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A general mathematical model of stenoses

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A GENERAL MATHEMATICAL MODEL OF STENOSES

by

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Bachelor of Science
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June 2001

A thesis submitted in partial fulfillment
of the requirements of the

Master of Science Degree in Mathematical Sciences
Department of Mathematical Sciences
College of Science

Graduate College
University of Nevada, Las Vegas
August 2007
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Entitled

A General Mathematical Model of Stenoses

is approved in partial fulfillment of the requirements for the degree of

Master of Science in Mathematical Sciences

Examination Committee Chair

Dean of the Graduate College

Graduate College Faculty Representative
ABSTRACT

A General Mathematical Model of Stenoses

by

Jordan Jimmy Crabbe

Dr. Dieudonné D. Phanord, Examination Committee Chair
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Incidence and the prevalence rate of stroke in the United States has been of great concern to the Centers for Disease Control and prevention (CDC), National Center for Health Statistics (NCHS), the American Society of Stroke, American Stroke Association and the U.S government as a whole.

Many mathematical models of the stroke problem have been proposed and solved using diverse methods and computations. Most of the models are based on the stenoses, occlusion or rupture of the artery and the formation of cerebral aneurysms, which all lead to the occurrence of stroke.

This paper utilizes a direct approach to change the motion and the continuity equation of the blood flow to a Bessel equation and then solve to obtain the properties of the blood under stenoses. The model is pertinent to the stenoses of the artery, which occurs as a result of debris in the blood stream, fatty deposits (lipids), cholesterol, a calcium deposit or a blood clot.
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ACKNOWLEDGEMENTS

This thesis has come this far not by my singular effort and it would therefore sound ungrateful if I do not acknowledge the contributions and assistance of some people.

I would first of all thank the Almighty God for what He has done for me and what He has caused me to do.

I would like to express my profound gratitude to my advisor, Dr. Dieudonné D. Phanord for his encouragement, direction, guidance, constructive criticisms and patience throughout the duration of this thesis that help in refining and executing the thesis to this style. He has been my mentor and his excellent teaching and research skills have been a tremendous help to me.

I also wish to express my heartfelt gratitude to my responsible and respectable committee members, Dr. Rohan Dalpatadu, Dr. Dennis Murphy and Dr. Pradip Bhowmik for their time, dedication and invaluable contributions.

I will not have said it all if I do not render special thanks to my parents Mr. and Mrs. Samuel H. Crabbe, my wife Abigail, my daughter Jodrell and my siblings Dora, Victor and Flora for their love, support and contributions throughout my education. They have been part of what has brought me this far.
CHAPTER 1

INTRODUCTION

Over the years, the incidence and prevalence rate of stroke in the United States has been of great concern to the Centers for Disease Control and Prevention (CDC), National Center for Health Statistics (NCHS), the American Society of Stroke, American Stroke Association, and the U.S. government as a whole. Stroke, which is medically termed Cerebrovascular Accident (CVA) or cerebral apoplexy, is the third leading cause of death in the United States, causing more serious long-term disabilities than any other disease [1, 10 and 27]. Over 700,000 cases of stroke are reported in the United States each year. It has been estimated that, on the average, every 45 seconds someone suffers a stroke and every 3 minutes, someone dies of stroke in the United States. Nearly three-quarters of all strokes occur in people over the age of 65. For African-Americans, stroke is more common and more deadly even in young and middle-aged adults than for any ethnic or other racial group in the United States [27].

1.1 Description of stroke

Stroke or cerebrovascular accident (CVA) is an acute neurological injury whereby the blood supply to a part of the brain is interrupted, either by arterial blockage or rupture [27]. The part of the brain perfused by a blocked or burst artery can no longer receive...
oxygen and other nutrients carried by the blood. The brain cells are therefore damaged or die, impairing function from that part of the brain. This affects the entire body, resulting in mild to severe disabilities. The most serious of these is paralysis, which can affect different parts of the body. Paralysis of one limb is termed as monoplegia. When one side of the body is paralyzed including the limbs and the face, it is termed hemiplegia. The paralysis of the lower extremities and the lower trunk, including the spinal cord is called paraplegia. When all the limbs are paralyzed, it is termed quadriplegia or tetraplegia. Other disabilities include problems with thinking or speaking, emotional problems, pneumonia and vision loss. If a stroke affects the part of the brain responsible for language comprehension and speech production, then the person will be unable to understand words or speak; this condition is known as aphasia. If the stroke is severe enough, coma or death may occur [10 and 27].

1.1.1 Types of stroke

There are two major categories of stroke: ischemic stroke (also called infarctions) and hemorrhagic stroke. 88% of all strokes are ischemic. However, 8-12% of ischemic strokes and 37-38% of hemorrhagic strokes result in death within 30 days of incidence [10 and 27].

1.1.2 Ischemic stroke

During ischemic stroke, the normal flow of blood through the brain is either partially or totally blocked. The culprit is often a blood clot, a fatty deposit (lipids), cholesterol, or a calcium deposit. In this case, the brain tissue that lies “down stream” from the obstruction is denied life-giving nutrients in the form of blood, which causes the cells in that part of the brain to die within a few minutes. Ischemic stroke is commonly divided
into thrombolic stroke (thrombosis) and embolic stroke (embolism). In thrombolic stroke, a fatty deposit or blood clot narrows the lumen of the artery and impedes blood flow to the brain tissue. These clots usually form around the atherosclerotic plaques. Embolic stroke refers to the blockage of arterial access to a part of the brain by traveling particles or debris (embolus) in the arterial blood stream originating from elsewhere. An embolus can be a fatty deposit or blood clot formed elsewhere in the body that detaches and travels through increasingly narrow blood vessels, eventually becoming wedged in the neck or the brain [9, 25, and 27].

1.1.3 Hemorrhagic stroke

Hemorrhagic stroke occurs when a blood vessel in the brain suddenly ruptures, mainly due to uncontrolled high blood pressure. When delicate blood vessels can no longer contain the highly pressurized blood, they burst like an over-inflated inner tube. When this happens, blood that under normal circumstances would be neatly contained inside the arteries leaks into the brain or the surrounding tissue. Hemorrhagic stroke is considered the most serious and deadly form of stroke, with two major types: intracerebral hemorrhage and subarachnoid hemorrhage. In former, blood seeps directly into the brain tissue, forming a gradually enlarged pool of blood. This is usually caused by high blood pressure. Subarachnoid hemorrhage occurs between the surface of the brain and the inside of the skull, where blood leaks into the fluid-filled area of delicate tissue that surrounds the brain (subarachnoid space) and becomes trapped within the bony confines of the skull, putting excruciating pressure on soft and fragile brain tissue. Often the cause of subarachnoid hemorrhage is an aneurysm (an enlargement of the arterial wall
which lacks a layer of smooth and elastic tissue and subsequently becomes weak) [9, 26, and 27].

1.2 Symptoms of stroke

The symptoms of stroke depend on the type of stroke and the area of the brain affected. Ischemic strokes usually affect regional areas of the brain perfused by the blocked artery. Hemorrhage strokes can affect local areas, but can cause more global symptoms due to bleeding and increased intracranial pressure. If the area of the brain affected includes one of the three pathways of the prominent central nervous system (CNS) (the spinothalamic tract, corticospinal tract, or dorsal column), symptoms may include muscle weakness or numbness, reduction of pain or temperature sensation, or reduction in sensory or vibratory sensation. In addition to the above CNS pathways, the brainstem also consists of the twelve cranial nerves. A stroke affecting the brainstem can therefore produce symptoms relating to deficits in these cranial nerves. These may include altered smell, altered taste, altered hearing or vision, drooping of eyelids (ptosis), weakness of the ocular muscles, decreased reflexes (gag, swallow, pupil reactivity to light), decreased sensation and muscle weakness of the face, balance problems and nystagmus, altered breathing and heart rate, and weakness in the tongue. If the cerebral cortex is involved, the CNS pathway can again be affected, but also can produce the following symptoms: aphasia (inability to speak or understand language), apraxia (altered voluntary movement), disorganized thinking, confusion, hypersexual gestures, altered vision, and memory deficits. If the cerebellum is involved, the patient may have trouble walking, or experience altered movement, lack of coordination and dizziness. Loss of
consciousness, headache, and vomiting is more associated with hemorrhage stroke because of the increased intracranial pressure from the leaking blood compressing on the brain. If the symptoms are maximal at the onset, the cause is more likely to be a subarachnoid hemorrhage or an embolic stroke [27].

If the symptoms of stroke are resolved within 24 hours, the diagnosis is transient ischemic attack (TIA). TIA is a sudden impairment of blood flow to the brain that typically lasts for less a minute. The blockage is usually caused by a tiny blood clot or a piece of debris from fatty deposits in an artery. TIA repairs itself when the obstruction dissolves, causing no permanent tissue damage and no lasting effects. One of the more common scenarios is for a person to notice that their hand or foot has grown numb, feels prickly or is burning. The sensation then quickly spreads to the whole arm or leg or to an entire side of the body before disappearing. Sometimes, the person cannot see clearly, cannot speak or may feel dizzy or confused. This syndrome may be a warning sign of stroke and a large proportion of such patients develop a stroke in the future. Recent data indicate that there is about ten to fifteen percent chance of suffering a stroke in the year following a TIA, with half of that risk manifest in the first month. If the symptoms of a transient ischemic attack last for more than a day, the event is classified as a full-fledged ischemic stroke [1, 16 and 27].

1.3 Risk factors of stroke

A risk factor of stroke is any condition that increases a person’s chances of getting a stroke or makes someone a candidate for stroke. Some risk factors are physiological, having to do with the makeup of our bodies. Unfortunately, physiological risk factors are
not preventable because they are a part of who we are; for instance, we cannot change our
genes, gender, or age. In contrast, other risk factors have more to do with environment.
These risk factors are caused by one’s lifestyle and surroundings, which are preventable
by positive lifestyle choices. The most common risk factors for stroke include aging,
hypertension, diabetes, obesity, heart disease, smoking, gender, alcohol consumption, and
cocaine use [27].

The risk of suffering a stroke increases dramatically as people get older. In fact, the
chances of having a stroke double for every decade that a person lives over the age of 55.
In fact, about 90% of all strokes happen in people over the age of 55 [26]. One theory
posits that advancing age causes a breakdown in collagen, a protein that forms much of
the connective tissue in the body. Collagen breakdown in the skin cells explains why skin
wrinkles and sags as we get older. Collagen is also what gives blood vessels their
strength; therefore, when collagen begins to break down, the blood-vessel wall loses its
holding power and can easily be ruptured or blocked. Moreover, the risk of hypertension
increases with age. When blood pressure increases in older people, it tends to weaken the
blood vessels, increasing their vulnerability to stroke. Therefore, the most important
uncontrollable risk factor for stroke is age [27].

Untreated hypertension is an open invitation for stroke, greatly increasing the risk of
both ischemic and hemorrhagic stroke. Hypertension seems to accelerate atherosclerosis,
the thickening and narrowing of the arteries by fatty plaque. Healthy arteries are lined
with a protective layer of endothelial cells, underneath which lies collagen. In the arteries
of a person with hypertension, high blood pressure damages the endothelial layer,
exposing collagen to the blood stream. When this happens, the collagen attract platelets
(blood cells responsible for clotting) causing them to stick together and to secrete a chemical that induces more platelets, along with fibrous tissue, bloodstream debris and cholesterol from the blood, to clump together as well. This chain reaction eventually results in the accumulation of fatty material inside the arterial wall, where it thickens and narrows the artery. An artery in this form can easily be ruptured or blocked [9 and 27].

Diabetes is also a major risk factor for stroke. Diabetics have high incidence of diseases such as atherosclerosis, which lines arteries with fatty deposits. About 30% of persons with ischemic stroke have diabetes [27].

Obesity is defined as 20% or more over a person’s ideal weight. Obesity itself is not a risk factor of stroke, but overweight people in general frequently have hypertension or diabetes or both, which are two risk factors for stroke [9 and 27].

Mitrval valve prolapse, in which one of the heart’s valves fails to make a tight seal with the surrounding tissue, appears to increase the risk of ischemic stroke, because the condition gives rise to blood clots. Coronary artery disease, the thickening and narrowing of the arteries that serve the heart, is also tied to ischemic stroke because of the atherosclerosis that clogs the arteries. In general, persons with impaired hearts have twice the risk of stroke than persons with normal hearts [9 and 27].

Cigarette smoking has been called one of the greatest public health menaces in the world [27]. It can lead to diseases such as bronchitis, emphysema, lung cancer, heart disease, and stroke. Smoking has been found to be a significant risk factor for stroke, especially for women. People who smoke two or more packs of cigarette a day have twice the risk of stroke than those who smoke half a pack daily [27].
Alcohol, particularly in large amounts, has never been kind to the human body. In addition to cirrhosis, nerve damage in adults, and permanent birth defects in infants actually in mothers who abuse alcohol, excessive alcohol consumption seems to heighten the risk for stroke. This is because alcohol contributes to either hypertension or a tendency to bleed [9 and 27].

Even though there are more risk factors of stroke in men and the prevalence of stroke is higher in men than women, the stroke death rate is relatively higher in women than men every year. This is simply because women live longer, so by old age, there are more women at risk of stroke [9 and 27].

The use of cocaine has a devastating effect on the heart. Once cocaine is snorted, injected or smoked, it causes the heartbeat to accelerate. Blood pressure also shoots up and the blood vessels squeeze tight. A sudden burst in blood pressure can cause a stroke by hemorrhage. If the arteries being squeezed are located in the brain, ischemic stroke might result when the brain receives insufficient oxygen [9 and 27].
CHAPTER 2

EXISTING RESULTS

Many models have been developed concerning the occurrence of stroke in all its manifestations. Since the literature on the occurrence of stroke is so vast, we will focus on the following topics: "a mathematical model for the formation of cerebral aneurysms", "mathematical modeling of paired arterial stenoses", "blood flow through an axisymmetric stenosis", and "modeling arterial stenosis and its applications to blood diseases".

2.1 A mathematical model for the formation of cerebral aneurysms

In reference [6], a mathematical model for the formation of cerebral aneurysms was considered by Brown. Aneurysms are a main cause of subarachnoid hemorrhage stroke, in particular, ruptured aneurysms are a significant cause of stroke mortality and morbidity. Cerebral aneurysms are hypothesized to be acquired lesions resulting from a loss of static equilibrium in the apical region of the bifurcation (a place where something divides into two, causing the opening angle of the bifurcation of the artery to change during the cardiac cycle). Repeated dynamic cycling may disrupt the wall elements. Brown proposed the bifurcation geometry depicted below.
Theoretical model

Figure 2.1 above shows a simplified geometry of a bifurcation. Flow is from the parent artery of radius $r_0$ into daughter branches of radii $r_1$ and $r_2$. The larger branch has radius $r_1$ and makes an angle $\theta_1$ with the direction of the parent artery, while the smaller branch with radius $r_2$ makes an angle $\theta_2$ with the direction of the parent artery. Figure 2.1(b) indicates the longitudinal tensile force $T_L$ in each branch and in the parent artery.

Figure 2.2 shows the free-body diagram for circumferential and longitudinal forces acting on an artery of radius $R$, length $L$, and thickness $t$ distended by transmural pressure $P$. $T_L$ and $T_c$ represent longitudinal and circumferential tensile forces per unit length, respectively. From Figure 2.1(b), the apex A is the point of zero tension.
Mathematical model for the formation of cerebral aneurysms. Brown proposed the following mathematical models composed of the following equations listed below:

\[ T_1 \cos \theta_1 + T_2 \cos \theta_2 = T_0, \quad T_1 \sin \theta_1 = T_2 \sin \theta_2, \quad T_c = Pr, \quad T_L = \frac{(Pr)}{2}; \quad (1) \]

\[ P_1 r_1 \cos \theta_1 + P_2 r_2 \cos \theta_2 = P_0 r_0, \quad P_1 r_1 \sin \theta_1 = P_2 r_2 \sin \theta_2; \quad (2) \]

\[ \cos \theta_1 + (P_2/P_1) \alpha \cos \theta_2 = (P_0/P_1) \left( \frac{(1 + \alpha^2)}{\beta} \right)^{1/2}, \quad \sin \theta_1 = (P_2/P_1) \alpha \sin \theta_2; \quad (3) \]

\[ \left(1 - \left[(P_2/P_1) \alpha \sin \theta_2 \right]^2\right)^{1/2} + (P_2/P_1) \alpha \cos \theta_2 = (P_0/P_1) \left( \frac{(1 + \alpha^2)}{\beta} \right)^{1/2}; \quad (4) \]

\[ \left[1 - \alpha^2 \sin \theta_2 \right]^{1/2} + \alpha \cos \theta_2 = \left( \frac{(1 + \alpha^2)}{\beta} \right)^{1/2}, \]

where the parameter \( \alpha \) is defined as \( \alpha = \frac{r_2}{r_1} \) \((0 \leq \alpha \leq 1)\) and the area ratio \( \beta = \frac{(r_1^2 + r_2^2)}{r_0^2} \).

