Four local colorings of graphs

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FOUR LOCAL COLORINGS
OF GRAPHS

by

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ABSTRACT

Four Local Colorings of Graphs

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A proper coloring of a graph $G$ is a mapping $c : V(G) \rightarrow \mathbb{N}$ that assigns colors to the vertices of $G$ in such a way that adjacent vertices are labeled with different colors. A coloring $c$ is called a 4-local coloring if for every subset $S \subseteq V(G)$, with $2 \leq |S| \leq 4$ there are two vertices $u, v$ such that the difference between colors of $u$ and $v$, is greater than or equal to the number of edges in the subgraph induced by $S$. That is,

$$\forall S \exists u, v \in S \ni |c(u) - c(v)| \geq m_s,$$

where $m_s$ is the number of edges in the subgraph induced by $S$, $m_s = |E(< S >)|$. The maximum color assigned by a local coloring $c$ to a vertex of $G$ is

$$\chi_4(c) = \max\{c(v) | v \in V(G)\}.$$

The four local chromatic number of $G$ is defined as

$$\chi_4(G) = \min\{\chi_4(c)\} = \min\{\max\{c(G)|\text{where } c \text{ is a 4-local coloring of } G\}\}.$$ 

In this thesis, the four local chromatic number of some well known class of graphs will be determined.
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CHAPTER 1

INTRODUCTION

Historically graphs have helped to shed light on various mathematics problems. One such problem, described in [1], is the scheduling of courses at a college. Two requirements are stated in this problem;

1. No two courses can meet at the same time, in case a student plans to take both courses and
2. We want a schedule that is most efficient, a schedule requiring the fewest number of time periods.

The question is: what is the minimum number of time slots needed for such a schedule?

This scheduling problem is related to graphs by letting each class of the schedule be a vertex of a graph $G$. An edge connecting two vertices represents that some student is a member of both corresponding classes. The minimum number of time slots needed for such a schedule of classes is called the chromatic number of $G$.

The four-color mapping theorem described in [4] is a graph theory problem that involves chromatic numbers on planar graphs. A planar graph can be drawn on a plane in such a way that no two edges intersect except at a vertex. The four-color theorem states that the chromatic number of all planar graphs is less than or equal to four. In other words, any arbitrary map of counties (planar graph) can be colored, with at most four colors, in such a way that no two adjacent countries have the same color.
A function \( c : V(G) \rightarrow \mathbb{N} \) is called a proper coloring of a graph \( G \) if whenever \( u, v \in V(G) \) are adjacent, then \( c(u) \neq c(v) \). An \( n \)-coloring of \( G \) is a proper coloring using \( n \) colors. The chromatic number of a graph \( G \), denoted \( \chi(G) \), is defined as the minimum value \( n \) for which an \( n \)-coloring exists. That is, \( \chi(G) = \min\{n | G \text{ has an } n\text{-coloring}\} \). Another way to define the chromatic number is as follows. Consider the function \( f \) where \( f : V(G) \rightarrow \mathbb{N} \). Let \( f \) have the property that \( \forall S \subseteq V(G) \), with \( S = \{u, v\} \), then \( |f(u) - f(v)| \geq m_s \), where \( m_s \) is the number of edges in the subgraph induced by vertices \( u,v \). \( f \) is called a 2-local coloring of \( G \). \( \chi_2(f) = \max\{|f(v)| \text{ where } v \in V(G)\} \) and \( \chi_2(G) = \min\{\max\{f(G)\} \text{ where } f \text{ is a 2-local coloring}\} \). This yields an equivalent chromatic definition.

**Theorem 1.** \( \chi_2(G) = \chi(G) \).

**Proof.** Suppose \( f : V(G) \rightarrow \mathbb{N} \) is a 2-local coloring, then \( \forall S \subseteq V(G) \), with \( S = \{u, v\} \), we have \( |f(u) - f(v)| \geq m_s \). If \( u \) and \( v \) are adjacent, then \( |f(u) - f(v)| \geq m_s = 1 \). Therefore certainly \( f(u) \neq f(v) \) for all adjacent vertices. If \( u,v \) are not adjacent, then \( |f(u) - f(v)| \geq 0 \).

In either case \( f \) meets the requirements of a proper coloring of \( G \).

Conversely, let \( c \) be any proper coloring of \( G \). Let \( \chi(G) = n \), \( S = \{u, v\} \) and \( m_s = |E(<S>)| \). Then since \( c \) is a proper coloring, \( c(u) \neq c(v) \) for all adjacent vertices \( u \) and \( v \). If \( u \) and \( v \) adjacent, then \( |c(u) - c(v)| \geq 1 \). If \( u \) and \( v \) are not adjacent, then \( |c(u) - c(v)| \geq 0 \).

In either case \( |c(u) - c(v)| \geq m_s \). Hence \( c \) is a 2-local coloring of \( G \). Consequently the set of all 2-local colorings of \( G \) equals the set of all proper colorings of \( G \). That is,

\[
\{f | f \text{ is a 2-local coloring of } G\} = \{c | c \text{ is a proper coloring of } G\}
\]

\[
\max\{f | f \text{ is a 2-local coloring of } G\} = \max\{c | c \text{ is a proper coloring of } G\}
\]
\[
\min\{\max\{f \mid f \text{ is a } k\text{-local coloring of } G\}\} = \min\{\max\{c \mid c \text{ is a proper coloring of } G\}\}
\]

Henceforth, \(\chi_2(G) = \chi(G)\).

In view of this equivalency Chartrand et al [2] generalized the concept \(\chi(G)\) as follows:

Let \(k \geq 2\), define a function \(c : V(G) \rightarrow \mathbb{N}\) to be a \(k\)-local coloring if \(\forall S \subseteq V(G)\) with \(2 \leq |S| \leq k\), \(\exists u, v \in V(G) \ni |c(u) - c(v)| \geq m_s\), where \(m_s\) is the number of edges of the subgraph induced by \(S\). The \(k\)-local chromatic number is defined as \(\chi_k(G) = \min\{\max\{c(G) \mid c \text{ is a } k\text{-local coloring}\}\}\).

