implies that one element of each pair (w_j, z_j) must be zero. If one is positive then the other must be zero. Hence the pair (w_j, z_j) is called the j-th complementary pair of variables.

Denoted by $I_{\dot{j}}$ and $M_{\dot{j}}$ the j$-th$ columns of the identity matrix I and M, repectviely. Then we can rewrite (2.1) as follows

$$q = Iw + (-M)z = \sum_{j=1}^{n} w_j I_{\dot{j}} + \sum_{j=1}^{n} z_j(-M_{\dot{j}}). \tag{2.1}$$

Thus solving the LCP (q, M) can be interpreted as finding a complementary pair of nonnegative vectors w and z such that q is a linear combination of n vectors consisting of the column vectors of I and M. This intepretation leads to pivoting methods for the LCP (q, M).

We shall group the $2n$ variables $\{w, z\} = \{w_1, \ldots, w_n, z_1, \ldots, z_n\}$ into basic variables $\{y_1, \ldots, y_n\}$ and nonbasic variables $\{v_1, \ldots, v_n\}$. It follows from equation (2.1) that

$$w = q + Mz. \tag{2.2}$$

Here variables $\{w_1, \ldots, w_n\}$ are basic and $\{z_1, \ldots, z_n\}$ are nonbasic. That is, the basic variables are the variables that depends on the nonbasic ones. Consider the r-th equation of the system (2.2):

$$w_r = q_q + m_{r1}z_1 + \ldots + w_{rn}z_n.$$

If $m_{rs} \neq 0$, we can solve for z_s in terms of w_r and all the other nonbasic varibales z_j with $j \neq s$. Then we have

$$z_s = -\frac{q_r}{m_{rs}} + \sum_{j \neq s} \left(-\frac{m_{rj}}{m_{rs}} \right) z_j + \frac{1}{m_{rs}} w_r.$$

After substituting this expression for z_s into all the other eqaution in (2.2), we have

$$w_i = q_i - q_r \frac{m_{is}}{m_{rs}} + \sum_{j \neq s} \left(m_{ij} - m_{rj} \frac{m_{is}}{m_{rs}} \right) z_j + \frac{m_{is}}{m_{rs}} w_r, \quad i \neq r.$$

This operation is called simple pivoting, which exchanges the roles of w_s and z_s. Namely, w_s and z_s becomes nonbasic and basic, respectively. Now the basic variables are $\{w_1, \ldots, w_{s-1}, z_s, w_{s+1}, \ldots, w_n\}$ and the nonbasic variables are $\{z_1, \ldots, z_{s-1}, w_s, z_{s+1}, \ldots, z_n\}$. The LCP (q, M) can be represented by the following tableau:

w	z	
I	$-M$	q

$$w \geq 0, \quad z \geq 0 \tag{2.3}$$

A pivoting method consists of a sequences of pivoting steps to transform the above initial tableau. Let the resulting tableau be as follows:

v	y	
I	$-\widetilde{M}$	$\widetilde{q}$

$$v \geq 0, \quad y \geq 0 \tag{2.4}$$

where v is the vector for the basic variables and y is the vector for the nonbasic ones. If $\widetilde{q} \geq 0$, a solution has been found and it can be obtained by letting all the nonbasic variables be 0 and basic ones be equal to the corresponding elements of $\widetilde{q}$.

A detailed account in pivoting method can be found in Cottle et al [10]. For our purpose, we only need the Chandrasekaran and Lemke methods.

2.5 Chandrasekaran Method

The following Chandrasekaran's Method is a drect application of the above pivoting method to the LCP (q, M) when M is a Z-matrix.

Algorithm 1. Chandrasekaran's Method to solve LCP

Consider the LCP (q, M) as represented by the initial tableau (2.3) with
$w = (w_1, w_2, ..., w_n)$ as the initial complementary basic vector.
if $q \geq 0$, i.e. w is a feasible basis, then
    $(w, z) = (q, 0)$ (Complementary Basic Feasible Solution);
    break;
else do
Display the tableau: $tab = [eye(M) \ -M \ q]$
    $\bar{q}$ = tab(:,end);
    if $\bar{q} \geq 0$;
        Present basic vector is a complementary feasible basic vector;
    break;
    else do
        Find t, such that $\bar{q}_t \leq 0$;
        if $-m_{tt} \geq 0$
            No nonnegative solution or LCP (q, M) has no solution;
        break;
        else do
        Update the tableau by pivoting at row t and column $t + n$;
    end do

A matrix $M = (m_{ij})$ is a Z-matrix if all its off diagonal entries are nonpositive, that is $m_{ij} \leq 0$ for all $i \neq j$. It can be easily verified that in tableau (2.3) for any t = 1 to n, all the entries in row t are nonnegative except for the entry in column z_t. Hence, all the pivot elements encountered during the Chandrasekaran's algorithm are strictly negative. In addition, once a pivot has been performed in a row, the value of the updated right hand side constant remains negative for all subsequent steps. Moreoever, once a variable z_t has been made a basic variable, it stays as a basic variable and its value remain nonnegative in all subsequent steps. As at most one principal pivot step is performed in each row, hence the algorithm terminates in at most n pivot steps either with the conclusion of infeasibility or with a complementary feasible basis [11].

2.6 The Lemke Method

The Lemke method is a pivoting methods for the ALCP $(q, M; e, s)$, where s will be determined by the algorithm. The advantage of considering the ALCP $(q, M; e, s)$ instead of the LCP (q, M) is that the ALCP $(q, M; e, s)$ has a solution (see Theorem 2.2. Also, the Lemke method will either find a solution or indicate no solution for the LCP (q, M).

The Lemke method uses complementary pivoting schemes and provide a choice of driving variable. One of the major advantages of these complementary pivoting schemes is the very fact that these are relatively easy to state, more versatile and does not depend on the invariance of matrix classes under principal pivoting.

Algorithm 2. Lemke Method to solve LCP

Initialization Step:
if $q \geq 0$, then
    $(w, z) = (q, 0)$ (Complementary Basic Feasible Solution);
    break;
else do
    Display the tableau: $tab = [eye(M) \;\; -M \;\; -z_0 \;\; q]$
    let $q_s = min\{q_i : 1 \leq i \leq n\}$
    Update the tableau by pivoting at row s and column z_0
    $tab(s :,) = tab(s, :)./tab(s, t_m - 1)$

    for $i = 1, \ldots, m$, do

        if $i \neq s$, $tab(i, :) = tab(i, :) - tab(s, :) * tab(i, t_m - 1)/tab(s, t_m - 1)$

    end do

    Let $y_s = z_s$, GOTO Main Step
end do
Main Step
STEP 1: Let d_s be the updated column under variable y_s,
    while($d_s > 0$)
    Determine index r by the minimum ratio test:

$$\frac{\bar{q}_r}{d_{rs}} = \min_{1 \leq i \leq m} \left\{ \frac{\bar{q}_i}{d_{is}} : d_{is} > 0 \right\}$$

    If the basic variable at row r is z_0, GOTO STEP 3
    else GOTO STEP 2.

STEP 2: Update the tableau by pivoting at row r and column y_s
    if the variable leaving the basis is w_l, then let $y_s = z_l$

    else if the variable leaving the basis is z_l, then let $y_s = w_l$

    GOTO STEP 1
STEP 3: Update the tableau by pivoting at row y_s column and z_0 row,
    break; (Complementary Basic Feasible Solution)
STEP 4: Ray $R = \{(w, z, z_0) + \lambda d : \lambda \geq 0\}$,
    where every point in R satisfies equations (2.1), (2.2), and (2.3)

    end do (Almost Complementary Basic Feasible Solution)

We have the following results about the convergence of the Lemke method.