Equations 1 to 4 of this work are analogous to equations 1 to 11, and 15 of [6]. Here, \( P_1 \) and \( P_2 \) are the pressure in the branches with radii \( r_1 \) and \( r_2 \) respectively and \( P_0 \) is the pressure in the parent artery. In addition, theoretical predictions of the angles \( \theta_1 \) and \( \theta_2 \), and the opening angle \((\theta_1 + \theta_2)\) are plotted as a function of the parameter \( \alpha \) in Figures 2.3 to 2.8 below. These figures illustrate graphical solutions of the problem. Moreover, the model presented here focuses on different cases depending on the value of the pressures.
Case 1: \( P_1 = P_2 = P_0 \) (equal transmural pressure in parent and daughter arteries)

![Figure 2.3](image)

Figure 2.3: Angle larger branch, \( \theta_1 \) as a function of \( \alpha \) at different area ratios \( \beta \).

From Figure 2.3, the larger branch makes a smaller angle with the direction of the parent artery. As the area ratio \( \beta \) decreases, the angle of the branch increases.

![Figure 2.4](image)

Figure 2.4: Angle of smaller branch, \( \theta_2 \) as a function of \( \alpha \), at different area ratios \( \beta \).

Figure 2.4 shows that, the smaller branch makes a larger angle with the direction of the parent artery. As the area ratio \( \beta \) decreases, the angle of the branch decreases.
From Figure 2.5, as the area ratio $\beta$ decreases, the opening angle decreases. When $\beta = 1$, the opening angle is about $90^\circ$. In addition, changes in the opening angles are much greater for a smaller $\alpha$ if the bifurcation is unstable.

Consequently, the opening angle of a bifurcation is stable only if the parameters $\alpha$ and $\beta$ are constant during cardiac cycle. This condition requires that the incremental strain (percentage change in radius) in the daughter and parent arteries must be the same for any increment in blood pressure.

Case 2: $P_1 = P_2 \neq P_0, \beta = 1.0$: Area ratio $\beta = 1.0$ and the transmural pressures in the branches do not equal that in the parent artery ($P_1 = P_2 = P; P_0 \neq P$).

In this case, the effect on the branching angles of a 10% difference in pressure between the parent and daughter arteries was discussed.
Figure 2.6: Graph of the angle $\theta_1$ as a function of $\alpha$, solid line, $P = P_0$; dash line, $\left( \frac{P_0}{P} \right) = 0.9$; dotted line $\left( \frac{P_0}{P} \right) = 1.1$.

Figure 2.7 Graph of the angle $\theta_2$ as a function of $\alpha$, $P = P_0$; dash line, $\left( \frac{P_0}{P} \right) = 0.9$; dotted line $\left( \frac{P_0}{P} \right) = 1.1$. 

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Figure 2.8: Graph of the opening angle $(\theta_1 + \theta_2)$ as a function of $\alpha$, $P = P_0$; dash line, $(\frac{P_0}{P}) = 0.9$; dotted line $(\frac{P_0}{P}) = 1.1$.

If the transmural pressure in the daughter branches exceeds that in the parent artery, the opening angle is greater than when the pressures are equal, and conversely for lower pressures in the daughter branches. The same result is true for the individual angles of the daughter branches with the direction of the parent artery.

The model displayed above is strictly applicable to surface membranes, whereas arteries may have thick walls relative to their radii. Because it is unlikely that arteries are under compressive stress, Laplace’s Law is considered a reasonable first approximation even for a thick vessel.

The mathematical model predicts the geometry of bifurcations caused by the interaction of hemodynamic forces and the elastic properties of the wall. Experimental measurements in extra cranial arteries show that the angle the larger branch makes with the direction of the parent artery is less than that of the smaller branch, in agreement with
the model. Bifurcations with smaller area ratios $\beta$ are predicted to have smaller opening angles. When the transmural pressure in the daughter arteries exceeds that in the parent artery, the opening angle is greater than when the transmural pressures are equal, and conversely for lower transmural pressures in the daughter arteries. Different transmural pressures in the parent and daughter vessels imply reflection in the bifurcation.

Finally, Brown proposed that repeated cycling of the opening due to an unstable equilibrium may disrupt the wall elements in a manner analogous to a wire breaking with repeated bending, resulting in the formation of an aneurysm.

2.2 Mathematical modeling of paired stenoses

In [12], mathematical modeling of paired stenoses was considered by Johnson and Kilpatrick. To determine the effect of paired stenoses and the influence of the separation between stenoses, a mathematical model of blood flow in the coronary arteries was developed. The model, which assumes that the artery and stenoses are axis-symmetric and that blood is a Newtonian fluid, is described by the two-dimensional cylindrical Navier-Stokes and continuity equations. These equations, being highly non-linear, are solved using the finite element method with FIDAP, a computational fluid dynamics software package.

Mathematical modeling of arterial blood flow provides a powerful yet simple tool to study the effect of any of the varied stenoses morphologies which may occur in the patients with coronary artery disease.

Johnson and Kilpatrick set out to model the combined effect of two stenoses in series. The modeling was performed in a two-dimensional axi-symmetric fashion and solved
using the finite element method. Blood flow through an axi-symmetric stenosis was modeled in two dimensions using a cylindrical coordinate system.

The governing continuity and Navier-Stokes equations for this system are:

\[
\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z} = 0, \tag{5}
\]

\[
v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} = -\frac{1}{\rho} \frac{\partial P}{\partial r} + \nu \left[ \frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r} \frac{\partial v_r}{\partial r} - \frac{v_z}{r^2} + \frac{\partial^2 v_r}{\partial z^2} \right], \tag{6}
\]

\[
v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} = -\frac{1}{\rho} \frac{\partial P}{\partial z} + \nu \left[ \frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r} \frac{\partial v_z}{\partial r} + \frac{\partial^2 v_z}{\partial z^2} \right]. \tag{7}
\]

Here, equations 5-7 of this work are the analogue to equations 1-3 of [12].

In addition, \( r \) and \( z \) are the physical coordinates with the \( z \)-axis located along the symmetry axis of the artery. No secondary or swirling flows have been allowed; i.e. the equations are independent of \( \theta \) so that the total velocity is defined by the radial and axial components \( v_r \) and \( v_z \), respectively. Finally, the quantities \( P, \rho \) and \( \nu \) are the pressure, density and kinematic viscosity, respectively. These equations assume that blood is an incompressible, Newtonian fluid, a reasonable assumption at the flow rates encountered in living arteries whose internal diameter is greater than 500\( \mu m \).
The boundary of the stenoses is represented by the following equations:

\[
R(z) = \begin{cases} 
  R_0, & z < -Z_0 \\
  R_0 - \frac{\delta_1}{2} \left( 1 + \cos \left( \frac{\pi z}{Z_1} \right) \right), & -Z_1 < z < Z_1 \\
  R_0 - \frac{\delta_2}{2} \left( 1 + \cos \left( \pi \left( z - \frac{Z_0 + Z_1 + Z_2}{2} \right) \right) \right), & Z_0 + Z_1 < z < Z_0 + Z_1 + 2Z_2 \\
  R_0, & z > Z_0 + Z_1 + 2Z_2
\end{cases}
\]  

where \( \delta_1 \) and \( \delta_2 \) are the lengths of the region between the two stenoses. When \( \delta_1 \) and \( \delta_2 \) are both zero, this form reduces to a single stenosis. In addition, the origin coincides with the origin of the cylindrical coordinate system. The severity of each stenosis is varied by changing the values of \( \delta_1 \) and \( \delta_2 \), whereas the separation changes with \( Z_0 \).
peak to peak separation of the stenoses is then given by \( Z_0 + Z_1 + Z_2 \). The velocity boundary conditions on the arterial wall are the usual no slip conditions
\[
v_r = v_z = 0 \quad \text{at} \quad r = R(z).
\]

Flow upstream from the stenoses was assumed to have a velocity profile corresponding to Hagan-Poiseulle flow through a long circular tube of constant cross-section
\[
v_z = U \left( 1 - \frac{r^2}{R_0^2} \right); \quad v_r = 0,
\]
where \( U \) is the maximum velocity of the parabolic profile.

The governing equations, being highly nonlinear, were solved numerically using the FIDAP computational fluid dynamics package, utilizing the finite element method. The above model was solved for many combinations of \( \delta_1 \) and \( \delta_2 \). In all cases, the stenoses were of the same length and equal to twice the arterial radius (i.e., \( Z_1 = Z_2 = R_0 \)). The values of \( \delta_1 \) and \( \delta_2 \) varied from 0 to 0.8 (0 to 96% areal occlusion). For each combination of \( \delta_1 \) and \( \delta_2 \), the model was solved for \( Z_0 = R_0 \) and \( Z_0 = 3R_0 \). Also, for identical paired stenoses with \( \delta_1 (= \delta_2) \) ranging from 0.3 to 0.8, the model was again solved for peak to peak separation varying from 0 to \( 5R_0 \). In all cases, the Reynolds number \( R_e = \frac{UR_0\rho}{\mu} \) was set at 1.00. [From Lin and Segel, the Reynolds number is an estimate of the ratio between the convective acceleration terms and viscous terms] [15]. This is typical for a 2mm radius artery containing blood flowing at about 30cm/s.
2.3 Blood flow through an axi-symmetric stenosis

In reference [22], blood flow through an axi-symmetric stenosis with a steady axi-symmetric flow in a constricted rigid tube was studied by Pontrelli. The equation of motion is written in vorticity-streamfunction formulation and is solved numerically by a finite difference scheme. He then computed the flow pattern with the distributions of pressure and shear stress at the wall. A mathematical model for blood.

Pontrelli proposed the stress-strain rate relationship as

\[ T = -pI + \mu \left( \gamma \right) A_i \]  \hspace{1cm} (10)

where \( A_i = (\nabla v) + (\nabla v)^T \), is the rate of deformation tensor, and

\[ \gamma = \left[ \frac{1}{2} \text{tr} \left( A_i^2 \right) \right]^{\frac{1}{2}} \]  \hspace{1cm} (11)

where \( p \) is the pressure, \( T \) is the Cauchy tensor stress, \( v \) is the velocity, \( \gamma \) is the shear stress and \( \mu \) is the viscosity of the fluid.

Pontrelli also assumed the blood to be isotropic, homogeneous and incompressible along a continuum, having constant density \( \rho \), and the vessel walls were considered rigid and incompressible. The equation of motion is

\[ \rho \left( \frac{\partial v}{\partial t} + v \cdot \nabla v \right) = \text{div} T \]  \hspace{1cm} (12)
where $v$ is the velocity vector and the body force is assumed to be negligible.

Pontrelli later wrote the equation of motion in scalar form with $v$ having components $u$ and $w$, and then used numerical methods to compute the flow velocities, pressure losses and the wall shear stress at the stenotic region of the artery. He concluded that mathematical models and numerical simulations offer an alternative and non-invasive tool for obtaining detailed and realistic descriptions of complex arterial flows. He found from his simulations and numerical analysis that the non-Newtonian character of the blood modifies the flow pattern, even beyond the contracted region, reduces the pressure drop and the shear stress at the wall across the stenosis.

2.4 Modeling arterial stenosis and its applications to blood diseases

In reference [24], modeling arterial stenosis and its applications to blood diseases was considered by Pralhad and Schultz. They studied blood flow in a stenosed tube, and represented the blood flow by a couple stress fluid. They then computed flow parameters such as velocity, resistance to flow, and shear stress distribution.

Analysis

Pralhad and Schultz assumed the blood flow to be a homogenous and incompressible couple stress fluid of constant velocity $\mu$ and density $\rho$. The constitutive equations and equations of motion for the couple stress fluid were expressed in tensors of rank 2 as

\begin{align}
T_{ij} &= \rho \frac{dV_i}{dt}; & e_{ij}T_{jk} + M_{ij} &= 0 \\
I_{ij} &= -p\delta_{ij} + 2\mu d_{ij}; & \mu_{ij} &= 4\eta w_{ij} + 4\eta' w_{ij}
\end{align}

(13) (14)
Equations 13 to 14 of this work are the analogue of equations 1 to 4 of [23],

where \( V \) is the velocity vector, \( I_{ij} \) and \( T_{ij}^{A} \) are the symmetric and antisymmetric part of the stress tensor \( T_{ij} \), respectively. \( M_{ij} \) is the couple stress tensor, \( \mu_{ij} \) is the derivative part of \( M_{ij} \), \( \omega_{ij} \) is the vorticity vector, \( d_{ij} \) is the symmetric part of the velocity gradient, \( \eta \) and \( \eta' \) are the constants associated with the couple stress, and \( p \) is pressure. Pralhad and Schultz then wrote the equations of motions together with the continuity equation as

\[
\rho \left( v \frac{\partial u}{\partial r} + u \frac{\partial u}{\partial z} \right) + \frac{\partial p}{\partial z} = \nabla^2 (\mu u - \eta \nabla u^2) \tag{15}
\]

\[
\rho \left( v \frac{\partial v}{\partial r} + u \frac{\partial v}{\partial z} \right) + \frac{\partial p}{\partial r} = \nabla^2 (\mu v - \eta \nabla v^2) \tag{16}
\]

\[
\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} + \frac{v}{r} = 0 \tag{17}
\]

where \( \nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) \), [24, equation numbers 6-9]

where \( v \) and \( u \) are the radial and axial velocity components in the \( r \) and \( z \) directions, respectively. Pralhad and Schultz approximated the equation of motion as

\[
u \frac{\partial u}{\partial z} + v \frac{\partial u}{\partial r} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \frac{1}{\rho} \nabla^2 \Psi ; \quad \frac{\partial p}{\partial r} = 0 \tag{18}
\]

with \( \Psi = \mu u - \eta \nabla u^2 \).

By integrating and applying the boundary conditions, they were able to obtain the components of the velocity. They then applied numerical methods to compute the resistance to flow and the shear stress distribution.
CHAPTER 3

APPROXIMATE SOLUTION OF STENOSES

In reference [17] “A non-Newtonian fluid model for blood flow through arteries under stenotic conditions”, a non-exact solution of stenoses was proposed by Misra, Patra and Misra whose research was concerned with the dynamics of blood flow through an arterial segment having a mild stenosis. They treated the artery as a thin-walled, initially stressed orthotropic non-linear viscoelastic cylindrical tube filled with a non-Newtonian fluid representing the blood. They derived the flow rate or the speed of blood flow in different directions along with pressure with which blood flows in the stenotic region.

The paper investigated the propagation of forced waves, which are harmonic in $z$ and $t$ and considered the velocity components and the pressure of the blood flow in the form

$$u = \bar{u}(r)\exp\left[i\omega\left(t - \frac{z}{c}\right)\right]; \quad v = \bar{v}(r)\exp\left[i\omega\left(t - \frac{z}{c}\right)\right];$$

$$p = \bar{p}(r)\exp\left[i\omega\left(t - \frac{z}{c}\right)\right],$$

[17, equations number 2].
The quantities $u$ and $v$ denote the components of the velocity of blood in the increasing directions of $r$ and $z$, respectively. The pressure is denoted by $p$, $\omega$ is the circular frequency and $c$ is the wave propagation velocity.

On the basis of long wave approximation, they provided the equations of motion of the flow as

$$
\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial r} + 2f^* \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2} \right),
$$

(20)

$$
\rho \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial z} + f^* \left( \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} \right),
$$

(21)

together with equation of continuity

$$
\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial v}{\partial z} = 0
$$

(22)

where $f^*$ is the complex viscosity of the fluid [17, equation numbers 3-5].

Wall motion and boundary conditions

On the basis of experimental findings, they assumed the arteries to be in a prestressed state and the wall tissues to be orthotropic, non-linear and viscoelastic. Taking into account the inertial forces, the surface forces and the forces of constraint, which are the reactions of the surrounding tissues, the authors derived the equations of motion of the arterial wall by employing the principle of superimposition of a small additional deformation of the arterial state of finite deformation of the arterial wall. These equations are
In equations 23 and 24, $\xi$ and $\tau$ are the physical components of the superimposed displacement in the radial and axial direction, $R_0$ and $h_0$ denote the mean surface radius and the wall thickness of the segment of the blood vessel, $\lambda_1$ and $\lambda_2$ are the axial and circumferential stretch ratios, and $\rho_0$ is the volume density of the wall tissue. The specific expressions for $\Psi_0, \phi, \beta_{\theta\theta}, \beta_{\theta\phi},$ and $\beta_{\phi\phi}$ are given in Appendix B, and $a_0, b_0$ represent the constant finite strains.