**Theorem 2.** If \(f\) is any \(k\)-local coloring of \(G\) that generates the \(k\)-local chromatic number, then there exists a vertex \(v\) in \(V(G)\) such that \(f(v) = 1\) and a vertex \(u\) in \(V(G)\) such that \(f(u) = \chi_k(G)\). The numbers 1 and \(\chi_k(G)\) are always used in the \(k\)-local coloring that produces the \(k\)-local chromatic number.

**Proof.** Suppose that \(\chi_k(G) = s\) and that function \(f\) provides this chromatic number. Suppose that \(t = \min\{f(v)\mid v \in V(G)\}\) and \(t > 1\). Since \(s\) is the chromatic number, \(s = \max\{f(v)\mid v \in V(G)\}\) and \(s = \min\{\max\{c(G) \mid c \text{ is a } k\text{-local coloring}\}\}\). Then the function \(h : V(G) \rightarrow \mathbb{N}\) defined by \(h(v) = f(v) - 1\) is a \(k\)-local coloring of \(G\) with \(\max\{h(v)\mid v \in V(G)\}\) = \(s - 1\). Hence \(\max\{h(v)\mid v \in G\}\) < \(\max\{f(v)\mid v \in G\}\) thus contradicting \(s = \min\{\max\{c(G) \mid c \text{ is a } k\text{-local coloring}\}\}\). Hence \(f\) does not produce \(\chi_k(G)\) when \(t > 1\). Therefore \(t = 1\).

**Theorem 3.** If \(H\) is a subgraph of \(G\), then \(\chi_k(H) \leq \chi_k(G)\).

**Proof.** Suppose \(\chi_k(H) > \chi_k(G)\) and \(H\) is a subgraph of \(G\). Suppose the function \(f\) provides the local chromatic number for \(G\) and \(H\), then \(\max\{f(v)\mid v \in G\}\) < \(\max\{f(v)\mid v \in H\}\). Hence there is at least one vertex \(v\) in \(V(H)\) that is not in \(V(G)\). This contradicts \(H\) as a subgraph of \(G\). Therefore \(\chi_k(H) \leq \chi_k(G)\) is necessary for \(H\) a subgraph of \(G\).
CHAPTER 2

ESTABLISHED RESULTS

Let $G$ be a graph with finite vertices. $V(G)$ is the set of vertices in $G$. The size of $V(G)$ is called the order of $G$. $E(G)$ is the set of edges in $G$. Adjacent and coincident are analogous terms meaning that two vertices have an edge in common. The degree of a vertex is defined to be the number of adjacent vertices. The compliment of $G$ is denoted $G^c$ and has the same set of vertices, $V(G) = V(G^c)$ and $u$ and $v$ are adjacent in $G$, if and only if $u$ and $v$ are not adjacent in $G^c$. If $G_1$ and $G_2$ are isomorphic graphs, then the degrees of the vertices of $G_1$ are exactly the degrees of $G_2$. If $u$ and $v$ be two vertices of a tree $G$. The diameter of a tree graph $G$ is the max $u$-$v$ path in $G$. A famous result from Euler is if $G$ is a connected plane graph with $p$ vertices, $q$ edges and $r$ regions, then $p - q + r = 2$.

The following results for three local chromatic colorings on finite graphs are due to [2]. A 3-local coloring is defined as follows; Let $f$ be a function such that $f : V(G) \rightarrow \mathbb{N}$, $\forall S \subset V(G)$, with $|S| \leq 3$, $\forall u, v \in V(G)$ $|f(u) - f(v)| \geq m$, $m = E(<S>)$. Let $c(f) = \max \{f(v) \mid v \in V(G)\}$, then the three local chromatic number is defined as $\chi_3(G) = \min \{c(f) \mid f$ is a 3-local coloring $\}$.

**Theorem 4.** For every graph $G$ of order at least 3, $\chi_2(G) \leq \chi_3(G) \leq 2\chi_3(G) - 1$.

**Theorem 5.** For $n \geq 3$, $\chi_3(P_n) = 3$.

**Theorem 6.** For $n \geq 3$, $\chi_3(K_n) = \left\lfloor \frac{3n-1}{2} \right\rfloor$. 
Theorem 7. Let $G = K_{n_1, n_2, \ldots, n_k}$, where $k \geq 2$ and $n_i \geq 2$ for all $i$ with $1 \leq i \leq k$.

Then $\chi_3(G) = 2k - 1$. 

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CHAPTER 3

FOUR LOCAL CHROMATIC NUMBER

In this chapter, we consider the subsets with $2 \leq |S| \leq 4$. The function $f : V(G) \rightarrow \mathbb{N}$ is said to be a 4-Local Coloring (4LC) if for any $S \subseteq V(G)$, with $2 \leq |S| \leq 4$ there are vertices $u, v \in S$ such that $|f(u) - f(v)| \geq m_s$, where $m_s$ is the number of edges in the subgraph induced by $S$. The maximum color assigned by a 4-local coloring $c$ to a vertex of $G$ is $n(f) = \max\{f(v) | v \in V(G)\}$. The four local chromatic number of $G$ is defined as $\chi_4(G) = \min\{n(f) | f$ is a 4-local coloring$\}$. We define a 4-Local Chromatic Coloring (4LCC) of $G$ to be a 4LC that provides the $\chi_4(G)$. In the Figure 3.1 all possible graphs with order four are illustrated. Note that all Graphs $G$ are paired with their complement $G^c$, except for the Path graph $P_3$. If $G = P_3$, then $G = G^c$.

![Figure 3.1: Four Local Chromatic Colorings on Graphs of Order 4.](image-url)
Four Local Colorings of Trees

A Tree is a connected graph that contains no cycle subgraphs. A path is a simple example of a Tree. Paths are Trees with no branches. We first establish a few results for Paths.

**Theorem 8.** If \( n \geq 4 \), then \( \chi_4(P_n) = 4 \).

**Proof.** Let \( v_1, v_2, \ldots, v_n \) be the vertices of \( P_n \) as illustrated in figure 3.2

![Figure 3.2: Graph of a Typical Path of Order n.](image)

Consider the function \( f : V(P_n) \to \mathbb{N} \) defined by

\[
    f(v_i) = \begin{cases} 
        1 & \text{if } i \text{ is odd}, \\
        4 & \text{if } i \text{ is even}.
    \end{cases}
\]

where \( v_i \in V(P_n), 1 \leq i \leq n \). The function \( f \) is a 4LC. Therefore \( \chi_4(P_n) \leq 4 \).

