Friendly index sets of starlike graphs

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FRIENDLY INDEX SETS OF
STARLIKE GRAPHS

by

Daniel Anthony Corral

Bachelor of Science
University of Nevada, Las Vegas
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A thesis submitted in partial fulfillment
of the requirements for the

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Department of Mathematical Sciences
College of Sciences

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ABSTRACT

Friendly Index Sets of
Starlike Graphs

by

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For a graph $G = (V, E)$ and a coloring (labeling) $f : V(G) \rightarrow \mathbb{Z}_2$ let $v_f(i) = |f^{-1}(i)|$. The coloring $f$ is said to be friendly if $|v_f(1) - v_f(0)| \leq 1$. The coloring $f : V(G) \rightarrow \mathbb{Z}_2$ induces an edge labeling $f^* : E(G) \rightarrow \mathbb{Z}_2$ defined by $f^*(xy) = f(x) + f(y) \pmod{2}$. Let $e_f(i) = |f^{*-1}(i)|$. The friendly index set of the graph $G$, denoted by $FI(G)$, is defined by

$$FI(G) = \{ |e_f(1) - e_f(0)| : f \text{ is a friendly vertex labeling of } G \}.$$

In this thesis the friendly index sets of certain classes of trees, called starlike graphs, will be determined.
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CHAPTER 1

INTRODUCTION

Given a graph $G$ with set of vertices $V(G)$, set of edges $E(G)$, and an abelian group $A$: a labeling $f : V(G) \to A$ induces an edge labeling $f^\ast$ defined by $f^\ast(xy) = f(x) + f(y)$. For $i \in A$, let $v_f(i) = |f^{-1}(i)|$, $e_f(i) = |f^\ast(i)|$, and let $X = \{|e_f(i) - e_f(j)| : i, j \in A\}$. If $|v_f(i) - v_f(j)| \leq 1$ for every $i, j \in A$ then $f$ is called $A$-friendly. If $X$ is a $(0,1)$-matrix then $f$ is also called $A$-cordial. Hovery was the first to study $A$-cordial labelings [2]. Cahit produced several results, including: every tree is cordial; $K_n$ is cordial if and only if $n \leq 3$; $W_n$, the wheel is cordial if and only if $n \not\equiv 3 \pmod{2}$; $C_n$ is cordial if and only if $n \not\equiv 2 \pmod{2}$ [8]. Cahit also demonstrated that if the cardinality of a Eulerian graph is congruent to 2 (mod 2), then the graph is not cordial. Benson and Lee [1] found a good number of cordial regular windmill graphs. The construction of cordial graphs by composition and Cartesian product was researched by Ho, Lee and Shee [7].

Given a graph $G = (V,E)$ with a coloring $f : V(G) \to \{0,1\}$, let $v_f(i) = |f^{-1}(i)|$. Then $f$ is said to be friendly provided that $|v_f(1) - v_f(0)| \leq 1$. This coloring induces an edge labeling $f^\ast : E(G) \to \{0,1\}$ defined by $f^\ast(xy) \equiv f(x) + f(y) \pmod{2}$ for every $x, y \in E(G)$. Also let $e_f(i) = |f^\ast(i)|$. For a graph $G$, the set $FI(G) = \{|e_f(1) - e_f(0)| : f$ is a friendly labeling$\}$ is called friendly index set of the graph $G$. 

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We use figure 1 to illustrate the concept of a friendly labeling. In this figure we have 3 graphs with 6 vertices which are colored in a friendly manner.

![Example of friendly colored graphs](image)

Figure 1.1: EXAMPLES OF FRIENDLY COLORED GRAPHS.

We note that a graph $G$ is cordial if 0 or 1 be in the friendly index set $FI(G)$ [11]. Therefore the study of friendly index sets is a generalization of the concept of cordiality. Our goal will be to investigate $A$-friendly labelings where $A = \mathbb{Z}_2$ and $FI(G) = \{|e_f(1) - e_f(0)| : f$ is a friendly labeling of $G\}$ will denote the friendly index set. Occasionally we will drop the subscript $f$ in the case of clear context.

The computational complexity in determining whether a graph has a cordial labeling is NP-complete. This fact was proven by Cairne and Edwards [5]. In fact the smaller scale question of whether or not a connected graph of diameter 2 admits a cordial labeling is also NP-complete. Therefore Cairne and Edwards have concluded that it is difficult to determine the friendly index sets of graphs. Sin-Min Lee and Ho Kuen Ng made the first attempt at this problem [11]. Currently it is known that elements of a friendly index set $FI(G)$ do not necessarily form an arithmetic progression.
Lee and Ng have conjectured that the elements of the friendly index set of any tree will form an arithmetic progression [12]. Further conjectures have included that the common difference of the arithmetic progression be 2 [13]. This conjecture has been verified for several classes of trees, including: trees with perfect matching, Fibonacci, and Lucas trees [13]. In this thesis the friendly index sets of certain graphs will be explored. The established results will be used to verify the validity of the conjecture that the elements of friendly index set of any tree form an arithmetic progression.

For a friendly labeling, an induced colored edge cannot be both labeled 0 and 1. So the set of all edges labeled 0 and the set of all edges labeled 1 will be disjoint. As a result \( q = e_f(0) + e_f(1) \). Immediately we see that \( e_f(1) = q - e_f(0) \) which yields \( e_f(1) - e_f(0) = q - e_f(0) - e_f(0) = q - 2e_f(0) \). A similar computation gives us \( e_f(0) - e_f(1) = q - 2e_f(1) \). Note that if \( q \) is even then \( e_f(0) - e_f(1) \) is even and if \( q \) is odd then \( e_f(0) - e_f(1) \) is odd. Observe that \( -q \leq e_f(0) - e_f(1) = q - 2e_f(1) \leq q \) since \( e_f(1) \) may equal \( q \). Consequently \( |e_f(1) - e_f(0)| \leq q \). Hence \( FI(G) \subseteq \{0, 2, 4, \ldots, q\} \) if \( q \) is even and \( FI(G) \subseteq \{1, 3, 5, \ldots, q\} \) if \( q \) is odd. This observation will be recorded as a lemma.

**Lemma 1.** [11] Suppose that \( G \) is a graph that has \( q \) edges and friendly index set \( FI(G) \). If \( q \) is even then \( FI(G) \subseteq \{0, 2, 4, \ldots, q\} \) and if \( q \) is odd then \( FI(G) \subseteq \{1, 3, 5, \ldots, q\} \).

We have a more cosmetic version of this result. By the above paragraph \( e_f(0) \) and \( e_f(1) \) may assume any integer value between 0 and \( q/2 \) if \( q \) is even; 0 and \( (q - 1)/2 \) if \( q \) is odd. Hence \( e_f(0) \) and \( e_f(1) \) may take on integer values between 0 and \( [q/2] \). So
depending on the graph $|e_f(0) - e_f(1)|$ may take on all values in the set $\{q - 2i : i = 0, 1, 2, ..., \lfloor q/2 \rfloor \}$. Therefore $FI(G) \subseteq \{q - 2i : i = 0, 1, 2, ..., \lfloor q/2 \rfloor \}$. We now state this result as a lemma.

**Lemma 2.** [13] For any graph $G$ with $q$ edges, $FI(G) \subseteq \{q - 2i : i = 0, 1, 2, ..., \lfloor q/2 \rfloor \}$.

Another useful result is the friendly index set of a path $P_n$. In fact there are several places later in the thesis where we will need to know the friendly index set of the path $P_n$. A path $P_n$ has $n$ vertices, hence $n - 1$ edges. Figure 2 shows the paths: $P_1, P_2, P_3,$ and $P_4$.

![Figure 1.2: THE GRAPHS OF $P_1, P_2, P_3, P_4$.](image)

The friendly index set for the path $P_n$ was initially computed by Salehi and Lee [13]. They derived this result using methods discovered during their work dealing with friendly index sets of graphs with perfect matching. A graph with perfect matching will be discussed later. For now we present a different proof that uses techniques that will prove to be useful in the study of the friendly index set $ST(n; b^n)$. We will have two cases depending upon the parity of $n$.

For the first case we consider $n$ to be odd, so then there is an integer $k$ such that $n = 2k+1$. Without loss of generality we label the left most vertex by 0. There will be
2k more vertices to label. Since we have a friendly labeling a possible coloring could be; of these remaining vertices: k vertices will be labeled 0 and k vertices labeled 1.

We can pair these 2k remaining vertices so that we get any combination of 0’s and 1’s sitting next to each other occurs: (0 and 1), (1 and 0), (1 and 1), and (0 and 0). Hence $e_f(0)$ ranges from the integers 0 to 2k. Likewise $e_f(1)$ ranges from the integers 0 to 2k. Now suppose $e_f(0) = i$ and $e_f(1) = j$ then $i + j = 2k$. So then

\[
|e_f(0) - e_f(1)| = |i - j| = |2k - j - j| = |2(k - j)|
\]

and $j \in Z \cap [0, 2k]$. Since $h(j) = 2(k - j)$ is a decreasing bijection from $Z \cap [0, k]$ to $\{0, 2, 4, ..., 2k\}$ thus $\{0, 2, 4, ..., 2k\} \subseteq FI(P_{2k+1})$. By lemma 1.1, $\{0, 2, 4, ..., 2k\} \subseteq FI(P_{2k+1}) \subseteq \{0, 2, 4, ..., 2k\}$. Therefore $FI(P_{2k+1}) = \{0, 2, 4, ..., 2k\}$.

Now we consider when $n = 2k$ for some integer $k$. For friendly labeling to occur we have $k$ vertices labeled 0 and $k$ of vertices labeled 1. We can pair these 2k vertices so that any combination of 0’s and 1’s sitting next to each other occurs: (0 and 1), (1 and 0), (1 and 1), and (0 and 0). So then $e_f(0)$ ranges from 0 to $2k - 1$ and $e_f(1)$ ranges from 0 to $2k - 1$. Let $e_f(0) = i$ and $e_f(1) = j$ then $i + j = 2k - 1$. So then

\[
|e_f(0) - e_f(1)| = |i - j| = |2k - 1 - j - j| = |2(k - j - 1) |
\]
and \( j \in \mathbb{Z} \cap [0, 2k - 1] \). Since \( h(j) = 2(k - j) - 1 \) is a decreasing bijection from \( \mathbb{Z} \cap [0, k - 1] \) to \( \{1, 3, 5, \ldots, 2k - 1\} \), as a result \( \{1, 3, 5, \ldots, 2k - 1\} \subseteq FI(P_{2k}) \subseteq \{1, 3, 5, \ldots, 2k - 1\} \). Therefore \( FI(P_{2k}) = \{1, 3, 5, \ldots, 2k - 1\} \).

The following theorem is the result of the above discussion:

**Theorem 3.** For any path \( P_n \),

\[
FI(P_n) = \begin{cases} 
\{0, 2, 4, \ldots, 2k\} & \text{if } n \text{ is odd;} \\
\{1, 3, 5, \ldots, 2k - 1\} & \text{if } n \text{ is even.}
\end{cases}
\]

**Remark 4.** It is easy to observe that any friendly index of \( P_n \) can be obtained by a coloring with \( e_f(1) \geq e_f(0) \).
Here we discuss the friendly index sets of complete bipartite graphs. We start with the definition of a complete graph and provide its friendly index set. Then we present the work done by Lee and Ng concerning complete bipartite graphs.

A graph $G$ is complete if every two distinct vertices are joined by an edge (adjacent). A complete graph of order $n$ is denoted by $K_n$ and has $(n(n - 1))/2$ edges. Figure 3 illustrates the complete graphs: $K_1$, $K_2$, $K_3$, and $K_4$.

Lee and Ng found the friendly index set for a complete graph [11]:

**Theorem 5.** For $n > 2$ the friendly index set of the complete graph $K_n$ is $FI(K_n) = \{[n/2]\}$.

**Proof.** We will have two cases based upon the parity of $n$. 
Case 1. Suppose that $n$ is even. So then there exists an integer $k$ such that $n = 2k$.

Since we have a friendly labeling we must color $k$ vertices by 0 and color $k$ vertices by 1. In this situation each "zero" edge paired with $k - 1$ other "zero" edges, avoiding double counting, gives us $k(k - 1)/2$ induced "zero" edges. Likewise each "one" edge is paired with $k - 1$ other "one" edges, yielding $k(k - 1)/2$ more "zero" edges. Hence

$$e_f(0) = k(k-1)/2 + k(k-1)/2 = 2k(k-1)/2 = k^2 - k.$$  

Each "zero" vertex is connected by an edge to $k$ "one" vertices resulting in $k^2/2$ "one" induced edges, similarly each "one" vertex is paired with $k$ "zero" vertices giving $k^2/2$ more "one" induced edges. Consequently $e_f(1) = k^2/2 + k^2/2 = k^2$. Therefore $|e_f(1) - e_f(0)| = k$.