To solve for the velocities in the $r$ and $z$ directions (that is, $u$ and $v$) and the pressure of the blood, they imposed boundary conditions and applied perturbation technique. In addition, $X$ and $Y$, the radial and longitudinal components of the external forces acting on the wall tissues, respectively are expressed as

$$X = \left( p - 2f^* \frac{\partial u}{\partial r} \right)_{r=a} \tag{25}$$

$$Y = -f^* \frac{\partial v}{\partial r} \bigg|_{r=a} \tag{26}$$
For boundary conditions, they used conditions from fluid theory that require the velocity of blood at the inner surface of the arterial segment be taken to be equal to the velocity of the inner surface of the arterial wall. These may be stated mathematically as

\[ u(r, z, t) \big|_{r=R(z)} = \frac{\partial \xi}{\partial t} \]  

(27)

and

\[ v(r, z, t) \big|_{r=R(z)} = \frac{\partial \eta}{\partial t} . \]  

(28)

In order to solve for \( u \), \( v \), and \( p \), the authors expanded the dependent variables involved in the equations 27 and 28 as power series in terms of a small parameter \( \delta \)

\( \left( \frac{\xi}{a} \right) \) that reduces all dependent variables of the problem to their known values. They expanded the dependent variables as follow as:

\[ u = u_1 \delta + u_2 \delta^2 + \ldots; v = v_1 \delta + v_2 \delta^2 + \ldots; p = p_0 + p_1 \delta + p_2 \delta^2 + \ldots \]  

(29)

\[ \xi = \xi_1 \delta + \xi_2 \delta^2 + \ldots; \eta = \eta_1 \delta + \eta_2 \delta^2 + \ldots; R(z) = a + R_1(z) \delta + R_2(z) \delta^2 + \ldots \]  

(30)

where \( p_0 \) and \( a \) are constants which define the initial state of the system.

They then used power series expansion on the dependent variables together with the boundary conditions given in equation 23 and 24 to come up with

\[ u_1 = B \left[ \beta_0 \left( x + iy \right) J_1 \left( \alpha_0 \frac{r}{a} \right) \right] + \frac{1}{\rho c} \left[ J_1 \left( \beta_0 \frac{r}{a} \right) e^{i \left( \delta + \delta_2 \right)} \right] \]  

(31)
\begin{align}
  v_i &= B \left[ \frac{x + iy}{\delta_1^2 + \delta_2^2} J_0 \left( \frac{3}{2} \frac{r}{a} \right) + \frac{1}{\rho c} J_0 \left( \frac{r}{a} \right) \right] e^{i\theta \left( \frac{1}{c} \right)} \\
  p_1 &= BJ_0 \left( \frac{r}{a} \right) e^{i\theta \left( \frac{1}{c} \right)}
\end{align}

(32) \quad (33)

where \( J_0 \) and \( J_1 \) are the Bessel functions of zeroth and first order, respectively. These are first order correction terms, and the approximations used in this reference generated some results numerically.
CHAPTER 4

THE PROPOSED GENERAL FORM OF THE SOLUTION OF THE PROBLEM

From the previous references mentioned above, most strokes (about 88%) are caused by total blockage or stenoses of the artery. How the stenoses of the artery cause stroke is our area of concern. One way to study stroke is to design a mathematical model of the disease in order to better understand the various phenomena that compose it and their relative importance. In this thesis, we propose a mathematical model for blood flow to the brain in the presence of stenoses.

Blood is a highly specialized circulating tissue consisting of several types of cells suspended in a fluid medium known as plasma. The average adult has about 5 liters of blood coursing through their vessels, delivering essential elements and removing harmful wastes. Normally, 7-8% of human weight is from blood. The normal pH of human arterial blood is approximately 7.40, with normal range is 7.35-7.45, a weak alkaline solution. Human blood density is around 1060kg/m³. The constituents of the blood are red blood cells, white blood cells, platelets, and plasma.

The red blood cells (erythrocytes) are relatively large microscopic cells, lacking nuclei and organelles. They normally make up 40-50% of the total blood volume, transporting oxygen from the lungs to all of the living tissues of the body and carrying away carbon dioxide. The red color of the blood is primarily due to oxygenated red cells.
The white blood cells (leukocytes) exist in variable numbers and types but make up a very small part of the blood’s volume, normally only about 1%. Some white blood cells (lymphocytes) are the first responders of the immune system. They fight diseases, and seek out, identify, and bind to alien proteins on bacteria, viruses, and fungi so that they can be removed.

Platelets (thrombocytes) are cells that clot blood at the site of wounds. They change fibrinogen into fibrin, creating a mesh onto which red blood cells collect and clot. This clot stops more blood from leaving the body and also helps to prevent bacteria from entering the body.

Plasma is a relatively clear liquid protein and salt solution, which carries the red blood cells, the white blood cells, and the platelets. Normally, 55% of blood volume is made up of plasma. About 92% of blood plasma is water, the rest consisting of blood protein and trace amounts of other materials.

With the exception of pulmonary and umbilical arteries and their corresponding veins, arteries carry oxygenated blood away from the heart and deliver it to the body via arterioles and capillaries, where the oxygen is consumed; afterwards, venules and veins carry deoxygenated blood back to the heart. When cholesterol, fatty substances and other materials in the blood are deposited on the walls of the artery, it forms a plaque, which narrows the arteries and impedes the flow of blood through the artery. This partial or total occlusion of the artery (stenosis) leads to either ischemic or hemorrhagic stroke.

This paper studies the flow of blood in an arterial segment having stenosis. The artery is modeled as an initially stressed, orthotropic, elastic cylindrical tube filled with a viscous incompressible fluid, blood, which is assumed to be non-Newtonian. The analysis
is restricted to propagation of small amplitude harmonic waves whose length is large compared to the radius of the arterial segment.

For the equations of motion of the arterial wall, consideration is made of a pair of appropriate equations derived by using suitable constitutive relations and the principle of superimposition of a state known as finite deformation.

In order to have a fuller understanding of the development of stenosis, an accurate knowledge of the mechanical properties of the vascular wall together with that of the flow characteristics of blood are indispensable. Most of these studies are based on the assumption that blood behaves like a Newtonian fluid. However, experimental observations reveal that blood does not behave as a Newtonian fluid under certain conditions [17 and 18]. It is generally accepted that blood, being a suspension of red cells in plasma, behaves like a non-Newtonian fluid at a low shear rate, which is usually low in the downstream side of the stenosis. Thus, the analysis of the flow pattern near the stenosis should include the non-Newtonian property of blood.

4.1 Formulation of the problem

Consider an axially symmetrical laminar and unsteady flow of blood through a circulatory cylindrical arterial segment with a mild stenosis developed in an axially symmetric manner.
The geometry of the stenosis is described mathematically as

\[
\frac{R(z)}{a} = \begin{cases} 
1 - \frac{\varepsilon}{2a} \left[ 1 + \cos \left( \frac{2\pi}{L_0} \left( z - d - \frac{L_0}{2} \right) \right) \right], & d \leq z \leq L_0 + d \\
1, & \text{otherwise}
\end{cases}
\]  

(34)

where \( L_0 \) and \( \varepsilon \) are, respectively, the length and the maximum height of the stenosis, \( d \) indicates its location and \( R(z) \) represents the radius of the arterial segment under consideration at an axial distance \( z \) from one of its ends. Here, \( \varepsilon \) is taken to be quite small compared to the radius \( a \) of the normal artery (outside the stenotic region) [8, 16, 17, and 18].

Considering the propagation of forced waves which are harmonic in \( z \) and \( t \), the velocity components and pressure are of the form
\[ u = \bar{u}(r) \exp \left[ i\omega \left( t - \frac{z}{c} \right) \right]; \quad v = \bar{v}(r) \exp \left[ i\omega \left( t - \frac{z}{c} \right) \right]; \]
\[ p = \bar{p}(r) \exp \left[ i\omega \left( t - \frac{z}{c} \right) \right], \quad (35) \]

where \( u \) and \( v \) denotes the components of the velocity of blood in the increasing directions of \( r \) and \( z \), respectively, \( p \) is the pressure, \( \omega \) is the circular frequency and \( c \) is the wave propagation velocity [17].

Owing to the observation that blood viscoelasticity and the non-Newtonian behavior of whole blood are quite prominent in large blood vessels, blood will be treated here as a viscoelastic fluid and the following equations derived on the basis of long-wave approximation will be used to describe the motion of blood.

4.2 Derivation of motion and continuity equations for the blood flow

The momentum equation of fluid flow is given by [13] as

\[ \rho \frac{Du}{Dt} = p\mathbf{f} - \nabla p + \mu \nabla^2 u, \]
where $\rho$ is the density of the fluid, $p$ is the pressure, $\mu$ is the viscosity of the fluid, $f$ is the body force of the fluid and $\frac{D}{Dt}$ is the material derivative. In cylindrical coordinates, the above equation can be written as

$$
\rho \left[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{u_\theta}{r} \frac{\partial u}{\partial \theta} + v \frac{\partial u}{\partial z} - \frac{u^2}{r} \right] = \rho f - \left[ \frac{\partial p}{\partial r} + \frac{1}{r} \frac{\partial p}{\partial \theta} + \frac{\partial p}{\partial z} \right] 
+ \mu \left[ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} - \frac{u}{r^2} \frac{\partial u}{\partial \theta} \right].
$$

(36)

Since the velocity of the flow is independent of $\theta$, equation (36) becomes

$$
\rho \left[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + v \frac{\partial u}{\partial z} \right] = \rho f - \left[ \frac{\partial p}{\partial r} + \frac{1}{r} \frac{\partial p}{\partial \theta} + \frac{\partial p}{\partial z} \right] + \mu \left[ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} - \frac{u}{r^2} \right].
$$

(37)

Taking the velocity of flow in the direction of $r$, we have

$$
\rho \left[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} \right] = \rho f - \left[ \frac{\partial p}{\partial r} + \frac{1}{r} \frac{\partial p}{\partial \theta} + \frac{\partial p}{\partial z} \right] + \mu \left[ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2} \right].
$$

(38)

If the pressure is constant in the direction of $z$, neglecting body forces, on the basis of long-wave approximation [17], equation (38) becomes

$$
\rho \frac{\partial u}{\partial t} = - \frac{\partial p}{\partial r} + \mu \left[ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2} \right].
$$

(39)
Letting $2f^* = \mu$, the motion equation in the direction of the velocity $u$ is
\[ \rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial r} + 2f^* \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2} \right). \] (40)

Similarly, the momentum equation of fluid flow is given by
\[ \rho \frac{Dv}{Dt} = \rho f - \nabla p + \mu \nabla^2 v. \]

In the cylindrical coordinate system and in the direction of the velocity $v$, the above equation can be written as
\[ \rho \left[ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial r} + v \frac{\partial v}{\partial z} + u_r \frac{\partial v}{\partial \theta} \right] = \rho f - \frac{\partial p}{\partial r} + \frac{1}{r} \frac{\partial p}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2 p}{\partial \theta^2} + \mu \left[ \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} + \frac{1}{\partial z^2} \right]. \] (41)

Since the velocity of the flow is independent of $\theta$, equation (41) becomes
\[ \rho \left[ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial r} + v \frac{\partial v}{\partial z} \right] = \rho f - \frac{\partial p}{\partial r} + \frac{1}{r} \frac{\partial p}{\partial z} + \frac{1}{r^2} \frac{\partial^2 p}{\partial \theta^2} + \mu \left[ \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{1}{\partial z^2} \right]. \] (42)

For an axially symmetric solution, that is, taking the velocity of flow in the direction of $r$, equation (42) becomes
$\rho \left[ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial r} \right] = \rho f - \left[ \frac{\partial p}{\partial r} + \frac{\partial p}{\partial \theta} \right] + \mu \left[ \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} \right].$ \hspace{1cm} (43)

If the pressure is constant in the direction of $r$, neglecting body forces, on the basis of long-wave approximation \[16\], equation (43) becomes

$$\rho \frac{\partial v}{\partial t} = - \frac{\partial p}{\partial z} + \mu \left( \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} \right).$$ \hspace{1cm} (44)

Letting $f^* = \mu$, the motion equation in the direction of the velocity $v$ is

$$\rho \frac{\partial v}{\partial t} = - \frac{\partial p}{\partial z} + f^* \left( \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} \right).$$ \hspace{1cm} (45)

The continuity equation for the motion is given by \[12\] as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho V) = 0,$$

where $\rho = \text{density}$, $V = \text{fluid velocity}$ and $t = \text{time}$. Since the blood is an incompressible fluid, $\nabla \cdot V = 0$ \[13\]. In cylindrical coordinates,

$$\nabla \cdot V = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial (u h_1 h_2)}{\partial q_1} + \frac{\partial (u h_1 h_2)}{\partial q_2} + \frac{\partial (v h_1 h_2)}{\partial q_3} \right].$$
where \( h_r = 1, h_\theta = r, h_z = 1 \) and \( q_1 = r, q_2 = \theta \) and \( q_3 = z \).

Hence we obtain

\[
\frac{1}{r} \left[ \frac{\partial (ur)}{\partial r} + \frac{\partial (u_\theta)}{\partial \theta} + \frac{\partial (vr)}{\partial z} \right] = 0,
\]

which leads to

\[
\frac{1}{r} \left[ \frac{r \partial u}{\partial r} + u + \frac{\partial (u_\theta)}{\partial \theta} + \frac{r \partial v}{\partial z} + \frac{\partial r}{\partial z} \right] = 0. \tag{46}
\]

Since the direction of flow is independent of \( \theta \) and \( r \) is not a function of \( z \), equation (46) becomes

\[
\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial v}{\partial z} = 0, \tag{47}
\]

where \( f^* \) is the complex viscosity of the fluid. Equations (40) and (45) bear the potential to account for the influence of the viscoelastic relaxation phenomena of blood on the wave propagation in a blood vessel of the circulatory system [17].

Hence, the model used here to present the non-Newtonian nature of blood is realistic.

These equations are particularly applicable for the case when the shear rate is small or when relaxation and retardation times are almost equal.
4.3 Method of solution

Consider the motion equations and the continuity equations below:

\[
\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial r} + 2f^* \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2} \right)
\]

\[
\rho \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial z} + f^* \left( \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} \right)
\]  \hspace{1cm} (48)

\[
\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial v}{\partial z} = 0
\]

Substituting equation (35) into equation (48), and taking equation (35) into consideration, \( \frac{\partial}{\partial r} \rightarrow \frac{d}{dr} \) so that

\[
(i\omega) \rho \vec{u} = -\frac{dp}{dr} + 2f^* \left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2} \right] \vec{u}
\]

\[
(i\omega) \rho \vec{v} = \left( \frac{i\omega}{c} \right) \vec{p} + f^* \left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right] \vec{v}
\]  \hspace{1cm} (49)

\[
\left( \frac{d}{dr} + \frac{1}{r} \right) \vec{u} = \frac{i\omega}{c} \vec{v}
\]

From the third part of equation (49), we solve for \( \vec{v} \)

\[
\vec{v} = \left( \frac{c}{i\omega} \right) \left[ \frac{d}{dr} + \frac{1}{r} \right] \vec{u} (r)
\]  \hspace{1cm} (50)
Working with equation (50) and the second part of equation (49), we obtain

\[
\bar{p} = (c\rho)\bar{v}(r) + \left(\frac{ic}{\omega}\right) f' \left[\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr}\right] \bar{v}(r). \tag{51}
\]

Substituting equation (50) into equation (51), we have

\[
\bar{p}(r) = (c\rho) \left\{ \frac{c}{i\omega} \left[ \frac{d}{dr} + \frac{1}{r} \right] \bar{u}(r) + \left(\frac{ic}{\omega}\right) f' \left[\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr}\right] \left[\left(\frac{d}{dr} + \frac{1}{r}\right) \bar{u}(r)\right] \right\},
\]

which after simplifying, gives

\[
\bar{p}(r) = \frac{\rho c^2}{i\omega} \left[ \frac{d\bar{u}}{dr} + \frac{\bar{u}}{r} \right] + f' \left[\frac{c^2}{\omega^2} \left[\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr}\right] \left[\left(\frac{d}{dr} + \frac{1}{r}\right) \bar{u}(r)\right] \right],
\]

which leads to

\[
\bar{p}(r) = \frac{\rho c^2}{i\omega} \left[ \frac{d\bar{u}}{dr} + \frac{\bar{u}}{r} \right] + f' \left[\frac{c^2}{\omega^2} \left[\frac{d^3}{dr^3} + \frac{2}{r} \frac{d^2}{dr^2} \left(\frac{\bar{u}}{r}\right)\right] + \frac{c^2}{\omega^2} \left[\frac{1}{r} \frac{d^2\bar{u}}{dr^2} + \frac{1}{r} \frac{d}{dr} \left(\frac{\bar{u}}{r}\right)\right] \right]. \tag{52}
\]

Observe that equation (52) has three segments and each segment consists of two terms.