On the other hand if \( g : V(P_4) \to \mathbb{N} \) is any 4LC of \( G \) then we choose \( S = \{v_1, v_2, v_3, v_4\} \).

There should be vertices \( v_i, v_j \) with \( |g(v_i) - g(v_j)| \geq 3 \). Therefore \( \max\{g(v)\mid v \in V(G)\} \geq 4 \).

Hence \( 4 \leq \chi_4(P_n) \). Therefore \( 4 \leq \chi_4(P_n) \leq 4 \). Henceforth \( \chi_4(P_n) = 4 \).

**Theorem 9.** In any 4LC of \( P_4 \) at least \( n \) vertices are colored 4 and at least \( n \) vertices are colored 1.

**Proof.** We proceed by induction on \( n \).

For \( n = 1 \), \( \chi_4(P_4) = 4 \) and by Theorem 2, the colors 1 and 4 must be used.

Assume the statement is true for \( n = k \), we wish to show that it is also true for \( n = k + 1 \).
Suppose that if $c$ is any 4LCC of $P_{4k}$, then at least $k$ vertices are colored 4 and $k$ vertices are colored 1. To show that for any 4LCC of $P_{4(k+1)}$ at least $k + 1$ vertices are colored 4 and $k + 1$ vertices are colored 1, we note that $P_{4(k+1)}$ is just $P_{4k+4}$, which is $P_{4k}$ along with $P_4$. We know from the Inductive step that if $c$ is any 4LCC of $P_{4k}$, then at least $k$ vertices are colored 4 and $k$ vertices are colored 1. From the Base step, if $c$ is any 4LCC of $P_4$ then at least 1 vertex is colored with a 4 and 1 vertex is colored with a 1. Thus $k + 1$ vertices are colored 4 and $k + 1$ vertices are colored 1. Therefore in any 4LCC of $P_{4(k+1)}$ at least $k + 1$ vertices are colored 4 and $k + 1$ vertices are colored 1. Hence, in any 4LCC of $P_{4n}$ at least $n$ vertices are colored 4 and $n$ vertices are colored 1. □

Trees $T_n$ have paths as subgraphs. The diameter of $T_n$ is the max u-v path in $T_n$. Let $P_n$ represent the diameter of $T_n$. Note that $P_n$ need not be unique. Choose either terminal vertex of $P_n$ to be labeled $v_1$. Let $v_1, v_2, ..., v_n$ be the vertices of $P_n$ with $v_j$ adjacent to $v_{j+1}$, $1 \leq j \leq n - 1$ as illustrated in Figure 3.3. The vertex labeled $v_1$ is considered the initial vertex and acts as the root of $T_n$. Define the depth of a vertex $v \in V(T_n)$ to be the number of edges, that lie in a $v$-$v_1$ path. Let $d(v)$ denote the depth of a vertex; the number of edges in the unique path $v - v_1$.

Clearly $d(v_1) = 0$, and diameter of a tree is equal to the maximum depth. The vertex $v_n \in V(P_n) \subseteq V(T_n)$ has depth $n - 1$, $d(v_n) = n - 1$. Thus the diameter of $T_n$ is $n - 1$. The vertex $v_h \in V(P_n)$ has depth $h - 1$, $d(v_h) = h - 1$. If the vertex $v_i \in V(T_n)$ has depth $k$, $d(v_i) = k$, then $k$ number of edges (and $(k - 2)$ vertices that form a path) separate $v_i$ from $v_1$. In Figure 3.3, the vertices not on $P_n$ with depth $k$ are labeled $d_k$.

**Theorem 10.** $\chi_4(T_n) = 4$, $n \geq 4$.

*Proof.* Suppose, $n \geq 4$. To show that $\chi_4(T_n) \leq 4$, consider the function $c : V(T_n) \rightarrow \mathbb{N}$,
defined for $v \in V(T_n)$ as

$$c(v) = \begin{cases} 
1 & \text{if } d(v) \text{ is even;} \\
4 & \text{if } d(v) \text{ is odd.}
\end{cases}$$

In order to show that this function $c$ provides a 4LC of Tree $T_n$, choose any subset $S \subseteq V(T_n)$ with $2 < |S| \leq 4$. Then the induced subgraph $\langle S \rangle$ has at most 3 edges. If $S$ contains one or more adjacent vertices then $\exists u, v \in S \ni c(u) = 1$ and $c(v) = 4$, hence $|c(u) - c(v)| = |1 - 4| = 3 \geq m_s = |E(\langle S \rangle)|$. If there are no vertices coincident in $S$ then $m_s = |E(\langle S \rangle)| = 0$, it follows that $\exists u, v \in S \ni |c(u) - c(v)| \geq 0 = m_s = |E(\langle S \rangle)|$. In either case $c$ provides a coloring such that $\forall S \subseteq V(T_n)$ with $2 \leq |S| \leq 4$, $\exists u, v \in S \ni |c(u) - c(v)| \geq m_s = |E(\langle S \rangle)|$. Therefore the function $c$ is a FLC, thus $\chi_4(T_n) \leq 4$. 

Figure 3.4: 4LC of a Tree with Diameter 6.
The four local chromatic number for paths along with the use Theorem 3 will set the lower bound for Tree graphs. To show $\chi_4(T_n) \geq 4$, $n \geq 4$, note that all trees have paths as subgraphs, then using the Theorem 3 we have, if $P_k$ is a subgraph of $T_n$ for $4 \leq k \leq n$, then $\chi_4(P_k) \leq \chi_4(T_n)$. By Theorem 8 we have that $4 = \chi_4(P_n) \leq \chi_4(T_n)$. Therefore $4 \leq \chi_4(T_n) \leq 4$, Hence $\chi_4(T_n) = 4$. 