Case 2. Now suppose that $n$ is odd. In this case there is an integer $k$ where $n = 2k + 1$. We have two situations which may occur: we label $k$ vertices by 0 or we label $k+1$ vertices by 0. In the first situation we have $k$ "zero" vertices and $k+1$ "one" vertices. Each "zero" vertex is connected to $k - 1$ other "zero" vertices and each "one" vertex is paired with $k$ "zero" vertices, thus $e_f(0) = k(k-1)/2 + (k + 1)k/2 = k^2$.

Each "zero" vertex is paired with $k + 1$ "one" vertices and each "one" vertex is connected to $k$ "zero" vertices, hence $e_f(1) = k(k + 1)/2 + (k + 1)k/2 = k^2 + k$.

So then $|e_f(1) - e_f(0)| = k$. In the other situation we color $k + 1$ vertices by 0 and color $k$ vertices by 1. So then $e_f(0) = (k + 1)k/2 + k(k - 1)/2 = k^2$ and $e_f(1) = (k + 1)k/2 + k(k + 1)/2 = k^2 + k$. Hence $|e_f(1) - e_f(0)| = k$. □

A graph $G$ is called bipartite if $V(G)$ can be partitioned into two disjoint subsets $S$ and $T$, called partite sets, such that every edge of $G$ joins a vertex of $S$ and a vertex of $T$. A graph that is both complete and bipartite is called a complete bipartite graph.
A complete bipartite graph such that $|S| = m$ and $|T| = n$ is denoted by $K(m, n)$. Figure 4 displays the complete bipartite graphs $K(2, 3)$ and $K(3, 3)$.

Figure 4: TWO COMPLETE BIPARTITE GRAPHS.

The following theorem and corresponding proof that are due to Lee and Ng [11] will pertain to the friendly index set for a complete bipartite graph.

**Theorem 6.** Suppose that $K_{m,n}$ is a complete bipartite graph where $m \leq n$. When $m + n$ is even and $m$ is even, then $FI(K_{m,n}) = \{m^2, (m - 2)^2, (m - 4)^2, \ldots, 2^2, 0\}$. In the case where $m + n$ is even and $m$ is odd, then $FI(K_{m,n}) = \{m^2, (m - 2)^2, (m - 4)^2, \ldots, 3^2, 1\}$. When $m + n$ is odd then $FI(K_{m,n}) = \{(m+1)m, m(m-1), (m-1)(m-2), \ldots, 2, 0\}$.

**Proof.** Suppose that $f$ is an arbitrary friendly coloring for $K_{m,n}$ with partite sets $M$ and $N$. We will assume that $|M| = m$ and $|N| = n$ where $m \leq n$. So then $K_{m,n}$ has $m + n$ vertices. This proof will be broken into 2 cases depending upon the parity of $m + n$.

**Case A.** $m + n$ is even. Since $f$ is a friendly labeling we know that $(m + n)/2$ vertices of $K_{m,n}$ will be labeled 0 and $(m + n)/2$ vertices of $K_{m,n}$ will be labeled 1.

Note that $m + n$ is even, so there exists an integer $k$ such that $m + n = 2k$. Let $i$
denote the number of vertices of $M$ that are colored 0. Since there are $m$ vertices in $M$; it must be the case that $m - i$ vertices of $M$ are colored 1. As a result there are $k - i$ vertices of $N$ colored 0 and $k - m + i$ vertices of $N$ colored 1. Now then $e_f(0) = i(k - i) + (m - i)(k - m + i)$ and $e_f(1) = i(k - m + i) + (m - i)(k - i)$. Hence

$$|e_f(0) - e_f(1)| = |i(2i - m) + (m - i)(m - 2i)|$$

$$= |-i(m - 2i) + (m - i)(m - 2i)|$$

$$= |(m - 2i)(m - 2i)|$$

$$= |(m - 2i)^2|.$$ (2.1)

Moreover, $i \leq (m+n)/2$ and $m - i \leq (m+n)/2$. So then $(m-n)/2 \leq i \leq (m+n)/2$. Observe that it is also true that $0 \leq i \leq m$. Hence, the friendly indices obtained would be

$$\{(|(m - 2i)^2| : c_i \leq i \leq C_i\},$$ (2.2)

where $c_i = \max\{0, (m - n)/2\}$ and $C_i = \min\{m, (m + n)/2\}$. By assumption we have $m \leq n$ so then the friendly index set is $FI(K_{m,n}) = \{|(m-2i)^2| : 0 \leq i \leq m\} = \{|(m-2i)^2| : 0 \leq i \leq |m/2|\}$. Note that when $m$ is even $m - 2i$ is even and when $m$ is odd $m - 2i$ is odd. So then, if $m$ is even then $FI(K_{m,n}) = \{m^2, (m - 2)^2, (m - 4)^2, ..., 2^2, 0\}$ and if $m$ is odd then $FI(K_{m,n}) = \{m^2, (m - 2)^2, (m - 4)^2, ..., 3^2, 1\}$.

**Case B.** $m+n$ is odd. In this situation there is an integer $k$ such that $m+n = 2k+1$. There are two subcases that need to be analyzed.

**Subcase B1.** Let $v_f(0) = k$ and $v_f(1) = k+1$. Again we color $i$ vertices of $M$ by 0, consequently labeling $m-i$ vertices of $M$ by 1. Now then $k-i$ vertices of $N$ are colored...
0 and \(k + 1 - m + i\) vertices labeled 1. So then \(e_f(0) = i(k - i) + (m - i)(k + 1 - m + i)\) and \(e_f(1) = i(k + 1 - m + i) + (m - i)(k - i)\). Thus

\[
|e_f(0) - e_f(1)| = |i(2i - m + 1) + (m - i)(m - 2i - 1)|
= |-i(m - 2i - 1) + (m - i)(m - 2i - 1)|
= |(m - 2i - 1)(m - 2i)|. \tag{2.3}
\]

Observe that \(i \leq (m+n-1)/2\) and \(m-i \leq (m+n+1)/2\), which implies \((m-n-1)/2 \leq i \leq (m + n - 1)/2\). So the friendly index set is

\[
\{(m - 2i - 1)(m - 2i) : c_i \leq i \leq C_i\}, \tag{2.4}
\]

where \(c_i = \max\{0, (m-n-1)/2\}\) and \(C_i = \min\{m, (m+n-1)/2\}\). Since \(m \leq n\) we have \(FI(K_{m,n}) = \{(m - 2i - 1)(m - 2i) : 0 \leq i \leq m\}\). Notice that \(m - 2i - 1\) and \(m - 2i\) are consecutive integers and have opposite parity, hence \((m - 2i - 1)(m - 2i)\) is even. Therefore \(FI(K_{m,n}) = \{(m + 1)m, m(m - 1), (m - 1)(m - 2), \ldots, 2, 0\}\).

**Subcase B2.** Let \(v_f(0) = k + 1\) and \(v_f(1) = k\). Here we label \(i\) of the vertices of \(M\) by 0, which implies \(m - i\) vertices of \(M\) are colored 1. Hence \(k + 1 - i\) vertices of \(N\) are colored 0 and \(k - m + i\) vertices of \(N\) are labeled 1. As a result \(e_f(0) = \)
\[ i(k + 1 - i) + (m - i)(k - m + i) \] and \[ e_f(1) = i(k - m + i) + (m - i)(k + 1 - i). \] Therefore

\[
|e_f(0) - e_f(1)| = |i(2i - m - 1) + (m - i)(m - 2i + 1)|
\]

\[
= | - i(m - 2i + 1) + (m - i)(m - 2i + 1)|
\]

\[
= |(m - 2i + 1)(m - 2i)|. \tag{2.5}
\]

In this instance \( i \leq (m + n + 1/2) \) and \( m - i \leq (m + n - 1)/2 \), yielding \((m - n + 1)/2 \leq i \leq (m + n + 1)/2\). Consequently

\[
\{(m - 2i + 1)(m - 2i) : c_i \leq i \leq C_i \}, \tag{2.6}
\]

where \( c_i = \max\{0, (m - n + 1)/2\} \) and \( C_i = \min\{m, (m + n + 1)/2\} \). Since \( m \leq n \) we have \( FI(K_{m,n}) = \{(m - 2i + 1)(m - 2i) : 0 \leq i \leq m\} = \{(m + 1)m, m(m - 1), (m - 1)(m - 2), \ldots, 2, 0\}. \]

Friendly Index Sets Of Trees

The study of friendly index sets of trees was initiated by Lee and Ng [11]. They studied the following trees: \( L_n \), \( P_n \odot K_1 \), and full binary trees. The \( L_n \) tree is the tree with vertex set \( V(L_n) = \{u_1, u_2, \ldots, u_{n-1}\} \cup \{v_1, v_2, \ldots, v_n\} \) and edge set \( E(L_n) = \{(u_i, v_i) | i \in [1, n - 1] \cap Z\} \cup \{(v_i, v_{i+1}) | i \in [1, n - 1] \cap Z\} \). This tree has \( 2(n - 1) + 1 \) vertices hence \( 2(n - 1) \) edges. Since \( L_n \) has an even edge number, by lemma 1.1, \( FI(L_n) \subset \{0, 2, 4, \ldots, 2(n - 1)\} \). So if one presents friendly colorings
that generate every element in \{0, 2, 4, ..., 2(n-1)\} then this will show that \(FI(L_n) = \{0, 2, 4, ..., 2(n-1)\}\). This is precisely what Lee and Ng did. They labeled all the “top” vertices up to a certain point \(k\), \(\{u_1, u_2, ..., u_k \mid k \in [0, n-1] \cap Z\}\), by 1. They colored all the “bottom” vertices up to \(k\), \(\{v_1, v_2, ..., v_k \mid k \in [0, n-1] \cap Z\}\) by 0. From \(k + 1\) onward they are colored: \(f(v_{k+1}) = 0, f(v_{k+2}) = 1, f(v_{k+3}) = 0, f(v_{k+4}) = 1, ..., f(v_n)\) such that

\[
f(u_n) = \begin{cases} 
0 & \text{if } n - k \text{ is odd;} \\
1 & \text{if } n - k \text{ is even.}
\end{cases}
\]

Also form \(k + 1\) onward they are colored \(f(u_{k+1}) = 1, f(u_{k+2}) = 0, f(u_{k+3}) = 1, f(u_{k+4}) = 0, ..., f(u_{n-1})\) such that

\[
f(u_{n-1}) = \begin{cases} 
1 & \text{if } n - 1 - k \text{ is odd;} \\
0 & \text{if } n - 1 - k \text{ is even.}
\end{cases}
\]

This labeling tells us that the only induced 0-labeled edges are the “bottom” edges \(\{(v_i, v_{i+1}) \mid i \leq k\}\). So then \(e_f(0) = k\) and \(e_f(0) = 2(n-1) - k\). Hence \(|e_f(1) - e_f(0)| = 2(n - 1) - 2k = 2(n - 1 - k)\) which is a decreasing bijection from \([0, n-1] \cap Z\) to \(\{0, 2, 4, ..., 2(n-1)\}\). Consequently \(\{0, 2, 4, ..., 2(n-1)\} \subseteq FI(L_n) \subseteq \{0, 2, 4, ..., 2(n-1)\}\). Therefore \(FI(L_n) = \{0, 2, 4, ..., 2(n-1)\}\).

We now state this as a theorem.

**Theorem 7.** [11] For \(L_n\) the friendly index set is given by \(FI(L_n) = \{0, 2, 4, ..., 2(n-1)\}\).

Lee and Ng also examined the friendly index set for the coronation of two graphs.
$G$ and $H$, which is denoted by $G \odot H$ [11]. The coronation of $G$ with $H$ is created by first forming the disjoint union of a copy of $H$ for each vertex in $G$. Then each vertex of $G$ is connected to every vertex in its copy of $H$. Lee and Ng focused their attention on $P_n \odot K_1$. For this graph all that is needed is an edge from each vertex of $P_n$ to a single “exterior” vertex. In figure 5 we illustrate the graph $P_n \odot K_1$.

![Figure 2.3: $FI(P_n \odot K_1)$](image)

The graph $P_n \odot K_1$ has $2n$ vertices, hence $2n - 1$ edges. Again by lemma 1.1, the friendly index set $FI(P_n \odot K_1) \subseteq \{1, 3, 5, ..., 2n - 1\}$. So in order to find $FI(P_n \odot K_1)$ it suffices to find a set of colorings which will produce the the friendly indices $\{1, 3, 5, ..., 2n - 1\}$. To present such a coloring: label the “top” vertices up to a certain point $k$, 
\{u_1, u_2, ..., u_k | k \in [0, n - 1] \cap Z\}, by 0. Next color the “bottom” vertices up to $k$, 
\{v_1, v_2, ..., v_k | k \in [0, n - 1] \cap Z\} by 1. After $k + 1$ we label in the following manner: 
$f(u_{k+1}) = 1, f(u_{k+2}) = 0, f(u_{k+3}) = 1, f(u_{k+4}) = 0, ... , f(u_n)$ such that 

$$
 f(v_n) = \begin{cases} 
 1 & \text{if } n - k \text{ is odd;} \\
 0 & \text{if } n - k \text{ is even.}
\end{cases}
$$

In a similar fashion we color: $f(u_{k+1}) = 0, f(u_{k+2}) = 1, f(u_{k+3}) = 0, f(u_{k+4}) = 14$.
such that

\[
f(u_{n-1}) = \begin{cases} 
0 & \text{if } n - k \text{ is odd;} \\
1 & \text{if } n - k \text{ is even.}
\end{cases}
\]

The only edges colored by 0 is the set \(\{(v_i, v_{i+1}) | i \leq k\}\). As a result \(e_f(0) = k\), thus \(e_f(0) = 2n - 1 - k\). Therefore \(|e_f(1) - e_f(0)| = 2(n - k) - 1\), and this is a bijection from \([0, n - 1] \cap \mathbb{Z}\) to \(\{1, 3, 5, ..., 2n - 1\}\).