Beginning with the second term of the second segment, we obtain
\[
\frac{d^2}{dr^2} \left( \frac{\bar{u}}{r} \right) = \frac{d}{dr} \left[ \frac{d}{dr} \left( \frac{\bar{u}}{r} \right) \right] = \frac{d}{dr} \left[ \frac{1}{r^2} \bar{u} + \frac{1}{r} \frac{d\bar{u}}{dr} \right],
\]

which leads to

\[
\frac{d^2}{dr^2} \left( \frac{\bar{u}}{r} \right) = \frac{2}{r^3} \bar{u} - \frac{2}{r^2} \frac{d\bar{u}}{dr} + \frac{1}{r} \frac{d^2\bar{u}}{dr^2}.
\] (53)

Similarly, the second term of the third segment of equation (52) yields

\[
\frac{1}{r} \frac{d}{dr} \left( \frac{\bar{u}}{r} \right) = -\frac{1}{r^3} \bar{u} + \frac{1}{r^2} \frac{du}{dr}.
\] (54)

Substituting equations (53) and (54) back into equation (52) leads to

\[
\bar{p}(r) = \frac{\rho c^2}{i\omega} \left[ \frac{d\bar{u}}{dr} + \bar{u} \right] + \frac{c^2}{\omega^2} \left[ \frac{d^3\bar{u}}{dr^3} + \frac{2}{r^3} \bar{u} - \frac{2}{r^2} \frac{d\bar{u}}{dr} + \frac{1}{r} \frac{d^2\bar{u}}{dr^2} \right] + \frac{c^2}{\omega^2} \left[ \frac{1}{r^2} \frac{d^2\bar{u}}{dr^2} - \frac{1}{r^3} \bar{u} + \frac{1}{r^2} \frac{du}{dr} \right].
\] (55)

Simplifying equation (55) gives

\[
\bar{p}(r) = \left( \frac{\rho c^2}{i\omega} \right) \left[ \frac{d\bar{u}}{dr} + \bar{u} \right] + \frac{c^2}{\omega^2} \left[ \frac{d^3\bar{u}}{dr^3} + \frac{1}{r^3} \bar{u} - \frac{1}{r^2} \frac{d\bar{u}}{dr} + \frac{2}{r} \frac{d^2\bar{u}}{dr^2} \right].
\] (56)

Now computing \( \frac{d}{dr} \) (56), we obtain

39
\[
\frac{\partial p}{\partial r} = \left( \frac{pc^2}{i\omega} \right) \left[ \frac{d^2\bar{u}}{dr^2} - \frac{1}{r^2}\bar{u} + \frac{1}{r}\frac{d\bar{u}}{dr} \right] + \left( \frac{c^2}{\sigma^2} f^* \right) \left[ \frac{d^4\bar{u}}{dr^4} + \frac{2}{r^3}\frac{d^3\bar{u}}{dr^3} - \frac{3}{r^2}\frac{d^2\bar{u}}{dr^2} + \frac{3}{r}\frac{d\bar{u}}{dr} - \frac{3}{r^4}\bar{u} \right].
\]

Therefore,

\[
\frac{\partial p}{\partial r} = \left( \frac{pc^2}{i\omega} \right) \left[ \frac{d^2\bar{u}}{dr^2} + \frac{1}{r}\frac{d\bar{u}}{dr} - \frac{1}{r^2}\bar{u} \right] + \left( \frac{c^2}{\sigma^2} f^* \right) \left[ \frac{d^4\bar{u}}{dr^4} + \frac{2}{r^3}\frac{d^3\bar{u}}{dr^3} - \frac{3}{r^2}\frac{d^2\bar{u}}{dr^2} + \frac{3}{r}\frac{d\bar{u}}{dr} - \frac{3}{r^4}\bar{u} \right]. \tag{57}
\]

Now, substituting equation (57) into equation (49) gives

\[
(i\omega) \rho \bar{u} = -\frac{\partial p}{\partial r} + 2f^* \left[ \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2} \right] \bar{u}
\]

\[
= \left( \frac{pc^2}{i\omega} \right) \left[ \frac{d^2\bar{u}}{dr^2} + \frac{1}{r}\frac{d\bar{u}}{dr} - \frac{1}{r^2}\bar{u} \right] - \left( \frac{c^2}{\sigma^2} f^* \right) \left[ \frac{d^4\bar{u}}{dr^4} + \frac{2}{r^3}\frac{d^3\bar{u}}{dr^3} - \frac{3}{r^2}\frac{d^2\bar{u}}{dr^2} + \frac{3}{r}\frac{d\bar{u}}{dr} - \frac{3}{r^4}\bar{u} \right] +
\]

\[
2f^* \left[ \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2} \right] \bar{u}.
\]

\[
\tag{58}
\]

Next, we find

\[
\left[ \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2} \right] \left[ \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2} \right] \bar{u}
\]

40
The first term of equation (59) can be simplified to

\[
\frac{d^4 \bar{u}}{dr^4} + \frac{1}{r} \frac{d^3 \bar{u}}{dr^3} - \frac{3}{r^2} \frac{d^2 \bar{u}}{dr^2} + \frac{6}{r^3} \frac{d \bar{u}}{dr} - \frac{6}{r^4} \bar{u} = 1.
\]

The second term of equation (59) can also be simplified to

\[
= \frac{1}{r} \frac{d^3 \bar{u}}{dr^3} - \frac{2}{r^2} \frac{d^2 \bar{u}}{dr^2} + \frac{1}{r^3} \frac{d \bar{u}}{dr} + \frac{2}{r^4} \bar{u} = II.
\]

Adding I and II gives

\[
I + II = \frac{d^4 \bar{u}}{dr^4} + \frac{2}{r} \frac{d^3 \bar{u}}{dr^3} + \frac{4}{r^2} \frac{d^2 \bar{u}}{dr^2} - \frac{2}{r^3} \frac{d \bar{u}}{dr} - \frac{4}{r^4} \bar{u}.
\]

Now adding equation (60) to the rest of the terms in equation (59) leads to

\[
\frac{d^4 \bar{u}}{dr^4} + \frac{2}{r^2} \frac{d^3 \bar{u}}{dr^3} + \frac{3}{r} \frac{d^2 \bar{u}}{dr^2} + \frac{3}{r^2} \frac{d \bar{u}}{dr} - \frac{3}{r^3} \bar{u} = \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2} \left[ \frac{d^2 \bar{u}}{dr^2} + \frac{1}{r} \frac{d \bar{u}}{dr} - \frac{\bar{u}}{r^2} \right].
\]
Comparing equations (61) and (57), and working with equation (61), equation (58) is modified to obtain

\[
(\rho i \omega) \ddot{u} = \left( -\frac{\rho c^2}{i \omega} \right) \Psi(r) - \frac{c^2}{\omega^2} f^* \left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2} \right] \Psi(r) + 2f^* \Psi(r), \tag{62}
\]

where \( \Psi(r) = \frac{d^2 \ddot{u}}{dr^2} + \frac{1}{r} \frac{d \ddot{u}}{dr} - \frac{1}{r^2} \ddot{u}. \tag{63} \)

Re-arranging equation (62) gives

\[
(\rho i \omega) \ddot{u} = \left( -\frac{i \rho c^2}{\omega} + 2f^* \right) \Psi(r) - \frac{c^2}{\omega^2} f^* \left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2} \right] \Psi(r)
\]

which leads to

\[
\left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2} \right] \Psi(r) - \left( \frac{2f^*}{\omega} + \frac{i \rho c^2}{\omega^2} \right) \frac{\omega^2}{c^2 f^*} \Psi(r) = -\frac{\rho i \omega}{c^2 f^*} \ddot{u}.
\]

Making the change \( \nabla^2 = \frac{d^2 \ddot{u}}{dr^2} + \frac{1}{r} \frac{d \ddot{u}}{dr} - \frac{1}{r^2} \ddot{u} \) and simplifying again, we have

\[
\nabla^2 \Psi(r) + \left( \frac{\rho c^2}{i \omega} - 2f^* \right) \frac{\omega^2}{c^2 f^*} \Psi(r) = -\frac{\rho i \omega}{c^2 f^*} \ddot{u}
\]

\[
\nabla^2 \Psi(r) + \left( \frac{\rho \omega}{if^*} - \frac{2\omega^2}{c^2} \right) \Psi(r) = -\frac{\rho i \omega}{c^2 f^*} \ddot{u}. \tag{64}
\]

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To continue with this derivation, a general solution of equation (63) must be obtained. Equation (63) is a non-homogenous second order ordinary differential equation. Its solution does follow standard technique. The method of solution is provided here for the sake of completeness. A well-trained mathematical reader can go straight to equation (77). Now, from equation (63),

$$
\Psi(r) = \frac{d^2 \bar{u}}{dr^2} + \frac{1}{r} \frac{d \bar{u}}{dr} - \frac{1}{r^2} \bar{u}.
$$

For the complementary solution,

$$
\frac{d^2 \bar{u}}{dr^2} + \frac{1}{r} \frac{d \bar{u}}{dr} - \frac{1}{r^2} \bar{u} = 0. \quad (65)
$$

Let \( \bar{u} = r^n \). Then

$$
\frac{d \bar{u}}{dr} = nr^{n-1}, \quad (66)
$$

$$
\frac{d^2 \bar{u}}{dr^2} = n(n-1)r^{n-2}. \quad (67)
$$

Substituting equations (66) and (67) into equation (65) gives
\[ n(n-1)r^{n-1} + \frac{1}{r} nr^{n-1} - \frac{1}{r^2} r^n = 0. \]

Multiplying through by \( r^2 \) gives

\[ n(n-1)r^n + nr^n - r^n = 0, \]
\[ r^n \left[ n^2 - n + n - 1 \right] = 0, \]
\[ r^n \left[ n^2 - 1 \right] = 0. \]

But \( r^n \neq 0 \), so \( n^2 - 1 = 0 \). Therefore, \( n = 1 \) and \( n = -1 \).

The complementary solution of equation (63) is therefore given by

\[ \tilde{u}_c = c_1 r + c_2 \frac{1}{r}. \]  

(68)

The particular solution is of the form

\[ \tilde{u}_p = h_1(r) u_1 + h_2(r) u_2 \]
\[ = h_1(r) r + h_2(r) \frac{1}{r} \]

(69)

Its derivative is given by
\[ \tilde{u}_p = h_1'(r) r + h_1(r) - \frac{1}{r^2} h_2(r) + \frac{1}{r} h_2'(r) \]
\[ = \left[ rh_1'(r) + \frac{1}{r} h_2'(r) \right] + \left[ h_1(r) - \frac{1}{r^2} h_2(r) \right]. \]

Imposing the condition of variation of parameters, we have

\[ rh_1'(r) + \frac{1}{r} h_2'(r) = 0, \quad (70) \]
\[ \tilde{u}_p = h_1'(r) - \frac{1}{r^2} h_2(r) \]
\[ \tilde{u}_p = h_1'(r) + \frac{2}{r^3} h_2(r) - \frac{1}{r^2} h_2'(r). \quad (71) \]

Substituting equations (69), and (71) into equation (65) gives

\[ \Psi(r) = h_1'(r) + \frac{2}{r^3} h_2(r) - \frac{1}{r^2} h_2'(r) + \frac{1}{r} \left[ h_1(r) - \frac{1}{r^2} h_2(r) \right] - \frac{1}{r^2} \left[ rh_1(r) + \frac{1}{r} h_2(r) \right], \]
\[ = h_1'(r) + \frac{2}{r^3} h_2(r) - \frac{1}{r^2} h_2'(r) + \frac{1}{r} h_1(r) - \frac{1}{r^3} h_2(r) - \frac{1}{r^2} h_2(r) - \frac{1}{r^3} h_2(r), \]
\[ = h_1'(r) - \frac{1}{r^2} h_2'(r). \quad (72) \]

Now solving equations (70) and (72),
By applying Cramer’s rule, this $2 \times 2$ linear system is solved to obtain

\[ h_1(r) = \frac{\Psi(r)}{2}, \]
\[ h_2(r) = -\frac{r^2}{2} \Psi(r). \]  

Therefore,

\[ h_1(r) = \int h_1(r) \, dr = \int \frac{\Psi(r)}{2} \, dr = \frac{1}{2} \int \Psi(r) \, dr, \]  

and

\[ h_2(r) = \int h_2(r) \, dr = -\frac{1}{2} \int r^2 \Psi(r) \, dr. \]

From equation (69), we have

\[ \bar{u}_r = h_1(r) r + h_2(r) \frac{1}{r}. \]
Therefore, \( \bar{u}_p = r \left[ \frac{1}{2} \int \Psi(r) \, dr \right] + \frac{1}{r} \left[ -\frac{1}{2} \int r^2 \Psi(r) \, dr \right] \)

which leads to

\[
\bar{u}_p = \frac{r}{2} \int \Psi(r) \, dr - \frac{1}{2r} \int r^2 \Psi(r) \, dr.
\] (76)

Therefore, the total solution of \( \bar{u} \) is given by

\[
\bar{u}(r) = c_1 r + c_2 + \frac{1}{r} \int \Psi(r) \, dr - \frac{1}{2r} \int r^2 \Psi(r) \, dr.
\] (77)

Since the solution must be continuous at 0, \( \frac{1}{r} \int \Psi(r) \, dr \) cannot be part of the solution. Therefore, the total solution given by equation (77) becomes

\[
\bar{u}(r) = c_1 r + \frac{1}{2} r \int \Psi(r) \, dr
\] (78)

Equation (78) is a significant step in this whole endeavor. It permits us to recast

the entire problem into one differential equation in one function \( \Psi(r) \).

From equation (64), we have
\[ \nabla^2 \Psi(r) + \left( \frac{\rho \omega}{i \beta^{\ast}} - \frac{2 \omega^2}{c^2} \right) \Psi(r) = -\frac{\partial \omega^3}{\partial^2 \beta^{\ast}} \mathbf{u} . \]

Letting \( \frac{\rho \omega}{i \beta^{\ast}} - \frac{2 \omega^2}{c^2} = \lambda^2 \) and \( -\frac{\partial \omega^3}{\partial^2 \beta^{\ast}} = \beta^2 \), equation (64) becomes

\[ \beta^2 \mathbf{u} = \nabla^2 \Psi(r) + \lambda^2 \Psi(r) \tag{79} \]

\[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2} \Psi(r) + \lambda^2 \Psi(r) . \tag{80} \]

Substituting equation (78) into equation (80) gives

\[ \frac{d^2 \Psi(r)}{dr^2} + \frac{1}{r} \frac{d \Psi(r)}{dr} - \frac{1}{r^2} \Psi(r) + \lambda^2 \Psi(r) = \beta^2 \left[ c_r + \frac{r}{2} \left[ \Psi(r) dr \right] . \tag{81} \]

Multiplying equation (81) by \( r^2 \), we have

\[ r^2 \Psi'(r) + r \Psi'(r) - \Psi(r) + \lambda^2 r^2 \Psi(r) = \beta^2 c_r r^3 + \frac{\beta^2}{2} r^3 \left[ \Psi(r) dr \right] . \]

and by re-arranging terms, it leads to

\[ r^2 \Psi'(r) + r \Psi'(r) + \left( \lambda^2 r^2 - 1 \right) \Psi(r) = \beta^2 c_r r^3 + \frac{\beta^2}{2} r^3 \left[ \Psi(r) dr \right] . \tag{82} \]
Using an ultrasound-echo ranging device, the systolic diameter of the artery (carotid) varies between 0.77 and 0.97 centimeters. This means the average radius of the artery (carotid) is $0.435cm \left( 4.35 \times 10^{-3}m \right) [3].$

Now, the term $\left\| r^3 \int \Psi (r) \, dr \right\| \leq \| r^3 \|| \int \Psi (r) \, dr \|.$

where $\Psi (r)$ is bounded and therefore $\int \Psi (r) \, dr$ is bounded [25].