Trees of Diameter 2 (Stars)

\[
\begin{align*}
&\text{Figure 3.5: } ST_4 \text{ and a 4-local Coloring of } ST_4. \\
&\text{Let } ST_n \text{ be a star with } n \text{ vertices. } ST_n \text{ has a central vertex that is adjacent to } n - 1 \\
&\text{terminal vertices. Label the terminal vertices in } ST_n \text{ as } v_1, v_2, ..., v_{n-1} \text{ and label the central } \\
&\text{vertex } v_n \text{ as illustrated in Figure 3.6. Note that } ST_2 = P_2 \text{ and } ST_3 = P_3 \text{ and their 4-local} \\
&\text{chromatic number have already been determined. So, for } ST_2 \text{ we may assume } n \geq 4. \\
&\text{Figure 3.6: Graph of } ST_n. \\
\end{align*}
\]

Also, since $ST_n$ is a Tree with diameter two, by Theorem 10, if $n \geq 4$, then $\chi_4(ST_n) = 4$. 

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The function $c : V(ST_n) \rightarrow \mathbb{N}$, defined by:

$$c(v_i) = \begin{cases} 
1 & \text{if } 1 \leq i \leq n - 1; \\
4 & \text{if } i = n.
\end{cases}$$

Provides a four local coloring for $ST_n$.

Figure 3.7: Proper Colorings on $ST_{17}$ and $ST_n$.

Four Local Colorings of Stars with order 17 and order $n$ are illustrated in 3.7.

Special case: Star $*ST_5$

$*ST_5$ is defined as the Star with the three terminal vertices colored with fours and one terminal vertex colored with a one.

Figure 3.8: Star $*ST_5$ and a Proper Coloring of $*ST_5$

Theorem 11. $\chi_4(*ST_5) = 7.$
Proof. Let \( c \) be a function such that \( c : V(*ST_5) \to \mathbb{N} \). Let \( c \) be a proper coloring of \(*ST_5\).

Let \( V(*ST_5) = \{v_1, v_2, v_3, v_4, v_5\} \), with \( c(v_1) = c(v_2) = c(v_3) = 4 \) and \( c(v_4) = 1 \). This implies \( v_1, v_2, v_3 \) and \( v_4 \) share no edges. Thus \( v_5 \) is the so called central vertex that is coincident to other four terminal vertices. Notice \( |E(*ST_5)| = 4 \). To find an upper bound on \( \chi_4(*ST_5) \), determination of \( c(v_5) \) is needed. What does the central vertex \( v_5 \) have to be colored with in order \( c \) be a proper coloring? Let \( H_i \) represent one of the 5 subgroups of \(*ST_5\) with \( |H_i| = 4 \), \( 1 \leq i \leq 5 \). Let \( V(H_1) = \{v_5, v_2, v_3, v_4\} \), \( V(H_2) = \{v_1, v_5, v_3, v_4\} \), \( V(H_3) = \{v_1, v_2, v_5, v_4\} \), \( V(H_4) = \{v_1, v_2, v_3, v_5\} \), and \( V(H_5) = \{v_1, v_2, v_3, v_4\} \). We conclude \( |E(H_1)| = |E(H_2)| = |E(H_3)| = |E(H_4)| = 3 \) and \( |E(H_5)| = 0 \). Consider the subgroup \( H_4 \), which has the three terminal vertices colored with fours and the coloring \( c(v_5) \) is undetermined. Essentially \( \chi_4(H_4) \) is desired. For the function \( c \) to be a proper coloring, the following inequality must be satisfied.

\[
|c(v_5) - c(v_i)| \geq m_{H_i} = |E(H_i)| = 3, \text{ with } 1 \leq i \leq 3. \text{ Hence } |c(v_5) - 4| \geq 3. \text{ Thus there are two options; 1. } c(v_5) - 4 \leq -3. \text{ Which yields } c(v_5) \leq 1, \text{ in which } c(v_5) = 1. \text{ This contradicts } \text{c being a proper coloring; since } v_4 \text{ and } v_5 \text{ are coincident, they must be colored with different colors. Hence } c(v_5) \neq 1. \text{ Or 2. } c(v_5) - 4 \geq 3. \text{ Which yields } c(v_5) \geq 7. \text{ Let } c(v_5) = 7 \text{ This c is an optimal efficient proper coloring of } *ST_5, \text{ since we need } \chi_4(G) = \min \{ \chi_4(c) | \text{where } c \text{ is a 4-local coloring} \}. \text{ Hence } \chi_4(*ST_5) \leq 7.
\]

An ad absurdum argument shows 7 is a lower bound if we assume falsely \( \chi_4(*ST_5) < 7 \). Thus \( \chi_4(*ST_5) \leq 6 \), this implies the existence of a coloring such that \( c(v_5) = 6 \). Which gives the result of the maximum difference between colors is \( |c(v_5) - 4| = |6 - 4| = 2 \geq 3 = m_{H_4} \). Thus there does not exist a function \( c \) that is a proper coloring of \(*ST_5\), such that \( c(v_5) = 6 \). Hence \( 6 < \chi_4(*ST_5) \) and thus \( 7 \leq \chi_4(*ST_5) \). Therefore \( 7 \leq \chi_4(*ST_5) \leq 7 \), henceforth \( \chi_4(*ST_5) = 7. \)

\[ \square \]
Trees of Diameter 3 (Double Stars)

Double stars are Trees of diameter three, Hence $\chi_4(DS_n) = 4$. Figure 3.9 shows a 4LC of $DS_n$ graph.

![Figure 3.9: 4LCC of a Double Star](image)

Trees Containing $P_4$ as Subgraphs (Caterpillars)

Since Caterpillar are Trees, then by theorem 10, $\chi_4(Caterpillar) = 4$. Figure 3.10 shows a Four Local Coloring of a Caterpillar graph.

![Figure 3.10: 4LCC of Caterpillar with diameter n](image)
CHAPTER 4

FOUR LOCAL COLORINGS OF CYCLE RELATED GRAPHS

Theorem 12. If \( n \geq 3 \), then

\[
\chi_4(C_n) = \begin{cases} 
4 & \text{if } n \neq 4; \\
5 & \text{if } n = 4.
\end{cases}
\]

Proof. Case \( n = 3 \). Let \( v_1, v_2, v_3 \) and be the vertices of \( C_3 \) as illustrated in Figure 4.1.

![Figure 4.1: Cycle \( C_3 \) and its 4LC.](image)

To show that \( \chi_4(C_3) \leq 4 \), consider the function \( c : V(C_3) \to \mathbb{N} \), defined by \( c(v_1) = 1 \), \( c(v_2) = 2 \) and \( c(v_3) = 4 \). This function \( c \) provides a 4LC of \( C_3 \). Therefore \( \chi_4(C_3) \leq 4 \).