This result is summarized in the following theorem.

**Theorem 8.** [11] For the friendly index set is given by \(FI(P_n \oplus K_1) = \{1, 3, 5, ..., 2n - 1\}\).

The friendly index set for the coronation \(T \oplus K_1\) was calculated by Salehi and Lee [13]. Their strategy was to study trees with perfect matching. They successfully computed the friendly index sets of such trees. Since \(T \oplus K_1\) is a tree with perfect matching so its friendly index set is derived as a corollary of the result for trees with perfect matching. The next several pages will detail Salehi and Lee’s discoveries concerning friendly index sets for graphs with perfect matching.

A set \(M\) of edges of a graph \(G\) that do not have any common endpoints is called a matching for \(G\). Let \(V(M)\) denote the vertex set for the matching \(M\). A matching with the property that every vertex of the graph is an element of \(V(M)\) is called a perfect matching. An example of a tree with perfect matching is illustrated in Figure 6.
Figure 2.4: A TREE WITH PERFECT MATCHING.

Suppose \( M \) is a perfect matching for \( G \). By definition a matching is a subset of \( E(G) \), thus \( V(M) \subseteq V(G) \). Since every vertex of a graph \( G \) with perfect matching \( M \) is in \( V(M) \), we have \( V(G) \subseteq V(M) \). Consequently \( V(G) = V(M) \). Note that \( M \) is a collection of disjoint edges, each with two vertices. Hence \( |V(M)| = |V(G)| \) is an even number. In fact \( |V(G)| = |V(M)| = 2|M| \).

**Lemma 9.** [13] A graph \( G \) with perfect matching has an even number of vertices.

An immediate consequence of this lemma is that a tree \( T \) with perfect matching must have an odd number of edges. Now by lemma 1.1, \( FI(T) \subseteq \{1,3,5,\ldots,q\} \), where \( q = E(T) \).

**Lemma 10.** [13] For a tree \( T \) with perfect matching \( FI(T) \subseteq \{1,3,5,\ldots,q\} \) where \( q = E(T) \).

For a perfect matching \( M \) the fact that every vertex of the graph is in \( V(M) \) guarantees that every terminal edge of a tree is in \( M \). To see this suppose to the contrary that there is a tree with perfect matching such that a terminal edge is not in \( M \). So
then the terminal vertex of this edge is not in \( V(M) \), but this is a contradiction since 
\( V(G) = V(M) \). As a result every terminal edge of a tree with perfect matching is in 
\( M \) [13]. Another fact of importance is that every graph with perfect matching has a 
copy of \( P_3 \) as a pendant [13]. Salehi and Lee also noted that the two terminal sections 
of any longest path in the graph contain \( P_3 \) pendants.

Now we can state the key result by Salehi and Lee about the friendly index set of 
a graph with perfect matching.

**Theorem 11.** [13] The friendly index set of a tree \( T \) with perfect matching and \( q \) 
edges is \( FI(T) = \{1, 3, 5, ..., q\} \).

**Proof.** Suppose that \( T \) is a tree with a perfect matching \( M \). By lemma 2.6, we just 
need to find a coloring of \( T \) which will generate all the elements in \( \{1, 3, 5, ..., q\} \).
Salehi and Lee proceed by induction on the cardinality of the matching \( M \). Note 
for \( |M| = 1 \) that \( |V(T)| = |V(M)| = 2 \), which implies that \( T \) is merely the tree 
\( P_2 \). We know by theorem 1.3, that \( FI(P_1) = \{1\} \), so the result to be proved holds 
for \( |M| = 1 \). Observe that when \( |M| = 2 \) that \( T = P_4 \) since the two edges in \( M \) 
cannot share a common vertex and \( T \) has a copy of \( P_3 \) as a subgraph. Again by 
theorem 1.3, we have \( FI(P_2) = \{1, 3\} \) as desired. So the induction is anchored at 
\( |M| = 2 \). Assume that \( FI(T) = \{1, 3, 5, ..., q\} \) is true for any tree \( T \) with perfect 
matching where \( 3 \leq |M| \leq n \). Now suppose that \( T \) is a tree with a perfect matching 
such that \( |M| = n + 1 \). We utilize the fact that \( T \) has a terminal copy of \( P_3 \) as a 
subgraph. We call the vertices of this path \( u, v, \) and \( w \). Let \( P \) denote the terminal 
path through \( v \) and \( w \). Notice that the tree \( T' \) such that \( V(T') = V(T) - \{v, w\} \)
is a tree with the perfect matching $M - P$ such that $|M - P| = n$. Observe that $|V(T')| = |V(M - P)| = 2|M - P| = 2n$ so then $E(T') = 2n - 1$. Now the induction statement provides $FI(T') = \{1, 3, 5, ..., 2n - 1\}$. Suppose that $f : V(T') \rightarrow \{0, 1\}$ is a friendly labeling of $T'$. Define the following labeling $\phi : V(T) \rightarrow \{0, 1\}$ by

$$
\phi(x) = \begin{cases} 
  f(x) & \text{if } x \neq v, w; \\
  f(u) & \text{if } x = v; \\
  1 - f(u) & \text{if } x = w.
\end{cases}
$$

This labeling leaves every vertex of $T'$ to retain the same coloring done by $f$. The labeling $\phi$ of $T$ provides $f(u)$ and $f(v)$ with the same coloring: either 0 and 0, or 1 and 1. So the induced coloring of the edge $f(u)f(v)$ is 0. Similarly $f(v)$ and $f(w)$ have opposite colorings: either 0 and 1, or 1 and 0. As a result the induced coloring of $f(v)$ is 1. Therefore the coloring $\phi$ has added both a 1 and a 0 to the induced edge labelings of $f$. So then $e_\phi(0) = e_f(0) + 1$ and $e_\phi(1) = e_f(1) + 1$. So then

$$
|e_\phi(0) - e_\phi(1)| = |e_f(0) + 1 - e_f(1) - 1| = |e_f(0) - e_f(1)|
$$

Equation (2.7) shows us that every friendly index element for $f$ will also be a friendly index element for $\phi$. Hence $\{1, 3, 5, ..., 2n - 1\} = FI(T') \subseteq FI(T)$. Now we need to demonstrate that $2(n + 1) - 1 = 2n + 1 \in FI(T)$. In order to achieve this goal we present a friendly coloring of $T$ which will produce a friendly index element $2n + 1$. We start by using a maximal friendly labeling $g$ of $T'$. Then we define the following
labeling of $T$, $\psi : V(T) \to \{0, 1\}$

$$\psi(x) = \begin{cases} 
    f(x) & \text{if } x \neq v, w; \\
    1 - f(u) & \text{if } x = v; \\
    f(u) & \text{if } x = w.
\end{cases}$$

Since $g$ is a maximal labeling for $T'$: $e_g(0) = 0$ and $e_g(1) = 2n - 1$. The labeling $\psi$ labels $u$ and $v$ with opposite colorings, adding a 1 to the induced edge labelings. Also $\psi$ labels $v$ and $w$ with opposite colorings, adding another 1 to the induced edge labelings. Note that no additional 0 edge colorings are created. Thus $e_\psi(0) = e_g(1) = 0$ and $e_\psi(1) = e_g(1) + 2 = 2n + 1$. So for this labeling we have

$$|e_\psi(0) - e_\psi(1)| = 2n + 1$$

(2.8)

Hence $2n + 1 \in FI(T)$. By lemma 2.6, $\{1, 3, 5, ..., 2(n + 1) - 1\} \subseteq FI(T) \subseteq \{1, 3, 5, ..., 2(n + 1) - 1\}$. Therefore $FI(T) = \{1, 3, 5, ..., 2(n + 1) - 1\}$, which is the result we desired to prove. Now by the principal of mathematical induction: given any tree $T$ with $q$ edges and a perfect matching, $M$, $FI(T) = \{1, 3, 5, ..., q\}$. \hfill \Box

We are now prepared to find the friendly index set for the coronation of a tree with $K_1$. Suppose that $T$ is a tree with $q$ edges. The graph $T \odot K_1$ is described by connecting every vertex of $T$ with a copy of $K_1$. This is the same as joining every vertex of $T$ by an edge to a single exterior vertex. These “exterior” edges form a matching $M$ for $T \odot K_1$ since they do not share any common endpoints. Furthermore
every vertex of $T \odot K_1$ is an endpoint of an edge in $M$. Hence $M$ is a perfect matching for $T \odot K_1$. Since $T$ has $q + 1$ vertices, to connect each to a copy of $K_1$ we need $q + 1$ edges in $M$. As a result $|V(T \odot K_1)| = |V(M)| = 2(q + 1)$, thus $|E(T \odot K_1)| = 2(q + 1) - 1 = 2q + 1$. Now by theorem 2.7 $FI(T \odot K_1) = \{1, 3, 5, ..., 2q + 1\}$. We will now state this as a corollary.

**Corollary 12.** [13] For a tree with $q$ edges $FI(T \odot K_1) = \{1, 3, 5, ..., 2q + 1\}$.

It is now time to talk about the friendly index sets of full binary trees. We denote a full binary tree by $BT$. The friendly index sets of full binary trees has been determined by Lee and Ng [11]. Defining a full binary tree is an inductive process. We form a full binary tree with depth 1 by connecting a root to two other vertices by two edges. The full binary tree with depth 2 is created by connecting each of the two terminal vertices in the full binary tree of depth 1 to two other vertices via two new edges. Continuing inductively a “full binary tree” with depth $d + 1$ is formed by connecting the roots of two binary trees of depth $d$ to a single new root by two new edges [11]. By definition a binary tree has an odd number of vertices, hence an even number of edges. So by lemma 1.1, $FI(BT) \subseteq \{0, 2, 4, ..., q\}$ where $q$ is the number of edges in the full binary tree.

Lee and Ng have a unique set of notations to discuss full binary trees. They denote a friendly coloring by $(v, x, g)$. We say $v = v^+$ when there is one more vertex colored by 1 then colored by 0. When $v = v^-$ there is one more 0-coloring than there are 1 colorings. Also $x = 0$ indicates that the root is colored by 0, and $x = 1$ denotes a root coloring by 1. Finally $g = e_f(1) - e_f(0)$. When the coloring is not
friendly, Lee and Ng use \( vx \) in the first coordinate of \((v, x, g)\). Also to avoid confusion if the coloring is not necessarily friendly then we write \( g = e(1) - (0) \). By example we will show how Lee and Ng use their notation to document the construction of full binary trees with depth \( d + 1 \) from full binary trees with depth \( d \). We suppose that \((v-, 0, 0)\) and \((v+, 1, g)\) are two friendly labeled full binary trees with depth \( d \). We construct \((v-, 0, g)\) by adding a new root and connecting to both of these full binary trees of depth \( d \). To preserve the friendly labeling we decide to color the root of the new full binary tree by \( x = 0 \). All of this can be described by the following equation \((v-, 0, 0) + (v+, 1, g) + 0 = (v-, 0, g)\). In their proof Lee and Ng use equations like this to describe the inductive construction of labelings that will generate certain friendly indices.

The strategy of Lee and Ng will be to find a maximal set which contains the friendly index set of a full binary tree. From there they show which element(s) are not in this maximal set, then they show all the remaining element(s) can be obtained by certain colorings of the full binary tree. Lee and Ng start with several lemmas, which we now discuss.

**Lemma 13.** [11] A full binary tree with depth \( d \) will always have a vertex coloring where \( g = 2^{d+1} - 2 \). This coloring might not be friendly.

**Proof.** It will be necessary to compute the number of vertices in the full binary tree. Note that \( V(BT) = \sum_{n=0}^{d} 2^n = (2^{d+1} - 1)/(2 - 1) = 2^{d+1} - 1 \). So then \( E(BT) = 2^{d+1} - 2 \). There are two cases to consider depending on the coloring of the root.
Case A. The root is colored by 0. We start with \( x = 0 \). Now color the vertices at the second level of vertices by 1's. Then color the third level by 0's. We continue these alternate colorings until the \( d+1 \)th level: where we color by 0's if \( d+1 \) is odd and by 1 if \( d+1 \)'s is even. It is immediately apparent that all the edges will be induced labeled by 1. Consequently \( e(0) = 0 \) and \( e(1) = 2^{d+1} - 2 \). So then \( e(1) - e(0) = 2^{d+1} - 2 \). Which is the desired result.

