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FOUR LOCAL COLORINGS

OF GRAPHS

by

James Katseanes

Bachelor of Science
Westminster College, Salt Lake, Utah
2002

A thesis submitted in partial fulfillment
of the requirements for the

Master of Science Degree in Mathematical Sciences
Department of Mathematical Sciences
College of Sciences

Graduate College
University of Nevada, Las Vegas
August 2007

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Examination Committee Chair

Dean of the Graduate College

Examination Committee Member

Examination Committee Member

Graduate College Faculty Representative

1017-53

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ABSTRACT

Four Local Colorings of Graphs

by

James Katseanes

Dr. Ebrahim Salehi, Examination Committee Chair
Associate Professor of Mathematics
University of Nevada, Las Vegas

A proper coloring of a graph G is a mapping $c : V(G) \longrightarrow \mathbb{N}$ that assigns colors to the vertices of G in such a way that adjacent vertices are labeled with different colors. A coloring c is called a 4-local coloring if for every subset $S \subseteq V(G)$, with $2 \leq |S| \leq 4$ there are two vertices u, v such that the difference between colors of u and v, is greater than or equal to the number of edges in the subgraph induced by S. That is,

$$\forall S \; \exists u, v \in S \ni |c(u) - c(v)| \geq m_s,$$

where m_s is the number of edges in the subgraph induced by S, $m_s = |E(< S >)|$. The maximum color assigned by a local coloring c to a vertex of G is

$$\chi_4(c) = \max\{c(v) | v \in V(G)\}.$$

The four local chromatic number of G is defined as

$$\chi_4(G) = \min\{\chi_4(c)\} = \min\{\max\{c(G) | \text{where c is a 4-local coloring of G}\}\}.$$

In this thesis, the four local chromatic number of some well known class of graphs will be determined.

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CHAPTER 1

INTRODUCTION

Historically graphs have helped to shed light on various mathematics problems. One such problem, described in [1], is the scheduling of courses at a college. Two requirements are stated in this problem;

1. No two courses can meet at the same time, in case a student plans to take both courses and

2. We want a schedule that is most efficient, a schedule requiring the fewest number of time periods.

The question is: what is the minimum number of time slots needed for such a schedule?

This scheduling problem is related to graphs by letting each class of the schedule be a vertex of a graph G. An edge connecting two vertices represents that some student is a member of both corresponding classes. The minimum number of time slots needed for such a schedule of classes is called the chromatic number of G.

The four-color mapping theorem described in [4] is a graph theory problem that involves chromatic numbers on planar graphs. A planar graph can be drawn on a plane in such a way that no two edges intersect except at a vertex. The four-color theorem states that the chromatic number of all planar graphs is less than or equal to four. In other words, any arbitrary map of counties (planar graph) can be colored, with at most four colors, in such a way that no two adjacent countries have the same color.

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A function $c : V(G) \longrightarrow \mathbb{N}$ is called a proper coloring of a graph G if whenever $u, v \in V(G)$ are adjacent, then $c(u) \neq c(v)$. An n-coloring of G is a proper coloring using n colors. The chromatic number of a graph G, denoted $X(G)$, is defined as the minimum value n for which an n-coloring exists. That is, $\chi(G)$ =min$\{n|$ G has an n-coloring$\}$. Another way to define the chromatic number is as follows. Consider the function f where $f : V(G) \longrightarrow \mathbb{N}$. Let f have the property that $\forall S \subseteq V(G)$, with $S = \{u, v\}$, then $|f(u) - f(v)| \geq m_s$, where m_s is the number of edges in the subgraph induced by vertices u,v. f is called a 2-local coloring of G. $\chi_2(f)$ =max$\{f(v)|$where $v \in V(G)\}$ and $\chi_2(G)$ =min$\{$max$\{f(G)|$where f is a 2-local coloring$\}\}$. This yields an equivalent chromatic definition.

Theorem 1. $\chi_2(G) = \chi(G)$.

Proof. Suppose $f : V(G) \longrightarrow \mathbb{N}$ is a 2-local coloring, then $\forall S \subset V(G)$, with $S = \{u, v\}$, we have $|f(u) - f(v)| \geq m_s$. If u and v are adjacent, then $|f(u) - f(v)| \geq m_s = 1$. Therefore certainly $f(u) \neq f(v)$ for all adjacent vertices. If u,v are not adjacent, then $|f(u) - f(v)| \geq 0$. In either case f meets the requirements of a proper coloring of G.

Conversely, let c be any proper coloring of G. Let $\chi(G) = n$, $S = \{u, v\}$ and $m_s = |E(< S >)|$. Then since c is a proper coloring, $c(u) \neq c(v)$ for all adjacent vertices u and v. If u and v adjacent, then $|c(u) - c(v)| \geq 1$. If u and v are not adjacent, then $|c(u) - c(v)| \geq 0$. In either case $|c(u) - c(v)| \geq m_s$. Hence c is a 2-local coloring of G. Consequently the set of all 2-local colorings of G equals the set of all proper colorings of G. That is,

$$\{f|\ f \text{ is a 2-local coloring of } G\} = \{c|\ c \text{ is a proper coloring of G}\}$$

$$\max\{f|\ f \text{ is a 2-local coloring of } G\} = \max\{c|\ c \text{ is a proper coloring of G}\}$$

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min$\{$max$\{f|\ f$ is a 2-local coloring of $G\}\}$ =min$\{$max$\{c|\ c$ is a proper coloring of G$\}\}$

Henceforth, $\chi_2(G) = \chi(G)$. ☐

In view of this equivalency Chartrand et al [2] generalized the concept $\chi(G)$ as follows: Let $k \geq 2$, define a function $c : V(G) \longrightarrow \mathbb{N}$ to be a k-local coloring if $\forall S \subseteq V(G)$ with $2 \leq |S| \leq k$, $\exists\ u, v \in V(G) \ni |c(u) - c(v)| \geq m_s$, where m_s is the number of edges of the subgraph induced by S. The k-local chromatic number is defined as $\chi_k(G)$ =min$\{$max$\{c(G) \ni$ c is a k-local coloring$\}\}$.