To show that \( \chi_4(C_3) \geq 4 \), suppose \( \chi_4(C_3) < 4 \), by letting \( \chi_4(C_3) = 3 \). This would imply that there exists a function \( c \) that is a proper coloring, with \( c : V(C_3) \to \mathbb{N} \), such that \( \max\{c(v_i) | v_i \in V(C_3)\} = 3 \). Thus \( \exists v_i \in V(C_3) \ni c(v_i) = 3 \) and \( \exists v_j \in V(C_3) \ni c(v_j) = 1 \).

The maximum difference of colors is \( |c(v_i) - c(v_j)| = |3 - 1| = 2 \geq m_4 \). This contradicts \( c \) being a proper coloring, thus there does not exist a \( c \) such that \( c(C_3) = 3 \). Thus \( 3 < \chi_4(C_3) \), which implies \( 4 \leq \chi_4(C_3) \). Hence \( 4 \leq \chi_4(C_3) \leq 4 \). Therefore \( \chi_4(C_3) = 4 \).
Case \( n = 4 \). The 4 local chromatic number of a cycle with four vertices is a special case since the number of edges, in the only subgroup with four vertices, is four; i.e. \( m_s = |E(C_4)| = 4 \).

Let \( v_1, v_2, v_3 \) and \( v_4 \) be the vertices of \( C_4 \) as illustrated in Figure 4.2.

![Figure 4.2: Cycle Graph \( C_4 \) and its 4-local coloring.](image)

To show that \( \chi_4(C_4) \leq 5 \), consider the function \( c : V(C_4) \to \mathbb{N}, \) defined by

\[
c(v_1) = 1, \quad c(v_2) = 2, \quad c(v_3) = 3 \quad \text{and} \quad c(v_4) = 5.
\]

This function \( c \) is a proper coloring of \( C_4 \).

Therefore \( \chi_4(C_4) \leq 5 \).

Suppose \( \chi_4(C_4) < 5 \), by letting \( \chi_4(C_4) = 4 \). This would imply that there exists a function \( c \) that is a proper coloring, with \( c : V(C_4) \to \mathbb{N}, \) such that \( \max\{c(v_i) | v_i \in V(C_4)\} = 4 \). Thus \( \exists v_i \in V(C_4) \) such that \( c(v_i) = 4 \) and \( \exists v_j \in V(C_4) \) such that \( c(v_j) = 1 \). The maximum difference of colors is

\[
|c(v_i) - c(v_j)| = |4 - 1| = 3 \geq 4 = m_s.
\]

This contradicts \( c \) being a proper coloring, thus there does not exist a \( c \) such that \( c(C_4) = 4 \). Thus \( 4 < \chi_4(C_4) \), which implies \( 5 \leq \chi_4(C_4) \). Hence \( 5 \leq \chi_4(C_4) \leq 5 \). Therefore \( \chi_4(C_4) = 5 \).

Case \( n \geq 5 \). Let \( v_1, v_2, \ldots, v_n \) be the vertices in \( V(C_n) \) with \( v_n \) adjacent to \( v_1 \) and \( v_i \) adjacent to \( v_{i+1} \), \( 1 \leq i \leq n - 1 \) as illustrated in Figure 4.3.
Consider the function $c : V(C_n) \rightarrow \mathbb{N}$ for $n \geq 5$ defined by

$$
c(u_i) = \begin{cases} 
1 & \text{if } i \text{ is odd}; \\
3 & \text{if } i \text{ is odd and } i = n; \\
4 & \text{if } i \text{ is even};
\end{cases}
$$

This function $c$ is a 4LC of $C_n$. Hence $\chi_4(C_n) \leq 4$.

Since $P_4$ is a subgraph of $C_n$ with $n \geq 4$, then from theorem 3, $\chi_4(P_4) \leq \chi_4(C_n)$. From Theorem 10 we have $4 = \chi_4(P_4) \leq \chi_4(C_n)$. Hence $4 \leq \chi_4(C_n)$. Hence $4 \leq \chi_4(C_n) \leq 4$. 

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Therefore \( \chi_4(C_n) = 4 \). Henceforth, if \( n \geq 3 \), then

\[
\chi_4(C_n) = \begin{cases} 
4 & \text{if } n \neq 4; \\
5 & \text{if } n = 4.
\end{cases}
\]

\( \square \)

Special case of \( C_4 \).

Let the special Cycle \( *C_4 \) have two vertices labeled with fours.

**Theorem 13.** \( \chi_4(*C_4) = 8 \).

**Proof.** Let the vertices of \( *C_4 \) be labeled \( v_1, v_2, v_3 \) and \( v_4 \). Let \( v_4 \) be adjacent to \( v_1 \) and let \( v_i \) be adjacent to \( v_{i+1} \), for \( 1 \leq i \leq 3 \). Notice \( 4 = m = |E(*C_4)| \). Consider the function \( c: V(*C_4) \rightarrow \mathbb{N} \), defined by \( c(v_2) = c(v_4) = 4 \) as shown on left in Figure 4.5.

![Figure 4.5: *C_4 and 4LC of *C_4.](image)

Let \( c(v_1) = c(v_3) = 8 \). This function \( c \) is a proper coloring of \(*C_4\), since \( \forall S \subseteq V(*C_4) \), with \( 2 \leq |S| \leq 4 \), \( \exists u, v \in S \exists |c(u) - c(v)| \geq m \). Therefore \( \chi_4(*C_4) \leq 8 \).