Case B. The root is colored by 1. Here we start with \( x = 1 \). Then we color the second level of vertices by 0's. Next we color the third level by 1's. This coloring procedure finishes at the \( d + 1 \)th level: where we color by 0's if \( d + 1 \) is even and by 1 if \( d + 1 \)'s is odd. It is obvious that all the edges are labeled by 1. Hence \( e(0) = 0 \) and \( e(1) = 2^{d+1} - 2 \), thus \( e(1) - e(0) = 2^{d+1} - 2 \). This again is the desired result. □

Lee and Ng observed that the maximal labeling described in lemma 2.9 cannot possibly be friendly. Hence \( FI(BT) \subseteq \{0, 2, 4, ..., 2^{d+1} - 4 \} \). They proved that \( FI(BT) = \{0, 2, 4, ..., 2^{d+1} - 4 \} \). They demonstrated that a full binary tree of depth \( d = 1 \) has a friendly index set \( FI(BT) = \{0, 2 \} \). They considered two cases depending upon the coloring of the root. Without loss of generality they labeled the root by 0. If the exterior vertices colored by 0 and 1 then \( g = e(1) - e(0) = 0 \). When the exterior vertices are colored by 1 and 1 we get \( g = e(1) - e(0) = 2 \). The case where the root is labeled by 1 is handled in a similar manner, in fact just swap 0 with 1 in the above argument.

In their paper they simply displayed all six possible cases for \( d = 2 \) and computed the values for \( g \) for each case. Since this is easily done we leave it the reader to
do the same computations. There result for a full binary tree of depth \( d = 2 \) is 
\[ FI(BT) = \{0, 2, 4\}. \] Then they proceed by induction on \( d \). We start the next theorem with \( d > 1 \).

**Theorem 14.** [11] For a full binary tree with depth \( d > 1 \), we have 
\[ FI(BT) = \{0, 2, 4, \ldots, 2^{d+1} - 4\}. \]

**Proof.** Lee and Ng anchored the induction at \( d = 2 \). Suppose the result holds for all full binary trees with depth \( d \). We discuss the full binary tree with depth \( d + 1 \). All that is needed is to provide friendly labelings which will generate all the indices in \( \{0, 2, 4, \ldots, 2^{d+1} - 4\} \). Lee and Ng considered 4 distinct cases.

**Case A.** Suppose that \( g \) is an even number between 0 and \( 2^{d+1} - 4 \) inclusive.

**Subcase A1.** When \( g \) is divisible by 4. Here \((v-, 0, 0) + (v+, 1, g) + 0 = (v-, 0, g)\) and \((v+, 1, 0) + (v-, 0, g) + 1 = (v+, 1, g)\) will be the labelings which generate the friendly indices.

**Subcase A2.** When \( g \) is not divisible by 4. In this situation the needed labelings are \((v-, 0, 0) + (v-, 1, g) + 1 = (v-, 1, g)\) and \((v+, 1, 0) + (v+, 0, g) + 0 = (v+, 0, g)\).

**Case B.** Suppose that \( g \) is an even number between \( 2^{d+1} - 2 \) and \( 2^{d+2} - 8 \) inclusive.

**Subcase B1.** When \( g \) is divisible by 4. Here \((v-, 0, g - (2^{d+1} - 4)) + (v+, 1, 2^{d+1} - 4) + 0 = (v-, 0, g)\) and \((v+, 1, g - (2^{d+1} - 4)) + (v-, 0, 2^{d+1} - 4) + 1 = (v+, 1, g)\) generate the friendly indices.

**Subcase B2.** When \( g \) is not divisible by 4. In this situation the needed labelings are \((v-, 1, g - (2^{d+1} - 4)) + (v-, 0, 2^{d+1} - 4) + 1 = (v-, 1, g)\) and \((v+, 0, g - (2^{d+1} - 4)) + (v+, 1, 2^{d+1} - 4) + 0 = (v+, 0, g)\).
Case C. Suppose $g = 2^{d+2} - 6$. Notice that $2^{d+2} - 6$ is not divisible by 4. The following labelings will yield the necessary result: $(v-, 0, 2^{d+1} - 4) + (v-, 0, 2^{d+1} - 4) + 1 = (v-, 1, 2^{d+2} - 6)$ and $(v+, 1, 2^{d+1} - 4) + (v+, 1, 2^{d+1} - 4) + 0 = (v+, 0, 2^{d+2} - 6)$.

Case D. Suppose $g = 2^{d+2} - 4$. Observe that $2^{d+2} - 4$ is divisible by 4. All the needed trees will come from the proof of lemma 2.9. The following labelings will yield the necessary result: $(vx, 0, 2^{d+1} - 2) + (vx, 1, 2^{d+1} - 2) + 0 = (v-, 0, 2^{d+2} - 4)$, and $(vx, 0, 2^{d+1} - 2) + (vx, 1, 2^{d+1} - 2) + 1 = (v+, 1, 2^{d+2} - 4)$.

The four cases above show us that the result holds for $d + 1$. Now by the principal of mathematical induction: for a full binary tree with depth $d > 1$, $FI(BT) = \{0, 2, 4, \ldots, 2^{d+1} - 4\}$. $\square$

The friendly index sets for Fibonacci trees was computed by Salehi and Lee [13]. Like full binary trees Fibonacci trees will need to be defined by an inductive process.

The $n^{th}$ Fibonacci trees is represented by $FT_n$. First $FT_1$ is defined to be $P_1$, then $FT_2$ represents $P_2$. Now when $n \geq 3$, $FT_n$ is the binary tree with root $r_n$ and whose left and right children are $FT_{n-1}$ and $FT_{n-2}$ respectively. Figure 7 shows the first 5 Fibonacci trees.

Fibonacci trees and Fibonacci numbers have a close relationship, hence the name. Fibonacci numbers determine the cardinality of $V(FT_n)$.

Lemma 15. [13] A Fibonacci tree has $A_{n+2} - 1$ vertices where $A_n$ is the $n^{th}$ Fibonacci number.

Proof. Figure 7 shows that $|V(FT_n)| = A_{n+2} - 1$ holds for $n = 1, 2, 3, 4, \text{and} \ 5$. 24
So we anchor the induction at \( n = 5 \). Now suppose \( |V(FT_m)| = A_{m+2} - 1 \) for all integers \( 5 \leq m \leq n \). Now by construction \( FT_{n+1} \) has \( FT_n \) and \( FT_{n-1} \) as left and right children respectively. By the induction hypothesis \( |V(FT_n)| = A_{n+2} - 1 \) and \( |V(FT_{n-1})| = A_{n+1} - 1 \). So then \( |V(FT_{n+1})| = A_{n+2} - 1 + A_{n+1} - 1 + 1 = A_{(n+1)+2} - 1 \) which is the desired result. By the principal of mathematical induction: a Fibonacci tree has \( A_{n+2} - 1 \) vertices where \( A_n \) is the \( n^{th} \) Fibonacci number.

The lemma just proved will be useful in proving the following theorem about the friendly index set of the Fibonacci tree \( FT_n \). Since \( |V(FT_n)| = A_{n+2} - 1 \), we know \( |E(FT_n)| = A_{n+2} - 2 \). Now by lemma 1.2, \( FI(FT_n) \subseteq \{ |E(FT_n)| - 2i : i = 0, 1, 2, \ldots \lfloor |E(FT_n)|/2 \rfloor \} \). The strategy of Salehi and Lee is finding a friendly labeling which will generate all the values \( \{ |E(FT_n)| - 2i : i = 0, 1, 2, \ldots \lfloor |E(FT_n)|/2 \rfloor \} \). We
now state their result for the friendly index set of the $n^{th}$ Fibonacci tree.

**Theorem 16. [13]** Given a Fibonacci tree $FT_n$ where $n \geq 3$ then $FI(FT_n) = \{ |E(FT_n)| - 2i : i = 0, 1, 2, ..., [|E(FT_n)|/2] \}$.

**Proof.** Since $FT_n$ is defined inductively, the best method is to use proof by induction. By definition $FT_1 = P_1$, $FT_2 = P_2$, and $FT_3 = P_4$. So then theorem 1.3 guarantees the desired result for $n = 1, 2, 3$. Hence we anchor the induction at $n = 3$. Now suppose for every Fibonacci tree $FT_n$ where $3 \leq k \leq n$ that $FI(FT_k) = \{ |E(FT_k)| - 2i : i = 0, 1, 2, ..., [|E(FT_k)|/2] \}$ We wish to calculate $FI(FT_{n+1})$. By definition $FT_{n+1}$ has left and right children $FT_n$ and $FT_{n-1}$ respectively. The induction hypothesis tells us that $FI(FT_n) = \{ |E(FT_n)| - 2i : i = 0, 1, 2, ..., [|E(FT_n)|/2] \}$ and $FI(FT_{n-1}) = \{ |E(FT_{n-1})| - 2i : i = 0, 1, 2, ..., [|E(FT_{n-1})|/2] \}$. Take the friendly labeling $\phi : V(FT_n) \rightarrow \{0, 1\}$ where $e(1) - e(0) = b_1$. We also utilize the friendly labeling $\psi : V(FT_{n-1}) \rightarrow \{0, 1\}$ such that $e(1) - e(0) = b_2$. These labelings will be used to define a typical friendly labeling of $FT_{n+1}$. This labeling is $f : V(FT_{n+1}) \rightarrow \{0, 1\}$ defined by

\[
f(x) = \begin{cases} 
\phi(x) & \text{if } x \in V(FT_n); \\
\psi(x) & \text{if } x \in V(FT_{n-1}); \\
j & \text{if } x = r_{n+1},
\end{cases}
\]

such that $j$ can be labeled either 0 or 1. Recall that the parities of the Fibonacci sequence follow the pattern: odd, odd, even, odd, odd, even, so forth and so on. Since $|V(FT_n)| = A_{n+2} - 1$, the parity for vertex numbers for a Fibonacci tree are odd,
even, even, odd, even, even, and so on. Therefore the parties for the pair $|V(FT_n)|$ and $|V(FT_{n-1})|$ are either: even, even; odd, even; or even, odd. Hence there are two general cases to discuss.

**Case A.** Suppose that $|V(FT_n)|$ and $|V(FT_{n-1})|$ are both even. As a result their friendly labelings have an equal number of zeros and ones. Therefore labeling $r_{n+1}$ by 0 or 1 will still cause a friendly labeling for $FT_{n+1}$. In the situation where we label both $r_n$ and $r_{n-1}$ with the opposite labeling of $r_{n+1}$ we get two additional induced labelings. So then $e_f(1) = e_\phi(1) + e_\psi(1) + 2$ and $e_f(0) = e_\phi(0) + e_\psi(0)$. Hence $e_f(1) - e_f(0) = e_\phi(1) - e_\phi(0) + e_\psi(1) - e_\psi(0) + 2 = b_1 + b_2 + 2$. When one of $r_n$ and $r_{n-1}$ labeled with the opposite coloring of $r_{n+1}$ and the other has the same coloring of $r_{n+1}$ then we create an additional 0 edge labeling and an additional 1 edge labeling. Thus $e_f(1) - e_f(0) = e_\phi(1) - e_\phi(0) + e_\psi(1) - e_\psi(0) = b_1 + b_2$. Notice that the friendly labelings $\phi$ and $\psi$ were chosen arbitrary, so $b_1$ generates all the values in $\{1, 3, 5, ..., A_{n+2} - 2\}$ and $b_2$ generates all the values in $\{1, 3, 5, ..., A_{n+1} - 2\}$. Consequently $b_1 + b_2 + 2$ and $b_1 + b_2$ will generate all the values in the set $\{2, 4, 6, ..., A_{(n+1)+2} - 2\}$. We need to demonstrate that $0 \in FI(FT_{n+1})$. Label $r_{n-1}$, $r_n$, and $r_{n+1}$ all with the same color. This will produce two additional 0 induced edge labelings. As a result $e_f(1) - e_f(0) = b_1 + b_2 - 2$. Letting $b_1 = 1$ and $b_2 = 1$ gives us $e_f(1) - e_f(0) = 0$. Thus $0 \in \{0, 2, 4, ..., A_{n+2} - 2\}$. So then $\{0, 2, 4, ..., A_{(n+1)+2} - 2\} \subseteq FI(FT_{n+1}) \subseteq \{0, 2, 4, ..., A_{(n+1)+2} - 2\}$. Therefore for this case $FI(FT_{n+1}) = \{0, 2, 4, ..., A_{(n+1)+2} - 2\}$. Which is the desired result.

**Case B.** Suppose that $|V(FT_n)|$ and $|V(FT_{n-1})|$ have opposite parity. In this case we have two subcases to consider depending upon which of $|V(FT_n)|$ and $|V(FT_{n-1})|$
is odd.