Theorem 2. *If f is any k-local coloring of G that generates the k-local chromatic number, then there exists a vertex v in $V(G)$ such that $f(v) = 1$ and a vertex u in $V(G)$ such that $f(u) = \chi_k(G)$. The numbers 1 and $\chi_k(G)$ are always used in the k-local coloring that produces the k-local chromatic number.*

Proof. Suppose that $\chi_k(G) = s$ and that function f provides this chromatic number. Suppose that $t = \min\{f(v)|v \in V(G)\}$ and $t > 1$. Since s is the chromatic number, s =max$\{f(v)|v \in V(G)\}$ and s =min$\{$max$\{c(G) \ni c$ is a k-local coloring$\}\}$. Then the function $h : V(G) \longrightarrow \mathbb{N}$ defined by $h(v) = f(v) - 1$ is a k-local coloring of G with $\max\{h(v)|v \in V(G)\} = s - 1$. Hence $\max\{h(v)|v \in G)\} < \max\{f(v)|v \in G)\}$ thus contradicting s =min$\{$max$\{c(G) \ni c$ is a k-local coloring$\}\}$. Hence f does not produce $\chi_k(G)$ when $t > 1$. Therefore $t = 1$. ☐

Theorem 3. *If H is a subgraph of G, then $\chi_k(H) \leq \chi_k(G)$.*

Proof. Suppose $\chi_k(H) > \chi_k(G)$ and H is a subgraph of G. Suppose the function f provides the local chromatic number for G and H, then $\max\{f(v)|v \in G)\} < \max\{f(v)|v \in H)\}$. Hence there is a at least one vertex v in V(H) that is not in V(G). This contradicts H as a subgraph of G. Therefore $\chi_k(H) \leq \chi_k(G)$ is necessary for H a subgraph of G. ☐

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CHAPTER 2

ESTABLISHED RESULTS

Let G be a graph with finite vertices. $V(G)$ is the set of vertices in G. The size of $V(G)$ is called the order of G. $E(G)$ is the set of edges in G. Adjacent and coincident are analogous terms meaning that two vertices have an edge in common. The degree of a vertex is defined to be the number of adjacent vertices. The compliment of G is denoted G^c and has the same set of vertices, $V(G) = V(G^c)$ and u and v are adjacent in G, if and only if u and v are not adjacent in G^c. If G_1 and G_2 are isomorphic graphs, then the degrees of the vertices of G_1 are exactly the degrees of G_2. If u and v be two vertices of a tree G. The diameter of a tree graph G is the max u-v path in G. A famous result from Euler is if G is a connected plane graph with p vertices, q edges and r regions, then $p - q + r = 2$.

The following results for three local chromatic colorings on finite graphs are due to [2]. A 3-local coloring is defined as follows; Let f be a function such that $f : V(G) \longrightarrow \mathbb{N}$, $\forall S \subset V(G)$, with $|S| \leq 3.$, $\exists\ u, v \in V(G) \ni |f(u) - f(v)| \geq m_s = E(< S >)$. Let $c(f)$=max$\{f(v)|$ $v \in V(G)\}$, then the three local chromatic number is defined as $\chi_3(G)$ =min$\{c(f)|$ f is a 3-local coloring$\}$.

Theorem 4. *For every graph G of order at least 3,* $\chi_2(G) \leq \chi_2(G) \leq 2\chi_3(G) - 1$.

Theorem 5. *For* $n \geq 3$, $\chi_3(P_n) = 3$.

Theorem 6. *For* $n \geq 3$, $\chi_3(K_n) = \lfloor \frac{3n-1}{2} \rfloor$.

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Theorem 7. *Let* $G = K_{n_1,n_2,\ldots,n_k}$, *where* $k \geq 2$ *and* $n_i \geq 2$ *for all* i *with* $1 \leq i \leq k$. *Then* $\chi_3(G) = 2k - 1$.

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CHAPTER 3

FOUR LOCAL CHROMATIC NUMBER

In this chapter, we consider the subsets with $2 \leq |S| \leq 4$. The function $f : V(G) \longrightarrow \mathbb{N}$ is said to be a 4-Local Coloring (4LC) if for any $S \subseteq V(G)$, with $2 \leq |S| \leq 4$ there are vertices $u, v \in S$ such that $|f(u) - f(v)| \geq m_s$, where m_s is the number of edges in the subgraph induced by S. The maximum color assigned by a 4-local coloring c to a vertex of G is $n(f) = \max\{f(v)|v \in V(G)\}$. The four local chromatic number of G is defined as $\chi_4(G) = \min\{n(f)| f \text{ is a 4-local coloring}\}$. We define a 4-Local Chromatic Coloring (4LCC) of G to be a 4LC that provides the $\chi_4(G)$. In the Figure 3.1 all possible graphs with order four are illustrated. Note that all Graphs G are paired with their complement G^c, except for the Path graph P_3. If $G = P_3$, then $G = G^c$.

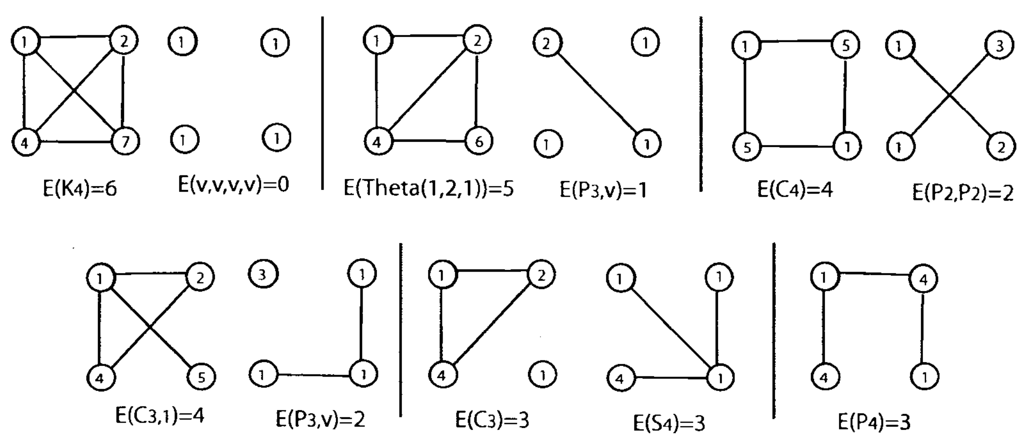

Figure 3.1: Four Local Chromatic Colorings on Graphs of Order 4.

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Four Local Colorings of Trees

A Tree is a connected graph that contains no cycle subgraphs. A path is a simple example of a Tree. Paths are Trees with no branches. We first establish a few results for Paths.

Theorem 8. *If $n \geq 4$, then $\chi_4(P_n) = 4$.*

Proof. Let $v_1, v_2, ..., v_n$ be the vertices of P_n as illustrated in figure 3.2

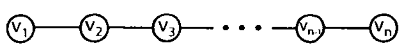

Figure 3.2: Graph of a Typical Path of Order n.

Consider the function $f : V(P_n) \longrightarrow \mathbb{N}$ defined by

$$f(v_i) = \begin{cases} 1 & \text{if i is odd,} \\ 4 & \text{if i is even.} \end{cases}$$

where $v_i \in V(P_n), 1 \leq i \leq n$. The function f is a 4LC. Therefore $\chi_4(P_n) \leq 4$.

On the other hand if $g : V(P_4) \longrightarrow \mathbb{N}$ is any 4LC of G then we choose $S = \{v_1, v_2, v_3, v_4\}$. There should be vertices $v_i, v_j \ni |g(v_i) - g(v_j)| \geq 3$. Therefore $\max\{g(v)|v \in V(G)\} \geq 4$.

Hence $4 \leq \chi_4(P_n)$. Therefore $4 \leq \chi_4(P_n) \leq 4$. Henceforth $\chi_4(P_n) = 4$ □

Theorem 9. *In any 4LCC of P_{4n} at least n vertices are colored* 4 *and at least n vertices are colored* 1.

Proof. We proceed by induction on n.

For $n = 1$, $\chi_4(P_4) = 4$ and by Theorem 2, the colors 1 and 4 must be used.

Assume the statement is true for $n = k$, we wish to show that it is also true for $n = k + 1$.