Theorem 2.3. *When applied to a nondegenerate instance of* (q, d, M)*, Lemke's Algorithm will terminate in finitely many steps with either a secondary ray or else a complementary feasible solution of* (q, d, M) *and hence with a solution of* (q, M) *[10].*

When Lemke's algorithm terminates with a secondary ray, it usually requires the strict positivity of the covering vector d.

Theorem 2.4. *If Lemke's Algorithm applied to* (q, d, M) *terminates with a secondary ray, then* M *reverses the sign of some nonzero nonnegative vecotr* $\bar{(}z)$ *[10], that is*

$$\bar{z}_i(M\bar{z}_i) \leq 0 \qquad i = 1, ..., n. \tag{2.1}$$

Hence the above theorem implies that the Lemke's Algorithm cannot terminate in a secondary ray when $M \in P$, as in a P the sign of a nonzero vector is never reversed [10]. Thus for any nondegenrate linear complimentarity problem of the P-matrix type, Lemke's Algorithm will obtain its solution.

CHAPTER 3

A FINITE DIFFERENCE LATTICE METHOD

In this chapter, we extend the simple lattice method proposed in [12] to compute the option prices for the given stock price S_0 and time to the expiration date T_0. Since the method is based on the forward Euler scheme for parabolic problems, we call it the Finite Difference lattice method. We only consider the call option problem since the put option problem can be treated in the same fashion.

Consider the variable transforms

$$S = Ke^x, \quad C_i(S, T-t) = Ku_i(x,t), \quad i = 1, \ldots, n.$$

The variational inequality problem (1.2)-(1.6) can be reformulated into

$$\frac{\partial u_i}{\partial t} + \mathcal{B}_i u_i - \sum_{j=1}^{n} \xi_{ij} u_i \geq 0, \quad -\infty < x < \infty, 0 \leq t < T, \tag{3.1}$$

$$u_i(x,t) \geq f(x), \quad -\infty < x < \infty, 0 \leq t < T, \tag{3.2}$$

$$\left(\frac{\partial u_i}{\partial t} + \mathcal{B}_i u_i - \sum_{j=1}^{n} q_{ij} u_i\right)(u_i(x,t) - f_i(x)) = 0, \quad -\infty < x < \infty, 0 \leq t < T, \tag{3.3}$$

$$u_i(-\infty, t) = 0, \quad 0 \leq t \leq T, \tag{3.4}$$

$$u_i(x, 0) = f_i(x), \quad -\infty < x < \infty, \tag{3.5}$$

for $i = 1, \ldots, n$, where

$$\mathcal{B}_i u_i = -\gamma_i \frac{\partial^2 u_i}{\partial x^2} + \nu_i \frac{\partial u_i}{\partial x} + r_i u_i,$$
$$\gamma_i = \frac{1}{2}\sigma_i^2, \quad \nu_i = \gamma_i - \mu_i, \quad f_i(x,t) = (e^x - 1)^+.$$

For a given positive integer M, let $k = T_0/M$ and $t_m = mk$ for $m = 0, 1, \ldots, M$.

For a positive number σ, let $h = \sigma\sqrt{k}$ be the mesh size in x, and let

$$x_j = \log\left(\frac{S_0}{K}\right) + jh, \quad j = -M, ..., M.$$

Denote by $u_{i,j}^m$ be the approximation of $u(x_j, t_m)$. Discretizing (3.1)–(3.3) using the finite difference methods, we have the following LCP:

$$\frac{u_{i,j}^m - u_{i,j}^{m-1}}{k} + \mathcal{L}_i u_{i,j}^{m-1} + r_i u_{i,j}^m - \sum_{j=1}^{n} \xi_{ij} u_{i,j}^m \geq 0, \quad u_{i,j}^m \geq f_{i,j},$$

$$\left(\frac{u_{i,j}^m - u_{i,j}^{m-1}}{k} + L_i u_{i,j}^{m-1} + r_i u_{i,j}^m - \sum_{j=1}^{n} \xi_{ij} u_{i,j}^m\right)\left(u_{i,j}^m - f_{n,j}\right) = 0,$$

for $i = 1, 2, ..., n$, where $f_{i,j} = f_i(x_j)$ and

$$L_i u_{i,j}^{m-1} = -\gamma_i \frac{u_{i,j+1}^{m-1} - 2u_{i,j}^{m-1} + u_{i,j-1}^{m-1}}{h^2} + \nu_i \frac{u_{i,j+1}^{m-1} - u_{i,j-1}^{m-1}}{2h}.$$

The above LCP can be rewritten into the following matrix form:

$$AU_j^m \geq G_j^m, \quad U_j^m \geq F_j, \tag{3.6}$$

$$\left(AU_j^m - G_j^m\right)\left(U_j^m - F_j\right) = 0, \tag{3.7}$$

where

$$A = \begin{bmatrix} 1 + k(r_1 + q_{11}) & -kq_{12} & \cdots & -kq_{1n} \\ -kq_{21} & 1 + k(r_2 + q_{22}) & \cdots & -kq_{2n} \\ \vdots & \vdots & & \vdots \\ -kq_{n1} & -kq_{n2} & \cdots & 1 + k(r_n + q_{nn}) \end{bmatrix},$$

$$U_j^m = \begin{bmatrix} u_{1,j}^m \\ u_{2,j}^m \\ \cdots \\ u_{n,j}^m \end{bmatrix}, \quad F_j = \begin{bmatrix} f_{1,j} \\ f_{2,j} \\ \cdots \\ f_{n,j} \end{bmatrix}, \quad G_j^{m-1} = \begin{bmatrix} g_{1,j}^{m-1} \\ g_{2,j}^{m-1} \\ \cdots \\ g_{n,j}^{m-1} \end{bmatrix},$$

$$g_{i,j}^{m-1} = P_i^+ u_{i,j+1}^{m-1} + P_i^0 u_{i,j}^{m-1} + P_i^- u_{i,j-1}^{m-1},$$

$$P_i^+ = \frac{\gamma_i}{\sigma^2} - \frac{\sqrt{k}\nu_i}{2\sigma}, \quad P_i^0 = 1 - \frac{2\gamma_i}{\sigma^2}, \quad P_i^- = \frac{\gamma_i}{\sigma^2} + \frac{\sqrt{k}\nu_i}{2\sigma}.$$

Notice that

$$P_i^+ + P_i^0 + P_i^- = 1, \quad \forall i = 1, 2, \ldots, n. \tag{3.8}$$

We can regard P_i^- and P_i^+ as the probabilities for which the stock price goes down and up and P_i^0 as the probability for which the stock price does not change when the underlying economy is in the i-th state. To this end, we shall choose σ and M such that

$$P_i^- \geq 0, \quad P_i^0 \geq 0, \quad P_i^+ \geq 0,$$

which are equivalent to the following constraints on σ and M:

$$\sigma \geq \max_{1\leq i\leq n} \sigma_i, \qquad M \geq \sigma^2 T_0 \max_{1\leq i\leq n} \frac{\nu_i^2}{\sigma_i^4}. \tag{3.9}$$

Let $S_j = S_0 e^{x_j}$ for $j = -M, \ldots, M$. Denote by $C_{i,j}^m$ the approximation of $C_i(S_j, T - t_m)$. Let

$$\widetilde{C}_{i,j}^m = P_i^+ C_{i,j+1}^{m-1} + P_i^0 C_{i,j}^{m-1} + P_i^- C_{i,j-1}^{m-1}, \quad i = 1, \ldots, n. \tag{3.10}$$