**Subcase B1.** When \(|V(FT_n)|\) is odd. For \(V(FT_n)\) there is either: there is one more 0 labeling than 1-colorings; or there is one more 1 labeling than 0-colorings. We start with: one more 1 labeling than 0-colorings and \(r_n\) is colored by 1. Since \(|V(FT_{n-1})|\) is even we know that it contributes an equal number of 1's an 0's to the friendly labeling of \(FT_{n+1}\). As a result we have one more 1 than 0's before we even consider the coloring of \(r_{n+1}\). So in order to preserve the friendly labeling of \(FT_{n+1}\) we must color \(r_{n+1} = 0\).

If \(r_{n-1}\) is colored by 1 then \(e_f(1) - e_f(0) = e_\phi(1) - e_\phi(0) + e_\psi(1) - e_\psi(0) + 2 = b_1 + b_2 + 2\).

If \(r_{n-1}\) is colored by 0 then \(e_f(1) - e_f(0) = e_\phi(1) - e_\phi(0) + e_\psi(1) - e_\psi(0) = b_1 + b_2\).

Note that \(b_1\) takes on all values in the set \(\{0, 2, 4, ..., A(n+1)+2\}\). Also \(b_2\) achieves all the values in \(\{1, 3, 5, ..., A(n+1)+2\}\). So then \(b_1 + b_2 + 2\) and \(b_1 + b_2\) will generate all the values in the set \(\{1, 3, 5, ..., A(n+1)+2\}\). If we instead color \(r_n\) by 0, then labeling \(r_{n-1} = 0\) yields \(e_f(1) - e_f(0) = b_1 + b_2 - 2\); and labeling \(r_{n-1} = 1\) results in \(e_f(1) - e_f(0) = b_1 + b_2\). These labelings do not add any additional elements to the friendly index set. When studying the situation where there there is one more 0 labeling than 1-colorings, we just swap 0 with 1 in the above argument. This gives the same result, namely \(e_f(1) - e_f(0)\) generates \(\{1, 3, 5, ..., A(n+1)+2\}\). This subcase has demonstrated that \(\{1, 3, 5, ..., A(n+1)+2\} \subseteq FI(FT_{n+1}) \subseteq \{1, 3, 5, ..., A(n+1)+2\}\).

**Subcase B2.** When \(|V(FT_n)|\) is even. In this case \(|V(FT_n)|\) is even and \(|V(FT_{n-1})|\) is odd. We use the same proof of subcase B1, just interchanging the roles of \(|V(FT_n)|\) and \(|V(FT_{n-1})|\). Hence \(e_f(1) - e_f(0)\) generates \(\{0, 2, 4, ..., A(n+1)+2\}\). Consequently
\{1, 3, 5, \ldots, A_{(n+1)+2} - 2\} \subseteq FI(FT_{n+1}) \subseteq \{1, 3, 5, \ldots, A_{(n+1)+2} - 2\}. Thus \(FI(FT_{n+1}) = \{1, 3, 5, \ldots, A_{(n+1)+2} - 2\}\).

Combined cases \(A\) and \(B\) yield

\(FI(FT_{n+1}) = \{\left\lfloor E(FT_{n+1}) \right\rfloor - 2i : i = 0, 1, 2, \ldots, \left\lfloor |E(FT_{n+1})|/2 \right\rfloor\}\), which is the desired result. Now by the principal of mathematical induction: \(FI(FT_n) = \{\left\lfloor E(FT_n) \right\rfloor - 2i : i = 0, 1, 2, \ldots, \left\lfloor |E(FT_n)|/2 \right\rfloor\}\) for every \(n\).

Salehi and Lee also computed the friendly index sets of Lucas trees. These trees are another collection of inductively defined trees. The first Lucas tree is just \(P_1\) and is denoted by \(LT_1\). The second Lucas tree, \(LT_2\), is \(P_3\). Furthermore for \(n \geq 3\) the \(n^{th}\) Lucas tree, \(LT_n\), is defined by connecting a single vertex, \(r_n\), to left and right children are \(LT_{n-1}\) and \(LT_{n-2}\) respectively [13]. Figure 8 illustrates the first five Lucas trees: \(LT_1, LT_2, LT_3, LT_4,\) and \(LT_5\).

Like Fibonacci trees, Lucas Trees vertex number also has an intimate relationship with the Fibonacci sequence. We now state this relationship as a lemma.

**Lemma 17.** [13] A Lucas tree has \(2A_{n+1} - 1\) vertices where \(A_n\) is the \(n^{th}\) Fibonacci number.

**Proof.** We will proceed by induction. Figure 8 demonstrates that the desired result holds for \(LT_1, LT_2, LT_3, LT_4,\) and \(LT_5\). Now suppose that for every Lucas tree \(LT_k\) where \(5 \leq k \leq n\) that \(V(LT_n) = 2A_{n+1} - 1\). We wish to study \(V(LT_{n+1})\).

Recall that \(LT_{n+1}\) has \(LT_n\) and \(LT_{n-1}\) as left and right children respectively. By the induction hypothesis \(|V(LT_n)| = 2A_{n+1} - 1\) and \(|V(LT_{n-1})| = 2A_n - 1\). So then
By the principal of mathematical induction: a Lucas tree has $2A_{n+1} - 1$ vertices where $A_n$ is the $n^{th}$ Fibonacci number. □

This lemma tells us that a Lucas tree has an odd number of vertices, hence an even number of edges. Now by lemma 1.1, $FI(LT_n) \subseteq \{0, 2, 4, ..., E(LT_n)\}$. Salehi and Lee use this fact and provide a labeling which generates every value in this maximal set, as a result showing $FI(LT_n) = \{0, 2, 4, ..., E(LT_n)\}$. Now we are prepared to state the theorem concerning the friendly index sets of Lucas trees.

**Theorem 18.** [13] Given a Lucas tree $LT_n$ where $n \geq 3$ then

$FI(LT_n) = \{0, 2, 4, ..., |E(LT_n)|\}$. Also all the values of $FI(LT_n)$ are obtained by using friendly labelings satisfying the following four properties: (a). $f(r_n) = 1$; (b).
Proof. Salehi and Lee proceed by induction on $n$. They observed that the theorem held for $n$ where $1 \leq n \leq 4$. The induction is anchored at $n = 3$. Now suppose that the result is true for all $k$ such that $3 \leq k \leq n$. We wish to study $LT_{n+1}$. Recall that $LT_{n+1}$ has left and right children: $LT_n$ and $LT_{n-1}$ respectively. We start by labeling both of these children in a friendly manner. The induction hypothesis tells us that $FI(LT_n) = \{0, 2, 4, \ldots, |E(LT_n)|\}$ and $FI(LT_{n-1}) = \{0, 2, 4, \ldots, |E(LT_{n-1})|\}$. Take the friendly labeling $\phi : V(LT_n) \to \{0, 1\}$ where $e_\phi(1) - e_\phi(0) = b_1$. We also utilize the friendly labeling $\psi : V(LT_{n-1}) \to \{0, 1\}$ such that $e_\psi(1) - e_\psi(0) = b_2$. Furthermore, the induction hypothesis tells us we get every desired index by labeling the roots of both $FI(LT_n)$ and $FI(LT_{n-1})$ by 1. These labelings will be used to define a typical friendly labeling of $LT_{n+1}$. It is necessary to demonstrate that 0 is a friendly index. This is accomplished by fixing $b_1 = 0$ and $b_2 = 2$. Note this implies $b_1 \equiv 0 \pmod{4}$ and $b_2 \equiv 2 \pmod{4}$. Then by the induction hypothesis: $v_\phi(1) = v_\phi(0) + 1$ and $v_\psi(0) = v_\psi(1) + 1$. We use the labeling $f : V(LT_{n+1}) \to \{0, 1\}$ defined by

$$f(x) = \begin{cases} 
\phi(x) & \text{if } x \in V(LT_n); \\
\psi(x) & \text{if } x \in V(LT_{n-1}); \\
1 & \text{if } x = r_{n+1}.
\end{cases}$$

Clearly this labeling is friendly. Furthermore, $e_f(1) - e_f(0) = b_1 + b_2 - 2 = 0 + 2 - 2 = 0$. Hence $0 \in FI(LT_{n+1})$. We now need to produce the set $\{2, 4, 6, \ldots, |E(LT_{n+1})|\}$. The
strategy of Salehi and Lee will be to demonstrate that $b_1 + b_2$ and $b_1 + b_2 + 2$ are members of $FI(LT_{n+1})$. Suppose that one of $b_1$ or $b_2$ is non-zero. Salehi and Lee choose without loss of generality $b_1 \neq 0$. Then they define a new labeling for $LT_n$ where $e_\lambda(1) - e_\lambda(0) = b_1 - 2$. They consider two cases.

**Case A.** Here $b_1 \equiv 0 \pmod{4}$ or $b_2 \equiv 0 \pmod{4}$.

Without loss of generality suppose that $b_2 \equiv 0 \pmod{4}$. Here $v_\psi(1) = v_\psi(0) + 1$. Use the friendly labeling $g : V(LT_{n+1}) \to \{0, 1\}$ defined by

$$g(x) = \begin{cases} 
\phi(x) & \text{if } x \in V(LT_n); \\
\psi(x) & \text{if } x \in V(LT_{n-1}); \\
0 & \text{if } x = r_{n+1}.
\end{cases}$$

For this labeling the roots of $LT_n$ and $LT_{n+1}$ are both labeled 1. This creates two new 1 induced edge labelings for $LT_{n+1}$. So then $e_g(1) - e_g(0) = b_1 + b_2 + 2$. The index $b_1 + b_2$ is created by instead considering the friendly labeling $h : V(LT_{n+1}) \to \{0, 1\}$ defined by

$$h(x) = \begin{cases} 
\lambda(x) & \text{if } x \in V(LT_n); \\
\psi(x) & \text{if } x \in V(LT_{n-1}); \\
0 & \text{if } x = r_{n+1}.
\end{cases}$$

Here we again label the roots of $LT_n$ and $LT_{n+1}$ by 1. So we have two extra 1 induced edge labelings. Hence $e_h(1) - e_h(0) = b_1 - 2 + b_2 + 2 = b_1 + b_2$. Salehi and Lee stated that the inverse labelings of $g$ and $h$ will obey the four inductive properties (a) – (d).

**Case B.** Here $b_1 \equiv 2 \pmod{4}$ and $b_2 \equiv 2 \pmod{4}$.  

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In this case use the friendly labeling $h : V(LT_{n+1}) \to \{0, 1\}$ defined by

$$h(x) = \begin{cases} 
\lambda(x) & \text{if } x \in V(LT_n); \\
\psi(x) & \text{if } x \in V(LT_{n-1}); \\
0 & \text{if } x = r_{n+1}.
\end{cases}$$

Here we again label the roots of $LT_n$ and $LT_{n+1}$ by 1. In this situation we have $b_1 - 2$ divisible by 4, thus $v_\lambda(1) = v_\lambda(0) + 1$. We have $e_h(1) - e_h(0) = b_1 - 2 + b_2 + 2 = b_1 + b_2$.

Salehi and Lee again state that $h$ will obey the four inductive properties (a) – (d).

We have shown all the properties hold for $LT_{n+1}$. Now by the principle of mathematical induction: $FI(LT_n) = \{0, 2, 4, \ldots, E(LT_n)\}$ for every $n$. 


Friendly Index Sets Of Stars

In this section we define the tree referred to as a star. Finally we present its friendly index set. A star is a tree where every “exterior” vertex is connected by an edge to a single common point which is called the center. Stars are the only trees of diameter 2. A star with $n$ edges denoted by $ST(n)$ has $n + 1$ vertices. Figure 9 is an example of star with 7 edges and 8 vertices. The following result is due to Lee-Ng [11].

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Figure 2.7: THE STAR WITH 7 EDGES.

**Theorem 19.** For any star $ST(n)$,

$$FI(ST(n)) = \begin{cases} \{0, 2\} & \text{if } n \text{ is even;} \\ \{1\} & \text{if } n \text{ is odd.} \end{cases}$$

**Proof.** [11] Since all the "exterior" vertices of the star $ST(n)$ are connected to the center, $ST(n)$ is a complete bipartite graph where $|M| = 1$ and $|N| = n$. Hence $ST(n) = K_{1,n}$. If $n$ is even then $n + 1$ is odd, thus by theorem 2.2, we have $FI(ST(n)) = \{(1 + 1)1, 1(1 - 1),\} = \{0, 2\}$. On the overhand if $n$ is odd then $n + 1$ is even. Now since $n + 1$ is even and $m = 1$ is odd, by theorem 2.2, $FI(K_{m,n}) = \{1^2\} = \{1\}$. □

In mathematics it is often useful to have a different proof that uses techniques that can be extended to other situations. The study of friendly index sets is no different. Another proof for $FI(ST(n))$ is by a counting argument.