Since \( *C_4 \) has the same structure as \( C_4 \), then the requirement that two vertices must be labeled with a 4 could only increase the value of the 4 chromatic number. Therefore by Theorem 3, \( \chi_4(C_4) \leq \chi_4(*C_4) \). From section 10, \( \chi_4(C_4) = 5 \). Thus \( 5 = \chi_4(C_4) \leq \chi_4(*C_4) \). Suppose \( 7 \leq \chi_4(*C_4) \). This implies the existence of \( c \), a function that is a proper coloring
of \(*C_4\) such that \(c : V(*C_4) \rightarrow \mathbb{N}\) with \(\exists v_i \in V(*C_4) \ni 7 = \max\{c(v_i) | v_i \in V(*C_4) = (v_1, v_2, v_3, v_4)\}\). Let \(c(v_1) = 7\) and let \(c(v_2) = c(v_4) = 4\). The maximum difference between colors is \(|c(v_1) - c(v_2)| = |7 - 4| = 3 \geq 4 = m_2\), thus there does not exist a function \(c\) such that \(7 = \max\{c(v_i) | v_i \in V(*C_4) = (v_1, v_2, v_3, v_4)\}\). Thus there does not exist vertices \(u, v \in V(*C_4) \ni |c(u) - c(v)| \geq m_2 = 4\). This contradicts that \(c\) is a 4LC of \(*C_4\), and the supposition \(7 \leq \chi_4(*C_4)\). Hence \(7 < \chi_4(*C_4)\). This implies that \(8 \leq \chi_4(*C_4)\). Therefore \(8 \leq \chi_4(*C_4) \leq 8\), hence \(\chi_4(*C_4) = 8\). □

Cycles with P Chords

\(CC_n\) is formed by connecting the vertices of \(C_n\) in such a way as to divide \(CC_n\) in subgraphs of \(C_4\) when \(n\) is even and subgraphs of \(C_4\) and \(C_3\) when \(n\) is odd.

Case A. When \(n\) is even then \(CC_n\) can be organized is terms of only \(C_4\) subgraphs. Figure 4.6 illustrates a 4LC of \(CC_n\) when \(n\) is even.

In this organization there will be four vertices with degree 2 (adjacent to two vertices). The remaining \(n - 4\) vertices all have degree 3 (adjacent to three vertices). Label one of the four vertices as \(v_1\) and label the vertex that is adjacent to \(v_1\) and has degree 3 as \(v_2\). Label the vertex that is adjacent to \(v_1\) and has degree 2 as \(u_1\) and the vertex that is adjacent
to $u_1$ and has degree 3 $u_2$. Note that $v_2$ and $u_2$ are adjacent. The un-labeled vertex, (not $u_2$), that is adjacent to $v_2$ shall be called $v_3$. The un-labeled vertex that is adjacent to $u_2$ shall be called $u_3$. Label a vertex $v_{i+1}$ if it is adjacent to $v_i$ and label the vertex $u_{i+1}$ if it is adjacent to $u_i$ for $1 \leq i \leq m - 1$. Continuation of this labeling method creates a ladder where the sides of the ladder are two paths $P_m$, and $P'_m$. Where $V(P_m) = \{v_1, v_2, v_3, ..., v_m\}$ and $V(P'_m) = \{u_1, u_2, u_3, ..., u_m\}$. The vertices $v_m$ and $u_m$ are the other two vertices that have both have degree 2. Notice that $m = \frac{n}{2}$. The vertices $v_i$ and $u_i$ are adjacent. The consecutive vertices $v_i$ and $v_{i+1}$ are adjacent $1 \leq i \leq m - 1$. We also have that $u_i$ is adjacent to $u_{i+1}$, for $1 \leq i \leq m - 1$. Thus the structure of Cycle Chord graphs is that of two paths.

**Case B.** When $n$ is odd, then $CC_n$ cannot be organized in terms of only $C_4$ subgraphs. When there is an odd outer cycle, then we need the odd subgraph $C_3$ with the rest of the $C_4$ subgraphs. Figure 4.7 illustrates a FLC of $CC_n$ when $n$ is odd.

![Graph Containing an Odd Cycle and a Proper Coloring of $CC_n$](image)

Figure 4.7: Graph Containing an Odd Cycle and a Proper Coloring of $CC_n$

In this organization there will be three vertices with degree 2 (adjacent to two vertices). The remaining $n - 3$ vertices all have degree 3 (adjacent to three vertices). Label the unique vertex $v_0 \in V(CC_n)$ that is adjacent to two vertices that both have degree three. The two remaining vertices that have degree 2 shall be labeled $v_m$ and $u_m$. Label the two vertices that are adjacent to $v_0$ as $v_1$ and $u_1$. Label the vertex that is adjacent to $v_1$ and has degree
3 as \( v_2 \) and the vertex that is adjacent to \( u_1 \) and has degree 3 \( u_2 \). Note that \( v_2 \) and \( u_2 \) are adjacent. Continue this pattern labeling a vertex \( v_{i+1} \) if it is adjacent to \( v_i \) and labeling \( u_{i+1} \) if it is adjacent to \( u_i \), for \( 1 \leq i \leq m - 1 \).

**Theorem 14.** \( X_4(CC_n) = 5 \)

*Proof.* To show \( X_4(CC_n) \leq 5 \), let function \( c(V(CC_n)) \rightarrow \mathbb{N} \) be defined, for \( v \in V(CC_n) \), as

\[
c(v) = \begin{cases} 
1 & \text{if } v = v_i, \text{ for } i \text{ is even;} \\
2 & \text{if } v = v_0; \\
5 & \text{if } v = v_i, \text{ for } i \text{ is odd;} \\
& \text{for } i \text{ is even.}
\end{cases}
\]

This function \( c \) is a proper coloring since \( \forall S \subseteq V(CC_n) \) with \( 2 \leq |S| \leq 4 \), \( \exists v_i, v_j \in S \ni |c(v_i) - c(v_j)| = m_S = |E(<S>)|. \) Hence \( X_4(CC_n) \leq 5 \).

To show \( X_4(CC_n) \geq 5 \), we use Theorem 12 and the Subgraph Theorem 3. Since \( C_4 \) is a subgraph of \( CC_n \) then \( X_4(C_4) \leq X_4(CC_n) \). Thus we have that \( 5 = X_4(C_4) \leq X_4(CC_n) \).

Therefore \( 5 \leq X_4(CC_n) \leq 5 \), hence \( X_4(CC_n) = 5 \). \( \square \)

**Books**

A Book \((B_{n,k})\) is constructed by taking \( k \) multiple \( C_n \) graphs that share one edge and thus two vertices. Consider only Books \((B_{n,k})\) with \( n \geq 5 \) and \( k \geq 2 \).