Recall that $C(S, T-t; e_i) = K u_i(x, t)$ for $x = \log(S/K)$. The LCP for U_j^m becomes the following LCP for $C_j^m = \left(C_{1,j}^m, \ldots, C_{n,j}^m\right)^T$:

$$AC_j^m \geq \widetilde{C}_j^m, \quad C_j^m \geq \Phi_j, \tag{3.11}$$

$$\left(AU_j^m - \widetilde{C}_j^m\right)\left(U_j^m - \Phi_j\right) = 0, \tag{3.12}$$

where

$$\widetilde{C}_j^m = \left(\widetilde{C}_{1,j}^m, \ldots, \widetilde{C}_{n,j}^m\right)^T, \quad \Phi_j = \left((S_j - K)^+, \ldots, (S_j - K)^+\right)^T.$$

We have the following algorithm to compute $C_{i,0}^M$, the approximation of $C_i(S_0, T - T_0)$:

Algorithm 3. A Finite Difference lattice algorithm for the American call

1. Set
   $C_{i,j}^M = (S_j - K)^+, \quad j = -M, \dots, M, \quad i = 1, \dots, n.$

2. For $m = 1, 2, \dots, M$, do
   For $j = -(M-m), \dots, M-m$, do
   (1) Compute $\widetilde{C}_j^m$ by (3.10).
   (2) Solve the LCP (3.11)–(3.12) for C_j^m by Algorithm 1 or 2.
   End do
   End do

The inequalities in (3.1) and (3.2) become equalities for the European call option problem. Then we have the following algorithm to compute $c_{i,0}^M$, the approximation of the European call price $c_i(S_0, T - T_0)$:

Algorithm 4. A Finite Difference lattice algorithm for the European call

1. Set
   $c_{i,j}^M = (S_j - K)^+, \quad j = -M, \dots, M, \quad i = 1, \dots, n.$

2. For $m = 1, 2, \dots, M$, do
   For $j = -(M-m), \dots, M-m$, do
   (i) Compute $\tilde{c}_j^m$ by
   $\tilde{c}_{i,j}^m = P_i^+ c_{i,j+1}^{m-1} + P_i^0 c_{i,j}^{m-1} + P_i^- c_{i,j-1}^{m-1}, \quad i = 1, \dots, n.$
   (ii) Solve the following equation for c_j^m:
   $A c_j^m = \tilde{c}_j^m.$
   End do
   End do

CHAPTER 4

MONTE CARLO METHODS

In this chapter, we develop two new methods that are based on the Monte Carlo simulation of the markov chain. In particular, the method will be named as the Monte Carlo lattice method (the MC lattice method for simplicity) for American options. Again, we only consider the call option problem since the put option problem can be treated in the same fashion.

4.1 American call options

Consider the American call options with strike price $\$K$ and expiration date T years. Its price is denote by $C(S_0, t_0)$ when the stock price is equal to S_0 at time t_0.

We shall follow the idea in the introduction section of [6]. For a given sample path $X(t)$ of the Markov chain, we let

$$\sigma(t) = \sigma_{X(t)}, \quad \mu(t) = \mu_{X(t)}.$$

Solving the following SDE

$$dS(t) = S(t)\left(\mu(t)dt + \sigma(t)dW(t)\right),$$

we get

$$S(T) = S(t)\exp\left(\int_t^T \left(\mu(s) - \frac{1}{2}\sigma(s)^2\right)ds + \int_t^T \sigma(s)dW(t)\right)$$

Then the American call price at time t when $X(t) = i$ and $S(t) = S$ is given by

$$C_i(S,t) = \mathrm{E}\left[C(S,t,X(\cdot))\middle|\mathcal{G}_T\right],$$

where the $c(S,t,X(\cdot)$ is the American call price with given sample path $X(\cdot)$ and $\mathcal{G}_T = \sigma\{X(s):\ t \le s \le T\}$. As usual, we have

$$C(S,t,X(\cdot)) = \max_{t\le\tau\le T} \mathrm{E}\left[\exp\left(-\int_t^\tau r(s)ds\right)(S(\tau)-K)^+ \middle| \mathcal{F}_t\right], \tag{4.1}$$

where τ is a stopping time taking value in interval $[t,T]$.

Now let us show how to compute $C(S,t,X(\cdot))$ by a lattice method. Let $Y(t) = \log(S(x))$. It follows from Itō's Lemma that

$$dY(t) = \nu(t)dx + \sigma(t)dW(t), \tag{4.2}$$

where

$$\nu(t) = \mu(t) - \frac{1}{2}\sigma(t)^2.$$

Recall that the sample path $X(t)$ is a piecewise right-continuous functon with values in the set $\{1,\ldots,n\}$. Let

$$t_0 < t_1 < \ldots < t_m = T$$

be a partion of the interval $[t_0,T]$, where M is a positive integer. Here we have assumed that the discontinuity of $X(t)$ occurs at the partition nodes. Discretizing the SDE (4.2) by the Euler-Maruyama scheme, we have

$$Y_m - Y_{m-1} = \nu\left(t_{m-1}\right)\Delta t + \sigma\left(t_{m-1}\right)\sqrt{\Delta t}\xi_{m-1}, \quad m = 1,2,\ldots,M,$$

where Y_m is the approximation of $Y\left(t_m\right)$ and $\xi_m \sim N(0,1)$. Let Δy be positive number. Assume that P_m^+, P_m^0 and P_m^- are the probabilities under which Y_m takes values $Y_{m-1}+\Delta y$, Y_{m-1} and $Y_{m-1}-\Delta y$, repectively. Then we have by matching the mean and variance of the change $Y_m - Y_{m-1}$:

$$P_m^+ + P_m^0 + P_m^- = 1,$$

$$(\Delta y)P_m^+ + (0)P_m^0 + (-dy)P_m^- = \nu\left(t_{m-1}\right)\Delta t,$$

$$(\Delta y)^2 P_m^+ + (0)^2 P_m^0 + (-dy)^2 P_m^- = \sigma\left(t_{m-1}\right)^2 \Delta t.$$

Sovling the above system for P_m^+, P_m^0 and P_m^-, we obtain

$$\begin{aligned}
P_m^+ &= \frac{\sigma\left(t_{m-1}\right)^2 \Delta t}{2\Delta y^2} + \frac{\nu\left(t_{m-1}\right)\Delta t}{2\Delta y}, \\
P_m^0 &= 1 - \frac{\sigma\left(t_{m-1}\right)^2 \Delta t}{\Delta y^2} \\
P_m^- &= \frac{\sigma\left(t_{m-1}\right)^2 \Delta t}{2\Delta y^2} - \frac{\nu\left(t_{m-1}\right)\Delta t}{2\Delta y}.
\end{aligned}$$

If Δy is chosen such that

$$\Delta y \geq \bar{\sigma}\sqrt{\Delta t} \quad \text{and} \quad \bar{\sigma}^2 \leq \bar{\nu}\Delta y, \tag{4.3}$$

where $\bar{\sigma} = \max_{t_0 \leq t \leq T} \sigma(t)$ and $\bar{\nu} = \max_{t_0 \leq t \leq T} |\nu(t)|$. Then P_m^+, P_m^0 and P_m^- are nonnegative.

Let $S_j = S_0 e^{j\Delta y}$ for $j = -M, \ldots, M$. Denote by C_j^m the approximations of option price $C\left(S_j, t_m, X\right)$. Let $r_m = r\left(t_m\right)$, where $r(t)$ is the interest rate a time t. We have Algorithm 5 to compute C_0^M, the approximation of $C(S_0, t_0, X)$. Furthermore, we have Algorithm 6 to compute the approximation of call price $C_i(S_0, t_0)$. We should point out that these algorithms can be applied to European options. Indeed, we just need to remove step (ii) in Algorithm 5.