**Proof.** Note that all the friendly labelings of a star with the central vertex labeled
0 will produce the same friendly index set as the same star with the center labeled 1. So without loss of generality we will label the center 0. There are two cases: \( n \) is even, or \( n \) is odd. If \( n \) is even then there exists a positive integer \( k \) such that \( n = 2k \). Hence we have \( 2k + 1 \) vertices to work with. One way to have friendly labeling is: \( k \) non-center vertices labeled 0 and \( k \) non-center vertices labeled 1. So then \( e_f(0) = k \) and \( e_f(1) = k \) which yields \( |e_f(0) - e_f(1)| = 0 \). The only other friendly labeling is to label \( k - 1 \) non-center vertices by 0 and to label \( k + 1 \) non-center vertices by 1 resulting in \( e_f(0) = k - 1 \) and \( e_f(1) = k + 1 \), thus \( |e_f(0) - e_f(1)| = 2 \). Therefore if \( n \) is even then \( FI(ST(n)) = \{0, 2\} \). On the other hand, if \( n \) is odd then there is an \( m \) such that \( n = 2m - 1 \). So then we will have \( 2m \) vertices to work with. The only possible friendly labeling is for \( m - 1 \) non-center vertices to be labeled 0 and \( m \) non-center vertices to be labeled 1. So then \( e_f(0) = m - 1 \) and \( e_f(1) = m \), thus \( |e_f(0) - e_f(1)| = 1 \). Hence if \( n \) is odd then \( FI(ST(n)) = \{1\} \).

The proof given above for the \( FI(ST(n)) \) is an example of the counting techniques we will need to use to study double-star’s, \( P_n \) stars, and caterpillars. In the next section we start our discussion about friendly index sets of star-like graph.
CHAPTER 3

STARLIKE GRAPHS

In this chapter we talk about the friendly index sets of star-like graphs. The term star-like will be used to mean that the graphs is created by joining stars together. We will fist discuss the friendly index sets of double stars and end the section with a result about the friendly index set for $G_{m,n}$.

Friendly Index Sets Of Double Stars

Double stars are graphs which have two central vertices $a$ and $b$ with their corresponding leaves. Double stars have diameter 3 and are denoted by $DS(m,n)$ where $\text{deg } a = m$ and $\text{deg } b = n$. A double star where $\text{deg } m = 1$ or $\text{deg } n = 1$ reduces to a star. So we will be taking $m, n > 1$. Suppose that $n \geq m$. The only edge that is shared in common with the $a$-star and the $b$-star is the single edge connecting $a$ and $b$. If we exclude this center edge then the remaining leaves of the $a$-star and the remaining leaves of the $b$-star form disjoint sets. Since $\text{deg } a = m$ there are $m - 1$ edges connecting $a$ to vertices other than $b$. We will refer to these edges as $a$-leaves. Similarly $\text{deg } b = n$ implies that there are $n - 1$ edges connecting $b$ to vertices other than $a$ and we will call these $b$-leaves. Therefore the total number of edges of a double
star would be \( m + n - 1 \). Hence there are \( m + n \) vertices in \( DS(m, n) \). Figure 3.1 shows us the double star \( DS(4, 8) \).

![Diagram of double star DS(4, 8)](image)

Figure 3.1: THE DOUBLE STAR \( DS(4, 8) \).

We now wish to determine the friendly index set for a double star \( DS(m, n) \). Let \( a \) be labeled 0 and \( b \) be labeled 1, or \( f(a) = 0 \) and \( f(b) = 1 \). Now for the \( a \)-leaves label \( i \) non-central vertices by 0 and \( m - 1 - i \) non-central vertices by 1. That is, \( f(u_1) = f(u_2) = f(u_3) = \cdots = f(u_i) = 0 \). Also for the \( b \)-leaves we will label \( j \) non-central vertices by 0 and \( n - 1 - j \) non-central vertices by 1, that is \( f(v_1) = f(v_2) = f(v_3) = \cdots = f(v_j) = 0 \). The induced labeling yields \( i \) zero edge labelings for the \( a \)-leaves and \( n - 1 - j \) zero edge labelings for the \( b \)-leaves resulting in \( e_f(0) = i + n - 1 - j \). There are \( m - 1 - i \) edges labeled 1 for the \( a \)-leaves, \( j \) edges labeled 1 for the \( b \)-leaves, and central edge connecting \( a \) and \( b \) is labeled 1: giving us \( e_f(1) = m - i + j \). Thus

\[
|e_f(0) - e_f(1)| = |n - m + 2i - 2j - 1|. \tag{3.1}
\]

The two cases will be: \( m + n \) is even or \( m + n \) is odd.
Case A. We first consider the case where \( m + n \) is even. If \( m + n \) is even then for a friendly labeling we need to have \( (m + n)/2 \) vertices labeled 0 and \( (m + n)/2 \) vertices labeled 1. Note that the number of vertices labeled 0 and 1 are equal so whether we choose \( f(0) = 0 \) and \( f(1) = 1 \) (or \( f(0) = 1 \) and \( f(1) = 0 \)) does not matter. So the labeling set up above will handle all the variant friendly labelings when \( m + n \) is even. Since there are \( i + j + 1 \) vertices labeled 0 then

\[
m + n = 2(i + j + 1).
\]  

Therefore,

\[
|e_f(0) - e_f(1)| = |2(i + j + 1) - m - m + 2i - 2j - 1|
= |-2m + 4i + 1|
= |2m - 4i - 1|.
\]  

(3.3)

Notice that \( i \) was counting the number of non-center 0-vertices for the \( a \)-leaves and as a result: \( 0 \leq i \leq m - 1 \). For all possible friendly labelings \( i \) will take on all the integer values in this interval. Also the function \( g(i) = |2m - 4i - 1| \) is a decreasing bijection from \( \mathbb{Z} \cap [0, m - 1] \) to \( \{1, 3, 5, ..., 2m - 1\} \). Hence \( FI(DS(m, n)) = \{1, 3, 5, ..., 2m - 1\} \) when \( m + n \) is even.

Case B. The other case we consider is when \( m + n \) is odd. Then either there is one more vertex labeled 0 or one more labeled 1. So now the coloring of \( a \) and \( b \) matters and as a result we will have two subcases to consider: \( f(a) = 0 \) and \( f(b) = 1 \); \( f(a) = 1 \) and \( f(b) = 0 \).
**Subcase B1.** We will first consider the friendly labelings where \( f(a) = 0 \) and \( f(b) = 1 \). So the coloring of the non-center vertices above works just fine. Given that \( m + n \) is odd we know that \((m + n - 1)/2\) vertices are labeled 0 and \((m + n + 1)/2\) vertices are labeled 1. Note that labeling \((m + n + 1)/2\) vertices 0 and \((m + n - 1)/2\) vertices 1 would yield the same friendly labelings. Now then

\[
m + n - 1 = 2(i + j + 1)
\]

Hence

\[
|e_f(0) - e_f(1)| = |2i + 2j + 2 - m + 1 - m + 2i - 2j - 1|
\]
\[
= | - 2m + 4i + 2|
\]
\[
= |2m - 4i - 2|.
\]

Therefore \( FI(DS(m, n)) = \{0, 2, 4, \ldots, 2(m - 1)\} \).

**Subcase B2.** Now we consider the labelings where \( f(a) = 1 \) and \( f(b) = 0 \). We have the same number of 0-vertices and 1-vertices. Note there are \( m - 1 - i \) zero edge labelings for the a-leaves and \( j \) zero edge labelings for the b-leaves. Thus \( e_f(0) = m - 1 - i + j \). There are \( i \) edges labeled 1 for the a-leaves, \( n - 1 - j \) edges labeled 1 for the b-leaves, and the center edge is labeled 1. Hence \( e_f(1) = i + n - j \). Therefore

\[
|e_f(0) - e_f(1)| = |n - m + 2i - 2j + 1|
\]
\[
= |2i + 2j + 2 - m + 1 - m + 2i - 2j + 1|
\]
\[
= | - 2m + 4i|
\]
\[
= |2m - 4i|.
\]
So then $FI(DS(m, n)) = \{0, 2, 4, ..., 2m\}$.

We will now summarize cases A to B as a theorem:

**Theorem 20.** In a double star $D(m, n)$, let $\text{dega} = m$ and $\text{degb} = n$ where $m \leq n$.

Then

$$FI(D(m, n)) = \begin{cases} 
\{1, 3, 5, ..., 2m - 1\} & \text{if } m + n \text{ is even;} \\
\{0, 2, 4, ..., 2m\} & \text{if } m + n \text{ is odd.}
\end{cases}$$

**Friendly Index Sets Of $ST(n; b^n)$**

We define $ST(n; b_1, b_2, b_3, ..., b_n)$ to be a graph which is formed by adjoining the paths $P_{b_1}, P_{b_2}, ..., P_{b_n}$ all sharing a common end vertex (which will be the center). In this section we will study a special form of these graphs, $ST(n; b^n)$. Here we have $n$ copies of $P_b$, the path with $b$ vertices, all sharing a common end vertex (center).

First note that $ST(n; 2^n)$ reduces to a star. So we will be looking at this problem for $b \geq 3$.

For $ST(n; 3^n)$ we have $n$ copies of $P_3$ attached by the center. The graph $ST(n; 3^n)$ has $n$ copies of the second vertices of $P_3$, $n$ copies of the third vertex of $P_3$, and the center vertices. So this graph has $2n + 1$ vertices.

Observe that $FI(ST(n; 3^n)) \subseteq \{0, 1, 2, ..., 2n\}$ since the number of edges is $q = 2n$.

Since $ST(n; 3^n)$ has an odd number of vertices we know in order to have friendly labeling there must be one more vertex labeled 0 than labeled 1 (or vice versa).

Without loss of generality we will assume the center vertex to be labeled 0. Then to
have friendly labeling we will have \( n \) non-center vertices labeled 0 and \( n \) non-center vertices labeled 1. We start with a friendly labeling where the sequence of "first" vertices are labeled by 1 and the sequence of "second" vertices are labeled 0. That is \( f(u_1) = f(u_2) = \cdots = f(u_n) = 1 \) and \( f(v_1) = f(v_2) = \cdots = f(v_n) = 0 \). For this labeling \( e_f(0) = 0 \) and \( e_f(1) = 2n \): so then \( |e_f(0) - e_f(1)| = 2n \). By design this is the maximum friendly coloring of \( ST(n;3^n) \). Now to produce all the other possible friendly labelings we just swap out one at a time the 0-non center vertices with the 1-non center vertices. To present the index \( 2(n - j) \) where \( 1 \leq j \leq n \) we consider the coloring \( f(c) = 0 \) and

\[
    h_j(x_i) = \begin{cases} 
        1 - f(x_i) & \text{if } 1 \leq i \leq j \\
        f(x_i) & \text{if } j < i \leq n 
    \end{cases}
\]

where \( x_i \) stands for \( u_i \) or \( v_i \). Note that each time we swap a 0 non center vertex with a 1 non center vertex that that our maximum friendly coloring looses an edge labeled by a 1 and gains an edge labeled by a 0. Also up this point we have gained \( j \) zero edge labelings. Hence \( e_f(0) = j \) and \( e_f(1) = 2n - j \). So then
\[ |e_f(0) - e_f(1)| = |2(n - j)|. \tag{3.7} \]

Notice that \( j \) assumes all values form 0 to \( n \). Since \( h(j) = 2(n - j) \) is a decreasing bijection from \( Z \cap [0, n] \) to \( \{0, 2, 4, ..., 2n\} \) thus \( FI(ST(n; 3^n)) \subseteq \{0, 2, 4, ..., 2n\} \subseteq FI(ST(n; 3^n)) \). Therefore \( FI(ST(n; 3^n)) = \{0, 2, 4, ..., 2n\} \).

The method used to find the friendly index set of \( ST(n; 3^n) \) needs to be altered. The bijectivity of the swapping function occurred only because we were dealing with two rows of 0's and 1's being swapped. So when we study \( ST(n; (2k + 1)^n) \) we will actually start by studying the friendly index sets of the path \( P_{2k+1} \). Since there are an odd number of vertices to label either 0 or 1 we can without loss of generality label the left most vertex of the path 0. When we form \( ST(n; (2k + 1)^n) \) this vertex will be the vertex in common with all the adjoined copies of \( P_{2k+1} \). From now on we will denote this central vertex by \( c \).

Now we can begin our analysis of \( ST(n; (2k + 1)^n) \). Suppose each copy of the path of \( P_{2k+1} \) is friendly labeled. Then \( ST(n; (2k + 1)^n) \) will be labeled friendly as well. So the friendly index set formed by such a version of \( ST(n; (2k + 1)^n) \) will form a subset of \( FI(ST(n; (2k + 1)^n)) \). For the the \( l^{th} \)-copy of \( P_{2k+1} \) denote the friendly labeling by \( l_f \).