If we let $\left\| \int \Psi (r) \, dr \right\| = M$, then $\lim_{r \to 4.35 \times 10^{-3}m} \left\| r^3 \| \cdot M = \varepsilon ,

where $\varepsilon$ is very small [25]. Then, the term $r^3 \int \Psi (r) \, dr$ will be negligible.

Since the radius of the artery under consideration is very small, we assume that the contribution of $r^3 \int \Psi (r) \, dr$ is negligible in the foregoing discussion. In this event, equation (82) reduces to

$$r^2 \Psi' (r) + r \Psi'' (r) + \left( \lambda^2 r^2 - 1 \right) \Psi (r) = \beta^2 c, r^3$$

(83)

4.3.1 Transformation of motion and continuity equation into Bessel equation

For the complementary solution of equation (83), we write

$$r^2 \Psi' (r) + r \Psi'' (r) + \left( \lambda^2 r^2 - 1 \right) \Psi (r) = 0.$$  

(84)

Let $\rho = \lambda r$, or $r = \frac{\rho}{\lambda}$. Then

$$\frac{d}{dr} = \frac{d}{d \rho} \cdot \frac{d \rho}{dr} = \lambda \frac{d}{d \rho}$$  

(85)
and

\[
\frac{d^2}{dr^2} = \frac{d}{dr} \left[ \frac{d}{dr} \right] = \lambda^2 \frac{d^2}{d\rho^2}. \tag{86}
\]

Equation (84) can then be written as

\[
r^2 \frac{d^2 \Psi(r)}{dr^2} + r \frac{d \Psi(r)}{dr} + \left( \lambda^2 r^2 - 1 \right) \Psi(r) = 0. \tag{87}
\]

Now changing variables using equations (85) and (86), equation (87) becomes

\[
\frac{\rho^2}{\lambda^2} \frac{d^2}{d\rho^2} \left\{ \frac{\rho}{\lambda} \Psi \left( \frac{\rho}{\lambda} \right) \right\} + \frac{\rho}{\lambda} \frac{d}{d\rho} \left[ \frac{\rho}{\lambda} \Psi \left( \frac{\rho}{\lambda} \right) \right] + \left[ \rho^2 - 1 \right] \Psi \left( \frac{\rho}{\lambda} \right) = 0,
\]

which can be simplified to

\[
\rho^2 \frac{d^2}{d\rho^2} \Psi \left( \frac{\rho}{\lambda} \right) + \rho \frac{d}{d\rho} \Psi \left( \frac{\rho}{\lambda} \right) + \left[ \rho^2 - 1 \right] \Psi \left( \frac{\rho}{\lambda} \right) = 0. \tag{88}
\]

Equation (88) is a Bessel equation of order 1.

Hence, the solution to equation (88) is of the form

\[
\Psi_c = A J_1(\rho) + B Y_1(\rho). \tag{89}
\]
Since $Y(\rho)$ has a logarithmic singularity at zero, the solution will be unbounded or nonconvergent. Therefore, we neglect the term containing $Y(\rho)$ by setting $B_1 = 0$.

Then, equation (89) becomes

$$
\Psi_c = A_1 J_1(\rho)
$$

with $\rho = \lambda r$ and

$$
\Psi_c = A_1 J_1(\lambda r).
$$

For the particular solution, we use

$$
\Psi_p = Ar^4 + Br^3 + Cr^2 + Dr + E.
$$

Substituting into equation (83) and equating coefficients, we have

$$
A = B = C = E = 0 \quad \text{and} \quad D = \frac{\beta^2 c_1}{\lambda^2}.
$$

Substituting these into equation (91), we have
\[ \Psi_p = \frac{\beta^2 c}{\lambda^2} r. \]  \hspace{1cm} (92)

Therefore, the total solution \( \Psi (r) = \Psi_e(r) + \Psi_p(r) \) becomes

\[ \Psi (r) = A J_1(\lambda r) + \frac{\beta^2 c}{\lambda^2} r. \]  \hspace{1cm} (93)

Substituting equation (93) into equation (78) leads to

\[ \bar{u}(r) = c r \left[ 1 + \frac{1}{4} \left( \frac{\beta}{\lambda} \right)^2 r^2 \right] + \frac{1}{2} A r \int J_1(\lambda r) dr. \]

But \( \int J_1(\lambda r) dr = -\frac{1}{\lambda} J_0(\lambda r) \) \hspace{1cm} [see appendix A for details]. Therefore, with \( \rho = \lambda r \),

\[ \bar{u}(r) = c r \left[ 1 + \frac{1}{4} \left( \frac{\beta}{\lambda} \right)^2 r^2 \right] - \frac{1}{2} A \frac{r}{\lambda} J_0(\lambda r). \]  \hspace{1cm} (94)

From equation (35), we have

\[ u(r) = \bar{u}(r) e^{i\alpha (r-z)}, \]  which leads to

\[ u(r) = \left\{ c r \left[ 1 + \frac{1}{4} \left( \frac{\beta}{\lambda} \right)^2 r^2 \right] - \frac{1}{2} A \frac{r}{\lambda} J_0(\lambda r) \right\} e^{i\alpha (r-z)}. \]  \hspace{1cm} (95)
Since we know the functional form of $u$, we now use equation (50) to obtain

$$
\bar{v}(r) = \left( \frac{c}{i\omega} \right) \left[ \frac{d}{dr} + \frac{1}{r} \right] \bar{u}(r) = \left( \frac{c}{i\omega} \right) \left[ \frac{d}{dr} + \frac{1}{r} \bar{u} \right].
$$

Substituting equation (95) into equation (50) we have

$$
\bar{v}(r) = \frac{c}{i\omega} \left[ c_1 + \frac{3(\beta^2)}{4(\lambda)} c_1 r^2 - \frac{1}{2\lambda} A_1 \left\{ r \frac{d}{dr} J_0(\lambda r) + J_0(\lambda r) \right\} + c_1 + \frac{1(\beta^2)}{4(\lambda)} c_1 r^2 - \frac{1}{2\lambda} A_1 J_0(\lambda r) \right].
$$

Working with $\frac{d}{dr} J_0(\lambda r) = -\lambda J_1(\lambda r)$ [see appendix A], the expression for $v$ is given by

$$
\bar{v}(r) = \frac{c}{i\omega} \left[ 2c_1 + \left( \frac{\beta^2}{\lambda} \right) c_1 r^2 + \frac{1}{2\lambda} A_1 \lambda r J_1(\lambda r) - \frac{1}{\lambda} A_1 J_0(\lambda r) \right]. \tag{96}
$$

Similarly to equation (95), $v$ is finally written as

$$
v(r) = \frac{c}{i\omega} \left[ 2c_1 + \left( \frac{\beta^2}{\lambda} \right) c_1 r^2 + \frac{1}{2\lambda} A_1 \lambda r J_1(\lambda r) - \frac{1}{\lambda} A_1 J_0(\lambda r) \right] e^{i\left(\frac{r-\frac{\beta}{\lambda}}{c}\right)}. \tag{97}
$$

Equations (95) and (97) give $u$ and $v$, respectively. Now, we need to deduce the expression of the pressure $p$. Substituting equation (96) into equation (51), and after simplification, we obtain
\[
\begin{align*}
\bar{p} &= (c_p) \left\{ \frac{c}{i\omega} \left[ 2c_i + \left( \frac{\beta}{\lambda} \right)^2 c_i r^2 + \frac{1}{2\lambda} \lambda r A J_1(\lambda r) - \frac{1}{\lambda} \lambda J_0(\lambda r) \right] \right\} \\
&\quad + \left( \frac{ic}{\omega} \right) \left( \frac{c}{i\omega} \right) f \left\{ \left[ \left( \frac{\beta}{\lambda} \right)^2 c_i \frac{d^2}{dr^2} (r^2) + \frac{1}{2\lambda} \lambda r A \frac{d^2}{dr^2} (r J_1(\lambda r)) - \frac{1}{\lambda} \lambda J_0(\lambda r) \right] \right\}.
\end{align*}
\] (98)

where,
\[
\frac{d^2}{dr^2} \left[ r J_1(\lambda r) \right] = \lambda \left[ J_0(\lambda r) - \lambda r J_1(\lambda r) \right]
\]
\[
\frac{d^2}{dr^2} J_0(\lambda r) = \lambda^2 \left[ \frac{1}{\lambda r} J_1(\lambda r) - J_0(\lambda r) \right]
\]
\[
\frac{d}{dr} \left[ \lambda r J_1(\lambda r) \right] = \lambda^2 r J_0(\lambda r)
\]
\[
\frac{d}{dr} J_0(\lambda r) = -\lambda J_1(\lambda r).
\]

For more details, see Appendix A.

Using the definitions below equation (98), we have
\[
\begin{align*}
\bar{p} &= (c_p) \left\{ \frac{c}{i\omega} \left[ 2c_i + \left( \frac{\beta}{\lambda} \right)^2 c_i r^2 + \frac{1}{2\lambda} \lambda r A J_1(\lambda r) - \frac{1}{\lambda} \lambda J_0(\lambda r) \right] \right\} \\
&\quad + \left( \frac{ic^2}{\omega^2} \right) f \left\{ 4c_i \left( \frac{\beta}{\lambda} \right)^2 + 2\lambda A r J_0(\lambda r) - \frac{1}{2} \lambda^2 A r J_1(\lambda r) \right\}.
\end{align*}
\] (99)

Similarly to equations (95) and (97), the final form for the pressure \( p \) is written as
\[ p(r) = \left( c \rho \left[ \frac{c}{i\omega} \left[ 2c_i + \left( \frac{\beta}{\lambda} \right)^2 c_i r^2 + \frac{1}{2} A_i \lambda J_1(\lambda r) - \frac{1}{\lambda} A_i J_0(\lambda r) \right] \right] \right) e^{i\omega t}, \tag{100} \]

where \( A_i \) and \( c_i \) are arbitrary constants, \( \omega \) is a real constant, and \( c \) is the complex wave propagation velocity. Moreover, \( J_0(\lambda r) \) and \( J_1(\lambda r) \) denote the zeroth and first order Bessel functions of the first kind, respectively.

4.4 Equations of Motion of the arterial wall

At this point, one has to know the motion of the arterial wall. This work was done by Misra and Chouldhury in their paper “Effect of Initial Stresses on the Wave propagation in Arteries” [19]. They assume the motion of the arterial wall and that of the fluid (blood) to be axisymmetric. They then employed the equations of motion of the fluid and those of the wall, together with the equation of continuity, to derive a frequency equation. The above-mentioned frequency equation was made possible by exploiting the conditions of continuity of the velocity of the arterial wall and of the blood on the endosteal surface of the wall.

Blood flowing through an artery exerts forces due to its pressure and friction on the inner surface of the arterial wall. The longitudinal and radial components of those forces may be calculated through the use of the following expressions given in [18]:

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\[ q_1 = -\mu \left[ \frac{\partial v}{\partial r} + \frac{\partial u}{\partial z} \right]_{r=R}, \]

\[ q_3 = \left[ p + 2\mu \frac{\partial u}{\partial r} \right]_{r=R}. \]

The sign \[ (\cdot)_{r=R} \] indicates that the value of the quantity inside the bracket is evaluated at \( r = R \) and \( \mu \) is the viscosity of the blood [19].

Since the equation motion of the blood is linear and harmonic in \( t \) and \( z \), the equations of the arterial wall will also linear and harmonic in \( t \) and \( z \). Thus, we let

\[ X = q_u e^{\frac{i \alpha (t-z)}{c}}, \]

\[ Y = q_v e^{\frac{i \alpha (t-z)}{c}}, \]

where \( q_u \) and \( q_v \) are constants and \( X \) and \( Y \) are displacements.

The equations of motion of the arterial wall for the super-imposed state for the present problem were then derived as [19]

\[ \frac{\partial X}{\partial z} \left[ \frac{h_0 \beta_0}{\lambda_2} - \frac{h_0 \phi_0}{\lambda_2 R_0} \right] + \frac{\partial^2 Y}{\partial z^2} \left[ \beta_{\phi} + \Psi_0 \right] \frac{h_0}{\lambda_2} + q_1 = \frac{h_0 \rho_0}{\lambda_4 \lambda_2} \frac{\partial^2 Y}{\partial t^2}, \]

and

\[ \frac{\lambda_1 h_0}{\lambda_2} \frac{\partial^2 X}{\partial z^2} + \frac{X h_0}{R_0^2} \left( \beta_{\phi \phi} - 2\phi_0 \right) + \frac{\lambda_2 h_0}{R_0} \left( \beta_{\phi z} - \phi_0 \right) \frac{\partial Y}{\partial z} + q_3 = \frac{h_0 \rho_0}{\lambda_4 \lambda_2} \frac{\partial^2 X}{\partial t^2}. \]
4.5 Boundary Conditions

The motion can be described completely if the equations of motion are supplemented by some prescribed conditions on the arterial wall. As the blood flows through the vessel (artery), the particles of the blood adhere to the inner surface of the vessel. So by the adherence boundary condition, there must be a continuity of the velocities of the blood particles and those of the arterial wall. Thus, the velocity of the blood at the inner surface of the arterial segment may be taken to be equal to the velocity of the inner surface of the arterial wall. Mathematically,

\[ \frac{\partial X}{\partial t} \bigg|_{r=R(z)} = u; \]
\[ \frac{\partial Y}{\partial t} \bigg|_{r=R(z)} = v. \]  

(105)

From equation (102), \( X = q_a e^{i\omega(t-c)} \), so equation (105) becomes

\[ i\omega q_a e^{i\omega(t-c)} = u. \]  

(106)

But, from equation (95),

\[ u(r) = \left(c_r \left[ 1 + \frac{1}{4} \left( \frac{\beta}{\lambda} \right)^2 r^2 \right] - \frac{1}{2} A_i \frac{R(z)}{\lambda} J_0(\lambda R) \right) e^{i\omega(t-c)}. \]

Therefore equation (106) becomes

\[ i\omega q_a e^{i\omega(t-c)} = \left(c_r R(z) \left[ 1 + \frac{1}{4} \left( \frac{\beta}{\lambda} \right)^2 [R(z)]^2 \right] - \frac{1}{2} A_i \frac{R(z)}{\lambda} J_0(\lambda R(z)) \right) e^{i\omega(t-c)}, \]

which can be simplified to

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\[ i \omega q_u = \alpha R(z) \left[ 1 + \frac{1}{4} \left( \frac{\beta}{\lambda} \right)^2 \right] - \frac{1}{2} \lambda J_0 (\lambda R(z)) \]

which leads to

\[ i \omega q_u = \left[ R(z) + \frac{1}{4} \left( \frac{\beta}{\lambda} \right)^2 \left[ R(z) \right]^2 \right] c_1 + \frac{1}{2} \lambda J_0 (\lambda R(z)) A_i = 0. \tag{107} \]

Similarly, from equation (102), \( Y = q, e^{i \omega (\tau - \xi)} \). Equation (105) then becomes

\[ i \omega q, e^{i \omega (\tau - \xi)} = Y. \tag{108} \]

But, from equation (97),

\[ v(r) = \frac{c}{i \omega} \left[ 2c_1 + \left( \frac{\beta}{\lambda} \right)^2 c_1 r^2 + \frac{1}{2} \lambda J_1 (\lambda r) \right] e^{i \omega (\tau - \xi)}. \]

Therefore, equation (108) reads

\[ i \omega q, e^{i \omega (\tau - \xi)} = \frac{c}{i \omega} \left[ 2c_1 + \left( \frac{\beta}{\lambda} \right)^2 c_1 [R(z)]^2 + \frac{1}{2} \lambda J_1 (\lambda R(z)) \right] e^{i \omega (\tau - \xi)} \]

which is simplified to
\[ -\omega^2 q_v = \left[ 2c + \left( \frac{B}{\lambda} \right)^2 c[R(z)]^2 \right] c_1 + \left[ \frac{1}{2\lambda} c\lambda R(z) J_1(\lambda R(z)) - \frac{1}{\lambda} c J_0(\lambda R(z)) \right] A_1, \]

which leads to

\[ -\omega^2 q_v = \left[ 2c + \left( \frac{B}{\lambda} \right)^2 c[R(z)]^2 \right] c_1 - \left[ \frac{1}{2\lambda} c\lambda R(z) J_1(\lambda R(z)) - \frac{1}{\lambda} c J_0(\lambda R(z)) \right] A_1 = 0. \tag{109} \]