Algorithm 5. A lattice algorithm to compute $C(S_0, t_0, X)$

1. Set

   $C_j^M = (S_j - K)^+, \quad j = -M, \dots, M.$

2. For $m = 1, 2, \dots, M$, do

   For $j = -(M-m), \dots, M-m$, do

   (i) $\tilde{C}_j^m = e^{-r_j \Delta t} \left(P_m^- C_{j-1}^{m-1} + P_m^0 C_j^{m-1} + P_m^+ C_{j+1}^{m-1} \right)$,

   (ii) $C_j^m = \max \left(\tilde{C}_j^m, S_j - K \right)$.

   End do

   End do

Algorithm 6. A MC lattice algorithm for the American call

1. Input a positive integer N.
2. Set $C = 0$.
3. For $m = 1, 2, \dots, N$, do

   (i) Generate a sample path $\{X(s) : \ t_0 \le s \le T\}$ with $X(t_0) = i$.

   (ii) Compute $C(S_0, t_0, X(\cdot))$ by Algorithm 5.

   (iii) Let $C = C + C(S_0, t_0, X(\cdot))$.

   End do

4. The American call price at state i is given by $\dfrac{C}{N}$.

4.2 European call options

Consider the European with strike price K and expiration date T years. Denote by $c\,(S,t)$ the European call price when the stock price is equal to S at time t. For a given sample path $X(t)$ of the Markov chain, the European call price at time t is ([6]):

$$\begin{aligned} c(S(t),t,X(\cdot)) &= \mathrm{E}\left[\exp\left(-\int_t^T r(s)ds\right)(S(T)-K)^+\Big|\mathcal{F}_t\right] \\ &= S(t)N\left(d_1(t,T)\right)-\exp\left(-R(t,T)\right)N\left(d_2(t,T)\right), \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} &\mathcal{F}_t=\sigma\{W(s):\ 0\le s\le t\}, \\ &r(t)=r_{X(t)},\quad R(t,T)=\int_t^T r(s)ds, \\ &\Theta(t,T)=\int_t^T \mu(s)ds,\quad V(t,T)=\int_t^T \sigma(s)^2 ds, \\ &d_1(t,T)=\frac{\log(S(t)/K)+\Theta(t,T)+\frac{1}{2}V(t,T)}{\sqrt{V(t,T)}}, \\ &d_2(t,T)=d_1(t,T)-\sqrt{V(t,T)}. \end{aligned}$$

Hence, the European call price at time t when $X(t)=i$ and $S(t)=S$ is given by

$$c_i(S,t)=\mathrm{E}\left[c(S,t,X(\cdot))\big|\mathcal{G}_T\right].$$

We have the following Monte Carlo algorithm for the European option price $c_i(S,t)$.

Algorithm 7. An Monte Carlo algorithm for the European call

1. Input a positive integer N.
2. Set $c = 0$.
3. For $m = 1, 2, \ldots, N$, do

   (i) Generate a sample path $\{X(s) : \ t \leq s \leq T\}$ with $X(t) = i$.

   (ii) Compute $c(S, t, X(\cdot))$ by formula (4.1).

   (iii) Let $c = c + c(S_0, t, X(\cdot)$.

   End do
4. The European call price is given by $\dfrac{c}{N}$.

CHAPTER 5

NUMERICAL RESULTS

In this section, we examine our Finite Difference lattice method (FDLM), Monte Carlo lattice method (MCLM) amd Monte Carlo method (MCM) developed in the previous chapters. Again, we only consider call options as the put options follow a similar trend. The option expiration date is 1 year and the strike price is $100. Numerical results are presented when the number of states of economy is 2 and 4.

Since no exact solutions are available, we use the numerical soltuions obtained by the FDLM with 10000 steps as "exact values". The accuracy of our FDLM has been checked by using the finite element methods of [12].

For the MC lattice method, we set $N = 10M$ for the given positive integer M, the number of steps for the lattice method in Algorithm 5.

Example 5.1. In this example, we assume that there are two states of economy. The rate matrix for the Markov chain is assumed to be

$$Q = \begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix} \quad \text{and} \ .$$

The other parameters are as follows:

$$\sigma = \begin{bmatrix} 0.3 \\ 0.2 \end{bmatrix}, \quad r = \begin{bmatrix} 0.05 \\ 0.05 \end{bmatrix}, \quad d = 0.05.$$

It means that the stock price volatility changes as the economy switches from one state to the other while the interest rate keeps constant.

We display the computed option values and its maximum absolute error (MAE) at 9 stock prices in Tables 5.1 – 5.8. We observe that the FDLM converges linearly and the MCLM and MCM converge with the speed of $1/\sqrt{N}$, which is as expected.

Table 5.1. The FD lattice method for American call option: State 1

S	C_1	$M = 500$	$M = 1000$	$M = 2000$	$M = 4000$
80	2.6783	2.6793	2.6788	2.6785	2.6783
85	4.0308	4.0305	4.0312	4.0310	4.0309
90	5.7589	5.7592	5.7594	5.7593	5.7590
95	7.8734	7.8742	7.8747	7.8739	7.8737
100	10.3692	10.3681	10.3687	10.3689	10.3690
105	13.2281	13.2302	13.2289	13.2287	13.2283
110	16.4229	16.4241	16.4227	16.4232	16.4229
115	19.9212	19.9231	19.9221	19.9216	19.9215
120	23.6884	23.6899	23.6891	23.6887	23.6886
MAE		$2.15e-03$	$1.23e-03$	$5.77e-04$	$2.65e-04$

Table 5.2. The FD lattice method for American call option: State 2

S	C_2	$M = 500$	$M = 1000$	$M = 2000$	$M = 4000$
80	2.2304	2.2312	2.2309	2.2305	2.2303
85	3.4766	3.4766	3.4766	3.4768	3.4767
90	5.1142	5.1148	5.1143	5.1146	5.1144
95	7.1621	7.1631	7.1632	7.1626	7.1622
100	9.6187	9.6180	9.6184	9.6185	9.6186
105	12.4660	12.4682	12.4664	12.4665	12.4662
110	15.6736	15.6740	15.6735	15.6739	15.6736
115	19.2043	19.2063	19.2050	19.2045	19.2046
120	23.0185	23.0201	23.0189	23.0187	23.0187
MAE		$2.26e-03$	$1.11e-03$	$5.19e-04$	$2.54e-04$

Table 5.3. The MC lattice method for American call option: State 1

S	C_1	$M = 250$	$M = 500$	$M = 1000$	$M = 2000$
80	2.6783	2.6870	2.6907	2.6792	2.6757
85	4.0308	4.0407	4.0423	4.0411	4.0337
90	5.7589	5.7711	5.7884	5.7643	5.7696
95	7.8734	7.8857	7.8777	7.8733	7.8630
100	10.3692	10.3815	10.3882	10.3721	10.3806
105	13.2281	13.2482	13.2427	13.2290	13.2387
110	16.4229	16.4389	16.4446	16.4395	16.4269
115	19.9212	19.9466	19.9330	19.9262	19.9446
120	23.6884	23.7173	23.7100	23.7014	23.7092
MAE		$2.88e-02$	$2.95e-02$	$1.65e-02$	$2.34e-02$

Table 5.4. The MC lattice method for American call option: State 2

S	C_2	$M = 250$	$M = 500$	$M = 1000$	$M = 2000$
80	2.2304	2.1173	2.2352	2.2270	2.2260
85	3.4766	3.3362	3.4496	3.4718	3.4764
90	5.1142	4.9475	5.1273	5.1164	5.1177
95	7.1621	6.9811	7.1697	7.1701	7.1616
100	9.6187	9.4278	9.6007	9.6189	9.6203
105	12.4660	12.2782	12.4828	12.4800	12.4767
110	15.6736	15.4890	15.6917	15.6837	15.6757
115	19.2043	19.0434	19.2267	19.2241	19.2144
120	23.0185	22.8789	23.0272	23.0318	23.0308
MAE		$1.91e-01$	$2.70e-02$	$1.98e-02$	$1.23e-02$