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Suppose that if c is any 4LCC of P_{4k}, then at least k vertices are colored 4 and k vertices are colored 1. To show that for any 4LCC of $P_{4(k+1)}$ at least $k+1$ vertices are colored 4 and $k+1$ vertices are colored 1, we note that $P_{4(k+1)}$ is just P_{4k+4}, which is P_{4k} along with P_4. We know from the Inductive step that if c is any 4LCC of P_{4k}, then at least k vertices are colored 4 and k vertices are colored 1. From the Base step, if c is any 4LCC of P_4 then at least 1 vertex is colored with a 4 and 1 vertex is colored with a 1. Thus $k+1$ vertices are colored 4 and $k+1$ vertices are colored 1. Therefore in any 4LCC of $P_{4(k+1)}$ at least $k+1$ vertices are colored 4 and $k+1$ vertices are colored 1. Hence, in any 4LCC of P_{4n} at least n vertices are colored 4 and n vertices are colored 1. $\square$

Trees T_n have paths as subgraphs. The diameter of T_n is the max u-v path in T_n. Let P_n represent the diameter of T_n. Note that P_n need not be unique. Choose either terminal vertex of P_n to be labeled v_1. Let $v_1, v_2, ..., v_n$ be the vertices of P_n with v_j adjacent to v_{j+1}, $1 \leq j \leq n-1$ as illustrated in Figure 3.3. The vertex labeled v_1 is considered the initial vertex and acts as the root of T_n. Define the depth of a vertex $v \in V(T_n)$ to be the number of edges, that lie in a v-v_1 path. Let $d(v)$ denote the depth of a vertex; the number of edges in the unique path $v - v_1$.

Clearly $d(v_1) = 0$, and diameter of a tree is equal to the maximum depth. The vertex $v_n \in V(P_n) \subseteq V(T_n)$ has depth $n-1$, $d(v_n) = n-1$. Thus the diameter of T_n is $n-1$. The vertex $v_h \in V(P_n)$ has depth $h-1$, $d(v_h) = h-1$. If the vertex $v_i \in V(T_n)$ has depth k, $d(v_i) = k$, then k number of edges (and $(k-2)$ vertices that form a path) separate v_i from v_1. In Figure 3.3, the vertices not on P_n with depth k are labeled d_k.

Theorem 10. $\chi_4(T_n) = 4$, $n \geq 4$.

Proof. Suppose, $n \geq 4$. To show that $\chi_4(T_n) \leq 4$, consider the function $c : V(T_n) \longrightarrow \mathbb{N}$,

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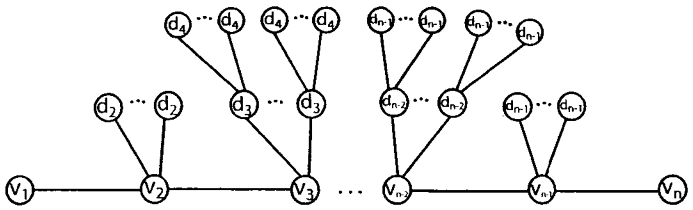

Figure 3.3: Tree Graph with Diameter n-1.

defined for $v \in V(T_n)$ as

$$c(v) = \begin{cases} 1 & \text{if } d(v) \text{ is even;} \\ 4 & \text{if } d(v) \text{ is odd.} \end{cases}$$

In order to show that this function c provides a 4LC of Tree T_n, Choose any subset $S \subseteq V(T_n)$ with $2 \leq |S| \leq 4\}$. Then the induced subgraph $< S >$ has at most 3 edges. If S contains one or more adjacent vertices then $\exists u, v \in S \ni c(u) = 1$ and $c(v) = 4$, hence $|c(u) - c(v)| = |1 - 4| = 3 \geq m_s = |E(< S >)|$. If there are no vertices coincident in S then $m_s = |E(< S >)| = 0$, it follows that $\exists u, v \in S \ni |c(u) - c(v)| \geq 0 = m_s = |E(< S >)|$. In either case c provides a coloring such that $\forall S \subseteq V(T_n)$ with $2 \leq |S| \leq 4$, $\exists u, v \in S \ni |c(u) - c(v)| \geq m_s = |E(< S >)|$. Therefore the function c is a FLC, thus $\chi_4(T_n) \leq 4$.

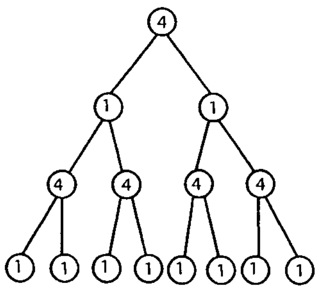

Figure 3.4: 4LC of a Tree with Diameter 6.

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The four local chromatic number for paths along with the use Theorem 3 will set the lower bound for Tree graphs. To show $\chi_4(T_n) \geq 4$, $n \geq 4$, note that all trees have paths as subgraphs, then using the Theorem 3 we have, if P_k is a subgraph of T_n for $4 \leq k \leq n$, then $\chi_4(P_k) \leq \chi_4(T_n)$. By Theorem 8 we have that $4 = \chi_4(P_n) \leq \chi_4(T_n)$. Therefore $4 \leq \chi_4(T_n) \leq 4$, Hence $\chi_4(T_n) = 4$. □

Trees of Diameter 2 (Stars)

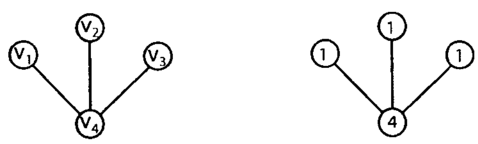

Figure 3.5: ST_4 and a 4-local Coloring of ST_4.

Let ST_n be a star with n vertices. ST_n has a central vertex that is adjacent to $n-1$ terminal vertices. Label the terminal vertices in ST_n as $v_1, v_2, \ldots, v_{n-1}$ and label the central vertex v_n as illustrated in Figure 3.6. Note that $ST_2 = P_2$ and $ST_3 = P_3$ and their 4-local chromatic number have already been determined. So, for ST_2 we may assume $n \geq 4$.

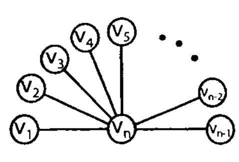

Figure 3.6: Graph of ST_n.

Also, since ST_n is a Tree with diameter two, by Theorem 10, if $n \geq 4$, then $\chi_4(ST_n) = 4$.

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The function $c : V(ST_n) \longrightarrow \mathbb{N}$, defined by.

$$c(v_i) = \begin{cases} 1 & \text{if } 1 \leq i \leq n-1; \\ 4 & \text{if } i = n. \end{cases}$$

Provides a four local coloring for ST_n.

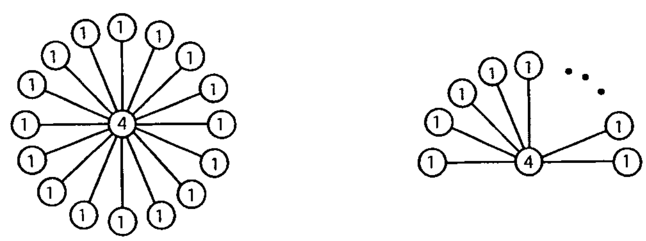

Figure 3.7: Proper Colorings on ST_{17} and ST_n.