Now, from the equations of motion of the arterial wall, equation (103) and (104) are

\[
\frac{\partial X}{\partial z} \left[ \frac{h_0 \beta_z}{\lambda^2} - \frac{h_0 \psi_0}{\lambda^2 R_0} \right] + \frac{\partial^2 Y}{\partial z^2} \left[ \beta_{,y} + \psi_0 \right] \frac{h_0}{\lambda_2} + q_i = \frac{h_0 \rho_0}{\lambda_1 \lambda_2} \frac{\partial^2 Y}{\partial t^2}
\]

\[
\frac{\lambda_2 h_0}{\lambda_2} \frac{\Psi_0}{\partial z^2} + \frac{X h_0}{R_0^2} (\beta_{,\phi} - 2\psi_0) + \frac{\lambda_2 h_0}{R_0} (\beta_{,z} - \psi_0) \frac{\partial Y}{\partial z} + q_3 = \frac{h_0 \rho_0}{\lambda_1 \lambda_2} \frac{\partial^2 X}{\partial t^2},
\]

respectively and the first part of equation (101) is

\[ q_1 = -\mu \left[ \frac{\partial v}{\partial r} + \frac{\partial u}{\partial z} \right]_{r=R}. \]

After carrying out all differentiations and substituting, we obtain
\[
q_t = -\mu \left[ \frac{\partial}{\partial r} \left\{ \frac{c}{i\omega} \left( 2c_1 + \left( \frac{\beta}{\lambda} \right)^2 c_1 r^2 + \frac{1}{2\lambda} A_t \lambda J_1(\lambda r) - \frac{1}{\lambda} A_1 J_0(\lambda r) \right) \right\} e^{i\alpha (z - \omega)} \right] \\
q_t = -\mu \left[ \frac{\partial}{\partial z} \left\{ c_1 r \left[ 1 + \frac{1}{4} \left( \frac{\beta}{\lambda} \right)^2 r^2 \right] - \frac{1}{2} A_1 \frac{r}{\lambda} J_0(\lambda r) \right\} e^{i\alpha (z - \omega)} \right] \\
\]  

which can be simplified as

\[
q_t = -\mu \left[ \left\{ \frac{c}{i\omega} \left( 2 \left( \frac{\beta}{\lambda} \right)^2 c_1 R(z) + \frac{1}{2\lambda} A_t \lambda^2 R(z) J_0(\lambda R(z)) + \frac{1}{\lambda} A_1 \lambda J_1(\lambda R(z)) \right) \right\} e^{i\alpha (z - \omega)} \right] \\
q_t = -\mu \left[ -\frac{i\omega}{c} \left\{ c_1 R(z) \left[ 1 + \frac{1}{4} \left( \frac{\beta}{\lambda} \right)^2 [R(z)]^2 \right] - \frac{1}{2} A_1 \frac{R(z)}{\lambda} J_0(\lambda R(z)) \right\} e^{i\alpha (z - \omega)} \right] \\
\]  

which leads to

\[
q_t = -\mu \left[ \left\{ \frac{c}{i\omega} \left( 2 \left( \frac{\beta}{\lambda} \right)^2 c_1 R(z) + \frac{1}{2} A_1 \lambda R(z) J_0(\lambda R(z)) + \frac{1}{\lambda} A_1 J_1(\lambda R(z)) \right) \right\} e^{i\alpha (z - \omega)} \right] \\
q_t = -\mu \left[ -\frac{i\omega}{c} \left\{ c_1 R(z) \left[ 1 + \frac{1}{4} \left( \frac{\beta}{\lambda} \right)^2 [R(z)]^2 \right] - \frac{1}{2} A_1 \frac{R(z)}{\lambda} J_0(\lambda R(z)) \right\} e^{i\alpha (z - \omega)} \right] \\
\]  

Similarly, the second part of equation (101)
\[ q_3 = \left[ p + 2\mu \frac{\partial u}{\partial r} \right]_r = R(z) . \]

After carrying out all differentiations and substituting, we obtain

\[
q_3 = \left[ \left( \frac{c^2 \rho}{\omega} \right) \left\{ \frac{c}{\omega} \left[ 2c_i + \left( \frac{\beta}{\lambda} \right)^2 c_r r^2 + \frac{1}{2} A_i r J_1(\lambda r) - \frac{1}{\lambda} A_i J_0(\lambda r) \right] \right\} \right] \\
\quad + c^2 \omega^2 \left[ 4c_i \left( \frac{\beta}{\lambda} \right)^2 + 2A_i \lambda J_0(\lambda r) - \frac{1}{2} \lambda^2 r A_i J_1(\lambda r) \right] \\
\quad - 2\mu \left( \frac{\partial}{\partial r} \left[ c_r \left( 1 + \frac{1}{4} \left( \frac{\beta}{\lambda} \right)^2 \right) \right] - \frac{1}{2} A_i \frac{r}{\lambda} J_0(\lambda r) \right) \right] \right|_r = R(z) \\
\]

which is simplified as

\[
q_3 = \left[ \left( \frac{c^2 \rho}{\omega} \right) \left\{ \frac{c}{\omega} \left[ 2c_i + \left( \frac{\beta}{\lambda} \right)^2 c_r r^2 + \frac{1}{2} A_i r J_1(\lambda r) - \frac{1}{\lambda} A_i J_0(\lambda r) \right] \right\} \right] \\
\quad + c^2 \omega^2 \left[ 4c_i \left( \frac{\beta}{\lambda} \right)^2 + 2A_i \lambda J_0(\lambda r) - \frac{1}{2} \lambda^2 r A_i J_1(\lambda r) \right] \\
\quad - 2\mu \left( c_r + \frac{3}{4} c_i \left( \frac{\beta}{\lambda} \right)^2 r^2 - \frac{1}{2} A_i \left[ J_0(\lambda r) - \lambda r J_1(\lambda r) \right] \right) \right] \right|_r = R(z) \\
\]

which leads to
\[
q_3 = \begin{bmatrix}
\frac{c_2 \rho}{i \omega} \left( 2c_1 + \left( \frac{\dot{c}}{\lambda} \right)^2 \right) c_i [R(z)]^2 + \frac{1}{2} A_i R(z) J_0(\lambda R(z)) - \frac{1}{\lambda} A_i J_0(\lambda R(z)) \\
+ \frac{c_2^2}{\omega^2} f^* \left( 4c_1 \left( \frac{\dot{c}}{\lambda} \right)^2 \right) + 2A_0 J_0(\lambda R(z)) - \frac{1}{2} \lambda^2 R(z) A_i J_1(\lambda R(z)) \\
-2\mu \left( c_i + \frac{3}{4} c_i \left( \frac{\dot{c}}{\lambda} \right)^2 \right) [R(z)]^2 - \frac{1}{2} \lambda A_i \left[ J_0(\lambda R(z)) - \lambda R(z) J_1(\lambda R(z)) \right]
\end{bmatrix} e^{i \omega (\zeta - c)}.
\]

(111)

Now, using equation (103)

\[
\frac{\partial X}{\partial z} \left[ \frac{h_0 \beta_{zc}}{\lambda_2} - \frac{h_0 \varphi_0}{\lambda_2 R_0} \right] + \frac{\partial^2 Y}{\partial z^2} \left[ (\beta_{zc} + \Psi_0) \frac{h_0}{\lambda_2} \right] + q_1 = \frac{h_0 \rho_0}{\lambda_1 \lambda_2} \frac{\partial^2 Y}{\partial t^2},
\]

after differentiation and substitution, we obtain

\[
\frac{h_0 \rho_0}{\lambda_1 \lambda_2} \left[ -\omega^2 q_i e^{i \omega (\zeta - c)} \right] = -\frac{i \omega}{c} \frac{h_0 \beta_{zc}}{\lambda_2^2 R_0} - \frac{h_0 \varphi_0}{\lambda_2^2 R_0} - \frac{\omega^2}{c^2} q_i e^{i \omega (\zeta - c)} \left( (\beta_{zc} + \Psi_0) \frac{h_0}{\lambda_2} \right),
\]

\[
= \frac{c}{i \omega} \left( 2 \left( \frac{\dot{c}}{\lambda} \right)^2 c_i R(z) + \frac{1}{2} A_i \lambda R(z) J_0(\lambda R(z)) + A_i J_1(\lambda R(z)) \right)
\]

\[
- \mu \left( c_i R(z) \left[ 1 + \frac{1}{4} \left( \frac{\dot{c}}{\lambda} \right)^2 [R(z)]^2 \right] - \frac{1}{2} A_i \frac{R(z)}{\lambda} \right) e^{i \omega (\zeta - c)}
\]

which is simplified to
\[-i\omega \left[ \frac{h_0}{\lambda^2 R_0} \frac{\partial_x}{\lambda^2 R_0} - \frac{h_0}{\lambda_2} \frac{\partial_x}{\lambda_2 R_0} \right] q_u + \frac{h_0}{\lambda x} \frac{\partial^2 q_y}{c^2} - \frac{\omega^2}{c^2} \left[ \beta_{\phi} + \Psi_0 \right] \frac{h_0}{\lambda^2} q_v \]

\[-\mu \frac{c}{i\omega} \left( \beta^2 \right) c_i R(z) + \frac{i\omega \mu}{c} \left[ \lambda R(z) + \frac{1}{4} \left( \beta^2 \right)^2 \left[ R(z) \right]^3 \right] c_i - \frac{c\mu}{2i\omega} A_i \lambda R(z) J_0 \left( \lambda R(z) \right) \]

\[-\frac{\mu c}{i\omega} A_i J_1 \left( \lambda R(z) \right) - \frac{i\omega \mu}{c} \frac{R(z)}{2 \lambda} J_0 \left( \lambda R(z) \right) = 0.\]

and leads to

\[-i\omega \left[ \frac{h_0}{\lambda^2 R_0} \frac{\partial_x}{\lambda^2 R_0} - \frac{h_0}{\lambda_2} \frac{\partial_x}{\lambda_2 R_0} \right] q_u + \frac{h_0}{\lambda x} \frac{\partial^2 q_y}{c^2} - \frac{\omega^2}{c^2} \left[ \beta_{\phi} + \Psi_0 \right] \frac{h_0}{\lambda^2} q_v \]

\[+ \frac{i\omega \mu}{c} \left( \beta^2 \right) c_i R(z) + \frac{1}{4} \left( \beta^2 \right)^2 \left[ R(z) \right]^3 \frac{2 \mu c}{i\omega} \frac{R(z)}{\lambda} \left( \beta^2 \right) R(z) \]

\[c_i - \frac{c\mu}{2i\omega} \lambda R(z) J_0 \left( \lambda R(z) \right) + \frac{\mu c}{i\omega} A_i J_1 \left( \lambda R(z) \right) - \frac{i\omega \mu R(z)}{2c \lambda} J_0 \left( \lambda R(z) \right) = 0. \tag{112} \]

Similarly, working with equation (104),

\[\frac{\lambda h_0}{\lambda_2} \Psi_0 \frac{\partial^2 X}{\partial z^2} + \frac{X h_0}{R_0^2} \left( \beta_{\phi} - 2 \varphi_0 \right) + \frac{\lambda_2 h_0}{R_0} \left( \beta_{\phi} - \varphi_0 \right) \frac{\partial Y}{\partial z} + q_3 = \frac{h_0}{\lambda x} \frac{\partial^2 X}{\partial t^2},\]

and after differentiation and substitution, we obtain
\[
\frac{\lambda h_0}{\lambda z} \Psi_0 \left[ \frac{-\omega^2}{c^2} q_u e^{i\omega (z/c)} \right] + q_u e^{i\omega (z/c)} \frac{h_0}{R_0^2} \left( \beta_{\theta \theta} - 2\varphi_0 \right) + \frac{\lambda h_0}{R_0} \left( \beta_{\theta z} - \varphi_0 \right) \left[ -i\omega \frac{h_0}{c} q_u e^{i\omega (z/c)} \right] \\
\left[ \frac{c^2 \rho}{i\omega} \left( 2c_1 \left( \frac{\beta}{\lambda} \right)^2 \right) + \frac{c^2}{\omega^2} f^* \left( 4c_1 \left( \frac{\beta}{\lambda} \right)^2 \right) + \frac{c^2 \rho}{i\omega} \left( \lambda R(z) \right) - \frac{1}{2} \lambda J_0 \left( \lambda R(z) \right) \right] \left[ -2\mu \left( c_1 + \frac{3\mu}{2} c_1 \left( \frac{\beta}{\lambda} \right)^2 \left[ R(z) \right]^2 - \frac{1}{2} \lambda J_0 \left( \lambda R(z) \right) - \lambda R(z) J_1 \left( \lambda R(z) \right) \right) \right] \\
= \frac{h_0 \rho_0}{\lambda \lambda_z} \left[ -\omega^2 q_u e^{i\omega (z/c)} \right],
\]

which is simplified to

\[
\frac{\lambda h_0}{\lambda z} \Psi_0 \left[ \frac{-\omega^2}{c^2} q_u e^{i\omega (z/c)} \right] + q_u e^{i\omega (z/c)} \frac{h_0}{R_0^2} \left( \beta_{\theta \theta} - 2\varphi_0 \right) q_u - \frac{\lambda h_0}{R_0} \left( \beta_{\theta z} - \varphi_0 \right) \frac{i\omega}{c} q_u \\
+ \frac{c^2 \rho}{i\omega} 2c_1 + \frac{c^2 \rho}{i\omega} \left( \frac{\beta}{\lambda} \right)^2 c_1 \left[ R(z) \right]^2 + \frac{c^2 \rho}{i\omega} \frac{1}{2} \lambda J_0 \left( \lambda R(z) \right) - \frac{c^2 \rho}{i\omega} \frac{1}{\lambda} \lambda J_0 \left( \lambda R(z) \right) \\
+ \frac{c^2}{\omega^2} f^* 4c_1 \left( \frac{\beta}{\lambda} \right)^2 + \frac{c^2}{\omega^2} f^* 2\lambda J_0 \left( \lambda R(z) \right) - \frac{c^2}{\omega^2} f^* \frac{1}{2} \lambda J_0 \left( \lambda R(z) \right) \\
- 2\mu c_1 - \frac{3\mu}{2} c_1 \left( \frac{\beta}{\lambda} \right)^2 \left[ R(z) \right]^2 + \frac{\mu A}{\lambda} \left[ J_0 \left( \lambda R(z) \right) - \lambda R(z) J_1 \left( \lambda R(z) \right) \right] \\
= -\omega^2 \frac{h_0 \rho_0}{\lambda \lambda_z} q_u
\]
and leads to

\[
\begin{aligned}
\left[ \frac{\omega^2}{\lambda_1\lambda_2} - \frac{h_0}{R_0^2} \left( \frac{\beta}{\lambda} \right)^2 \right] q_0 + \frac{\omega}{R_0} \left( \beta \phi_0 - 2\phi_0 \right) q_v = \frac{i\omega}{c} \frac{h_0}{R_0} \left( \beta \phi_0 - \phi_0 \right) q_v \\
+ \left[ \frac{2e^2}{i\omega} + \frac{c^2}{i\omega} \frac{\beta}{\lambda} \right] \left[ R(z) \right]^2 + \frac{e^2}{\omega^2} f^* \left[ \frac{\beta}{\lambda} \right]^2 - 2\mu - \frac{3\mu}{2} \right] \left[ R(z) \right]^2 c_i
\end{aligned}
\]  

(113)

Now, the following equations (107), (109), (112), and (113) form a linear system of four homogeneous equations in four unknowns: \( q_v, q_0, A_i \) and \( c_i \).

Recasting the system in matrix form leads to

\[
A\tilde{x} = 0 ,
\]

(114)

where
For a nontrivial solution of this set of equations to exist, the determinant of \( A \) must be equal to zero. The equation so obtained involving the sole unknown \( c \), (i.e. the complex wave propagation velocity), is the so-called frequency equation for the problem under consideration. The homogeneity of the algebraic equation (114) implies a family or families of solutions. Due to the complexity of the determinant involved. Some realistic approximations must be made.
We are interested in cases dealing with \( \lambda R \ll 1 \) and neglecting terms of power \( R^2 \) and above. We also impose the conditions that \( J_0 (\lambda R) = 1 \) and \( J_1 (\lambda R) = \frac{\lambda R}{2} \). [18]

Consequently, the 4×4 system can be reduced to

\[
\begin{bmatrix}
 i\omega & -R & \frac{1}{2\lambda} & 0 \\
 0 & -2c & \frac{1}{c} & -\omega^2 \\
 -\frac{i\omega}{c} \alpha_0 & c \left( R \frac{1}{2} \frac{1}{\lambda} \right) & -\frac{i\omega R}{2c\lambda} & \frac{h_0 \rho_0 \omega^2}{\lambda \lambda_0} \\
 \frac{i\omega}{c} \alpha_0 & \frac{2\mu c (\beta^2)}{i\omega} R & \frac{2\mu c (\beta^2)}{i\omega} R & \frac{2\mu c (\beta^2)}{i\omega} R \\
\end{bmatrix}
\begin{bmatrix}
 q_x \\
 c_1 \\
 A_1 \\
 q_z \\
\end{bmatrix}
= 0.
\]

(115)

where \( \alpha_0 = \frac{h_0 \beta_{\text{eff}}}{\lambda_0^2 R_0} - \frac{h_0 \rho_0}{\lambda_0^2 R_0} \).