Table 5.5. The FD lattice method for European call option: State 1

S	C_1	$M=500$	$M=1000$	$M=2000$	$M=4000$
80	2.6604	2.6613	2.6609	2.6606	2.6604
85	3.9997	3.9993	4.0000	3.9998	3.9998
90	5.7078	5.7079	5.7082	5.7081	5.7078
95	7.7932	7.7938	7.7944	7.7937	7.7934
100	10.2485	10.2471	10.2478	10.2482	10.2484
105	13.0531	13.0551	13.0538	13.0536	13.0533
110	16.1771	16.1780	16.1766	16.1773	16.1770
115	19.5852	19.5870	19.5861	19.5855	19.5855
120	23.2404	23.2419	23.2410	23.2407	23.2407
MAE		$2.02e-03$	$1.17e-03$	$5.40e-04$	$2.60e-04$

Table 5.6. The FD lattice method for European call option: State 2

S	C_2	$M=500$	$M=1000$	$M=2000$	$M=4000$
80	2.2180	2.2187	2.2184	2.2181	2.2179
85	3.4541	3.4540	3.4541	3.4542	3.4542
90	5.0759	5.0763	5.0759	5.0762	5.0760
95	7.1000	7.1008	7.1010	7.1005	7.1001
100	9.5226	9.5216	9.5221	9.5224	9.5225
105	12.3229	12.3250	12.3232	12.3233	12.3231
110	15.4675	15.4676	15.4672	15.4677	15.4675
115	18.9160	18.9178	18.9166	18.9161	18.9162
120	22.6253	22.6267	22.6256	22.6254	22.6255
MAE		$2.10e-03$	$1.02e-03$	$4.59e-04$	$2.39e-04$

Table 5.7. The MC method for European call option: State 1

S	C_1	$M = 250000$	$M = 500000$	$M = 1000000$	$M = 2000000$
80	2.6604	2.6050	2.6593	2.6604	2.6605
85	3.9997	3.9297	3.9980	3.9996	3.9988
90	5.7078	5.6255	5.7076	5.7082	5.7071
95	7.7932	7.7022	7.7940	7.7943	7.7934
100	10.2485	10.1530	10.2461	10.2483	10.2478
105	13.0531	12.9571	13.0541	13.0543	13.0527
110	16.1771	16.0845	16.1772	16.1768	16.1765
115	19.5852	19.4988	19.5843	19.5864	19.5839
120	23.2404	23.1622	23.2407	23.2409	23.2409
MAE		$9.60e-02$	$2.43e-03$	$1.22e-03$	$1.29e-03$

Table 5.8. The MC method for European call option: State 2

S	C_2	$M = 250000$	$M = 500000$	$M = 1000000$	$M = 2000000$
80	2.2180	2.1777	2.2186	2.2180	2.2187
85	3.4541	3.4002	3.4531	3.4534	3.4540
90	5.0759	5.0103	5.0737	5.0759	5.0747
95	7.1000	7.0261	7.0999	7.1009	7.0997
100	9.5226	9.4445	9.5204	9.5237	9.5219
105	12.3229	12.2449	12.3218	12.3206	12.3241
110	15.4675	15.3934	15.4675	15.4687	15.4684
115	18.9160	18.8485	18.9161	18.9159	18.9161
120	22.6253	22.5660	22.6247	22.6250	22.6254
MAE		$7.81e-02$	$2.20e-03$	$2.22e-03$	$1.23e-03$

Example 5.2. In this example, we assume that there are two states of economy. The rate matrix for the Markov chain is the same as in Example 5.1. The other parameters are as follows:

$$\sigma = \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix}, \quad r = \begin{bmatrix} 0.1 \\ 0.05 \end{bmatrix}, \quad d = 0.08.$$

It means that the interest rate changes as the economy switches from one state to the other while the the stock price volatility keeps constant.

We display the computed option values and its maximum absolute error at 9 stock prices in Tables 5.9 – 5.16. Again, we observe that the FDLM converges linearly and the MCLM and MCM converge with the speed of $1/\sqrt{N}$.

Table 5.9. The FD lattice method for American call option: State 1

S	C_1	$M = 500$	$M = 1000$	$M = 2000$	$M = 4000$
80	1.1570	1.1580	1.1574	1.1566	1.1571
85	2.1011	2.0997	2.1010	2.1011	2.1013
90	3.4807	3.4800	3.4816	3.4809	3.4809
95	5.3455	5.3454	5.3446	5.3460	5.3454
100	7.7118	7.7089	7.7104	7.7111	7.7115
105	10.5650	10.5668	10.5650	10.5658	10.5652
110	13.8670	13.8691	13.8683	13.8675	13.8669
115	17.5663	17.5664	17.5655	17.5667	17.5662
120	21.6055	21.6055	21.6061	21.6058	21.6055
MAE		$2.90e-03$	$1.45e-03$	$8.02e-04$	$3.60e-04$

Table 5.10. The FD lattice method for American call option: State 2

S	C_2	$M = 500$	$M = 1000$	$M = 2000$	$M = 4000$
80	1.0544	1.0553	1.0548	1.0540	1.0544
85	1.9351	1.9338	1.9350	1.9351	1.9353
90	3.2364	3.2358	3.2374	3.2366	3.2367
95	5.0132	5.0133	5.0124	5.0137	5.0132
100	7.2889	7.2860	7.2874	7.2882	7.2885
105	10.0565	10.0585	10.0566	10.0574	10.0568
110	13.2871	13.2894	13.2885	13.2877	13.2870
115	16.9392	16.9396	16.9385	16.9397	16.9391
120	20.9695	20.9697	20.9702	20.9698	20.9695
MAE		$2.82e-03$	$1.44e-03$	$8.39e-04$	$3.50e-04$

Table 5.11. The MC lattice method for Americanu call option: State 1

S	C_1	$M = 250$	$M = 500$	$M = 1000$	$M = 2000$
80	1.1570	1.1744	1.1619	1.1716	1.1749
85	2.1011	2.1307	2.1115	2.1251	2.1296
90	3.4807	3.5255	3.4958	3.5151	3.5218
95	5.3455	5.4041	5.3661	5.3902	5.4001
100	7.7118	7.7840	7.7345	7.7657	7.7790
105	10.5650	10.6499	10.5923	10.6263	10.6438
110	13.8670	13.9613	13.8949	13.9308	13.9546
115	17.5663	17.6609	17.5960	17.6285	17.6593
120	21.6055	21.6983	21.6377	21.6624	21.6998
MAE		$9.47e-02$	$3.22e-02$	$6.38e-02$	$9.43e-02$

Table 5.12. The MC lattice method for American call option: State 2

S	C_2	$M = 250$	$M = 500$	$M = 1000$	$M = 2000$
80	1.0544	1.0341	1.0561	1.0524	1.0301
85	1.9351	1.9029	1.9396	1.9327	1.8957
90	3.2364	3.1921	3.2444	3.2339	3.1781
95	5.0132	4.9540	5.0252	5.0114	4.9341
100	7.2889	7.2184	7.3047	7.2899	7.1898
105	10.0565	9.9826	10.0795	10.0641	9.9427
110	13.2871	13.2188	13.3160	13.3039	13.1685
115	16.9392	16.8862	16.9765	16.9701	16.8311
120	20.9695	20.9485	21.0199	21.0203	20.8924
MAE		$7.39e-02$	$5.04e-02$	$5.08e-02$	$1.19e-01$