Four Local Colorings of Stars with order 17 and order n are illustrated in 3.7.

<u>Special case: Star $*ST_5$</u>

$*ST_5$ is defined as the Star with the three terminal vertices colored with fours and one terminal vertex colored with a one.

Figure 3.8: Star $*ST_5$ and a Proper Coloring of $*ST_5$

Theorem 11. $\chi_4(*ST_5) = 7$.

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Proof. Let c be a function such that $c : V(*ST_5) \longrightarrow \mathbb{N}$. Let c be a proper coloring of $*ST_5$. Let $V(*ST_5) = \{v_1, v_2, v_3, v_4, v_5\}$, with $c(v_1) = c(v_2) = c(v_3) = 4$ and $c(v_4) = 1$. This implies v_1, v_2, v_3 andv_4 share no edges. Thus v_5 is the so called central vertex that is coincident to other four terminal vertices. Notice $|E(*ST_5)| = 4$. To find an upper bound on $\chi_4(*ST_5)$, determination of $c(v_5)$ is needed. What does the central vertex v_5 have to be colored with in order c be a proper coloring? Let H_i represent one of the 5 subgroups of $*ST_5$ with $|H_i| = 4$, $1 \leq i \leq 5$. Let $V(H_1) = \{v_5, v_2, v_3, v_4\}$, $V(H_2) = \{v_1, v_5, v_3, v_4\}$, $V(H_3) = \{v_1, v_2, v_5, v_4\}$, $V(H_4) = \{v_1, v_2, v_3, v_5\}$ and $V(H_5) = \{v_1, v_2, v_3, v_4\}$. We conclude $|E(H_1)| = |E(H_2)| = |E(H_2)| = |E(H_4)| = 3$ and $|E(H_5)| = 0$. Consider the subgroup H_4, which has the three terminal vertices colored with fours and the coloring $c(v_5)$ is undetermined. Essentially $\chi_4(H_4)$ is desired. For the function c to be a proper coloring, the following inequality must be satisfied. $|c(v_5) - c(v_i)| \geq m_{H_4} = |E(H_4)| = 3$, with $1 \leq i \leq 3$. Hence $|c(v_5) - 4| \geq 3$. Thus there are two options; 1. $c(v_5) - 4 \leq -3$. Which yields $c(v_5) \leq 1$, in which $c(v_5) = 1$. This contradicts c being a proper coloring; since v_4 and v_5 are coincident, they must be colored with different colors. Hence $c(v_5) \neq 1$. Or 2. $c(v_5) - 4 \geq 3$. Which yields $c(v_5) \geq 7$. Let $c(v_5) = 7$ This c is an optimal efficient proper coloring of $*ST_5$, since we need $\chi_4(G) = \min \{\chi_4(c) | \text{where c is a 4-local coloring}\}$. Hence $\chi_4(*ST_5) \leq 7$.

An ad absurdum argument shows 7 is a lower bound if we assume falsely $\chi_4(*ST_5) < 7$. Thus $\chi_4(*ST_5) \leq 6$, this implies the existence of a coloring such that $c(v_5) = 6$. Which gives the result of the maximum difference between colors is $|c(v_5) - 4| = |6 - 4| = 2 \ngeq 3 = m_{H_4}$. Thus there does not exist a function c that is a proper coloring of $*ST_5$, such that $c(v_5) = 6$. Hence $6 < \chi_4(*ST_5)$ and thus $7 \leq \chi_4(*ST_5)$. Therefore $7 \leq \chi_4(*ST_5) \leq 7$, henceforth $\chi_4(*ST_5) = 7$. □

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Trees of Diameter 3 (Double Stars)

Double stars are Trees of diameter three, Hence $\chi_4(DS_n) = 4$. Figure 3.9 shows a 4LC of DS_n graph.

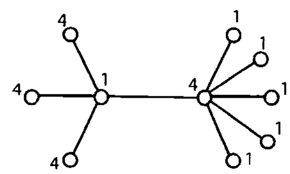

Figure 3.9: 4LCC of a Double Star

Trees Containing P_4 as Subgraphs (Caterpillars)

Since Caterpillar are Trees, then by theorem 10, $\chi_4(Caterpillar) = 4$. Figure 3.10 shows a Four Local Coloring of a Caterpillar graph.

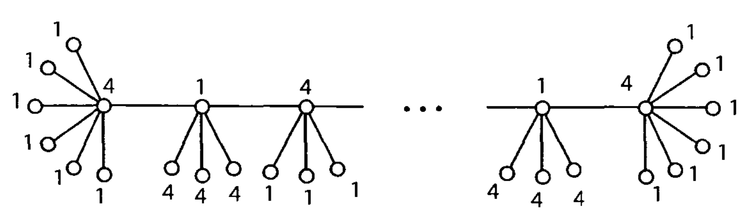

Figure 3.10: 4LCC of Caterpillar with diameter n

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CHAPTER 4

FOUR LOCAL COLORINGS OF CYCLE RELATED GRAPHS

Theorem 12. *If $n \geq 3$, then*

$$\chi_4(C_n) = \begin{cases} 4 & \text{if } n \neq 4; \\ 5 & \text{if } n = 4. \end{cases}$$

Proof. **Case $n = 3$.** Let v_1, v_2, v_3 and be the vertices of C_3 as illustrated in Figure 4.1.

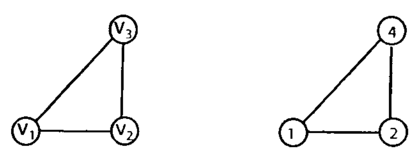

Figure 4.1: Cycle C_3 and its 4LC.

To show that $\chi_4(C_3) \leq 4$, consider the function $c : V(C_4) \longrightarrow \mathbb{N}$, defined by $c(v_1) = 1$, $c(v_2) = 2$ and $c(v_3) = 4$, This function c provides a 4LC of C_3. Therefore $\chi_4(C_3) \leq 4$. To show that $\chi_4(C_3) \geq 4$, suppose $\chi_4(C_3) < 4$, by letting $\chi_4(C_3) = 3$. This would imply that there exists a function c that is a proper coloring, with $c : V(C_3) \longrightarrow \mathbb{N}$, such that $\max\{c(v_i)|v_i \in V(C_3)\} = 3$. Thus $\exists v_i \in V(C_3) \ni c(v_i) = 3$ and $\exists\ v_j \in V(C_3) \ni c(v_j) = 1$. The maximum difference of colors is $|c(v_i) - c(v_j)| = |3 - 1| = 2 \gneqq= m_s$. This contradicts c being a proper coloring, thus there does not exist a c such that $c(C_3) = 3$. Thus $3 < \chi_4(C_3)$, which implies $4 \leq \chi_4(C_3)$. Hence $4 \leq \chi_4(C_3) \leq 4$. Therefore $\chi_4(C_3) = 4$.

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Case $n = 4.$; The 4 local chromatic number of a cycle with four vertices is a special case since the number of edges, in the only subgroup with four vertices, is four; i.e. $m_s = |E(C_4)| = 4$. Let v_1, v_2, v_3 and v_4 be the vertices of C_4 as illustrated in Figure 4.2.