For the \( l^{th} \) denote the number of edges induced labeled 0 by \( e_{lf}(0) \) and the number of edges induced labeled 1 by \( e_{lf}(1) \). Suppose further that \( e_{lf}(1) \leq e_{lf}(0) \). From above, we know for the \( l^{th} \)-copy we have that \( |e_{lf}(1) - e_{lf}(0)| = e_{lf}(1) - e_{lf}(0) \), which takes on every value in the set \( \{0, 2, 4, ..., 2k\} \). Let \( f \) denote the total friendly labeling by the
union of each of the friendly labeling’s $IJ$. So then $|e_f(1) - e_f(0)| = e_f(1) - e_f(0) = \sum_{i=0}^{n} e_{lf}(1) - e_{lf}(0)$ takes on every even number from 0 to $2kn$. So for this version of $ST(n; (2k + 1)^n)$ we have $FI(n; (2k + 1)^n) = \{0, 2, 4, ..., 2kn\}$. Since this is just one class of $ST(n; (2k + 1)^n)$ we see that $\{0, 2, 4, ..., 2kn\} \subseteq FI(ST(n; (2k + 1)^n))$. Hence we have $FI(ST(n; (2k + 1)^n)) \subseteq \{0, 2, 4, ..., 2kn\} \subseteq FI(ST(n; (2k + 1)^n))$. Therefore $FI(ST(n; (2k + 1)^n)) = \{0, 2, 4, ..., 2kn\}$.

Now we study $ST(n; (2k)^n)$. As above each copy of the path $P_{2k}$ is adjoined with a common vertex. Without loss of generality label the center 0. Since we are assuming that each copy of $P_{2k}$ will be labeled friendly, then we have that $ST(n; (2k)^n)$ will be labeled friendly as well. So the friendly index set formed by such a version of $ST(n; (2k)^n)$ will form a subset of $FI(ST(n; (2k)^n))$. Let $lf$ be the friendly labeling for the $l^{th}$ copy of $P_{2k}$. For the the $l^{th}$-copy of $P_{2k}$ denote number of edges labeled 0 by $e_{lf}(0)$ and the number of edges labeled 1 by $e_{lf}(1)$. Suppose further that $e_{lf}(1) \geq e_{lf}(0)$. Form above we know for the $l^{th}$-copy we have that $e_{lf}(1) - e_{lf}(0)$ takes on every value in the set $\{1, 3, 5, ..., 2k - 1\}$. So then $|e_f(1) - e_f(0)| = e_f(1) - e_f(0) = \sum_{i=0}^{n} e_{lf}(1) - e_{lf}(0)$ takes on every odd number from 1 to $(2k - 1)n$ if $n$ is odd. So for this version of $ST(n; (2k)^n)$ we have $FI(n; (2k + 1)^n) = \{1, 3, 5, ..., (2k - 1)n\}$. Since this is just one class of $ST(n; (2k)^n)$ we see that $\{1, 3, 5, ..., (2k - 1)n\} \subseteq FI(ST(n; (2k)^n))$. Hence we have for $n$ odd: $FI(ST(n; (2k)^n)) \subseteq \{1, 3, 5, ..., (2k - 1)n\} \subseteq FI(ST(n; (2k)^n))$. So when $n$ is odd we have $FI(ST(n; (2kn)^n)) = \{1, 3, 5, ..., (2k - 1)n\}$. However $|e_f(1) - e_f(0)| = \sum_{i=0}^{n} e_{lf}(1) - e_{lf}(0)$ takes on every even number from 0 to $(2k - 1)n$ where $n$ is even.
hence $FI(ST(n; (2kn)^n)) \subseteq \{0, 2, 4, \ldots, (2k - 1)n\} \subseteq FI(ST(n; (2kn)^n))$ and consequently $FI(ST(n; (2kn)^n)) = \{0, 2, 4, \ldots, (2k - 1)n\}$.

We can summarize the above discussion in the following theorem,

**Theorem 21.**

$$FI(ST(n; b^n)) = \begin{cases} 
\{0, 2, 4, \ldots, 2kn\} & \text{if } b = 2k + 1; \\
\{1, 3, 5, \ldots, (2k - 1)n\} & \text{if } b = 2k \text{ and } n \text{ is odd.} \\
\{0, 2, 4, \ldots, (2k - 1)n\} & \text{if } b = 2k \text{ and } n \text{ is even.}
\end{cases}$$

**Friendly Index Sets Of $G_{m,n}$**

The graph $G_{m,n}$ is formed connecting two stars by using a common end vertex of the two stars with degrees $m$ and $n$ respectively. This vertex is the central vertex will call it $c$. The two vertices which are the centers of the two stars themselves will be called $a$ and $b$. In the case where $m = \text{deg } a = 1$ or $n = \text{deg } b = 1$ then $G_{m,n}$ will become a double star. So we will be considering the cases where: $1 < m \leq n$. Also this graph has $m + n + 1$ vertices. It is sufficient to find the friendly index set of two special forms of $G_{m,n}$. The first form is where we have a friendly labeling where $f(a) = 0$, $f(b) = 1$, and $f(c) = 1$. The second form occurs when a friendly labeling such that $f(a) = 0$, $f(b) = 1$, and $f(c) = 0$ is used instead. As with the double stars the cases we consider will be based upon the parity of $m + n$.

**Case A.** If $m + n$ is odd then $m + n + 1$ is even. So for friendly labeling to occur
we must split the number of vertices labeled by 0 and 1 in half. This means that 
\((m + n + 1)/2\) vertices will be labeled 0 and \((m + n + 1)/2\) will be labeled 1. Since we 
have an equal number of vertices labeled by 0 and 1 it does not matter which of the two 
forms of \(G_{m,n}\) we are considering. As with the double star we label the end vertices: 
\[0 = f(u_1) = f(u_2) = f(u_3) = \ldots = f(u_i),\ 0 = f(v_1) = f(v_2) = f(v_3) = \ldots = f(v_j),\]
and the rest of the end vertices by 1. We want to find out how many edges are labeled 0. To get a 0 edge labeling on the left-hand star we would need to add a 0, so the 
left-hand star contributes \(i\) such 0-edges. To get a 0 edge labeling on the right-hand 
side we will need to add a 1 thus the right hand star contributes \(n - 1 - j\) such edges. 
By design we also have a central edge labeled by 0 (since \(1 + 1 = 0\)). In all we have 
that \(e_f(0) = i + n - 1 - j + 1 = i + n - j\). Now we need to find out how many edges 
are labeled 1. For a 1-edge labeling on the left-hand star to occur we need to add a 1 
hence there are \(m - 1 - i\) such edges. In order to get a 1-edge labeling from the right 
hand star we need to add a 0 so there will be \(j\) such edges. There is also a central 
edge labeled 1. All together we have that \(e_f(1) = m - 1 - i + j + 1 = m - i + j\). So then 
\[|e_f(1) - e_f(0)| = |m - n + 2(j - i)|. \quad (3.8)\]
Notice that in this choice of graph we have the number of 0-vertices is given by \(i+j+1\) thus
\[
m + n + 1 = 2(i + j + 1),
\]
which yields \(n = 2(i + j) - m + 1\). Hence
\[
|e_f(1) - e_f(0)| = |2m - 4i - 1|. \tag{3.10}
\]
As \(i\) ranges over the integers 0 to \(m - 1\) the previous equation gives us the set \(\{0, 3, 5, \ldots, 2m - 1\}\).

**Case B.** Now if \(m + n\) is even then \(m + n + 1\) is odd. So then for a friendly labeling to occur we have two situations: either we label \((m + n)/2\) vertices by 0 and \((m + n + 2)/2\) vertices by 1; or alternatively: label \((m + n)/2\) vertices by 1 and \((m + n + 2)/2\) vertices by 0. We will handle this situation and the alternate situation by computing the possible friendly index set for both forms of \(G_{m,n}\) in just one of the situations mentioned in the previous statement. With out loss of generality we label \((m + n)/2\) vertices by 0 and \((m + n + 2)/2\) vertices by 1.

**Subcase B1.** In the first subcase we consider the first form of the graph. Recall that in the previous case we computed:
\[
|e_f(1) - e_f(0)| = |m - n + 2(j - i)|. \tag{3.11}
\]
In this sub-case we have a different number of 0 vertices. Here we have that

\[ i + j + 1 = (m + n + 2)/2. \]  \hfill (3.12)

Which gives us \( n = 2(i + j) - m \). So then

\[ |e_f(1) - e_f(0)| = |2m - 4i|. \]  \hfill (3.13)

**Subcase B2.** In the second subcase we study the second form. Again we label according to: \( 0 = f(u_1) = f(u_2) = f(u_3) = \ldots = f(u_i), \ 0 = f(v_1) = f(v_2) = f(v_3) = \ldots = f(v_j) \), and the rest of the end vertices by 1. We again get \( e_f(0) = i + n - 1 - j + 1 = i + n - j \) and \( e_f(1) = m - 1 - i + j + 1 = m - i + j \). So then

\[ |e_f(1) - e_f(0)| = |m - n + 2(j - i)|. \]  \hfill (3.14)

However in this subcase

\[ i + j + 2 = (m + n + 2)/2 \]  \hfill (3.15)

which gives us \( n = 2(i + j + 1) - m \). Hence

\[ |e_f(1) - e_f(0)| = |2m - 4i - 2|. \]  \hfill (3.16)

We can summarize all of this into a theorem.
Theorem 22. Given $2 \leq m \leq n$. Then the friendly index set of $G_{m,n}$ is

$$FI(G_{m,n}) = \begin{cases} 
\{1, 3, 5, ..., 2m - 1\} & \text{if } m + n \text{ is odd;} \\
\{0, 2, 4, ..., 2m\} & \text{if } m + n \text{ is even.}
\end{cases}$$

The next section will discuss the friendly index sets of caterpillars.

Friendly Index Sets of Caterpillars

In this section we define the tree called caterpillar and derive its friendly index set. Caterpillar is a tree having the property that the removal of its end-vertices results in a path (the spine). We use $CR(a_1, a_2, \cdots, a_n)$ to denote the caterpillar with a $P_n$-spine, where the $i$th vertex of $P_n$ has degree $a_i$. Since $CR(1, a_1, \cdots, a_n, 1) = CR(a_1, \cdots, a_n)$ and $a_i \neq 1 \ (2 \leq i \leq n - 1)$, we will assume that $a_i \geq 2$.

![Figure 3.4: A CATERPILLAR OF DIAMETER $n + 1$ ($P_n$-SPINE).](image)

In this section we will concentrate on caterpillars whose spines are $P_3$ and will use the notation $G = CR(a, b, c)$, where $\deg u = a$, $\deg v = b$, and $\deg w = c$, as illustrated in Figure 3.4.
This caterpillar has $a+b+c-1$ vertices and $a+b+c-2$ edges. Assume $a, c > 1$ $b > 2$ and let $f : V(G) \to \{0, 1\}$ be a friendly labeling. We will consider two cases:

Case A. $a + b + c$ is odd.

Notice that any friendly labeling where $f(u) = f(v) = f(w) = 0$ will result in $|e(1) - e(0)| = 1$. This is the trivial situation. The other situations will be discussed in subcase $A1$ to subcase $A3$.

Subcase A1. Let $f(u) = 0$, and $f(v) = f(w) = 1$ be the labeling of the central vertices and all other labels be 1 except

$$
\begin{align*}
    f(u_i) &= f(u_2) = \cdots = f(u_i) = 0; \\
    f(v_1) &= f(v_2) = \cdots = f(v_j) = 0; \\
    f(w_1) &= f(w_2) = \cdots = f(w_k) = 0.
\end{align*}
$$

Then $v_f(0) = i + j + k + 1$ and $v_f(1) = a + b + c - i - j - k - 2$. For this labeling to be friendly we need

$$
i + j + k + 1 = \frac{a + b + c - 1}{2},
$$

which implies $|e(1) - e(0)| = |2a - 4i - 1|$. Moreover, $i + 1 \leq (a + b + c - 1)/2$ and
\[ a - i + 1 \leq (a + b + c - 1)/2, \text{ which provide the inequalities } (a - b - c + 3)/2 \leq i \leq (a + b + c - 3)/2. \text{ Therefore, the friendly indices obtained in this subcase would be } \]

\[ \{ |2a - 4i - 1| : m_i \leq i \leq M_i \}, \]

where \( m_i = \max\{0, (a - b - c + 3)/2\} \) and \( M_i = \min\{a - 1, (a + b + c - 3)/2\} \).

The label assignments \( f(u) = 1 \), and \( f(v) = f(w) = 0 \), will result in \( |e(1) - e(0)| = |2a - 4i - 3| \), which is the complementary labeling and will provide the same friendly indices.

Note that if \( a < b, c \), then \( 0 \leq i \leq a - 1 \) and \( \{1, 3, 5, \cdots, 2a - 1\} \subset FL(G) \).