Table 5.13. The FD lattice method for European call option: State 1

S	C_1	$M = 500$	$M = 1000$	$M = 2000$	$M = 4000$
80	1.1427	1.1438	1.1431	1.1423	1.1428
85	2.0708	2.0692	2.0705	2.0707	2.0709
90	3.4218	3.4208	3.4227	3.4220	3.4220
95	5.2398	5.2395	5.2388	5.2403	5.2397
100	7.5342	7.5307	7.5325	7.5334	7.5339
105	10.2830	10.2851	10.2829	10.2839	10.2833
110	13.4402	13.4426	13.4418	13.4408	13.4400
115	16.9469	16.9472	16.9457	16.9475	16.9467
120	20.7400	20.7399	20.7409	20.7404	20.7399
MAE		$3.55e-03$	$1.73e-03$	$8.36e-04$	$3.64e-04$

Table 5.14. The FD lattice method for European call option: State 2

S	C_2	$M = 500$	$M = 1000$	$M = 2000$	$M = 4000$
80	1.0412	1.0422	1.0416	1.0408	1.0413
85	1.9068	1.9054	1.9066	1.9067	1.9069
90	3.1807	3.1798	3.1817	3.1809	3.1809
95	4.9117	4.9114	4.9107	4.9122	4.9116
100	7.1152	7.1116	7.1134	7.1143	7.1148
105	9.7750	9.7770	9.7748	9.7758	9.7752
110	12.8498	12.8521	12.8514	12.8504	12.8495
115	16.2841	16.2842	16.2828	16.2846	16.2838
120	20.0160	20.0158	20.0168	20.0164	20.0159
MAE		$3.57e-03$	$1.74e-03$	$8.39e-04$	$3.67e-04$

Table 5.15. The MC method for European call option: State 1

S	C_1	$M = 250000$	$M = 500000$	$M = 1000000$	$M = 2000000$
80	1.1427	1.1602	1.1592	1.1437	1.1380
85	2.0708	2.0989	2.0973	2.0724	2.0629
90	3.4218	3.4631	3.4607	3.4241	3.4101
95	5.2398	5.2960	5.2928	5.2431	5.2239
100	7.5342	7.6061	7.6020	7.5386	7.5138
105	10.2830	10.3699	10.3650	10.2883	10.2580
110	13.4402	13.5412	13.5355	13.4464	13.4110
115	16.9469	17.0602	17.0539	16.9540	16.9139
120	20.7400	20.8637	20.8568	20.7477	20.7038
MAE		$1.24e-01$	$1.17e-01$	$7.73e-03$	$3.62e-02$

Table 5.16. The MC method for European call option: State 2

S	C_2	$M = 250000$	$M = 500000$	$M = 1000000$	$M = 2000000$
80	1.0412	1.0333	1.0357	1.0385	1.0570
85	1.9068	1.8939	1.8979	1.9022	1.9323
90	3.1807	3.1618	3.1676	3.1738	3.2184
95	4.9117	4.8860	4.8940	4.9021	4.9633
100	7.1152	7.0824	7.0926	7.1028	7.1814
105	9.7750	9.7351	9.7475	9.7595	9.8554
110	12.8498	12.8035	12.8180	12.8317	12.9437
115	16.2841	16.2320	16.2483	16.2635	16.3897
120	20.0160	19.9592	19.9770	19.9933	20.1317
MAE		$5.68e-02$	$3.90e-02$	$2.27e-02$	$1.16e-01$

Example 5.3. In this example, we assume that there are four states of economy. The rate matrix for the Markov chain is assumed to be

$$Q = \begin{bmatrix} -1.8 & 0.80 & 0.40 & 0.60 \\ 0.70 & -1.50 & 0.30 & 0.50 \\ 0.24 & 0.45 & -1.24 & 0.55 \\ 0.25 & 0.70 & 0.40 & -1.35 \end{bmatrix}.$$

The other parameters are as follows:

$$\sigma = \begin{bmatrix} 0.3 \\ 0.2 \\ 0.4 \\ 0.18 \end{bmatrix}, \quad r = \begin{bmatrix} 0.05 \\ 0.05 \\ 0.05 \\ 0.05 \end{bmatrix}, \quad d = 0.05.$$

As in Example 5.1, the stock price volatility changes as the economy switches from one state to the other while the interest rate keeps constant.

We display the computed option values for the first two states and their maximum absolute error at 9 stock prices in Tables 5.9 – 5.16. We have the same observation about the convergence of our methods as in previous examples.

Table 5.17. The FD lattice method for American call option: State 1

S	C_1	$M = 500$	$M = 1000$	$M = 2000$	$M = 4000$
80	2.8635	2.8647	2.8639	2.8631	2.8635
85	4.2271	4.2258	4.2269	4.2270	4.2272
90	5.9623	5.9613	5.9629	5.9623	5.9624
95	8.0819	8.0812	8.0809	8.0821	8.0817
100	10.5820	10.5790	10.5806	10.5814	10.5818
105	13.4453	13.4463	13.4450	13.4458	13.4455
110	16.6437	16.6450	16.6446	16.6440	16.6434
115	20.1435	20.1433	20.1425	20.1439	20.1434
120	23.9094	23.9090	23.9099	23.9096	23.9093
MAE		$3.01e-03$	$1.43e-03$	$6.34e-04$	$2.38e-04$

Table 5.18. The FD lattice method for American call option: State 2

S	C_2	$M = 500$	$M = 1000$	$M = 2000$	$M = 4000$
80	1.9741	1.9757	1.9748	1.9738	1.9742
85	3.1164	3.1152	3.1163	3.1164	3.1165
90	4.6622	4.6612	4.6629	4.6623	4.6623
95	6.6433	6.6426	6.6422	6.6436	6.6432
100	9.0659	9.0624	9.0643	9.0652	9.0657
105	11.9132	11.9144	11.9129	11.9138	11.9134
110	15.1501	15.1518	15.1513	15.1506	15.1499
115	18.7324	18.7324	18.7314	18.7329	18.7323
120	22.6125	22.6124	22.6132	22.6128	22.6124
MAE		$3.54e-03$	$1.68e-03$	$7.46e-04$	$2.80e-04$

Table 5.19. The MC lattice method for American call option: State 1

S	C_1	$M = 250$	$M = 500$	$M = 1000$	$M = 2000$
80	2.8635	2.8189	2.8052	2.6861	3.0678
85	4.2271	4.1706	4.1464	4.0105	4.4768
90	5.9623	5.8998	5.8662	5.7183	6.2503
95	8.0819	8.0104	7.9745	7.8186	8.3985
100	10.5820	10.5015	10.4714	10.3103	10.9147
105	13.4453	13.3840	13.3450	13.1811	13.7843
110	16.6437	16.5869	16.5602	16.3951	16.9777
115	20.1435	20.1031	20.0757	19.9204	20.4643
120	23.9094	23.8959	23.8714	23.7190	24.2117
MAE		$8.05e-02$	$1.11e-01$	$2.72e-01$	$3.39e-01$

Table 5.20. The MC lattice method for American call option: State 2

S	C_2	$M = 250$	$M = 500$	$M = 1000$	$M = 2000$
80	1.9741	2.0293	2.1358	1.9160	1.9107
85	3.1164	3.1874	3.3116	3.0459	3.0415
90	4.6622	4.7408	4.8893	4.5827	4.5790
95	6.6433	6.7381	6.8963	6.5582	6.5551
100	9.0659	9.1557	9.3300	8.9804	8.9766
105	11.9132	12.0143	12.1850	11.8333	11.8298
110	15.1501	15.2651	15.4223	15.0839	15.0763
115	18.7324	18.8540	19.0030	18.6823	18.6750
120	22.6125	22.7354	22.8776	22.5863	22.5760
MAE		$1.23e-01$	$2.72e-01$	$8.55e-02$	$8.93e-02$