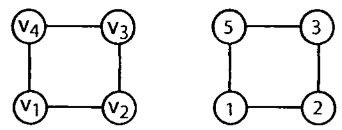

Figure 4.2: Cycle Graph C_4 and its 4-local coloring.

To show that $\chi_4(C_4) \leq 5$, consider the function $c : V(C_4) \longrightarrow \mathbb{N}$,defined by $c(v_1) = 1$, $c(v_2) = 2$, $c(v_3) = 3$ and $c(v_4) = 5$. This function c is a proper coloring of C_4. Therefore $\chi_4(C_4) \leq 5$.

Suppose $\chi_4(C_4) < 5$, by letting $\chi_4(C_4) = 4$. This would imply that there exists a function c that is a proper coloring, with $c : V(C_4) \longrightarrow \mathbb{N}$, such that $\max\{c(v_i)|v_i \in V(C_4)\} = 4$. Thus $\exists v_i \in V(C_4) \ni c(v_i) = 4$ and $\exists\ v_j \in V(C_4) \ni c(v_j) = 1$. The maximum difference of colors is $|c(v_i) - c(v_j)| = |4 - 1| = 3 \not\geq 4 = m_s$. This contradicts c being a proper coloring, thus there does not exist a c such that $c(C_4) = 4$. Thus $4 < \chi_4(C_4)$, which implies $5 \leq \chi_4(C_4)$. Hence $5 \leq \chi_4(C_4) \leq 5$. Therefore $\chi_4(C_4) = 5$.

Case $n \geq 5$. Let $v_1, v_2, ..., v_n$ be the vertices in $V(C_n)$ with v_n adjacent to v_1 and v_i adjacent to v_{i+1}, $1 \leq i \leq n-1$ as illustrated in Figure 4.3.

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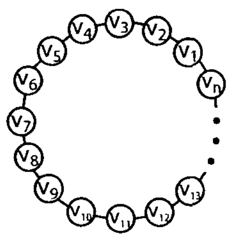

Figure 4.3: Cycle of Order n.

Consider the function $c : V(C_n) \longrightarrow \mathbb{N}$ for $n \geq 5$ defined by

$$c(v_i) = \begin{cases} 1 & \text{if i is odd;} \\ 3 & \text{if i is odd and i=n;} \\ 4 & \text{if i is even;} \end{cases}$$

This function c is a 4LC of C_n. Hence $\chi_4(C_n) \leq 4$.

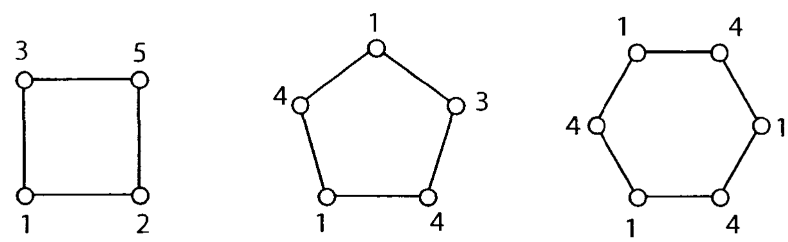

Figure 4.4: Four-local colorings on Cycles c_4, c_5, c_6.

Since P_4 is a subgraph of C_n with $n \geq 4$, then from theorem 3, $\chi_4(P_4) \leq \chi_4(C_n)$. From Theorem 10 we have $4 = \chi_4(P_4) \leq \chi_4(C_n)$. Hence $4 \leq \chi_4(C_n)$. Hence $4 \leq \chi_4(C_n) \leq 4$.

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Therefore $\chi_4(C_n) = 4$. Henceforth, If $n \geq 3$, then

$$\chi_4(C_n) = \begin{cases} 4 & \text{if } n \neq 4; \\ 5 & \text{if } n = 4. \end{cases}$$

□

Special case of C_4.

Let the special Cycle $*C_4$ have two vertices labeled with fours.

Theorem 13. $X_4(*C_4) = 8$.

Proof. Let the vertices of $*C_4$ be labeled v_1, v_2, v_3 and v_4. Let v_4 be adjacent to v_1 and let v_i be adjacent to v_{i+1}, for $1 \leq i \leq 3$. Notice $4 = m_s = |E(*C_4)|$. Consider the function $c : V(*C_4) \longrightarrow \mathbb{N}$, defined by $c(v_2) = c(v_4) = 4$ as shown on left in Figure 4.5.

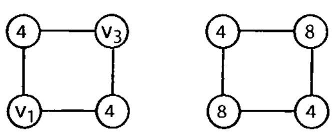

Figure 4.5: $*C_4$ and 4LC of $*C_4$.

Let $c(v_1) = c(v_3) = 8$. This function c is a proper coloring of $*C_4$, since $\forall S \subseteq V(*C_4)$, with $2 \leq |S| \leq 4$, $\exists u, v \in S \ni |c(u) - c(v)| \geq m_s$. Therefore $\chi_4(*C_4) \leq 8$.

Since $*C_4$ has the same structure as C_4, then the requirement that two vertices must be labeled with a 4 could only increase the value of the 4 chromatic number. Therefore by Theorem 3, $\chi_4(C_4) \leq \chi_4(*C_4)$. From section 10, $\chi_4(C_4) = 5$. Thus $5 = \chi_4(C_4) \leq \chi_4(*C_4)$. Suppose $7 \leq \chi_4(*C_4)$. This implies the existence of c, a function that is a proper coloring

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of $*C_4$ such that $c : V(*C_4) \longrightarrow \mathbb{N}$ with $\exists v_i \in V(*C_4) \ni 7 = \max\{c(v_i)|v_i \in V(*C_4) = (v_1, v_2, v_3, v_4)\}$. Let $c(v_1) = 7$ and let $c(v_2) = c(v_4) = 4$. The maximum difference between colors is $|c(v_1) - c(v_2)| = |7 - 4| = 3 \ngeq 4 = m_s$, thus there does not exist a function c such that $7 = \max\{c(v_i)|v_i \in V(*C_4) = (v_1, v_2, v_3, v_4)\}$. Thus there does not exist vertices $u, v \in V(*C_4) \ni |c(u) - c(v)| \geq m_s = 4$. This contradicts that c is a 4LC of $*C_4$, and the supposition $7 \leq \chi_4(*C_4)$. Hence $7 < \chi_4(*C_4)$. This implies that $8 \leq \chi_4(*C_4)$. Therefore $8 \leq \chi_4(*C_4) \leq 8$, hence $\chi_4(*C_4) = 8$. □

Cycles with P Chords

CC_n is formed by connecting the vertices of C_n in such a way as to divide CC_n in subgraphs of C_4 when n is even and subgraphs of C_4 and C_3 when n is odd.

Case A. When n is even then CC_n can be organized is terms of only C_4 subgraphs. Figure 4.6 illustrates a 4LC of CC_n when n is even.