As stated before, for the system to generate a non-trivial solution, the determinant of equation (115) must be zero. Therefore, we have
\[
\begin{vmatrix}
\omega & -R & \frac{1}{2} \frac{R}{\lambda} & 0 \\
0 & -2c & \frac{1}{c} & -\omega^2 \\
-\frac{i\omega}{c} \alpha_0 & -\frac{2\mu c (\beta^2)}{\omega^2} & \frac{i\omega R}{2c \lambda} & \left[\frac{h_0 \rho \omega^2}{\lambda_1 \lambda_2} \right] \\
\omega^2 \frac{h_0 \rho}{\lambda_1 \lambda_2} & \frac{2 c^2 \rho}{i \omega} & \frac{1}{c^2} + \frac{2 c^2 \rho}{\omega^2} - 2 \mu & -\frac{i \omega \lambda}{c} \frac{h_0}{R_0} (\beta_{\epsilon_0} - \varphi_0) \\
-\frac{\lambda_1 \lambda_2}{c^2} \omega^2 \Psi_0 & \frac{4 c^2 f^* (\beta^2)}{\omega^2} & \left[\frac{\mu}{c^2} + \frac{2 c^2 \rho}{\omega^2} f^* \lambda \right] & \frac{2 i \omega \lambda}{c} \frac{h_0}{R_0} (\beta_{\epsilon_0} - \varphi_0)
\end{vmatrix} = 0
\]

(116)

After all simplifications have been accomplished, the system leads to a fourth order equation in \(c\).

\[
\alpha_1 c^4 + \beta_2 c^2 + \gamma_1 = 0,
\]

(117)

where
\[
\alpha_i = -\frac{2\omega^2 \rho h_0 \rho_0}{\lambda \lambda'_2} + \frac{4i\omega f^* \lambda_2 \rho_0}{\lambda \lambda'_2} + \frac{2 \rho h_0 \rho_0 \omega^3}{\lambda \lambda'_2} \left( \frac{\beta}{\lambda} \right)^2 - \frac{2 \omega \mu_0}{\lambda} \left( \frac{\beta}{\lambda} \right)^2 R + 4 \mu f^* \lambda \left( \frac{\beta}{\lambda} \right)^2 R
\]

\[
\beta_i = \frac{2i\omega \mu h_0 \rho_0}{\lambda \lambda'_2} + \frac{2 \rho h_0 \omega^3}{\lambda} (\beta + \Psi_0) \frac{h_0}{\lambda'_2} - 4i\omega f^* \lambda (\beta + \Psi_0) \frac{h_0}{\lambda'_2} + \frac{4 \omega h_0}{\lambda_0} \left( \beta - \psi_0 \right) \left( \frac{\beta}{\lambda} \right)^2 R
\]

\[
-\frac{2\omega^2 \rho}{\lambda} (\beta + \Psi_0) \frac{h_0}{\lambda'_2} - 4i\omega f^* \left( \frac{\beta}{\lambda} \right) (\beta + \Psi_0) \frac{h_0}{\lambda'_2} - \frac{2 \mu i \omega h_0 \rho_0}{\lambda \lambda'_2} - \frac{2 \omega^3 \mu R}{i \lambda} + 2 \omega^3 \mu R^* \lambda
\]

\[
+ \frac{\omega^2 R \rho}{\lambda} \alpha_0 - 2i\omega f^* \lambda \alpha_0,
\]

and

\[
\gamma_i = \frac{2i\omega \mu R h_0}{\lambda_0} (\beta - \psi_0) - \frac{2i\omega \mu}{\lambda} (\beta + \Psi_0) \frac{h_0}{\lambda'_2} - 2 \omega^3 \mu R h_0}{\lambda_0} (\beta - \psi_0)
\]

\[
+ \frac{2 \omega^3 \mu i \lambda (\beta + \Psi_0) \frac{h_0}{\lambda'_2} + \frac{\omega^3 \mu R}{\lambda} \frac{i \omega R}{\lambda} - 2 \omega^3 \mu}{\lambda_0} \alpha_0 (\beta - \psi_0)
\]

\[
- \frac{h_0 \omega^3 R}{\lambda \lambda'_2} \Psi_0 - \frac{R \omega^3 h_0 \rho_0}{\lambda \lambda'_2} (\beta + \Psi_0) + \frac{R \omega^3 h_0}{\lambda_0} (\beta + \Psi_0) \Psi_0
\]

\[
- \frac{R \omega^3}{\lambda_0} (\beta + \Psi_0) (\beta - \psi_0) \frac{h_0}{\lambda'_2} + \frac{i \omega R \mu}{\lambda} \alpha_0.
\]

For more details, see appendix C.

The roots of equation (117) are complex and can be represented as

\[ c = \theta_1 + i \theta_2 \], where both \( \theta_1 \) and \( \theta_2 \) are both real numbers. It's evident that two roots of equation (117) differ from the other two only in sign. The two for which \( \theta_1 \) is positive represent velocities for the outgoing waves while the
other two represent velocities for incoming waves.

More about equation (114) and the constants $q_v, q_w, A_1$ and $c_1$.

The homogenous system $A\vec{x} = \vec{0}$ with $\det(A) = 0$ can be solved by using Gauss-Jordan elimination. The system is expected to generate more than one solution. Linear algebra can be used to obtain the null space of $A$. The constants involved can be determined subject to both boundary and initial conditions.
CHAPTER 5

CONCLUSIONS, APPLICATIONS AND FUTURE WORK

5.1 Conclusion

The main goal of this thesis is to find components of the velocities of blood flow, $u$ and $v$, in the $r$ and $z$ directions, respectively, and the pressure $p$ with which the blood flows under stenotic conditions.

In the paper "A non-Newtonian fluid model for blood flow through arteries under stenotic conditions" (Misra, Patra and Misra, 1993) described in chapter 3, the same quantities were solved, i.e. the components of the velocities $u$ and $v$ and the pressure $p$, by expanding $u$, $v$ and $p$ in power series and using the method of perturbation

$$u = u_1 \delta + u_2 \delta^2 + ...$$
$$v = v_1 \delta + v_2 \delta^2 + ...$$
$$p = p_0 + p_1 \delta + p_2 \delta^2 + ...$$

to yields the first order correction terms $u_1$, $v_1$ and $p_1$ given below:
The first order perturbation corrections $u_1$, $v_1$ and $p_1$ can be taken to be the components of the velocities $u$, $v$ and the pressure $p$, only when the remaining terms of the perturbations of the velocities and the pressure can be assumed to be very small. We can therefore say that they used $u_1$, $v_1$ and $p_1$ as approximated to $u$, $v$ and $p$.

In this thesis, we provided a general model for the problem, using a direct approach to generate the general analytical forms for the quantities of $u$, $v$ and $p$, i.e.
\[ u(r) = \left\{ c, r \left[ 1 + \frac{1}{4} \left( \frac{\beta}{\lambda} \right)^2 r^2 \right] - \frac{1}{2} A_1 \frac{r}{\lambda} J_0(\lambda r) \right\} e^{ia(t-\frac{z}{c})}, \]

\[ v(r) = \frac{c}{i\omega} \left[ 2c_1 + \left( \frac{\beta}{\lambda} \right)^2 c_1 r^2 + \frac{1}{2} A_1 \lambda r J_1(\lambda r) - \frac{1}{\lambda} A_1 J_0(\lambda r) \right] e^{ia(t-\frac{z}{c})}, \]

\[ p(r) = \left\{ \left( \frac{c}{i\omega} \right) \left[ 2c_1 + \left( \frac{\beta}{\lambda} \right)^2 c_1 r^2 + \frac{1}{2} A_1 \lambda r J_1(\lambda r) - \frac{1}{\lambda} A_1 J_0(\lambda r) \right] \right\} e^{ia(t-\frac{z}{c})}, \]

where the constants \( c_1 \) and \( A_1 \) are to be determined.

The solution exists and is bounded. The harmonic nature of the general propagation is preserved. Moreover, equation (114) provides a firm mechanism to evaluate both \( c_1 \) and \( A_1 \). As a result, the work delineated in this thesis is completely self-consistent.

In this entire process, we have been working with an artery of dimension \( r = 4.35 \times 10^{-3} m \) [3]. Even though the general solution was obtained subject to the restriction that \( r^3 \int \Psi(r) dr \) is negligible, for \( r = 4.35 \times 10^{-3} m \), the general physical characteristics of the problem have been preserved. Equations (95), (97), and (100) generate the exact analytical form for the components of the velocities \( u, v \) and the pressure \( p \).
5.2 Applicability

Owing to the expressions of the components of the velocities of the blood \( u, v \) and the pressure \( p \) in this paper, if the values of the radius of the artery are known, \( u, v \) and \( p \) can be evaluated. This in turn will help to compute the distance from the center of the artery to the edge of the stenotic region. The knowledge acquired by this process will be of great help to medical practitioners and researchers. They need to know which size of the stenoses corresponds to particular values of the components of the velocity \( u, v \) and the pressure \( p \) with which the blood flows. The accurate knowledge of these quantities will permit medical practitioners to alert the patients of their risk conditions which may, in the long run, save the life of a patient.

5.3 Future Work

In the future, numerical analysis will be used to simulate the stenoses. That is to say, the components of the velocity and pressure of the blood flow will be numerically estimated. We will also consider results of actual experiments that have been conducted on stenoses and then compare our numerical results with their data to help medical scientists calibrate their instruments.

In addition, we will look for an appropriate transformation to recast the main equation of stenoses into Helmholtz’s equation and then use Twersky’s scattering theory to find the solution to the problem.
APPENDIX A

Bessel functions for cylindrical coordinate system:

\[ J_n = X^2 \sum_{m=0}^{\infty} \frac{(-1)^m X^{2m}}{2^m m!(n+m)!} \]

\[ J_n(\rho) = \sqrt{\frac{2}{\pi}} \cos \left[ \rho - \frac{n\pi}{2} - \frac{\pi}{4} \right] \]

\[ J_0(X) = \sum_{m=0}^{\infty} \frac{(-1)^m X^{2m}}{2^m m^2 m!} = 1 - \frac{X^2}{2^2(1)!} + \frac{X^4}{2^4(2)!^2} - \frac{X^6}{2^6(3)!^2} + \cdots \]

\[ J_0(\rho) = \sqrt{\frac{2}{\pi}} \cos \left[ \rho - \frac{\pi}{4} \right] \]

Finding derivatives of terms involving Bessel functions of order zero and one as used in equations (96) to (99)

Let \( \rho = \lambda r \Rightarrow r = \frac{\rho}{\lambda} \), then

\[ \frac{d}{dr} = \frac{d}{d\rho} \cdot \frac{d\rho}{dr} = \lambda \frac{d}{d\rho} \]

and \( \frac{d^2}{dr^2} = \frac{d}{d\rho} \left[ \frac{d}{dr} \right] = \lambda^2 \frac{d^2}{d\rho^2} \).

Hence,
\[
\frac{d}{dr} J_0 (\lambda r) = \lambda \frac{d}{d\rho} J_0 (\rho) = - \lambda J_1 (\rho) = - \lambda J_1 (\lambda r)
\]

\[
\frac{d^2}{dr^2} \left[ r J_1 (\lambda r) \right] = \lambda^2 \frac{d^2}{d\rho^2} \left[ \frac{\rho}{\lambda} J_1 (\rho) \right] = \lambda \frac{d}{d\rho} \left[ \frac{d}{d\rho} \left\{ \rho J_1 (\rho) \right\} \right]
\]

\[
= \lambda \frac{d}{d\rho} \left[ \rho J_0 (\rho) \right] = \lambda \left[ J_0 (\rho) + \rho J'_0 (\rho) \right] = \lambda \left[ J_0 (\rho) - \rho J_1 (\rho) \right]
\]

\[
\frac{d^2}{dr^2} \left[ r J_1 (\lambda r) \right] = \lambda \left[ J_0 (\lambda r) - \lambda r J_1 (\lambda r) \right]
\]

\[
\frac{d^2}{dr^2} J_0 (\lambda r) = \lambda^2 \frac{d^2}{d\rho^2} J_0 (\rho) = \lambda^2 \frac{d}{d\rho} \left[ \frac{d}{d\rho} J_0 (\rho) \right] = \lambda^2 \frac{d}{d\rho} \left[ - J_1 (\rho) \right]
\]

But \( \frac{d}{d\rho} [- J_1 (\rho)] = \frac{1}{\rho} J_1 (\rho) - J_0 (\rho) \), so

\[
\frac{d^2}{dr^2} J_0 (\lambda r) = \lambda^2 \left[ \frac{1}{\lambda r} J_1 (\lambda r) - J_0 (\lambda r) \right]
\]

\[
\frac{d}{dr} \left[ \lambda r J_1 (\lambda r) \right] = \lambda \frac{d}{d\rho} \left[ \rho J_1 (\rho) \right] = \lambda \rho J_0 (\rho)
\]

\[
\frac{d}{dr} \left[ \lambda r J_1 (\lambda r) \right] = \lambda^2 r J_0 (\lambda r)
\]

\[
\left[ J_1 (\lambda r) \right] dr = \left[ J_1 (\rho) \right] \frac{d\rho}{\lambda} = \frac{1}{\lambda} \left[ J_1 (\rho) \right] d\rho
\]

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\[ -\frac{1}{\lambda} J_0(\rho) = -\frac{1}{\lambda} J_0(\lambda r) \]

\[ \int J_1(\lambda r) dr = -\frac{1}{\lambda} J_0(\lambda r) \]

\[ \frac{d}{dr} \{ r J_0(\lambda r) \} = \lambda \frac{d}{d\rho} \left\{ \frac{\rho}{\lambda} J_0(\lambda \rho) \right\} \]

\[ = \frac{\lambda}{\lambda} \frac{d}{d\rho} \{ \rho J_0(\rho) \} = J_0(\rho) - \rho J'_0(\rho) \]

\[ = J_0(\rho) - \rho J_1(\rho) \]

\[ \frac{d}{dr} \{ r J_0(\lambda r) \} = J_0(\lambda r) - \lambda r J_1(\lambda r) \]
APPENDIX B

Constants used by Misra and Patra as used in chapter 3.

\[ \phi_0 = (1 + 2a_0) \left[ a_0 K_2^2(\infty) + b_0 K_3^3(\infty) \right], \]

\[ \Psi_0 = (1 + 2b_0) \left[ a_0 K_2^2(\infty) + b_0 K_3^3(\infty) \right], \]

\[ \beta_{\theta_0} = (1 + 2a_0) (\phi_{\theta_0} + \phi_{\theta_0}), \]

\[ \beta_{\theta_0} = (1 + 2b_0) (\phi_{\theta_0} + \phi_{\theta_0}), \]

\[ \beta_\theta = (1 + 2a_0) (\Psi_{\theta} + \Psi_{\theta}), \]

\[ \beta_z = (1 + 2a_0) (\Psi_{z} + \Psi_{z}), \]

where

\[ \phi_{\omega_0} = (1 + 2a_0) K_2^2(0), \]

\[ \phi_{\omega_0} = (1 + 2a_0) \int_0^\infty \frac{dK_2^2(s)}{ds} e^{-ivs} ds, \]

\[ \phi_{\omega_0} = (1 + 2a_0) K_3^3(0), \quad \phi_{\omega_0} = 0, \]

\[ \Psi_{\omega_0} = (1 + 2b_0) K_3^3(0), \]

\[ \Psi_{\omega_0} = (1 + 2b_0) \int_0^\infty \frac{dK_3^3(s)}{ds} e^{-ivs} ds, \]

\[ \Psi_{\omega_0} = (1 + 2b_0) K_3^3(0), \]

\[ \Psi_{\omega_0} = (1 + 2b_0) \int_0^\infty \frac{dK_3^3(s)}{ds} e^{-ivs} ds, \]

where

\[ K_2^2(t) = 282 + 22 \exp(-0.47t^{0.47}), \quad K_3^3(t) = 179 \]