**Theorem 15.** \( \chi_4(B_{n,k}) = 4, n \geq 5 \).

*Proof.* To show \( \chi_4(B_{n,k}) \leq 4, n \geq 5 \), label each \( C_n \) page of \( B_{n,k} \) in the same manner as labeling \( C_n \). Label each of the \( k \) pages which are identical \( C_n \) subgraphs with the same color.
scheme on corresponding vertices. \( c(v_1) = 1, c(v_2) = 4, c(v_3) = 1, c(v_4) = 4, \ldots, c(v_n) = 4 \) (if \( n \) is even), \( c(v_n) = 3 \) (if \( n \) is odd). Consider the function \( c : V(B_{n,k}) \rightarrow \mathbb{N} \) for \( n \geq 5 \) defined by

\[
\begin{align*}
c(v_i) &= \begin{cases} 
1 & \text{if } i \text{ is odd;} \\
4 & \text{if } i \text{ is even;} \\
3 & \text{if } i \text{ is odd and } i = n.
\end{cases}
\end{align*}
\]

Since \( \forall S \subseteq V(B_{n,k}) \) with \( 2 \leq |S| \leq 4 \), \( \exists u, v \in S \exists |c(u) - c(v)| \geq m_s = |E(<S>)| \). This function \( c \) is a proper coloring of \( B_{n,k} \). Therefore \( \chi_4(B_{n,k}) \leq 4 \).

![Figure 4.8: 4LCs on Books \( B_{16,3} \) and \( B_{16,4} \).](image)

To show \( \chi_4(B_{n,k}) \geq 4 \), \( n \geq 5 \), we use the fact that \( P_4 \) is a subgraph of \( B_{n,k} \). Since \( P_4 \) is a subgraph of \( B_{n,k} \), then \( \chi_4(P_4) \leq \chi_4(B_{n,k}) \), by Theorem 3. From Theorem 8 we have \( 4 = \chi_4(P_4) \leq \chi_4(B_{n,k}) \). Hence \( 4 \leq \chi_4(B_{n,k}) \), and \( 4 \leq \chi_4(B_{n,k}) \leq 4 \). Therefore \( \chi_4(B_{n,k}) = 4 \). \( \square \)
Fans

Fan with $n$ vertices

Case $n=13$.

Define $F_{13}$ to be $P_{12}$ joined with $v_{13}$. Let $V(F_{13}) = (V(P_{12}), v_{13})$. Let $v_1, v_2, ..., v_{12}$ be the vertices of the path subgraph of $P_{12}$ with $v_i$ adjacent to $v_{i+1}$, $1 \leq i \leq 11$ as illustrated in Figure 4.9.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fan_of_order_13}
\caption{Fan of Order 13}
\end{figure}

To show that $\chi_4(F_{13}) \leq 7$, Consider the function $c : V(F_{13}) \rightarrow \mathbb{N}$, defined by

$$c(v_i) = \begin{cases} 
1 & \text{if } i \text{ is odd;} \\
4 & \text{if } i \text{ is even;} \\
7 & \text{if } i = 13.
\end{cases}$$

where $v_i \in V(F_{13})$, $1 \leq i \leq n$. This function is a proper coloring, hence $\chi_4(F_{13}) \leq 7$.

To show $\chi_4(F_{13}) \geq 7$, we use Theorem 3 together with Theorem 9 and Theorem 11. The vertex $u$ is adjacent to the vertices in $P_{12}$; therefore by 9, $u$ is adjacent to three vertices colored 4 and three vertices colored 1. There is a subgroup of $F_{13}$ where $u$ is adjacent to three vertices colored with fours and one that is colored with a one, hence $*S_5$ is a subgraph of $F_{13}$. Therefore applying Theorem 3 and Theorem 11, $7 = \chi_4(*S_5) \leq \chi_4(F_{13})$. \quad \therefore 7 \leq \chi_4(F_{13}) \leq 7.
Hence $\chi_4(F_{13}) = 7$.

Let $F_n$ be the graph defined by $P_{n-1}$ joined with a single vertex labeled $v_n$. Every vertex in $P_{n-1}$ is adjacent to $v_n$. Label the vertices on the $P_{n-1}$ part of $F_n$ as $v_1, v_2, ..., v_{n-1}$ where $v_i$ adjacent to $v_{i+1}, 1 \leq i \leq n-2$, as illustrated in Figure 4.10.

![Figure 4.10: Fan Graph of Order n.](image)

**Theorem 16.** $\chi_4(F_n) = 7, (n \geq 14)$.

**Proof.** Consider the function $c : V(F_n) \rightarrow \mathbb{N}, n \geq 14$, defined by,

$$c(v_i) = \begin{cases} 
1 & \text{if } i \text{ is odd and } i \neq n; \\
4 & \text{if } i \text{ is even and } i \neq n; \\
7 & \text{if } i = n.
\end{cases}$$

This function $c$ is a proper coloring of $F_n$, hence $\chi_4(F_n) \leq 7$. From Theorem 17 and Theorem 3, we have $7 = \chi_4(F_{13}) \leq \chi_4(F_n)$. Therefore $7 \leq \chi_4(F_n) \leq 7$, hence $\chi_4(F_n) = 7$. □

**Bi-Fan with $n$ vertices**

**Case $n=10$.**

Label the vertices on the $P_8$ part of $Bi-F_{10}$ as $v_1, v_2, ..., v_8$ with $v_i$ adjacent to $v_{i+1}, 1 \leq i \leq 7$. Let vertices $v_9$ and $v_{10}$ both be adjacent to the eight vertices in $P_8$, but not adjacent to each other as illustrated in Figure 4.11.
Figure 4.11: Bi-Fan of Order 10.

Theorem 17. $\chi_4(Bi-F_{10}) = 8$.

Proof. To show $\chi_4(Bi-F_{10}) \leq 8$, consider the function $c : V(Bi-F_{10}) \rightarrow \mathbb{N}$ defined as

$$c(v_i) = \begin{cases} 
1 & \text{if } i = 1, 3, 5, 7; \\
4 & \text{if } i = 2, 4, 6, 8; \\
7 & \text{if } i = 9; \\
8 & \text{if } i = 10.
\end{cases}$$

The function $c$ is a proper coloring of $Bi-F_{10}$. Hence $\chi_4(Bi-F_{10}) \leq 8$.