Table 5.21. The FD lattice method for European call option: State 1

S	C_1	$M = 500$	$M = 1000$	$M = 2000$	$M = 4000$
80	2.8438	2.8451	2.8443	2.8434	2.8439
85	4.1940	4.1927	4.1937	4.1939	4.1941
90	5.9091	5.9080	5.9097	5.9091	5.9092
95	7.9998	7.9992	7.9988	8.0001	7.9997
100	10.4601	10.4570	10.4586	10.4595	10.4599
105	13.2703	13.2715	13.2700	13.2708	13.2704
110	16.3995	16.4011	16.4007	16.4000	16.3993
115	19.8120	19.8122	19.8110	19.8125	19.8120
120	23.4699	23.4699	23.4707	23.4703	23.4698
MAE		$3.08e-03$	$1.46e-03$	$6.49e-04$	$2.43e-04$

Table 5.22. The FD lattice method for European call option: State 2

S	C_2	$M = 500$	$M = 1000$	$M = 2000$	$M = 4000$
80	1.9638	1.9654	1.9644	1.9634	1.9639
85	3.0977	3.0965	3.0976	3.0977	3.0978
90	4.6300	4.6289	4.6308	4.6301	4.6301
95	6.5903	6.5895	6.5891	6.5906	6.5901
100	8.9820	8.9783	8.9803	8.9812	8.9817
105	11.7853	11.7868	11.7850	11.7860	11.7855
110	14.9617	14.9637	14.9631	14.9622	14.9615
115	18.4630	18.4635	18.4620	18.4637	18.4630
120	22.2379	22.2381	22.2388	22.2383	22.2378
MAE		$3.63e-03$	$1.72e-03$	$7.64e-04$	$2.87e-04$

Table 5.23. The MC method for European call option: State 1

S	C_1	$M = 250000$	$M = 500000$	$M = 1000000$	$M = 2000000$
80	2.8438	2.9349	2.8451	2.8899	2.9006
85	4.1940	4.3068	4.1991	4.2531	4.2650
90	5.9091	6.0403	5.9177	5.9795	5.9920
95	7.9998	8.1443	8.0110	8.0783	8.0913
100	10.4601	10.6118	10.4726	10.5429	10.5562
105	13.2703	13.4225	13.2821	13.3530	13.3666
110	16.3995	16.5471	16.4097	16.4790	16.4928
115	19.8120	19.9508	19.8197	19.8856	19.8996
120	23.4699	23.5969	23.4745	23.5358	23.5496
MAE		$1.52e-01$	$1.25e-02$	$8.28e-02$	$9.63e-02$

Table 5.24. The MC method for European call option: State 2

S	C_2	$M = 250000$	$M = 500000$	$M = 1000000$	$M = 2000000$
80	1.9638	1.9626	1.9294	2.0065	2.0451
85	3.0977	3.0983	3.0550	3.1568	3.1974
90	4.6300	4.6323	4.5804	4.7033	4.7453
95	6.5903	6.5940	6.5360	6.6739	6.7168
100	8.9820	8.9866	8.9253	9.0707	9.1147
105	11.7853	11.7893	11.7281	11.8735	11.9186
110	14.9617	14.9647	14.9061	15.0449	15.0913
115	18.4630	18.4647	18.4108	18.5380	18.5854
120	22.2379	22.2377	22.1898	22.3023	22.3504
MAE		$4.59e-03$	$5.72e-02$	$8.88e-02$	$1.33e-01$

Example 5.4. In this example, we assume that there are four states of economy. The rate matrix for the Markov chain is the same as in Example 5.3. The other parameters are as follows:

$$\sigma = \begin{bmatrix} 0.2 \\ 0.2 \\ 0.2 \\ 0.2 \end{bmatrix}, \quad r = \begin{bmatrix} 0.05 \\ 0.10 \\ 0.08 \\ 0.05 \end{bmatrix}, \quad d = 0.08.$$

As in Example 5.2, the interest rate changes as the economy switches from one state to the other while the the stock price volatility keeps constant.

As in Example 3, we display the computed option values for the first two states and their maximum absolute error at 9 stock prices in Tables 5.25 – 5.32. Also, we observe that the FDLM converges linearly and the MCLM and MCM converge with the speed of $1/\sqrt{N}$, which is as expected.

Table 5.25. The FD lattice method for American call option: State 1

S	C_1	$M = 500$	$M = 1000$	$M = 2000$	$M = 4000$
80	0.9592	0.9602	0.9596	0.9588	0.9592
85	1.7828	1.7816	1.7827	1.7827	1.7829
90	3.0165	3.0161	3.0174	3.0167	3.0167
95	4.7232	4.7234	4.7224	4.7237	4.7231
100	6.9361	6.9338	6.9350	6.9356	6.9359
105	9.6600	9.6620	9.6601	9.6608	9.6602
110	12.8759	12.8782	12.8772	12.8764	12.8757
115	16.5516	16.5521	16.5512	16.5521	16.5516
120	20.6513	20.6514	20.6517	20.6515	20.6512
MAE		$2.32e-03$	$1.39e-03$	$7.79e-04$	$2.20e-04$

Table 5.26. The FD lattice method for American call option: State 2

S	C_2	$M = 500$	$M = 1000$	$M = 2000$	$M = 4000$
80	1.1488	1.1498	1.1492	1.1484	1.1489
85	2.0891	2.0875	2.0889	2.0890	2.0892
90	3.4653	3.4645	3.4661	3.4654	3.4654
95	5.3286	5.3285	5.3277	5.3290	5.3285
100	7.6966	7.6938	7.6953	7.6960	7.6964
105	10.5559	10.5575	10.5558	10.5566	10.5561
110	13.8679	13.8698	13.8691	13.8684	13.8678
115	17.5799	17.5799	17.5791	17.5803	17.5798
120	21.6320	21.6318	21.6325	21.6322	21.6319
MAE		$2.81e-03$	$1.33e-03$	$6.94e-04$	$2.21e-04$

Table 5.27. The MC lattice method for American call option: State 1

S	C_1	$M = 250$	$M = 500$	$M = 1000$	$M = 2000$
80	0.9592	0.9684	0.9434	0.9586	0.9563
85	1.7828	1.7981	1.7591	1.7821	1.7785
90	3.0165	3.0406	2.9836	3.0156	3.0558
95	4.7232	4.7527	4.6823	4.7215	4.7749
100	6.9361	6.9735	6.8857	6.9342	6.9984
105	9.6600	9.7048	9.6095	9.6592	9.7059
110	12.8759	12.9258	12.8298	12.8772	12.9308
115	16.5516	16.6039	16.5158	16.5586	16.6140
120	20.6513	20.7082	20.6382	20.6693	20.7201
MAE		$5.70e-02$	$5.05e-02$	$1.81e-02$	$6.88e-02$

Table 5.28. The MC lattice method for American call option: State 2

S	C_2	$M = 250$	$M = 500$	$M = 1000$	$M = 2000$
80	1.1488	1.1719	1.1450	1.1728	1.1559
85	2.0891	2.1245	2.0843	2.1266	2.1004
90	3.4653	3.5202	3.4609	3.5177	3.4699
95	5.3286	5.3978	5.3262	5.3945	5.3358
100	7.6966	7.7850	7.6964	7.7728	7.7069
105	10.5559	10.6593	10.5615	10.6373	10.5049
110	13.8679	13.9827	13.8802	13.9470	13.8171
115	17.5799	17.6947	17.6009	17.6516	17.5340
120	21.6320	21.7370	21.6587	21.6923	21.5958
MAE		$1.15e-01$	$2.67e-02$	$8.14e-02$	$5.10e-02$