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\[K_3^2(t) = 270 + 16\exp(-1.67t^{0.22}),\]

\[K_3^3(t) = 267 + 28\exp(-0.5t^{0.44}).\]
APPENDIX C

Finding the frequency equation of the problem by equating the determinant of the matrix A to zero as in equation (116)

\[
\begin{vmatrix}
    -2c & \frac{1}{\lambda} c & -\omega \dot{\lambda} \\
    \left(\frac{i\mu R}{c} - \frac{2\mu c}{\omega} \left(\frac{\beta}{\lambda}\right)^2\right) & \frac{i\omega \mu R}{2c \lambda} & \left\{\frac{\mu R}{\lambda \lambda_2} - \frac{\omega^2}{c^2} (\beta_{\omega} + \Psi_{\omega}) \frac{h_i}{\lambda_2}\right\} \\
    \left\{\frac{2c^2 \rho + 4c^2 f' \left(\frac{\beta}{\lambda}\right)^2}{i\omega} - 2\mu\right\} & \left\{-\frac{c^2 \rho}{i\omega} + \frac{2c^2}{\omega^2} f' \lambda + \mu\right\} & \frac{2i\omega \lambda h_i}{c R_0} (\beta_{\omega} - \Psi_{\omega}) \\
\end{vmatrix}
\]

\[
\begin{vmatrix}
    0 & \frac{1}{\lambda} c & -\omega \dot{\lambda} \\
    \frac{i\omega}{c} \alpha_0 & \frac{-i\omega \mu R}{2c \lambda} & \left\{\frac{\mu R}{\lambda \lambda_2} - \frac{\omega^2}{c^2} (\beta_{\omega} + \Psi_{\omega}) \frac{h_i}{\lambda_2}\right\} \\
    \left\{\frac{i\beta_0}{\lambda \lambda_2} - \frac{\lambda_i}{\lambda_2} \frac{\omega^2}{c^2} \Psi_{\omega}\right\} & \left\{-\frac{c^2 \rho}{i\omega} + \frac{2c^2}{\omega^2} f' \lambda + \mu\right\} & \frac{-2i\omega \lambda h_i}{c R_0} (\beta_{\omega} - \Psi_{\omega}) \\
\end{vmatrix}
\]

\[
\begin{vmatrix}
    0 & -2c & -\omega \dot{\lambda} \\
    \frac{1}{2\lambda} \frac{R}{\lambda} & \frac{-i\omega}{c} \alpha_i & \left\{\frac{i\mu R}{c} - \frac{2\mu c}{\omega} \left(\frac{\beta}{\lambda}\right)^2\right\} & \left\{\frac{\mu R}{\lambda \lambda_2} - \frac{\omega^2}{c^2} (\beta_{\omega} + \Psi_{\omega}) \frac{h_i}{\lambda_2}\right\} \\
    \left\{\frac{i\beta_0}{\lambda \lambda_2} - \frac{\lambda_i}{\lambda_2} \frac{\omega^2}{c^2} \Psi_{\omega}\right\} & \left\{\frac{2c^2 \rho + 4c^2 f' \left(\frac{\beta}{\lambda}\right)^2}{i\omega} - 2\mu\right\} & \frac{-2i\omega \lambda h_i}{c R_0} (\beta_{\omega} - \Psi_{\omega}) \\
\end{vmatrix} = 0
\]
\[ -2\alpha \left\{ \frac{\alpha^2 \mu R \lambda (\beta_{\text{sc}} - \varphi_0)}{c^2 R_0} - \frac{\alpha^2 (\beta_{\text{sc}} + \Psi_0) \lambda}{c^2 \lambda} \right\} \]

\[ + \frac{i \omega}{\lambda} \left\{ \frac{2 \alpha \mu R \lambda (\beta_{\text{sc}} - \varphi_0)}{c} - \frac{2 \mu c (\beta_{\text{sc}} + \Psi_0)}{c} \right\} \]

\[ - \frac{2 \alpha^2 \rho \lambda}{c^2 \lambda^2} \left\{ \frac{\alpha^2 (\beta_{\text{sc}} + \Psi_0) \lambda}{c^2 \lambda} \right\} \]

\[ + \left\{ \frac{-2 \alpha^2 \rho \lambda}{c^2 \lambda} \right\} \left\{ \frac{\alpha^2 (\beta_{\text{sc}} + \Psi_0) \lambda}{c^2 \lambda} \right\} \]

\[ = 0 \]
\[
\frac{2i\omega \mu R_h}{c R_0} (\beta_{\omega} - \phi_h) - \frac{2\omega \rho_i \rho_c c^2}{\lambda \lambda \lambda_2} + \frac{4i\omega \rho_i \rho_c c^2}{\lambda \lambda_2} + \frac{2i\omega' \rho_i \rho_c}{\lambda \lambda \lambda_2} + \frac{2\omega \rho_0^2}{\lambda} (\beta_{\omega} + \Psi_0) \frac{h_0}{\lambda_2} \]

\[-4i\omega f' \lambda c (\beta_{\omega} + \Psi_0) \frac{h_0}{\lambda_2} - \frac{2i\omega \mu}{c \lambda} (\beta_{\omega} + \Psi_0) \frac{h_0}{\lambda_2} - 2\omega^2 \frac{h_0}{R_0} (\beta_{\omega} - \phi_h) \left[ \frac{i \omega R}{c} \frac{2 \mu c}{\lambda} \left( \frac{\beta}{\lambda} \right)^2 R \right] \]

\[+ \left[ \frac{2 c^2 \rho}{\lambda} + \frac{4i c f'}{\lambda} \left( \frac{\beta}{\lambda} \right)^2 \right] \left[ - \frac{c^2 \rho}{i \omega \lambda} + \frac{2c^2 f' + \mu}{\lambda} \right] + \frac{i \omega R c}{\lambda} + \frac{2i \mu R c f'}{\lambda} \left( \frac{\beta}{\lambda} \right)^2 \frac{i \omega R}{c \lambda} \]

\[+ \frac{2 R i \omega}{c R_0} \alpha_0 (\beta_{\omega} - \phi_h) + \left[ \frac{\lambda h_0 \rho_i c R}{\lambda \lambda \lambda_2} - \frac{\alpha \omega}{c^2} (\beta_{\omega} + \Psi_0) \frac{h_0}{\lambda_2} \right] \left[ \frac{\omega h_0 \rho_i}{\lambda \lambda_2} + \frac{\lambda h_0 \rho_i \alpha}{c^2} \Psi_0 + \frac{h_0}{R_0} (\beta_{\omega} - 2 \phi_h) \right] \]

\[+ \frac{i \omega R}{c} \alpha_0 \left[ \frac{c^2 \rho}{i \omega \lambda} + \frac{2c^2 f' + \mu}{\lambda} \right] \left[ \frac{\omega h_0 \rho_i}{\lambda \lambda_2} + \frac{\lambda h_0 \rho_i \alpha}{c^2} \Psi_0 + \frac{h_0}{R_0} (\beta_{\omega} - 2 \phi_h) \right] \]

\[-\frac{2 \omega^2 R_0}{c} \alpha_0 (\beta_{\omega} - \phi_h) \alpha_0 \left[ \frac{h_0 \rho_i \rho_c c R}{\lambda \lambda \lambda_2} + \frac{\alpha \omega}{c^2} (\beta_{\omega} + \Psi_0) \frac{h_0}{\lambda_2} \right] \left[ \frac{\omega h_0 \rho_i}{\lambda \lambda_2} + \frac{\lambda h_0 \rho_i \alpha}{c^2} \Psi_0 + \frac{h_0}{R_0} (\beta_{\omega} - 2 \phi_h) \right] \]

\[\frac{\omega^2 R}{2 \lambda c} \alpha_0 \left[ \frac{2c^2 \rho}{i \omega} + \frac{4c^2 f'}{\omega^2} - 2 \mu \right] + \left[ \frac{i \omega R c^2}{2 \lambda c} \frac{\alpha c}{\lambda} \left( \frac{\beta}{\lambda} \right)^2 R \right] \left[ \frac{\omega h_0 \rho_i}{\lambda \lambda_2} + \frac{\lambda h_0 \rho_i \alpha}{c^2} \Psi_0 + \frac{h_0}{R_0} (\beta_{\omega} - 2 \phi_h) \right] = 0 \]

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\[
\frac{2i\omega \mu R h_i}{c R_0 (\beta_0 - \eta_0)} + \frac{2i\omega \rho \lambda^2}{\lambda^2} + \frac{4i\varphi \omega \lambda h_i}{\lambda^2} + \frac{2i\omega \rho \lambda}{\lambda} (\beta_0 + \Psi_0) \frac{h_i}{\lambda^2} + 2c \rho \lambda (\beta_0 + \Psi_0) \frac{h_i}{\lambda}
\]

\[
-4i\omega \lambda (\beta_0 + \Psi_0) \frac{h_i}{\lambda} - \frac{2i\omega \mu}{c \lambda} (\beta_0 + \Psi_0) \frac{h_i}{\lambda} - 2i\omega \frac{R h_i}{c R_0 (\beta_0 - \eta_0)} + 4i\varphi \rho \lambda h_i \frac{h_i}{\lambda} - 4i\omega \lambda (\beta_0 + \Psi_0) \frac{h_i}{\lambda}
\]

\[
+ \frac{2i\omega \mu}{\lambda} (\beta_0 + \Psi_0) \frac{h_i}{\lambda} - \frac{2i\omega \mu R \rho}{c \lambda} + \frac{2i\omega \mu R}{\lambda} + \frac{2i\omega \mu R}{\lambda} + \frac{2i\omega \mu R}{\lambda} + \frac{2i\omega \mu R}{\lambda} R + 4i\omega \lambda \left( \frac{\beta_0}{\lambda} \right) R
\]

\[
+ \frac{2i\omega \mu c}{\lambda} \left( \frac{\beta_0}{\lambda} \right) R + \frac{i\omega R c}{\lambda} + \frac{2i\omega R c}{\lambda} + \frac{i\omega R c}{\lambda} - 2i\omega \frac{h_i}{c R_0 (\beta_0 - \eta_0)} + \frac{\alpha (\beta_0 - \eta_0)}{\lambda c}
\]

\[
\frac{c c c}{\lambda^2 c} \left( \beta_0 + \Psi_0 \right) \frac{h_i}{\lambda^2} - \frac{R h_i}{\lambda^2 c} \frac{\beta_0}{\lambda^2} \frac{h_i}{\lambda^2} - \frac{R h_i}{\lambda^2 c} \frac{\beta_0}{\lambda^2} \frac{h_i}{\lambda^2} + \frac{\lambda h_i}{\lambda c} (\beta_0 + \Psi_0)
\]

\[
- \frac{R h_i}{\lambda^2 c} (\beta_0 + \Psi_0) \frac{h_i}{\lambda^2} - \frac{\alpha (\beta_0 - \eta_0)}{\lambda c} \alpha = 0
\]
\[ \begin{align*}
&\frac{2i\omega \mu R_h}{c R_0} (\beta_e - \omega_0) - \frac{2i\omega \rho h \rho c^2}{\lambda \lambda_2} + \frac{4i\omega \rho h \rho c^2}{\lambda_2} + \frac{2i\omega \rho h \rho c^2}{\lambda \lambda_2} + \frac{2e\rho \omega}{\lambda_2} (\beta_e + \Psi_0) \frac{h_0}{\lambda_2} \\
&- 4i\omega \rho \lambda c (\beta_e + \Psi_0) \frac{h_0}{\lambda_2} - \frac{2i\omega \rho \mu R_h}{c R_0} (\beta_e - \omega_0) + \frac{4i\omega \mu c R_h}{i R_0} (\beta_e - \omega_0) \left( \frac{\beta}{\lambda} \right)^2 R \\
&+ \frac{2\rho \rho h (\beta_e + \Psi_0) h_0}{\lambda_2} - \frac{2i\omega \rho c^2}{\lambda} (\beta_e + \Psi_0) \frac{h_0}{\lambda_2} + \frac{2i\omega \rho c^2}{\lambda_2} \left( \frac{\beta}{\lambda} \right)^2 R + 4i\alpha \omega \lambda c (\beta_e - \omega_0) \frac{h_0}{\lambda_2} \\
&+ \frac{2i\omega \rho c^2}{\lambda} \left( \frac{\beta}{\lambda} \right)^2 R + \frac{i \mu R_0}{\omega} \lambda + \frac{2i\omega \rho c^2}{\lambda} \left( \frac{\beta}{\lambda} \right)^2 R - 2i\alpha \omega \lambda c (\beta_e - \omega_0) + \frac{2\rho \rho h c^2}{\lambda_2} R \\
&- \frac{h_0^2 \omega h (\beta_e - \omega_0) R_0}{\lambda_2^2} + \frac{h_0^2 \rho \rho \omega^2 R_0}{\lambda_2^2} (\beta_e - \omega_0) + \frac{R \alpha h_0^2}{\lambda_2^2} (\beta_e + \Psi_0) \Psi_0 \\
&- \frac{R \alpha h_0^2}{\lambda_2^2} (\beta_e + \Psi_0) (\beta_e - \omega_0) \frac{h_0}{\lambda_2} + \frac{\omega^2 \rho c^2}{\lambda} R - 2i\omega \alpha \lambda c \frac{\omega_0}{\lambda} = 0
\end{align*} \]
Multiplying through by \( c \)

\[
\frac{2\omega \mu R h_0 (\beta_e - \varphi_i)}{R_0} - \frac{2\omega \varphi h_0 R c^4}{\lambda \lambda_2} + \frac{4i\omega \varphi \lambda h_0 R c^4}{\lambda \lambda_2} + \frac{2\omega \varphi h_0 R c^4}{\lambda \lambda_2} + \frac{2 i \omega \mu R h_0 (\beta_e - \varphi_i)}{\lambda \lambda_2} + \frac{4 i \omega \mu R h_0 (\beta_e - \varphi_i)}{\lambda \lambda_2} + \frac{2 \omega \varphi h_0 R c^4}{\lambda \lambda_2} \]

\[
-4i \omega \varphi \lambda (\beta_e + \varphi_i) \frac{h_0}{\lambda_2} c^2 - \frac{2 i \omega \mu (\beta_e + \varphi_i) h_0}{\lambda_2} \lambda h_0 R c^4 (\beta_e - \varphi_i) + \frac{4 i \omega \mu h_0 (\beta_e - \varphi_i)}{\lambda_2} (\beta_e - \varphi_i) \frac{(\beta_e - \varphi_i)^2}{\lambda_2} R c^4
\]

\[
+ \frac{2 i \omega \mu h_0}{\lambda_2} (\beta_e + \varphi_i) h_0 \frac{(\beta_e + \varphi_i)}{\lambda_2} \lambda h_0 R c^4 + \frac{4 i \omega \mu h_0}{\lambda_2} (\beta_e + \varphi_i) h_0 \frac{(\beta_e + \varphi_i)}{\lambda_2} \lambda h_0 R c^4 + \frac{2 i \omega \mu h_0}{\lambda_2} (\beta_e + \varphi_i) h_0 \frac{(\beta_e + \varphi_i)}{\lambda_2} \lambda h_0 R c^4
\]

\[
+ \frac{2 i \omega \mu (\beta_e + \varphi_i) h_0}{\lambda_2} (\beta_e + \varphi_i) h_0 \frac{(\beta_e + \varphi_i)}{\lambda_2} \lambda h_0 R c^4 + \frac{2 i \omega \mu (\beta_e + \varphi_i) h_0}{\lambda_2} (\beta_e + \varphi_i) h_0 \frac{(\beta_e + \varphi_i)}{\lambda_2} \lambda h_0 R c^4
\]

\[
- \frac{k \omega R h_0 \varphi o}{\lambda_2} (\beta_e - \varphi_i) c^2 - \frac{k \omega R h_0 \varphi o}{\lambda_2} (\beta_e - \varphi_i) c^2 - \frac{R \omega \lambda h_0 \varphi o}{\lambda_2} (\beta_e + \varphi_i) + \frac{R \omega \lambda h_0 \varphi o}{\lambda_2} (\beta_e + \varphi_i) \Psi_0
\]

\[
- \frac{R \omega \lambda h_0 \varphi o}{\lambda_2} (\beta_e + \varphi_i) (\beta_e - \varphi_i) - \frac{\omega \lambda h_0 \varphi o}{\lambda_2} c^2 - \frac{2 i \omega \varphi \lambda c^2}{\lambda_2} - \frac{i \omega \lambda h_0 \varphi o}{\lambda_2} c^2 = 0
\]
REFERENCES


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