To show that $\chi_4(Bi-F_{10}) \geq 8$, we make use of the fact that $Bi-F_{10}$ contains the subgraph $P_5$ as well as the subgraph $*C_4$ defined previously. From Theorem 3 for $n = 2$, in any 4LC of $P_5$ at least 2 vertices are colored 4 and 2 vertices are colored 1. Applying Theorem 3 and Theorem 13 yield $8 = \chi_4(*C_4) \leq \chi_4(Bi-F_{10})$. Therefore $8 \leq \chi_4(Bi-F_{10}) \leq 8$. Hence $\chi_4(Bi-F_{10}) = 8$. □

Let $Bi-F_n$ be the graph defined by $P_{n-2}$ joined with two vertices labeled $v_{n-1}$ and $v_n$. Every vertex in $P_{n-2}$ is adjacent to $v_{n-1}$. Label the vertices on the $P_{n-2}$ part of $Bi-F_n$ as $v_1, v_2, ..., v_{n-2}$ where $v_i$ adjacent to $v_{i+1}, 1 \leq i \leq n - 3$. Let vertices $v_{n-1}$ and $v_n$ both be adjacent to the $n - 2$ vertices in $P_{n-2}$, but not adjacent to each other as illustrated in Figure 4.12.
Theorem 18. $\chi_4(Bi-F_n) = 8$, for $n \geq 10$.

Proof. To show that $\chi_4(Bi-F_n) \leq 8$, $n \geq 10$, consider the function $c : V(Bi-F_n) \to \mathbb{N}$ defined by,

$$c(v_i) = \begin{cases} 
1 & \text{if } i \text{ is odd and } i \neq n, n-1; \\
4 & \text{if } i \text{ is even and } i \neq n, n-1; \\
7 & \text{if } i = n-1; \\
8 & \text{if } i = n.
\end{cases}$$

The function $c$ is a proper coloring of $Bi-F_n$. Hence $\chi_4(Bi-F_n) \leq 8$.

To show that $\chi_4(Bi-F_n) \geq 8$, $n \geq 10$, we use the fact that $Bi-F_n$, $n \geq 10$, contains the subgraph $Bi-F_{10}$. Applying Theorem 3, and Theorem 19 yields $8 = \chi_4(Bi-F_{10}) \leq \chi_4(Bi-F_n)$.

Therefore $8 \leq \chi_4(Bi-F_n) \leq 8$. Hence $\chi_4(Bi-F_{10}) = 8$. $\square$

Wheels

Wheel with $n$ vertices

Let a Wheel with $n$ vertices be $C_{n-1}$ coupled with a single vertex that is adjacent to every vertex in the cycle graph. In terms of vertices; $V(W_n) = (V(C_{n-1}), u)$. In terms of the number of edges; $|E(W_n)| = 2n$

Label the vertices on the $C_{n-1}$ part of $W_n$ as $v_1, v_2, ..., v_{n-1}$ where $v_{n-1}$ is adjacent to $v_1$, and $u$ adjacent to $v_{i+1}$, $1 \leq i \leq n-2$. Label the the lone vertex that is adjacent to all other
Theorem 19. $\chi_4(W_n) = 7, (n \geq 14)$

Proof. To show that $\chi_4(W_n) \leq 7, n \geq 14$, consider the function $c : V(W_n) \rightarrow \mathbb{N}$ for $n \geq 14$, defined by,

$$c(v_i) = \begin{cases} 
1 & \text{if } i \text{ is odd and } i \neq n; \\
3 & \text{if } i \text{ is odd and } i = n - 1; \\
4 & \text{if } i \text{ is even and } i \neq n; \\
7 & \text{if } i = n.
\end{cases}$$

This function $c$ is a proper coloring of $W_n$. Hence $\chi_4(W_n) \leq 7$.

To show $\chi_4(W_n) \geq 7, n \geq 14$, we use the fact that $F_{12}$ is a subgraph of $W_n$.

From Theorem 3 and Theorem 17 we have the following; since $F_{13}$ is a subgraph of $W_n$, then $7 = \chi_4(F_{13}) \leq \chi_4(W_n)$, with $n \geq 14$. Therefore $7 \leq \chi_4(W_n) \leq 7$. Hence $\chi_4(W_n) = 7$. 

$\square$

Bi-Wheels ($Bi-W_n$)

Bi-Wheels are composed of a Wheel joined with another hub vertex. $V(G) = V(C_{n-2}, u, v)$.

Label the vertices on the $C_{n-2}$ part of $Bi-W_n$ as $v_1, v_2, \ldots, v_{n-2}$ with $v_i$ adjacent to $v_{i+1}$, $1 \leq i \leq n - 2$. Let vertices $v_n$ and $v_{n-1}$ both be adjacent to the $n - 2$ vertices in $C_{n-2}$, but
not adjacent to each other as illustrated in Figure 4.14.

**Theorem 20.** \( \chi_4(Bi-W_n) = 8, \) for \( n \geq 11. \)

**Proof.** To show that \( \chi_4(Bi-W_n) \leq 8, n \geq 11, \) Consider the function \( c : V(Bi-W_n) \to \mathbb{N}, \) \( n \geq 10 \) defined as

\[
c(v_i) = \begin{cases} 
1 & \text{if } i \text{ is odd and } i \neq n, n-1; \\
3 & \text{if } i \text{ is odd and } i = n-2; \\
4 & \text{if } i \text{ is even and } i \neq n, n-1; \\
7 & \text{if } i = n-1; \\
8 & \text{if } i = n. 
\end{cases}
\]

The function \( c \) is a proper coloring, hence \( \chi_4(Bi-W_n) \leq 8. \)

To show that \( \chi_4(Bi-W_n) \geq 8, n \geq 11, \) we use Theorem 3 in conjunction with Theorem 20, which produces \( 8 = \chi_4(Bi-Fan) \leq \chi_4(Bi-W_n). \) Therefore \( 8 \leq \chi_4(Bi-Py_n) \leq 8. \) Hence \( \chi_4(Bi-Py_n) = 8. \) \( \square \)
CHAPTER 5

CONCLUSION

In conclusion I reiterate the fact that graphs shed light on various math problems. Vertices are representative of cities on a route map, atoms in a chemical compound, or microprocessors in a computer. The idea for four local chromatic colorings were started with the Four Color Mapping Theorem. All vertex labeling and graph colorings stem from this problem. Chartrand extended the definition of proper colorings and defined k-local colorings. In this paper we have determined the 4-Local Chromatic Number for some classes of well known graphs.
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