Table 5.29. The FD lattice method for European call option: State 1

S	C_1	$M = 500$	$M = 1000$	$M = 2000$	$M = 4000$
80	0.9412	0.9423	0.9416	0.9408	0.9413
85	1.7434	1.7420	1.7432	1.7433	1.7435
90	2.9378	2.9370	2.9387	2.9380	2.9380
95	4.5778	4.5776	4.5768	4.5783	4.5777
100	6.6849	6.6815	6.6833	6.6842	6.6846
105	9.2494	9.2516	9.2494	9.2503	9.2497
110	12.2351	12.2376	12.2368	12.2358	12.2349
115	15.5898	15.5903	15.5887	15.5906	15.5898
120	19.2543	19.2542	19.2552	19.2547	19.2542
MAE		$3.40e-03$	$1.66e-03$	$8.68e-04$	$2.68e-04$

Table 5.30. The FD lattice method for European call option: State 2

S	C_2	$M = 500$	$M = 1000$	$M = 2000$	$M = 4000$
80	1.1315	1.1325	1.1319	1.1310	1.1316
85	2.0522	2.0505	2.0518	2.0520	2.0522
90	3.3937	3.3926	3.3946	3.3938	3.3939
95	5.2008	5.2003	5.1996	5.2012	5.2006
100	7.4834	7.4797	7.4816	7.4826	7.4831
105	10.2205	10.2223	10.2203	10.2213	10.2207
110	13.3666	13.3686	13.3680	13.3671	13.3663
115	16.8632	16.8633	16.8620	16.8638	16.8631
120	20.6479	20.6474	20.6485	20.6482	20.6477
MAE		$3.67e-03$	$1.74e-03$	$7.80e-04$	$2.89e-04$

Table 5.31. The MC method for European call option: State 1

S	C_1	$M = 250000$	$M = 500000$	$M = 1000000$	$M = 2000000$
80	0.9412	0.9535	0.9507	0.9332	0.9419
85	1.7434	1.7636	1.7587	1.7299	1.7441
90	2.9378	2.9680	2.9604	2.9175	2.9385
95	4.5778	4.6197	4.6088	4.5497	4.5785
100	6.6849	6.7393	6.7248	6.6486	6.6854
105	9.2494	9.3160	9.2977	9.2044	9.2493
110	12.2351	12.3133	12.2913	12.1820	12.2344
115	15.5898	15.6786	15.6533	15.5297	15.5886
120	19.2543	19.3518	19.3234	19.1877	19.2522
MAE		$9.75e-02$	$6.91e-02$	$6.66e-02$	$2.14e-03$

Table 5.32. The MC method for European call option: State 2

S	C_1	$M = 250000$	$M = 500000$	$M = 1000000$	$M = 2000000$
80	1.1315	1.1044	1.1313	1.1403	1.1304
85	2.0522	2.0086	2.0515	2.0660	2.0500
90	3.3937	3.3302	3.3926	3.4137	3.3903
95	5.2008	5.1150	5.1991	5.2276	5.1960
100	7.4834	7.3746	7.4811	7.5173	7.4770
105	10.2205	10.0890	10.2172	10.2609	10.2121
110	13.3666	13.2145	13.3624	13.4130	13.3564
115	16.8632	16.6932	16.8583	16.9149	16.8515
120	20.6479	20.4627	20.6421	20.7036	20.6346
MAE		$1.85e-01$	$5.79e-03$	$5.57e-02$	$1.33e-02$

CHAPTER 6

CONCLUSIONS

We have considered the problems of pricing options under regime switching model of stock price processes. Since the option prices can be computed by either solving the variational inequality problem or evaluating the expectation by Monte Carlo simulation, we have proposed and implemented two numerical methods correspondingly. The advantage of these methods is their flexibility to compute the option prices for any given stock price at any given time.

The first method is based on discretizing the partial differential inequalities by the explicit finite difference scheme. The method is called the finite difference lattice method, which is studied in Chapter 3. In order to solve the resulting linear complimentary problems, we have given a detailed account for the Chandrasekaran and Lemke Methods in Chapter 2. The second method is based on the Monte Carlo simulation of the Markov chain. It is named as the Monte Carlo lattice method and studied in Chapter 4. Numerical examples are given to examine these methods in Chapter 5. It has been shown that the proposed methods provides fast and accurate approximations of option prices. Hence they should be helpful for practitioners working in this field.

The future work will be extended our methods for pricing of options under the regime switching model with jumps ([7]).

BIBLIOGRAPHY

[1] M. Broadie and P. Glasserman, *Pricing American-style securities using simulation*, J. Economic Dynamics and Control, 21, (1997), 13231352. C741

[2] L. Clewlow and C. Strickland, *Valuing Energy options in a one factor model fitted to forward prices*, Working paper, University of Technology, Sydney, 1999.

[3] F. A. Longstaff and E. S. Schwartz, *Valuing American options by simulation: a simple least-squares approach*, The Review of Financial Studies, 14 (2001), 113-147.

[4] D. Bates, *Jumps and stochastic volatility: Exchange rate processes implicit in Deutschemark options*, The Review of Financial Studies 9 (1996), pp. 69–108.

[5] S.L. Heston, *Closed-form solution for options with stochastic volatility with applications to bond and currency options*, The Review of Financial Studies 6 (1993), pp. 327-343.

[6] J. Buffington and R. J. Elliott, *American options with regime switching*, International Journal of Theoretical and Applied Finance, 5 (2002), pp. 497–514.

[7] Y. Huang, P. A. Forsyth, and G. Labahn, *Methods for pricing American options under regime switching*, SIAM J. Sci. Comput. 33(2011), pp. 2144-2168.

[8] A. D. Holmes and H. Yang, *A front-fixing finite element method for the valuation of American options*, SIAM J. Scientific Computing 30 (2008), pp. 2158–2180.

[9] Y. L. Zhu, X. Wu, and I-Liang Chern, Derivative Securities and Difference Methods, Springer, 2004.

[10] R. W. Cottle, J. S. Pang and R. E. Stone, *The Linear Complementarity Problem*, 1992

[11] K. G. Murty and V. F. Yu, Linear Complementarity, Linear and Nonlinear Programming, Internet Edition, http://ioe.engin.umich.edu/people/fac/books/murty/linear_complementarity_webbook/.

[12] H. Yang, *A numerical analysis of American options with regime switching*, J. Sci. Comput. 44 (2010), pp. 69–91.

[13] A. Bermon and R. J. Plemmons, Nonnegative Matrices in the Mathematical Sciences, Society for Industrial and Applied Mathematics, Philadelphia, 1994.

[14] Anthony D. Holmes, Shuhua Zhang, and Hongtao Yang, *A front-fixing finite element method for the valuation of American options with regime switching*, International Journal of Computer Mathematics 89 (2012), pp. 1094–1111.

VITA

Graduate College

University of Nevada, Las Vegas

Atul Sancheti

Degrees:

Bachelor of Technology, Electronics & Communication Engineering, 2010
Indian Institute of Technology Guwahati

Thesis Title:

Lattice Methods for the Valuation of Options with Regime Switching

Thesis Examination Committee:

Chairperson, Hongtao Yang, Ph.D.

Committee Member, Dr. Xin Li, Ph.D.

Committee Member, Dr. Amei Amei, Ph.D.

Graduate Faculty Representative, Dr. Pushkin Kachroo, Ph.D.