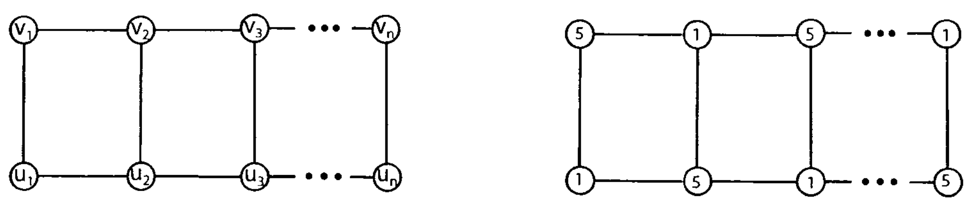

Figure 4.6: Graph Containing an Even Outer Cycle and a 4LC of CC_n.

In this organization there will be four vertices with degree 2 (adjacent to two vertices). The remaining $n - 4$ vertices all have degree 3 (adjacent to three vertices). Label one of the four vertices as v_1 and label the vertex that is adjacent to v_1 and has degree 3 as v_2. Label the vertex that is adjacent to v_1 and has degree 2 as u_1 and the vertex that is adjacent

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to u_1 and has degree 3 u_2. Note that v_2 and u_2 are adjacent. The un-labeled vertex, (not u_2), that is adjacent to v_2 shall be called v_3. The un-labeled vertex that is adjacent to u_2 shall be called u_3. Label a vertex v_{i+1} if it is adjacent to v_i and label the vertex u_{i+1} if it is adjacent to u_i for $1 \leq i \leq m-1$. Continuation of this labeling method creates a ladder where the sides of the ladder are two paths P_m, and P'_m. Where $V(P_m) = \{v_1, v_2, v_3, ..., v_m\}$ and $V(P'_m) = \{u_1, u_2, u_3, ..., u_m\}$. The vertices v_m and u_m are the other two vertices that have both have degree 2. Notice that $m = \frac{n}{2}$. The vertices v_i and u_i are adjacent. The consecutive vertices v_i and v_{i+1} are adjacent $1 \leq i \leq m-1$. We also have that u_i is adjacent to u_{i+1}, for $1 \leq i \leq m-1$. Thus the structure of Cycle Chord graphs is that of two paths.

Case B. When n is odd, then CC_n cannot be organized is terms of only C_4 subgraphs. When there is an odd outer cycle, then we need the odd subgraph C_3 with the rest of the C_4 subgraphs. Figure 4.7 illustrates a FLC of CC_n when n is odd.

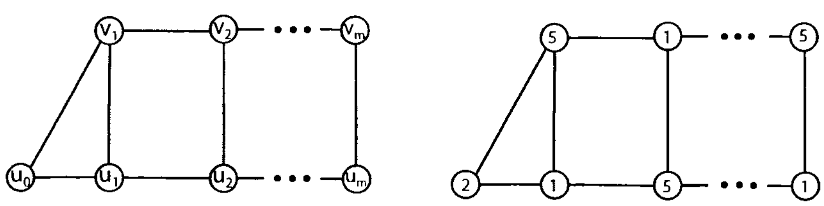

Figure 4.7: Graph Containing an Odd Cycle and a Proper Coloring of CC_n

In this organization there will be three vertices with degree 2 (adjacent to two vertices). The remaining $n-3$ vertices all have degree 3 (adjacent to three vertices). Label the unique vertex $v_0 \in V(CC_n)$ that is adjacent to two vertices that both have degree three. The two remaining vertices that have degree 2 shall be labeled v_m and u_m. Label the two vertices that are adjacent to v_0 as v_1 and u_1. Label the vertex that is adjacent to v_1 and has degree

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3 as v_2 and the vertex that is adjacent to u_1 and has degree 3 u_2. Note that v_2 and u_2 are adjacent. Continue this pattern labeling a vertex v_{i+1} if it is adjacent to v_i and labeling u_{i+1} if it is adjacent to u_i, for $1 \leq i \leq m-1$.

Theorem 14. $X_4(CC_n) = 5$

Proof. To show $X_4(CC_n) \leq 5$, let function $c(V(CC_n)) \longrightarrow \mathbb{N}$ be definded ,for $v \in V(CC_n)$, as

$$c(v) = \begin{cases} 1 \quad \text{if} \begin{cases} v = v_i, \text{ for i is even;} \\ v = u_i, \text{ for i is odd;} \end{cases} \\ 2 \quad \text{if} \ \ v = v_0; \\ 5 \quad \text{if} \begin{cases} v = v_i, \text{ for i is odd;} \\ v = u_i, \text{ for i is even.} \end{cases} \end{cases}$$

This function c is a proper coloring since $\forall S \subseteq V(CC_n)$ with $2 \leq |S| \leq 4$, $\exists v_i, v_j \in S \ni |c(v_i) - c(v_j)| \geq m_s = |E(< S >)|$. Hence $X_4(CC_n) \leq 5$.

To show $X_4(CC_n) \geq 5$, We use Theorem 12 and the Subgraph Theorem 3. Since C_4 is a subgraph of CC_n then $X_4(C_4) \leq X_4(CC_n)$. Thus we have that $5 = X_4(C_4) \leq X_4(CC_n)$. Therefore $5 \leq X_4(CC_n) \leq 5$, hence $X_4(CC_n) = 5$. □

Books

A Book $(B_{n,k})$ is constructed by taking k multiple C_n graphs that share one edge and thus two vertices. Consider only Books $(B_{n,k})$ with $n \geq 5$ and $k \geq 2$.

Theorem 15. $\chi_4(B_{n,k}) = 4$, $n \geq 5$.

Proof. To show $\chi_4(B_{n,k}) \leq 4, n \geq 5$, label each C_n page of $B_{n,k}$ in the same manner as labeling C_n. Label each of the k pages which are identical C_n subgraphs with the same color

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scheme on corresponding vertices. $c(v_1) = 1$, $c(v_2) = 4$, $c(v_3) = 1$, $c(v_4) = 4$,...,$c(v_n) = 4$(if n is even), $c(v_n) = 3$(if n is odd). Consider the function $c : V(B_{n,k}) \longrightarrow \mathbb{N}$ for $n \geq 5$ defined by

$$c(v_i) = \begin{cases} 1 & \text{if i is odd;} \\ 4 & \text{if i is even;} \\ 3 & \text{if i is odd and } i = n. \end{cases}$$

Since $\forall S \subseteq V(B_{n,k})$ with $2 \leq |S| \leq 4$, $\exists u, v \in S \ni |c(u) - c(v)| \geq m_s = |E(< S >)|$. This function c is a proper coloring of $B_{n,k}$ Therefore $\chi_4(B_{n,k}) \leq 4$.

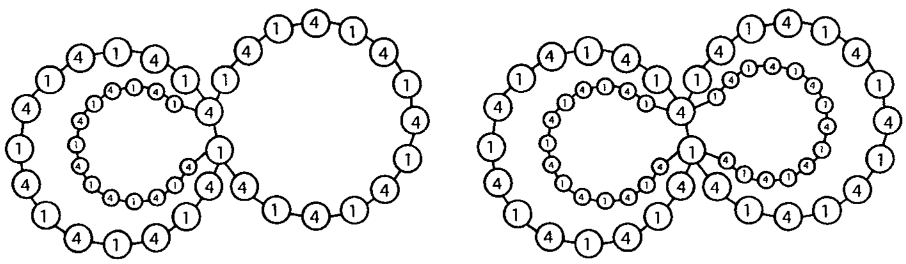

Figure 4.8: 4LCs on Books $B_{16,3}$ and $B_{16,4}$.