**Subcase A2.** Suppose that we label central vertices: \( f(v) = 0 \), and \( f(u) = f(w) = 1 \) and label all other vertices by 1 except

\[ f(u_1) = f(u_2) = \cdots = f(u_i) = 0; \]
\[ f(v_1) = f(v_2) = \cdots = f(v_j) = 0; \]
\[ f(w_1) = f(w_2) = \cdots = f(w_k) = 0. \]

Then \( v_f(0) = i + j + k + 1 \) and \( v_f(1) = a + b + c - i - j - k - 2 \). For friendly labeling to occur it must be that

\[ i + j + k + 1 = \frac{a + b + c - 1}{2}, \]

thus \( |e(1) - e(0)| = |2b - 4j - 1| \). Now then, \( j + 1 \leq (a + b + c - 1)/2 \) and \( b - j \leq \)
\[\frac{(a+b+c-1)}{2},\] which yield the inequalities \[\frac{(a-b-c+3)}{2} \leq j \leq \frac{(a+b+c-3)}{2}.\]

Hence, the friendly indices obtained in this subcase would be

\[\{|2b-4j-1|: m_j \leq j \leq M_j\}, \quad (3.22)\]

where \(m_j = \max\{0, (b-a-c+1)/2\}\) and \(M_j = \min\{b-2, (a+b+c-3)/2\}\).

The following labeling for the central vertices: \(f(v) = 1\), and \(f(u) = f(w) = 0\), will result in \(|e(1) - e(0)| = |2b-4j-7|\), which is the complementary labeling and will provide the same friendly indices.

When \(b \leq a, c\), it is seen that \(0 \leq j \leq b-2\) and \(\{1, 3, 5, \ldots, 2b-1\} \subset FI(G)\).

Subcase A3. Let \(f(w) = 0\), and \(f(u) = f(v) = 1\) be the labeling of the central vertices and all other labels be 1 except

\[
\begin{align*}
    f(u_1) &= f(u_2) &= \cdots &= f(u_i) = 0; \\
    f(v_1) &= f(v_2) &= \cdots &= f(v_j) = 0; \\
    f(w_1) &= f(w_2) &= \cdots &= f(w_k) = 0.
\end{align*}
\]

(3.23)

Then \(v_f(0) = i + j + k + 1\) and \(v_f(1) = a + b + c - i - j - k - 2\). For this labeling to be friendly we need

\[i + j + k + 1 = \frac{a+b+c-1}{2}, \quad (3.24)\]

which implies \(|e(1) - e(0)| = |2c-4k-1|\). Moreover, \(k + 1 \leq \frac{(a+b+c-1)}{2}\) and \(c - k + 1 \leq \frac{(a+b+c-1)}{2}\), which provide the inequalities \(\frac{(a-b-c+3)}{2} \leq k \leq \frac{(a+b+c-3)}{2}\).
\[(a + b + c - 3)/2. \text{ Therefore, the friendly indices obtained in this subcase would be}
\[
\{\lfloor 2c - 4k - 1 \rfloor : m_k \leq k \leq M_k \},
\]

where \( m_k = \max\{0, (a - b - c + 3)/2 \} \) and \( M_k = \min\{c - 1, (a + b + c - 3)/2 \} \).

The label assignments \( f(w) = 1 \), and \( f(v) = f(w) = 0 \), will result in \(|e(1) - e(0)| = |2c - 4k - 3|\), which is the complementary labeling and will provide the same friendly indices.

Note that if \( c \leq a, b \), then \( 0 < k < a - 1 \) and \( \{1, 3, 5, \cdots, 2c - 1\} \subseteq FI(G) \).

This case is summarized by the following theorem.

**Theorem 23.** Suppose \( a + b + c \) is odd. Then \( FI(CR(a, b, c)) = A \cup B \cup C \), where

\[
A = \{\lfloor 2a - 4i - 1 \rfloor : m_i \leq i \leq M_i \};
\]
\[
B = \{\lfloor 2b - 4j - 1 \rfloor : m_j \leq j \leq M_j \};
\]
\[
C = \{\lfloor 2c - 4k - 1 \rfloor : m_k \leq k \leq M_k \}.
\]

Here

\[
m_i = \max\{0, (a - b - c + 3)/2 \};
\]
\[
m_j = \max\{0, (b - a - c + 1)/2 \};
\]
\[
m_k = \max\{0, (a - b - c + 3)/2 \},
\]

and
\[ M_i = \min\{a - 1, (a + b + c - 3)/2\}; \]
\[ M_j = \min\{b - 2, (a + b + c - 3)/2\}; \]
\[ M_k = \min\{c - 1, (a + b + c - 3)/2\}. \]

\textbf{Case B.} \(a + b + c\) is even.

Notice that any friendly labeling where \(f(u) = f(v) = f(w) = 0\) will result in
\(|e(1) - e(0)| = 0\) when \(|v_f(0)| = a + b + c/2\). In the situation where \(f(u) = f(v) = f(w) = 0\) and \(|v_f(0)| = a + b + c - 2/2\) we get \(|e(1) - e(0)| = 2\). These are the trivial situations. The other situations will be discussed in subcase B1 to subcase B3.

\textbf{Subcase B1.} Suppose we label the central vertices by \(f(u) = 0\), and \(f(v) = f(w) = 1\) and label all other vertices by 1 except
\[ f(u_1) = f(u_2) = \cdots = f(u_i) = 0; \]
\[ f(v_1) = f(v_2) = \cdots = f(v_j) = 0; \]
\[ f(w_1) = f(w_2) = \cdots = f(w_k) = 0. \]

Then \(v_f(0) = i + j + k + 1\) and \(v_f(1) = a + b + c - i - j - k - 2\). For this labeling to be friendly we need
\[ i + j + k + 1 = \frac{a + b + c - 2}{2}, \]  
\[ (3.30) \]
or
\[ i + j + k + 1 = \frac{a + b + c}{2}, \]  
\[ (3.31) \]

The last equation yields \(|e(1) - e(0)| = |2a - 4i|\). In this situation, \(i + 1 \leq (a + b + c)/2\)
and \( a - i + 1 \leq (a + b + c - 2)/2 \), which provide the inequalities \((a - b - c + 4)/2 \leq i \leq (a + b + c - 2)/2\). Therefore, the friendly indices obtained in this subcase would be

\[
\{ |2a - 4i| : m_i \leq i \leq M_i \},
\]

where \( m_i = \max \{0, (a - b - c + 4)/2 \} \) and \( M_i = \min \{a - 1, (a + b + c - 2)/2 \} \). The parallel equation gives us \(|e(1) - e(0)| = |2a - 4i - 2|\). And this produces the same friendly elements.

The label assignments \( f(u) = 1 \), and \( f(v) = f(w) = 0 \), will result in \(|e(1) - e(0)| = |2a - 4i|\), and \(|e(1) - e(0)| = |2a - 4i - 4|\), which are the equations for the complementary labeling and will provide the same friendly indices.

When \( a \leq b - 3, c \), it is seen that \( 0 \leq i \leq a - 1 \) and \( \{0, 2, 4, \ldots, 2a\} \subset FI(G) \).

**Subcase B2.** Let \( f(v) = 0 \), and \( f(u) = f(w) = 1 \) be the labeling of the central vertices and all other labels be 1 except

\[
\begin{align*}
  f(u_1) &= f(u_2) = \cdots = f(u_i) = 0; \\
  f(v_1) &= f(v_2) = \cdots = f(v_j) = 0; \\
  f(w_1) &= f(w_2) = \cdots = f(w_k) = 0.
\end{align*}
\]

Then \( v_f(0) = i + j + k + 1 \) and \( v_f(1) = a + b + c - i - j - k - 2 \). Friendly labeling implies

\[
i + j + k + 1 = \frac{a + b + c - 2}{2},
\]

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\[ i + j + k + 1 = \frac{a + b + c}{2}, \]  

(3.35)

The previous equation gives us \( |e(1) - e(0)| = |2b - 4j| \). In this instance, \( j + 1 \leq (a+b+c)/2 \) and \( b-j \leq (a+b+c-2)/2 \), which provide the inequalities \( (b-a-c+2)/2 \leq j \leq (a+b+c-2)/2 \). Therefore, the friendly indices obtained in this subcase would be

\[ \{[\! 2b - 4j] : m_j \leq j \leq M_j \}, \]

(3.36)

where \( m_j = \max\{0, (b - a - c + 2)/2\} \) and \( M_j = \min\{b - 2, (a + b + c - 2)/2\} \).

The other equation yields \( |e(1) - e(0)| = |2b - 4j - 2| \). This result gives us the same friendly indices.

Using the labeling such that \( f(v) = 1 \), and \( f(u) = f(w) = 0 \), gives us \( |e(1) - e(0)| = |2b - 4j - 10| \), and \( |e(1) - e(0)| = |2b - 4j - 8| \). This is the complementary labeling and will provide the same friendly indices.

Note that if \( b \leq a, c \), then \( 0 \leq j \leq b - 2 \) and \( \{0, 2, 4, \cdots, 2b\} \subset FI(G) \).

**Subcase B3.** Suppose we label the central vertices by \( f(w) = 0 \), and \( f(u) = f(v) = 1 \)

and label all other vertices by 1 except

\[
\begin{align*}
    f(u_1) &= f(u_2) = \cdots = f(u_i) = 0; \\
    f(v_1) &= f(v_2) = \cdots = f(v_j) = 0; \\
    f(w_1) &= f(w_2) = \cdots = f(w_k) = 0.
\end{align*}
\]

(3.37)

Then \( v_f(0) = i + j + k + 1 \) and \( v_f(1) = a + b + c - i - j - k - 2 \). For this labeling to
be friendly we need

\[ i + j + k + 1 = \frac{a + b + c - 2}{2}, \]  

(3.38)

or

\[ i + j + k + 1 = \frac{a + b + c}{2}, \]  

(3.39)

The last equation yields \(|e(1) - e(0)| = |2c - 4k|\). In this situation, \(k + 1 \leq (a + b + c)/2\) and \(a - k + 1 \leq (a + b + c - 2)/2\), which provide the inequalities \((c - a - b + 4)/2 \leq k \leq (a + b + c - 2)/2\). Therefore, the friendly indices obtained in this subcase would be

\[ \{|2c - 4k| : m_k \leq k \leq M_k\}, \]  

(3.40)

where \(m_k = \max\{0, (c - a - b + 4)/2\}\) and \(M_k = \min\{c - 1, (a + b + c - 2)/2\}\). The parallel equation gives us \(|e(1) - e(0)| = |2c - 4k - 2|\). And this produces the same friendly elements.

The label assignments \(f(w) = 1\), and \(f(u) = f(v) = 0\), will result in \(|e(1) - e(0)| = |2c - 4k - 2|\), and \(|e(1) - e(0)| = |2a - 4k - 4|\), which are the equations for the complementary labeling and will provide the same friendly indices.

When \(c \leq a, b - 1\), it is seen that \(0 \leq k \leq c - 1\) and \(\{0, 2, 4, \ldots, 2c\} \subset FI(G)\).

This is summarized by the following theorem.

**Theorem 24.** Suppose \(a + b + c\) is even. Then \(FI(CR(a, b, c)) = A \cup B \cup C\), where
Here

\[ A = \{ |2a - 4i| : m_i \leq i \leq M_i \}; \]
\[ B = \{ |2b - 4j| : m_j \leq j \leq M_j \}; \]
\[ C = \{ |2c - 4k| : m_k \leq k \leq M_k \}. \]  

(3.41)

Here

\[ m_i = \max \{0, (a - b - c + 4)/2\}; \]
\[ m_j = \max \{0, (b - a - c + 2)/2\}; \]
\[ m_k = \max \{0, (c - a - b + 4)/2\}, \]  

(3.42)

and

\[ M_i = \min \{a - 1, (a + b + c - 2)/2\}; \]
\[ M_j = \min \{b - 2, (a + b + c - 2)/2\}; \]
\[ M_k = \min \{c - 1, (a + b + c - 2)/2\}. \]  

(3.43)
CHAPTER 4

CONCLUSION

This thesis has served as an expository discussion about the friendly index sets of a number of graphs. The study of friendly index sets was started by Lee and Ng [11]. They computed the friendly index sets of complete graphs, complete bipartite graphs, and a number of trees. Salehi and Lee continued the study of friendly index sets of trees [13]. They found the friendly index sets for number of trees including: trees with perfect matching, Fibonacci trees, and Lucas trees. The study of friendly index sets is a fresh field of interest and there are numerous open problems. In this thesis, we explored the friendly index sets of star-like graphs. We computed the friendly index sets of double-stars, \( ST(n; b^n) \), \( G_{m,n} \), and some caterpillars of diameter 4. Our future research goals include calculating the friendly index sets for the general caterpillar of diameter \( n+1 \). We believe that our proof for the friendly index set of \( ST(n; b^n) \) can be generalized for \( ST(n; b_1, b_2, ..., b_n) \). All the work done so far concerning friendly index sets of trees have led researchers to believe that the elements of the friendly index sets of trees form an arithmetic progression. Salehi and Lee have conjectured that the friendly indices of any tree form an arithmetic progression of common difference two [13]. Our study of the friendly indices of star-like graphs was aimed at verifying this conjecture.
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