To show $\chi_4(B_{n,k}) \geq 4$, $n \geq 5$, we use the fact that P_4 is a subgraph of $B_{n,k}$. Since P_4 is a subgraph of $B_{n,k}$, then $\chi_4(P_4) \leq \chi_4(B_{n,k})$, by Theorem 3. From Theorem 8 we have $4 = \chi_4(P_4) \leq \chi_4(B_{n,k})$. Hence $4 \leq \chi_4(B_{n,k})$, and $4 \leq \chi_4(B_{n,k}) \leq 4$. Therefore $\chi_4(B_{n,k}) = 4$. $\square$

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Fan with n vertices

Case n=13.

Define F_{13} to be P_{12} joined with v_{13}. Let $V(F_{13}) = (V(P_{12}), v_{13})$. Let $v_1, v_2, ..., v_{12}$ be the vertices of the path subgraph of F_{12} with v_i adjacent to v_{i+1}, $1 \leq i \leq 11$ as illustrated in Figure 4.9.

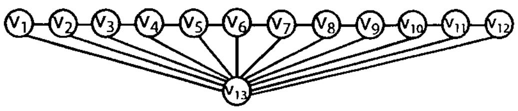

Figure 4.9: Fan of Order 13

To show that $\chi_4(F_{13}) \leq 7$, Consider the function $c : V(F_{13}) \longrightarrow \mathbb{N}$, defined by

$$c(v_i) = \begin{cases} 1 & \text{if i is odd;} \\ 4 & \text{if i is even;} \\ 7 & \text{if } i = 13. \end{cases}$$

where $v_i \in V(F_{13}), 1 \leq i \leq n$. This function is a proper coloring, hence $\chi_4(F_{13}) \leq 7$.

To show $\chi_4(F_{13}) \geq 7$, we use Theorem 3 together with Theorem 9 and Theorem 11. The vertex u is adjacent to the vertices in P_{12}, therefore by 9, u is adjacent to three vertices colored 4 and three vertices colored 1. There is a subgroup of F_{13} where u is adjacent to three vertices colored with fours and one that is colored with a one, hence $*S_5$ is a subgraph of F_{13}. Therefore applying Theorem 3 and Theorem 11, $7 = \chi_4(*S_4) \leq \chi_4(F_{13})$. $\therefore 7 \leq \chi_4(F_{13}) \leq 7$.

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Hence $\chi_4(F_{13}) = 7$.

Let F_n be the graph defined by P_{n-1} joined with a single vertex labeled v_n. Every vertex in P_{n-1} is adjacent to v_n. Label the vertices on the P_{n-1} part of F_n as $v_1, v_2, ..., v_{n-1}$ where v_i adjacent to v_{i+1}, $1 \leq i \leq n-2$, as illustrated in Figure 4.10.

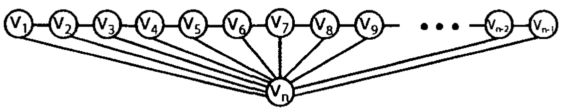

Figure 4.10: Fan Graph of Order n.

Theorem 16. $\chi_4(F_n) = 7, (n \geq 14)$.

Proof. Consider the function $c : V(F_n) \longrightarrow \mathbb{N}$, $n \geq 14$, defined by,

$$c(v_i) = \begin{cases} 1 & \text{if i is odd and } i \neq n; \\ 4 & \text{if i is even and } i \neq n; \\ 7 & \text{if } i = n. \end{cases}$$

This function c is a proper coloring of F_n, hence $\chi_4(F_n) \leq 7$. From Theorem 17 and Theorem 3, we have $7 = \chi_4(F_{13}) \leq \chi_4(F_n)$. Therefore $7 \leq \chi_4(F_n) \leq 7$, hence $\chi_4(F_n) = 7$. □

Bi-Fan with n vertices

Case n=10.

Label the vertices on the P_8 part of *Bi-*F_{10} as $v_1, v_2, ..., v_8$ with v_i adjacent to v_{i+1}, $1 \leq i \leq 7$. Let vertices v_9 and v_{10} both be adjacent to the eight vertices in P_8, but not adjacent to each other as illustrated in Figure 4.11.

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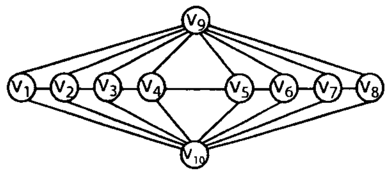

Figure 4.11: Bi-Fan of Order 10.

Theorem 17. $\chi_4(Bi\text{-}F_{10}) = 8$.

Proof. To show $\chi_4(Bi\text{-}F_{10}) \leq 8$, consider the function $c : V(Bi\text{-}F_{10}) \longrightarrow \mathbb{N}$ defined as

$$c(v_i) = \begin{cases} 1 & \text{if } i = 1, 3, 5, 7; \\ 4 & \text{if } i = 2, 4, 6, 8; \\ 7 & \text{if } i = 9; \\ 8 & \text{if } i = 10. \end{cases}$$

The function c is a proper coloring of $Bi\text{-}F_{10}$. Hence $\chi_4(Bi\text{-}F_{10}) \leq 8$.

To show that $\chi_4(Bi\text{-}F_{10}) \geq 8$, we make use of the fact that $Bi\text{-}F_{10}$ contains the subgraph P_8 as well as the subgraph $*C_4$ defined previously. From Theorem 3 for $n = 2$, in any 4LC of P_8 at least 2 vertices are colored 4 and 2 vertices are colored 1. Applying Theorem 3 and Theorem 13 yield $8 = \chi_4(*C_4) \leq \chi_4(Bi\text{-}F_{10})$. Therefore $8 \leq \chi_4(Bi\text{-}F_{10}) \leq 8$. Hence $\chi_4(Bi\text{-}F_{10}) = 8$. □

Let $Bi\text{-}F_n$ be the graph defined by P_{n-2} joined with two vertices labeled v_{n-1} and v_n. Every vertex in P_{n-2} is adjacent to v_{n-1}. Label the vertices on the P_{n-2} part of $Bi\text{-}F_n$ as $v_1, v_2, ..., v_{n-2}$ where v_i adjacent to v_{i+1}, $1 \leq i \leq n-3$. Let vertices v_{n-1} and v_n both be adjacent to the $n-2$ vertices in P_{n-2}, but not adjacent to each other as illustrated in Figure 4.12.

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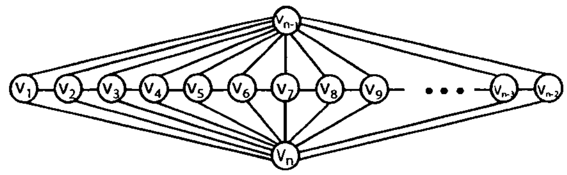

Figure 4.12: Bi-Fan Graph of Order n.

Theorem 18. $\chi_4(Bi\text{-}F_n) = 8$, *for* $n \geq 10$.

Proof. To show that $\chi_4(Bi\text{-}F_n) \leq 8$, $n \geq 10$, consider the function $c : V(Bi\text{-}F_n) \longrightarrow \mathbb{N}$ defined by,

$$c(v_i) = \begin{cases} 1 & \text{if i is odd and } i \neq n, n-1; \\ 4 & \text{if i is even and } i \neq n, n-1; \\ 7 & \text{if } i = n-1; \\ 8 & \text{if } i = n. \end{cases}$$

The function c is a proper coloring of $Bi\text{-}F_n$. Hence $\chi_4(Bi\text{-}F_n) \leq 8$.

To show that $\chi_4(Bi\text{-}F_n) \geq 8$, $n \geq 10$, we use the fact that $Bi\text{-}F_n$, $n \geq 10$, contains the subgraph $Bi\text{-}F_{10}$. Applying Theorem 3, and Theorem 19 yields $8 = \chi_4(Bi\text{-}F_{10}) \leq \chi_4(Bi\text{-}F_n)$. Therefore $8 \leq \chi_4(Bi\text{-}F_n) \leq 8$. Hence $\chi_4(Bi\text{-}F_{10}) = 8$. ☐

Wheels

Wheel with n vertices

Let a Wheel with n vertices be C_{n-1} coupled with a single vertex that is adjacent to every vertex in the cycle graph. In terms of vertices; $V(W_n) = (V(C_{n-1}), u)$. In terms of the number of edges; $|E(Wn)| = 2n$

Label the vertices on the C_{n-1} part of W_n as $v_1, v_2, ..., v_{n-1}$ where v_{n-1} is adjacent to v_1, and v_i adjacent to v_{i+1}, $1 \leq i \leq n-2$. Label the the lone vertex that is adjacent to all other

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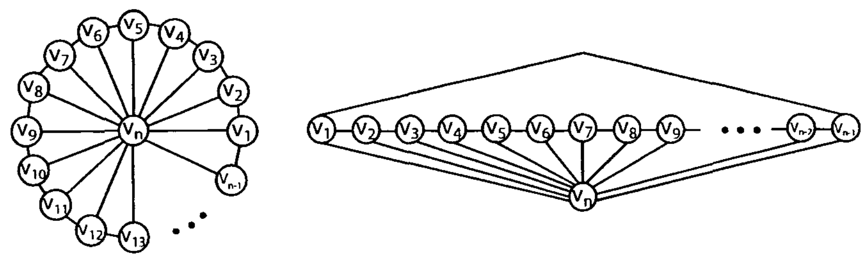

Figure 4.13: Two Different Looks of the Same Wheel of Order n.

vertices as v_n as illustrated in Figure 4.13.

Theorem 19. $\chi_4(W_n) = 7, (n \geq 14)$

Proof. To show that $\chi_4(W_n) \leq 7$, $n \geq 14$, consider the function $c : V(W_n) \longrightarrow \mathbb{N}$ for $n \geq 14$, defined by,

$$c(v_i) = \begin{cases} 1 & \text{if i is odd and } i \neq n; \\ 3 & \text{if i is odd and } i = n-1; \\ 4 & \text{if i is even and } i \neq n; \\ 7 & \text{if } i = n. \end{cases}$$

This function c is a proper coloring of W_n. Hence $\chi_4(W_n) \leq 7$.

To show $\chi_4(W_n) \geq 7$, $n \geq 14$, we use the fact that F_{12} is a subgraph of W_n. From Theorem 3 and Theorem 17 we have the following; since F_{13} is a subgraph of W_n, then $7 = \chi_4(F_{13}) \leq \chi_4(W_n)$, with $n \geq 14$. Therefore $7 \leq \chi_4(W_n) \leq 7$. Hence $\chi_4(W_n) = 7$.

□

Bi-Wheels (Bi-W_n)

Bi-Wheels are composed of a Wheel joined with another hub vertex. $V(G) = V(C_{n-2}, u, v)$.

Label the vertices on the C_{n-2} part of Bi-W_n as $v_1, v_2, ..., v_{n-2}$ with v_i adjacent to v_{i+1}, $1 \leq i \leq n-2$. Let vertices v_n and v_{n-1} both be adjacent to the $n-2$ vertices in C_{n-2}, but

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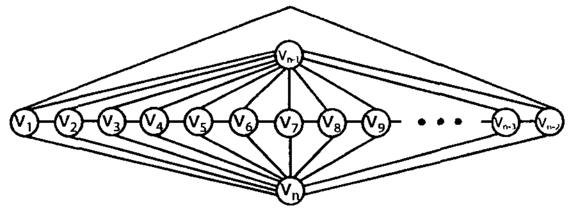

Figure 4.14: Bi-Wheel of Order n.

not adjacent to each other as illustrated in Figure 4.14.

Theorem 20. $\chi_4(Bi - W_n) = 8$, *for* $n \geq 11$.

Proof. To show that $\chi_4(Bi\text{-}W_n) \leq 8$, $n \geq 11$, Consider the function $c : V(Bi\text{-}W_n) \longrightarrow \mathbb{N}$, $n \geq 10$ defined as

$$c(v_i) = \begin{cases} 1 & \text{if i is odd and } i \neq n, n-1; \\ 3 & \text{if i is odd and } i = n-2; \\ 4 & \text{if i is even and } i \neq n, n-1; \\ 7 & \text{if } i = n-1; \\ 8 & \text{if } i = n. \end{cases}$$

The function c is a proper coloring, hence $\chi_4(Bi - W_n) \leq 8$.

To show that $\chi_4(Bi\text{-}W_n) \geq 8$, $n \geq 11$, we use Theorem 3 in conjunction with Theorem 20, which produces $8 = \chi_4(Bi - Fan) \leq \chi_4(Bi - W_n)$. Therefore $8 \leq \chi_4(Bi\text{-}Py_n) \leq 8$. Hence $\chi_4(Bi\text{-}Py_n) = 8$. □

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CHAPTER 5

CONCLUSION

In conclusion I reiterate the fact that graphs shed light on various math problems. Vertices are representative of cities on a route map, atoms in a chemical compound, or microprocessors in a computer. The idea for four local chromatic colorings were started with the Four Color Mapping Theorem. All vertex labeling and graph colorings stem from this problem. Chartrand extended the definition of proper colorings and defined k-local colorings. In this paper we have determined the 4-Local Chromatic Number for some classes of well known graphs.

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VITA

Graduate College
University of Nevada, Las Vegas

James Katseanes

Home Address: 2301 Malcolm Avenue
Los Angeles, CA 90064

Degrees:
    Bachelor of Science, Mathematics, 2002
    Westminster College

Special:
    Graduate College Representative
    University of Nevada, Las Vegas

Thesis Title: Four Local Colorings of Graphs

Thesis Examination Committee:
    Chairperson, Dr. Ebrahim Salehi, Ph.D.
    Committee Member, Dr. Rohan Dalpatadu, Ph.D.
    Committee Member, Dr. Xin Li, Ph.D.
    Graduate Faculty Representative, Dr. Ebrahim Saberinia, Ph.D.

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